Proceedings of the Edinburgh Mathematical Society (2017) **60**, 937–945
DOI:10.1017/S0013091516000638

A GENERALIZATION OF THE EISENSTEIN–DUMAS–SCHÖNEMANN IRREDUCIBILITY CRITERION

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(Received 22 August 2015)

Dedicated to Professor R. Sridharan on his 80th birthday

Abstract In 2013, Weintraub gave a generalization of the classical Eisenstein irreducibility criterion in an attempt to correct a false claim made by Eisenstein. Using a different approach, we prove Weintraub's result with a weaker hypothesis in a more general setup that leads to an extension of the generalized Schönemann irreducibility criterion for polynomials with coefficients in arbitrary valued fields.

Keywords: Eisenstein criterion; irreducibility; valued fields

2010 *Mathematics subject classification:* Primary 12E05; 11R09; 12J10

1. Introduction

The classical Schönemann irreducibility criterion [**13**] proved by Schönemann in 1846 states that if $g(x)$ is a monic polynomial with coefficients from the ring $\mathbb{Z}$ of integers that is irreducible modulo a prime number p, and if $F(x)$ belonging to $\mathbb{Z}[x]$ is a polynomial of the form $F(x) = g(x)^n + pM(x)$, where $M(x)$ belonging to $\mathbb{Z}[x]$ has degree less than that of $F(x)$ and is relatively prime to $g(x)$ modulo p, then $F(x)$ is irreducible over the field $\mathbb{Q}$ of rational numbers. It can be easily verified that a polynomial $F(x)$ belonging to $\mathbb{Z}[x]$ satisfies the hypothesis of the Schönemann irreducibility criterion if and only if the $g(x)$-expansion of $F(x)$ obtained on dividing it by successive powers of $g(x)$ given by

$$F(x) = \sum_{i=0}^{n} F_i(x)g(x)^i, \quad \deg F_i(x) < \deg g(x),\ F_n(x) \neq 0,$$

satisfies the following three conditions:

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(i) $F_n(x) = 1$,

(ii) p divides the content of each polynomial $F_i(x)$ for $0 \leqslant i \leqslant n-1$,

(iii) p^2 does not divide the content of $F_0(x)$.

This reformulation of the hypothesis shows that the well-known Eisenstein irreducibility criterion [**7**] is a particular case of the Schönemann irreducibility criterion by taking $g(x) = x$ besides leading to its generalization (see [**11**, Theorem 1.1], [**4**]). One of the early generalizations of the Eisenstein irreducibility criterion was by Dumas [**6**], which states that if $F(x) = a_n x^n + \cdots + a_0$ is a polynomial with coefficients in $\mathbb{Z}$, and if there exists a prime p whose exact power $v_p(a_i)$ dividing a_i (where $v_p(a_i) = \infty$ if $a_i = 0$) satisfies $v_p(a_n) = 0$, $v_p(a_i)/(n-i) \geqslant v_p(a_0)/n$ for $0 \leqslant i \leqslant n-1$, and $v_p(a_0)$, n are coprime, then $F(x)$ is irreducible over $\mathbb{Q}$. Over the years, these criteria have witnessed many variations and generalizations using prime ideals, valuations and Newton polygons (see [**12**], [**11**, Corollary 1.2], [**2**, Proposition 3.1], [**3**–**5**, **9**]). In 2013, Weintraub [**14**] gave the following simple but interesting generalization of the Eisenstein irreducibility criterion in an attempt to correct a false claim made by Eisenstein himself.

Theorem 1.1 (Weintraub [14]). *Let $F(x) = a_n x^n + \cdots + a_0 \in \mathbb{Z}[x]$ be a polynomial and suppose that there is a prime p such that p does not divide a_n, p divides a_i for $i = 0, 1, \ldots, n-1$, and, for some k with $0 \leqslant k \leqslant n-1$, p^2 does not divide a_k. Let k_0 be the smallest such value of k. If $F(x) = G(x)H(x)$ is a factorization in $\mathbb{Z}[x]$, then $\min(\deg G(x), \deg H(x)) \leqslant k_0$. In particular, if $k_0 = 0$ or if $k_0 = 1$ and $F(x)$ does not have a root in $\mathbb{Q}$, then $F(x)$ is irreducible over $\mathbb{Q}$.*

The above result is significant only when $k_0 < n/2$. Observe that the integer $k_0 < n/2$ in the above theorem is characterized by the property that k_0 is the smallest non-negative integer for which

$$\min_{0 \leqslant i \leqslant n-1} \left\{ \frac{v_p(a_i)}{n-i} \right\} = \frac{1}{n-k_0},$$

because if $0 \leqslant i < k_0$, then

$$\frac{v_p(a_i)}{n-i} \geqslant \frac{2}{n-i} > \frac{1}{n-k_0}.$$

In this paper we first prove Theorem 1.1 assuming a weaker hypothesis, namely, if k_0 is the smallest non-negative integer for which

$$\min_{0 \leqslant i \leqslant n-1} \left\{ \frac{v_p(a_i)}{n-i} \right\} = \frac{v_p(a_{k_0})}{n-k_0},$$

then the assumption that $v_p(a_{k_0}) = 1$ is replaced by the assumption that $v_p(a_{k_0})$ is coprime to $n - k_0$. Our result also extends the Eisenstein–Dumas irreducibility criterion because in the hypothesis of the latter criterion,

$$\min_{0 \leqslant i \leqslant n-1} \left\{ \frac{v_p(a_i)}{n-i} \right\} = \frac{v_p(a_0)}{n}$$

with $v_p(a_0)$ coprime to n. In fact, our results are proved for polynomials with coefficients in arbitrary valued fields. Using the theory of prolongations of valuations, the analogous extension of the Schönemann irreducibility criterion is proved in Theorem 1.5. As an immediate consequence, we obtain Corollary 1.6, which is the generalized Schönemann irreducibility criterion originally given by Brown [**4**].

Theorem 1.2. *Let v be a Krull valuation of arbitrary rank of a field K with value group G_v. Let $F(x) = a_n x^n + \cdots + a_0$ be a polynomial having coefficients in K with $v(a_n) = 0$. If $k \geqslant 0$ is the smallest non-negative integer for which*

$$\min_{0 \leqslant i \leqslant n-1} \left\{ \frac{v(a_i)}{n-i} \;\middle|\; 0 \leqslant i \leqslant n-1 \right\}$$

is $v(a_k)/(n-k)$ and $v(a_k) \notin dG_v$ for any number $d > 1$ dividing $n-k$, then for any factorization $F(x) = G(x)H(x)$ of $F(x)$ over K, one has $\min\{\deg G(x), \deg H(x)\} \leqslant k$.

Note that when $G_v = \mathbb{Z}$, the condition $v(a_k) \notin d\mathbb{Z}$ for any number $d > 1$ dividing $n-k$ is equivalent to saying that $v(a_k)$ and $n-k$ are coprime. So the following corollary extends Theorem 1.1 as well as the Eisenstein–Dumas irreducibility criterion.

Corollary 1.3. *Let v be a valuation of a field K with value group $\mathbb{Z}$. Let $F(x) = a_n x^n + \cdots + a_0$ be a polynomial having coefficients in K with $v(a_n) = 0$. If $k \geqslant 0$ is the smallest non-negative integer for which*

$$\min_{0 \leqslant i \leqslant n-1} \left\{ \frac{v(a_i)}{n-i} \;\middle|\; 0 \leqslant i \leqslant n-1 \right\} = \frac{v(a_k)}{n-k}$$

and $v(a_k)$, $n-k$ are coprime, then for any factorization $F(x) = G(x)H(x)$ of $F(x)$ over K, one has $\min\{\deg G(x), \deg H(x)\} \leqslant k$.

The following example with $k = 1$ is a quick application of Corollary 1.3.

Example 1.4. Let $F(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial of even degree $n \geqslant 4$ with coefficients from the ring $\mathbb{Z}$ of integers. Suppose that there exists a prime number p such that $p \nmid a_n$, $p^2 | a_i$ for $1 \leqslant i \leqslant n-1$, $p^3 \nmid a_1$ and $p^3 | a_0$. Then either $F(x)$ has a linear factor over $\mathbb{Q}$ or it is irreducible over $\mathbb{Q}$.

In what follows, for a valuation v of a field K we shall denote by v^x its Gaussian prolongation to a simple transcendental extension $K(x)$ of K defined on $K[x]$ by

$$v^x\left(\sum_i a_i x^i\right) = \min_i\{v(a_i)\}, \quad a_i \in K. \tag{1.1}$$

Theorem 1.5. *Let v be a Krull valuation of arbitrary rank of a field K with value group G_v and valuation ring R_v having maximal ideal M_v. Let $g(x) \in R_v[x]$ be a monic polynomial of degree m that is irreducible modulo M_v. Let $F(x)$ belonging to $R_v[x]$ be a polynomial having $g(x)$-expansion*

$$\sum_{i=0}^{n} F_i(x) g(x)^i \quad \text{with } F_n(x) = 1,\; F_i(x) \neq 0$$

https://doi.org/10.1017/S0013091516000638 Published online by Cambridge University Press

for some $i < n$. Let k be the smallest non-negative integer for which

$$\min_{0 \leqslant i \leqslant n-1} \left\{ \frac{v^x(F_i(x))}{n-i} \;\middle|\; 0 \leqslant i \leqslant n-1 \right\} = \frac{v^x(F_k(x))}{n-k} > 0.$$

Suppose that $v^x(F_k(x)) \notin dG_v$ for any number $d > 1$ dividing $n-k$. Then for any factorization $G(x)H(x)$ of $F(x)$ over K, $\min\{\deg G(x), \deg H(x)\} \leqslant km$.

Corollary 1.6 (generalized Schönemann irreducibility criterion). *Let v be a Krull valuation of a field K with value group G_v and valuation ring R_v having maximal ideal M_v. Let $g(x) \in R_v[x]$ be a monic polynomial of degree m that is irreducible modulo M_v. Assume that $F(x)$ belonging to $R_v[x]$ is a polynomial whose $g(x)$-expansion $\sum_{i=0}^{n} F_i(x)g(x)^i$ satisfies*

(i) *$F_n(x) = 1$, $F_0(x) \neq 0$;*

(ii) *$\dfrac{v^x(F_i(x))}{n-i} \geqslant \dfrac{v^x(F_0(x))}{n} > 0$ for $0 \leqslant i \leqslant n-1$; and*

(iii) *$v^x(F_0(x)) \notin dG_v$ for any number $d > 1$ dividing n.*

Then $F(x)$ is irreducible over K.

In Example 3.4 we give examples to show that the hypothesis $v^x(F_k(x))/(n-k) > 0$, as well as $F_n(x) = 1$, cannot be dispensed with.

It may be pointed out that Theorem 1.2 is not a particular case of Theorem 1.5 because the former is true for polynomials with coefficients that are not necessarily from the valuation ring of (K, v); furthermore, for the hypothesis of Theorem 1.5, we assume that $v^x(F_k(x))/(n-k) > 0$, whereas in Theorem 1.2 $v(a_k)/(n-k)$ may be positive, negative or zero.

2. Proof of Theorem 1.2

Set $\lambda = v(a_k)/(n-k)$; then $(n-k)\lambda \in G_v$. Our claim is that the hypothesis $v(a_k) \notin dG_v$ for any number $d > 1$ dividing $n-k$ implies that

$$r\lambda \notin G_v \quad \text{for any positive integer } r < n-k. \tag{2.1}$$

Otherwise, for some number $r < n-k$, $r\lambda \in G_v$ and $(n-k)\lambda \in G_v$, so $d_r\lambda \in G_v$, where d_r is the greatest common divisor of r and $n-k$. Set $d = (n-k)/d_r$ and observe that $d > 1$. Then $v(a_k) = (n-k)\lambda = ((n-k)/d_r)d_r\lambda = d(d_r\lambda)$ belongs to dG_v, contradicting the hypothesis.

Let w denote the mapping on $K[x]$ defined by

$$w\left(\sum_i c_i x^i\right) = \min_i\{v(c_i) + i\lambda\}, \quad c_i \in K.$$

As in [**8**, Theorem 2.2.1], it can be easily shown that w is a valuation on $K[x]$. Since

$$\lambda = \min_{0 \leqslant i \leqslant n-1} \left\{\frac{v(a_i)}{n-i}\right\},$$

it follows that $w(F(x)) = v(a_k) + k\lambda = n\lambda$. If $0 \leqslant i < k$, then

$$\frac{v(a_i)}{n-i} > \frac{v(a_k)}{n-k}$$

by choice of k; consequently, $v(a_i) + i\lambda > v(a_k) + k\lambda$. So k is the smallest index at which $w(F(x))$ is attained.

Suppose to the contrary that $F(x) = G_1(x)G_2(x)$ is a factorization of $F(x)$ into a product of polynomials over K with $\min\{\deg G_1(x), \deg G_2(x)\} > k$. Write

$$G_1(x) = \sum_{i=0}^{d_1} b_i x^i, \qquad G_2(x) = \sum_{j=0}^{d_2} c_j x^j.$$

Since $v(a_n) = 0$, we may assume that $v(b_{d_1}) = v(c_{d_2}) = 0$. Let k_i be the smallest index at which $w(G_i(x))$ is attained for $i = 1, 2$. Our claim is that

$$k_1 + k_2 = k. \tag{2.2}$$

By the choice of k_1 and k_2, we have

$$v(b_i) + i\lambda \geqslant w(G_1(x)), \qquad v(c_j) + j\lambda \geqslant w(G_2(x)) \tag{2.3}$$

for $0 \leqslant i \leqslant d_1$, $0 \leqslant j \leqslant d_2$ with strict inequality if $i < k_1$ or $j < k_2$. Keeping in mind (2.3), it can be easily checked that $k_1 + k_2$ is the smallest index at which $w(F(x)) = w(G_1(x)) + w(G_2(x))$ is attained, and hence (2.2) follows. By virtue of (2.2) and the assumption $\min\{d_1, d_2\} > k$, we see that $k_i \leqslant k < d_i$; consequently,

$$d_i - k_i > 0 \text{ for } i = 1, 2 \quad \text{and} \quad n - k = d_1 - k_1 + d_2 - k_2. \tag{2.4}$$

Keeping in mind (2.4) and (2.1), we will arrive at a contradiction and the theorem will be proved once it is shown that $(d_1 - k_1)\lambda \in G_v$. Note that $w(G_1(x)) = d_1\lambda$, for otherwise

$$n\lambda = w(F(x)) = w(G_1(x)) + w(G_2(x)) < d_1\lambda + d_2\lambda = n\lambda.$$

Thus, $d_1\lambda = w(G_1(x)) = v(b_{k_1}) + k_1\lambda$, which implies that $(d_1 - k_1)\lambda \in G_v$, as desired.

3. Proof of Theorem 1.5

We first introduce some notation and definitions.

Let v be a Krull valuation of a field K. We fix a prolongation $\tilde{v}$ of v to an algebraic closure $\tilde{K}$ of K. The image of an element ξ belonging to the valuation ring $R_{\tilde{v}}$ of $\tilde{v}$ under the canonical homomorphism from $R_{\tilde{v}}$ onto its residue field $R_{\tilde{v}}/M_{\tilde{v}}$ will be denoted by $\bar{\xi}$ and will be referred to as the $\tilde{v}$-residue of ξ. For a polynomial $f(x) \in R_v[x]$, $\bar{f}(x)$ will stand for the polynomial over R_v/M_v obtained by replacing each coefficient of $f(x)$ by its v-residue.

Definition 3.1. Let (K, v) be a Henselian-valued field of arbitrary rank and let $\tilde{v}$ be the unique prolongation of v to an algebraic closure $\tilde{K}$ of K with value group $G_{\tilde{v}}$. A pair (α, δ) belonging to $\tilde{K} \times G_{\tilde{v}}$ will be called a minimal pair (more precisely a (K, v)-minimal pair) if whenever β belongs to $\tilde{K}$ with $[K(\beta) : K] < [K(\alpha) : K]$, then $\tilde{v}(\alpha - \beta) < \delta$.

Example. If $g(x)$ belonging to $R_v[x]$ is a monic polynomial of degree $m \geqslant 1$ with $\overline{g}(x)$ irreducible over the residue field of v and $\alpha \in \tilde{K}$ is a root of $g(x)$, then (α, δ) is a (K, v)-minimal pair for each positive δ in $G_{\tilde{v}}$, because whenever β belongs to $\tilde{K}$ with $[K(\beta) : K] < m$, then $\tilde{v}(\alpha - \beta) \leqslant 0$, for otherwise $\bar{\alpha} = \bar{\beta}$, which, in view of the fundamental inequality [**8**, Theorem 3.3.4], would imply that $[K(\beta) : K] \geqslant [\bar{K}(\bar{\beta}) : \bar{K}] = [\bar{K}(\bar{\alpha}) : \bar{K}] = m$, leading to a contradiction.

Definition 3.2. Let (K, v), $(\tilde{K}, \tilde{v})$ be as in the above definition and let (α, δ) be a (K, v)-minimal pair. The valuation $\tilde{w}_{\alpha,\delta}$ of a simple transcendental extension $\tilde{K}(x)$ of $\tilde{K}$ defined by

$$\tilde{w}_{\alpha,\delta}\left(\sum_i c_i(x - \alpha)^i\right) = \min_i \{\tilde{v}(c_i) + i\delta\}, \quad c_i \in \tilde{K}, \tag{3.1}$$

will be referred to as the valuation with respect to the minimal pair (α, δ). The restriction of $\tilde{w}_{\alpha,\delta}$ to $K(x)$ will be denoted by $w_{\alpha,\delta}$.

Remark. With (α, δ) as above, if $g(x)$ is the minimal polynomial of α over K, then it is well known (see [**1**, Theorem 2.1]) that for any polynomial $F(x)$ belonging to $K[x]$ with $g(x)$-expansion $\sum_i F_i(x)g(x)^i$, $\deg F_i(x) < \deg g(x)$, one has

$$w_{\alpha,\delta}(F(x)) = \min_i \{\tilde{v}(F_i(\alpha)) + i w_{\alpha,\delta}(g(x))\}. \tag{3.2}$$

Notation 3.3. Let (α, δ), $w_{\alpha,\delta}$ and $g(x)$ be as in the above remark. For a polynomial $F(x)$ belonging to $K[x]$ with $g(x)$-expansion $\sum_{i=0}^n F_i(x)g(x)^i$, we shall respectively denote by $I_{\alpha,\delta}(F)$ and $S_{\alpha,\delta}(F)$ the minimum and the maximum integers belonging to the set

$$\{0 \leqslant i \leqslant n \mid w_{\alpha,\delta}(F(x)) = \tilde{v}(F_i(\alpha)) + i w_{\alpha,\delta}(g(x))\}.$$

Proof of Theorem 1.5. Since the value group and the residue field remain the same on replacing (K, v) by its Henselization, we may assume that (K, v) is Henselian. Let $\tilde{v}$ denote the unique prolongation of v to the algebraic closure $\tilde{K}$ of K. Set

$$\lambda = \frac{v^x(F_k(x))}{n - k}.$$

Then $v^x(F_k(x)) + k\lambda = n\lambda$. Let α be a root of $g(x)$ in $\tilde{K}$. Write $g(x) = c_m(x - \alpha)^m + \cdots + c_1(x - \alpha)$, $c_m = 1$. Define a positive element δ of the divisible closure $G_{\tilde{v}}$ of G_v by

$$\delta = \max_{1 \leqslant i \leqslant m} \left\{\frac{\lambda - \tilde{v}(c_i)}{i}\right\}.$$

Let $\tilde{w}_{\alpha,\delta}$ denote the valuation of $\tilde{K}(x)$ defined by (3.1). Then, by the choice of λ, we have

$$\tilde{w}_{\alpha,\delta}(g(x)) = \min_i \{\tilde{v}(c_i) + i\delta\} = \lambda. \tag{3.3}$$

We first show that, for any polynomial $A(x) = \sum_{i=0}^{m-1} a_i x^i$ belonging to $K[x]$ having degree less than m, one has

$$\tilde{v}(A(\alpha)) = v^x(A(x)). \tag{3.4}$$

Clearly, (3.4) needs to be verified when $m > 1$. Keeping in view that $\overline{g}(x)$ is irreducible over R_v/M_v of degree $m > 1$, it follows that $\tilde{v}(\alpha) = 0$. If (3.4) were false, then the triangle inequality would imply that $\tilde{v}(A(\alpha)) > \min_i\{\tilde{v}(a_i\alpha^i)\} = v(a_j)$ (say), which yields

$$\sum_{i=0}^{m-1} \left(\overline{\frac{a_i}{a_j}}\right)(\bar{\alpha})^i = \overline{0},$$

contradicting the fact that the minimal polynomial of $\bar{\alpha}$ over R_v/M_v is of degree m. Hence, (3.4) is proved. For any polynomial $T(x) \in K[x]$ with the $g(x)$-expansion $\sum_i T_i(x)g(x)^i$, it follows from (3.2)–(3.4) that

$$w_{\alpha,\delta}(T(x)) = \min_i \{\tilde{v}(T_i(\alpha)) + i\lambda\} = \min_i \{v^x(T_i(x)) + i\lambda\}. \tag{3.5}$$

Let $I_{\alpha,\delta}(F)$ and $S_{\alpha,\delta}(F)$ be as in Notation 3.3. We now show that

$$I_{\alpha,\delta}(F) = k, \qquad S_{\alpha,\delta}(F) = n. \tag{3.6}$$

Recall that $F_n(x) = 1$, so $v^x(F_n(x)) = 0$. Furthermore, keeping in view the hypothesis

$$\lambda = \frac{v^x(F_k(x))}{n-k} = \min_{0\leqslant i\leqslant n-1}\left\{\frac{v^x(F_i(x))}{n-i}\right\}$$

and using formula (3.5), it can be easily checked that

$$w_{\alpha,\delta}(F(x)) = \min_{0\leqslant i\leqslant n}\{v^x(F_i(x)) + i\lambda\} = v^x(F_k(x)) + k\lambda = v^x(F_n(x)) + n\lambda = n\lambda.$$

If $0 \leqslant i < k$, then

$$\frac{v^x(F_i(x))}{n-i} > \frac{v^x(F_k(x))}{n-k}$$

by choice of k, and hence $v^x(F_i(x)) + i\lambda > v^x(F_k(x)) + k\lambda = w_{\alpha,\delta}(F(x))$. Thus, (3.6) is proved.

Suppose to the contrary that $F(x) = G_1(x)G_2(x)$ is a factorization of $F(x)$ into a product of polynomials over K with

$$\min\{\deg G_1(x), \deg G_2(x)\} > km. \tag{3.7}$$

Since $F(x)$ is monic, we may assume that $G_1(x)$ and $G_2(x)$ are monic. Let

$$G_1(x) = \sum_{i=0}^{d_1} B_i(x)g(x)^i, \qquad G_2(x) = \sum_{j=0}^{d_2} C_j(x)g(x)^j$$

with $B_{d_1}(x)C_{d_2}(x) \neq 0$ be the $g(x)$-expansions of $G_1(x)$, $G_2(x)$. Denote $I_{\alpha,\delta}(G_i)$ by k_i for $i = 1, 2$. Then, by [**10**, Lemma 2.1], $I_{\alpha,\delta}(F) = I_{\alpha,\delta}(G_1) + I_{\alpha,\delta}(G_2)$. Therefore, in view of (3.6), we have

$$k = k_1 + k_2. \tag{3.8}$$

Again using [**10**, Lemma 2.1] and (3.6), we see that

$$n = S_{\alpha,\delta}(F) = S_{\alpha,\delta}(G_1) + S_{\alpha,\delta}(G_2) \leqslant d_1 + d_2 \leqslant n,$$

and hence

$$n = d_1 + d_2. \tag{3.9}$$

Recall that by hypothesis, $\deg F(x) = mn$. Therefore, in view of (3.9), we conclude that $\deg G_1(x) = d_1 m$, $\deg G_2(x) = d_2 m$. It now follows from (3.7)–(3.9) that

$$k_i \leqslant k < d_i, \qquad n - k = d_1 - k_1 + d_2 - k_2 \tag{3.10}$$

for $i = 1, 2$. Keeping in mind (3.10), we will arrive at a contradiction and the theorem will be proved once we show that $(d_1 - k_1)\lambda \in G_v$, because $r\lambda \notin G_v$ for any positive integer $r < n - k$ in view of the hypothesis and (2.1). Using (3.5) and the fact that $G_1(x)$ is monic, we see that $w_{\alpha,\delta}(G_1(x)) = v^x(B_{k_1}(x)) + k_1\lambda \leqslant d_1\lambda$; the last inequality will be equality, for otherwise

$$n\lambda = w_{\alpha,\delta}(F(x)) = w_{\alpha,\delta}(G_1(x)) + w_{\alpha,\delta}(G_2(x)) < d_1\lambda + d_2\lambda \leqslant n\lambda.$$

Thus, $d_1\lambda = w_{\alpha,\delta}(G_1(x)) = v^x(B_{k_1}(x)) + k_1\lambda$, which proves that $(d_1 - k_1)\lambda \in G_v$ and hence the theorem. □

The following example shows that each of conditions $v^x(F_k(x))/(n - k) > 0$ and $F_n(x) = 1$ are necessary for the above theorem.

Example 3.4. Take $K = \mathbb{Q}$ as the field of rational numbers with the 3-adic valuation v_3 defined by $v_3(3) = 1$ and $g(x) = x^2 + 1$. Consider $F(x) = x^2 - 1 = g(x) + F_0(x)$ and $F^*(x) = x(x^2+1)+3x = xg(x)+F_0^*(x)$. Here, $F_0(x) = 2$, $v_3^x(F_0(x)) = 0$ and $F_0^*(x) = 3x$, $v_3^x(F_0^*(x)) = 1$. With notation as in Theorem 1.5, $k = 0$ for both $F(x)$ and $F^*(x)$, but Theorem 1.5 holds neither for $F(x)$ nor for $F^*(x)$ as both are reducible over $\mathbb{Q}$.

Acknowledgements. Financial support from the National Board for Higher Mathematics, Mumbai and from the CSIR (Grant 09/135(0683)/2013-EMR-I) is gratefully acknowledged.

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