Bull. Aust. Math. Soc. **104** (2021), 45–58
doi:10.1017/S0004972720001483

ON GOOD APPROXIMATIONS AND THE BOWEN–SERIES EXPANSION

LUCA MARCHESE

(Received 10 September 2020; accepted 24 November 2020; first published online 25 January 2021)

Abstract

We consider the continued fraction expansion of real numbers under the action of a nonuniform lattice in $\mathrm{PSL}(2,\mathbb{R})$ and prove metric relations between the convergents and a natural geometric notion of good approximations.

2020 *Mathematics subject classification*: primary 11J70; secondary 20H10.

Keywords and phrases: continued fractions, Fuchsian groups.

1. Introduction

Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the *upper half plane* and, for $p/q \in \mathbb{Q}$, let $H_{p/q} \subset \mathbb{H}$ be the circle of diameter $1/q^2$ tangent at p/q. Set $H_\infty = \{z \in \mathbb{H} : \mathrm{Im}(z) > 1\}$ and consider the family $\{H_{p/q} : p/q \in \mathbb{Q} \cup \{\infty\}\}$ of *Ford circles*, which are the orbit of H_∞ under the projective action of the *modular group* $\mathrm{SL}(2,\mathbb{Z})$, that is, the group of 2×2 matrices with coefficients a, b, c, d in $\mathbb{Z}$ (the notation refers to (1.3) below). Any two circles are either disjoint or tangent. Figure 1 shows that for any irrational α there exist infinitely many $p/q \in \mathbb{Q}$ with $\alpha \in \Pi(H_{p/q})$, that is, $|\alpha - p/q| < (1/2)q^{-2}$, where $\Pi(x + iy) := x$.

This defines the sequence of *geometric good approximations* of α as the sequence of p_n/q_n in $\mathbb{Q}$ with $\alpha \in \Pi(H_{p_n/q_n})$. The same sequence arises from the continued fraction expansion $\alpha = a_0 + [a_1, a_2, \ldots]$ of α. Indeed, the *convergents* defined by $p_n/q_n := a_0 + [a_1, \ldots, a_n]$ have the property that

$$|\alpha - p/q| < (1/2)q^{-2} \Rightarrow p/q = p_n/q_n \quad \text{for some } n \geq 1. \tag{1.1}$$

The first $n + 1$ *partial quotients* $a_1, \ldots, a_{n+1}$ approximate α with error given by

$$\frac{1}{2 + a_{n+1}} \leq q_n^2 \cdot |\alpha - p_n/q_n| \leq \frac{1}{a_{n+1}} \quad \text{for any } n \in \mathbb{N}. \tag{1.2}$$

Rosen continued fractions were introduced in [9], in relation to diophantine approximation for *Hecke groups*, giving an extension of (1.2) which was later improved in [7]. Equation (1.1) was extended to Rosen continued fractions in [5] and the sharp constant replacing 1/2 was obtained in [10]. In this note we consider

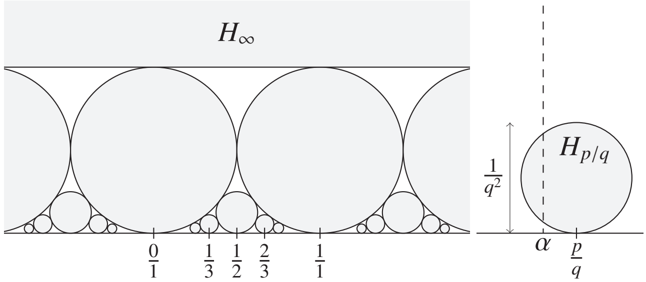

FIGURE 1. Balls $G(H_k)$, $k \in \mathbb{Z}$, tangent to $H_{p/q} = G(H_\infty)$, where $p/q = G \cdot \infty$.

diophantine approximation for a general nonuniform lattice Fuchsian group, in relation to the so-called *Bowen–Series expansion* of real numbers [3]. Our main theorem (Theorem 3.1) provides an extension of (1.1) and (1.2) to this setting. This result is used in [6] to approximate the dimension of sets of *badly approximable points* by the dimension of dynamically defined regular Cantor sets. The study of the higher part of the *Markov and Lagrange spectra* is also a natural application, in the spirit of [1, 2, 11]. Theorem 3.1 applies to many diophantine approximation problems, since it translates diophantine properties into ergodic properties of the Bowen–Series expansion.

Let $\mathrm{SL}(2, \mathbb{C})$ be the group of matrices

$$G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.3}$$

with $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. Any such G acts on points $z \in \mathbb{C} \cup \{\infty\}$ by

$$G \cdot z := \frac{az + b}{cz + d}. \tag{1.4}$$

Denote the coefficients of G in (1.3) by $a = a(G)$, $b = b(G)$, $c = c(G)$ and $d = d(G)$. The subgroup $\mathrm{SL}(2, \mathbb{R})$ of G with coefficients a, b, c, d in $\mathbb{R}$ acts by isometries on $\mathbb{H}$ via (1.4) and inherits a topology from the identification with the set of $(a, b, c, d) \in \mathbb{R}^4$ with $ad - bc = 1$. A *Fuchsian group* is a discrete subgroup $\Gamma < \mathrm{SL}(2, \mathbb{R})$. Referring to [4], we say that Γ is a *lattice* if it has a *Dirichlet region* $\Omega \subset \mathbb{H}$ with finite hyperbolic area. If Ω is not compact, then the lattice Γ is called *nonuniform*. In this case the intersection $\overline{\Omega} \cap \partial\mathbb{H}$ is a finite nonempty set, whose elements are called the vertices *at infinity* of Ω. A point $z \in \mathbb{R} \cup \{\infty\}$ is a parabolic fixed point for Γ if there exists a parabolic element $P \in \Gamma$ with $P(z) = z$. Let $\mathcal{P}_\Gamma$ be the set of parabolic fixed points of Γ. The set $\mathcal{P}_\Gamma$ is the orbit under Γ of the vertices at infinity of Ω and is dense in $\mathbb{R}$. Two points z_1 and z_2 in $\mathcal{P}_\Gamma$ are *equivalent* if $z_2 = G(z_1)$ for some $G \in \Gamma$. Any nonuniform lattice Γ has a finite number $p \geq 1$ of equivalence classes $[z_1], \ldots, [z_p]$ of parabolic fixed points, called the *cusps* of Γ.

Let Γ be a nonuniform lattice with $p \geq 1$ cusps. Fix a list $\mathcal{S} = (A_1, \ldots, A_p)$ of elements $A_k \in \mathrm{SL}(2, \mathbb{R})$ such that the points

$$z_k = A_k \cdot \infty \quad \text{for } k = 1, \ldots, p \tag{1.5}$$

form a complete set $\{z_1, \ldots, z_p\} \subset \mathcal{P}_\Gamma$ of inequivalent parabolic fixed points. A natural choice for $z_1, \ldots, z_p$ is a maximal set of nonequivalent vertices at infinity of a fundamental domain. Any element of $\mathcal{P}_\Gamma$ has the form $G \cdot z_k$ for some $G \in \Gamma$ and $k = 1, \ldots, p$. We have horoballs

$$B_k := A_k(\{z \in \mathbb{H} : \mathrm{Im}(z) > 1\}) \quad \text{with } k = 1, \ldots, p,$$

each B_k being tangent to $\mathbb{R} \cup \{\infty\}$ at z_k. We allow $A_k = \mathrm{Id}$, that is, $z_k = \infty$ and $B_k = H_\infty$. Thus, $G(B_k)$ is a ball tangent to the real line at $G \cdot z_k$ for any $G \in \Gamma$ with $G \cdot z_k \neq \infty$. These balls generalise Ford circles and we measure how their diameter shrinks to zero as G varies in Γ with the *denominator*

$$D(G \cdot z_k) := \begin{cases} 1/\sqrt{\mathrm{Diam}(G(B_k))} & \text{if } G \cdot z_k \neq \infty, \\ 0 & \text{if } G \cdot z_k = \infty. \end{cases}$$

For any $T > 0$ and any $G \in \mathrm{SL}(2, \mathbb{R})$ with $c(G) \neq 0$, using the notation in (1.3),

$$\mathrm{Diam}(G(\{z \in \mathbb{H} : \mathrm{Im}(z) > T\})) = \frac{1}{Tc^2(G)}. \tag{1.6}$$

Hence,

$$D(G \cdot z_k) = |c(GA_k)| \quad \text{for any } G \cdot z_k \in \mathcal{P}_\Gamma. \tag{1.7}$$

In [8], Patterson proved that there exists a constant $M = M(\Gamma, \mathcal{S}) > 0$ such that for any $Q > 0$ big enough and any $\alpha \in \mathbb{R}$ there exist $G \in \Gamma$ and $k \in \{1, \ldots, p\}$ with

$$|\alpha - G \cdot z_k| \leq \frac{M}{D(G \cdot z_k)Q} \quad \text{and} \quad 0 < D(G \cdot z_k) \leq Q.$$

For $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, $\mathcal{S} = \{\mathrm{Id}\}$ and $M = 1$, this is the classical Dirichlet theorem. In general, for any $\alpha \in \mathbb{R}$ we obtain infinitely many $G \cdot z_k \in \mathcal{P}_\Gamma$ with

$$|\alpha - G \cdot z_k| \leq \frac{M}{D^2(G \cdot z_k)}. \tag{1.8}$$

The *Bowen–Series expansion* [3] provides a coding $\alpha = [W_1, W_2, \ldots]$ of a real number α, where the symbols W_r for $r \geq 1$ are *cuspidal words* which belong to a countable alphabet $\mathcal{W}$ (definitions are in Sections 2 and 3). Cuspidal words $W \in \mathcal{W}$, introduced in [1, 2], label a subset of elements $\{G_W : W \in \mathcal{W}\}$ of Γ, which generalise the role played in the theory of classical continued fractions by the matrices

$$\begin{pmatrix} 1 & a_{2k+1} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ a_{2k} & 1 \end{pmatrix} \quad \text{with } a_{2k}, a_{2k+1} \in \mathbb{N}^* \text{ for any } k \in \mathbb{N}.$$

The coding is a continuous bijection $\Sigma \to \mathbb{R}$, where $\Sigma \subset \mathcal{W}^{\mathbb{N}}$ is a subshift with *aperiodic transition matrix* (see [6]). For $r \geq 1$, the first r symbols in the expansion of $\alpha = [W_1, W_2, \ldots]$ define $\zeta_r = \zeta_r(W_1, \ldots, W_r) \in \mathcal{P}_\Gamma$ (see (3.5)). This extends the classical notion of convergents p_n/q_n of α. The main result of this note is Theorem 3.1 in Section 3. We give the following preliminary statement (see also Remark 3.2).

THEOREM 1.1. *Fix* $\alpha = [W_1, W_2, \ldots]$ *which is not an element of* $\mathcal{P}_\Gamma$. *The convergents* $\zeta_r = \zeta_r(W_1, \ldots, W_r)$ *approximate* α *with error given by an analogue of (1.2). Moreover, there exists a constant* $\epsilon_0 > 0$ *such that any* $G \cdot z_k \in \mathcal{P}_\Gamma$ *satisfying (1.8) with* $M = \epsilon_0$ *belongs to the sequence* $(\zeta_r)_{r\geq 1}$.

2. The Bowen–Series expansion

We follow [6, Section 3], which is based on [1, Section 2.4] and [2, Section 2]. The original construction is the *Markov map* in [3], which is *orbit equivalent* to the action of a given finitely generated Fuchsian group of the first kind. In our setting the Markov map corresponds to an *acceleration* of the map in (2.7) below. This section describes the coding by *cuspidal words*. The same description appears in [6], where it is followed by the study of the combinatorial and metric properties of the subshift related to the coding. Consider the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and the map

$$\varphi : \mathbb{H} \to \mathbb{D}, \quad \varphi(z) := \frac{z - i}{z + i}. \tag{2.1}$$

The conjugate of $\mathrm{SL}(2, \mathbb{R})$ under φ is the group $\mathrm{SU}(1, 1)$ of $F \in \mathrm{GL}(2, \mathbb{C})$, where

$$F = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \quad \text{with } |\alpha|^2 - |\beta|^2 = 1. \tag{2.2}$$

Denote by $\alpha = \alpha(F)$ and $\beta = \beta(F)$ the coefficients of F as in (2.2).

2.1. Isometric circles. Consider $F \in \mathrm{SU}(1, 1)$ and $\alpha = \alpha(F)$, $\beta = \beta(F)$ as in (2.2). Assume that $\beta \neq 0$ and let $\omega_F := -\overline{\alpha}/\beta$ be the pole of F. The *isometric circle* I_F of F is the euclidean circle centred at ω_F with radius $\rho(F) := |\beta|^{-1}$, that is,

$$I_F := \{\xi \in \mathbb{C} : |\xi - \omega_F| = |\beta|^{-1}\}.$$

We have $F(I_F) = I_{F^{-1}}$, where $\rho(F) = \rho(F^{-1})$ and $|\omega_{F^{-1}}| = |\omega_F|$ (see [4, Theorem 3.3.2]). Moreover, $I_F \cap \mathbb{D}$ is a geodesic of $\mathbb{D}$ for any $F \in \mathrm{SU}(1, 1)$, by Theorem 3.3.3 in [4]. Denote by U_F the disc in $\mathbb{C}$ with $\partial U_F = I_F$, that is, the interior of I_F.

2.2. Labelled ideal polygon. Let $\Gamma \subset \mathrm{SU}(1, 1)$ be a nonuniform lattice. From [12], there exists a free subgroup $\Gamma_0 < \Gamma$ with finite index $[\Gamma_0 : \Gamma] < +\infty$ (see also [6, Section 2.2]). In particular, referring to (2.2), $\beta(F) \neq 0$ for any $F \in \Gamma_0$, so that the isometric circle I_F and the disc U_F are defined. The origin $0 \in \mathbb{D}$ is not a fixed point of any $F \in \Gamma_0$ and Theorem 3.3.5 in [4] implies that the set

$$\Omega_0 := \overline{\mathbb{D} \setminus \bigcup_{F \in \Gamma_0} U_F} \tag{2.3}$$

is a Dirichlet region for Γ_0. From [4], Ω_0 is an hyperbolic polygon with an even number $2d$ of sides, denoted by the letter s, and with $2d$ vertices, denoted by the letter ξ

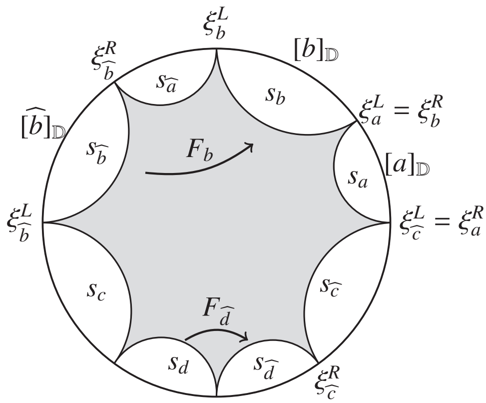

FIGURE 2. Ideal polygon labelled by $\mathcal{A} = \{a, b, c, d, \widehat{a}, \widehat{b}, \widehat{c}, \widehat{d}\}$.

(see also [6, Section 2.4]). All vertices of Ω_0 belong to $\partial\mathbb{D}$ because Γ_0 is free. Any side s is a complete geodesic in $\mathbb{D}$ and for any such s there exists a unique $F \in \Gamma$ such that $F(s)$ is another side of Ω_0 with $F(s) \neq s$. The sides s and $F(s)$ are thus *paired* as shown in Figure 2. The set of pairings generates Γ_0 [4, Theorem 3.5.4]. For a convenient labelling, consider two finite alphabets $\mathcal{A}_0$ and $\widehat{\mathcal{A}_0}$, both with d elements, and a map

$$\iota : \mathcal{A}_0 \cup \widehat{\mathcal{A}_0} \to \mathcal{A}_0 \cup \widehat{\mathcal{A}_0} \quad \text{with } \iota^2 = \mathrm{Id} \text{ and } \iota(\mathcal{A}_0) = \widehat{\mathcal{A}_0},$$

that is, an involution of $\mathcal{A}_0 \cup \widehat{\mathcal{A}_0}$ which exchanges $\mathcal{A}_0$ with $\widehat{\mathcal{A}_0}$. Set $\mathcal{A} := \mathcal{A}_0 \cup \widehat{\mathcal{A}_0}$ and, for any $a \in \mathcal{A}$, denote $\widehat{a} := \iota(a)$.

Label the sides of Ω_0 by the letters in $\mathcal{A}$, so that for any $a \in \mathcal{A}$ the sides s_a and $s_{\widehat{a}}$ are those which are paired by the action of Γ_0. For any pair of sides s_a and $s_{\widehat{a}}$ as above, let F_a be the unique element of Γ_0 such that

$$F_a(s_{\widehat{a}}) = s_a. \tag{2.4}$$

For any $a \in \mathcal{A}$, we have $F_{\widehat{a}} = F_a^{-1}$ and the latter form a set of generators for Γ_0.

In the following, we denote by $\Omega_{\mathbb{D}} := \Omega_0 \subset \mathbb{D}$ the labelled ideal polygon defined above and $\Omega_{\mathbb{H}} := \varphi^{-1}(\Omega_{\mathbb{D}}) \subset \mathbb{H}$ its pre-image under the map in (2.1).

2.3. The boundary map. Parametrise arcs $J \subset \partial\mathbb{D}$ by $t \mapsto e^{-it}$ with $t \in (x, y)$. Set $\inf J := e^{-ix}$ and $\sup J := e^{-iy}$. We say that J is *right open* if $\inf J \in J$ and $\sup J \notin J$. Let $\Gamma_0 < \Gamma$ be a finite-index free subgroup and $\Omega_{\mathbb{D}}$ be an ideal polygon for Γ_0 labelled by $\mathcal{A}$, as in Section 2.2.

For $a \in \mathcal{A}$, let F_a be the map in (2.4). Let I_{F_a} be the isometric circle of F_a and U_{F_a} its interior, as in Section 2.1. Recall that $s_{\widehat{a}} = I_{F_a} \cap \mathbb{D}$ and $s_a = I_{F_{\widehat{a}}} \cap \mathbb{D}$. Let $[a]_{\mathbb{D}}$ be the right open arc of $\partial\mathbb{D}$ cut by the side s_a, that is,

$$[a]_{\mathbb{D}} := U_{F_{\widehat{a}}} \cap \partial\mathbb{D}.$$

https://doi.org/10.1017/S0004972720001483 Published online by Cambridge University Press

Set $\xi_a^L := \inf[a]_{\mathbb{D}}$ and $\xi_a^R := \sup[a]_{\mathbb{D}}$. Figure 2 gives examples of this notation. In order to take account of the cyclic order in $\partial\mathbb{D}$ of the arcs $[a]_{\mathbb{D}}$, fix $a_0 \in \mathcal{A}$ and define a map $o : \mathcal{A} \to \mathbb{Z}/2d\mathbb{Z}$ by setting $o(a_0) := 0$ and

$$o(b) = o(a) + 1 \bmod 2d \quad \text{for } a, b \in \mathcal{A} \text{ with } \xi_a^R = \xi_b^L. \tag{2.5}$$

We have $F_a(I_{F_a}) = I_{F_{\widehat{a}}}$ for any $a \in \mathcal{A}$ and F_a sends the complement of $[\widehat{a}]_{\mathbb{D}}$ to $[a]_{\mathbb{D}}$, that is,

$$F_a(\partial\mathbb{D} \setminus [\widehat{a}]_{\mathbb{D}}) = [a]_{\mathbb{D}}. \tag{2.6}$$

The *Bowen–Series map* is the map $\mathcal{BS} : \partial\mathbb{D} \to \partial\mathbb{D}$ defined by

$$\mathcal{BS}(\xi) := F_a^{-1}(\xi) \quad \text{if and only if } \xi \in [a]_{\mathbb{D}}. \tag{2.7}$$

The *boundary expansion* of a point $\xi \in \partial\mathbb{D}$ is the sequence $(a_k)_{k\in\mathbb{N}}$ of letters $a_k \in \mathcal{A}$ with

$$\mathcal{BS}^k(\xi) \in [a_k]_{\mathbb{D}} \quad \text{for any } k \in \mathbb{N}. \tag{2.8}$$

By (2.6), any such sequence satisfies the so-called *no backtracking condition*

$$a_{k+1} \neq \widehat{a_k} \quad \text{for any } k \in \mathbb{N}. \tag{2.9}$$

A finite word $(a_0, \ldots, a_n)$ satisfying Condition (2.9) corresponds to a *factor* of the map $\mathcal{BS} : \partial\mathbb{D} \to \partial\mathbb{D}$, that is, a finite concatenation $F_{a_n}^{-1} \circ \cdots \circ F_{a_0}^{-1}$ arising from iterations of $\mathcal{BS}$. An *admissible word*, or simply a *word*, is any finite or infinite word in the letters of $\mathcal{A}$ satisfying Condition (2.9). We use the notation

$$F_{a_0,\ldots,a_n} := F_{a_0} \circ \cdots \circ F_{a_n} \in \Gamma_0.$$

Define the right open arc $[a_0, \ldots, a_n]_{\mathbb{D}}$ as the set of $\xi \in \partial\mathbb{D}$ such that $\mathcal{BS}^k(\xi) \in [a_k]_{\mathbb{D}}$ for any $k = 0, \ldots, n$, that is,

$$[a_0, \ldots, a_n]_{\mathbb{D}} := F_{a_0,\ldots,a_{n-1}}[a_n]_{\mathbb{D}} = F_{a_0,\ldots,a_n}(\partial\mathbb{D} \setminus [\widehat{a_n}]_{\mathbb{D}}). \tag{2.10}$$

Two such arcs satisfy $[a_0, \ldots, a_n]_{\mathbb{D}} \subset [b_0, \ldots, b_m]_{\mathbb{D}}$ if and only if $m \geq n$ and $a_k = b_k$ for any $k = 0, \ldots, n$. It is easy to see that $[a_0, \ldots, a_n]_{\mathbb{D}}$ shrinks to a point as $n \to \infty$ (see [6, Lemma 3.1] for a proof). A sequence $(a_k)_{k\in\mathbb{N}}$ satisfying Condition (2.9) corresponds to a point $\xi = [a_0, a_1, \ldots]_{\mathbb{D}}$ in $\partial\mathbb{D}$, where we use the notation

$$[a_0, a_1, \ldots]_{\mathbb{D}} := \bigcap_{n\in\mathbb{N}} [a_0 \ldots, a_n]_{\mathbb{D}}.$$

Conversely, if $(a_k)_{k\in\mathbb{N}}$ is the boundary expansion of $\xi \in \partial\mathbb{D}$, then $\xi = [a_0, a_1, \ldots]_{\mathbb{D}}$. The Bowen–Series map $\mathcal{BS}$ is the shift on the space of admissible infinite words.

2.4. Cuspidal words. Consider the map $o : \mathcal{A} \to \mathbb{Z}/2d\mathbb{Z}$ in (2.5). The definitions in Section 2.3 easily yield the following lemma (see [6, Lemma 3.2] for a proof).

LEMMA 2.1. *Let $(a_0, \ldots, a_n)$ be a word satisfying Condition (2.9) with $n \geq 1$ and $a_0 = a_n$. The map $F_{a_0,\ldots,a_{n-1}}$ is a parabolic element of Γ_0 fixing $\xi_{a_0}^R$ if and only if*

$$o(a_{k+1}) = o(\widehat{a_k}) - 1 \quad \textit{for any } k = 0, \ldots, n-1. \tag{2.11}$$

The map $F_{a_0,\dots,a_{n-1}}$ *is a parabolic element of* Γ_0 *fixing* $\xi^L_{a_0}$ *if and only if*

$$o(a_{k+1}) = o(\widehat{a_k}) + 1 \quad \textit{for any } k = 0, \dots, n-1. \tag{2.12}$$

Let $W = (a_0, \dots, a_n)$ be an admissible word. We say that W is a *cuspidal word* if it is the initial factor of an admissible word $(a_0, \dots, a_m)$ with $m \geq n$ such that $F_{a_0,\dots,a_m}$ is a parabolic element of Γ_0 fixing a vertex of $\Omega_{\mathbb{D}}$.

- If $n \geq 1$ and (2.11) is satisfied, we say that W is a *right cuspidal word*. In this case we define its type by $\varepsilon(W) := R$ and we set $\xi_W := \xi^R_{a_0}$.
- If $n \geq 1$ and (2.12) is satisfied, we say that W is a *left cuspidal word*. In this case we define its type by $\varepsilon(W) := L$ and we set $\xi_W := \xi^L_{a_0}$.
- If $n = 0$, that is, $W = (a_0)$ has just one letter, the type $\varepsilon(W)$ is not defined. By convention, $\xi_W := \xi^R_{a_0}$.

If $W = (a_0, \dots, a_n)$ is cuspidal with $n \geq 1$, Lemma 2.1 implies that $\xi^{\varepsilon(W)}_{a_k} = F_{a_k} \cdot \xi^{\varepsilon(W)}_{a_{k+1}}$ for any $k = 0, \dots, n-1$ and it follows that

$$\xi_W = \partial[a_0]_{\mathbb{D}} \cap \partial[a_0, a_1]_{\mathbb{D}} \cap \cdots \cap \partial[a_0, \dots, a_n]_{\mathbb{D}}, \tag{2.13}$$

that is, the $n+1$ arcs above share ξ_W as common end point (see also [2, Section 2.4] and [1, Section 4.3]). A sequence $(a_n)_{n\in\mathbb{N}}$ is called *cuspidal* if any initial factor $(a_0, \dots, a_n)$ with $n \in \mathbb{N}$ is a cuspidal word, and *eventually cuspidal* if there exists $k \in \mathbb{N}$ such that $(a_{n+k})_{n\in\mathbb{N}}$ is a cuspidal sequence.

2.5. The cuspidal acceleration. If $W = (b_0, \dots, b_m)$ and $W' = (a_0, \dots, a_n)$ are words with $a_0 \neq \widehat{b_m}$, define the word $W * W' := (b_0, \dots, b_m, a_0, \dots, a_n)$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence satisfying Condition (2.9) and not eventually cuspidal.

Initial step. Set $n(0) := 0$. Let $n(1) \in \mathbb{N}$ be the maximal integer $n(1) \geq 1$ such that $(a_0, \dots, a_{n(1)-1})$ is cuspidal; then set $W_0 := (a_0, \dots, a_{n(1)-1})$.

Recursive step. Fix $r \geq 1$ and assume that the instants $n(0) < \cdots < n(r)$ and the cuspidal words $W_0, \dots, W_{r-1}$ are defined. Define $n(r+1) \geq n(r) + 1$ as the maximal integer such that $[a_{n(r)}, \dots, a_{n(r+1)-1}]$ is cuspidal; then set

$$W_r := (a_{n(r)}, \dots, a_{n(r+1)-1}).$$

The sequence of words $(W_r)_{r\in\mathbb{N}}$ is called the *cuspidal decomposition* of $(a_n)_{n\in\mathbb{N}}$. Of course, $(a_0, a_1, a_2 \dots) = W_0 * W_1 * \cdots$. For any $\xi = [a_0, a_1, \dots]_{\mathbb{D}}$, if $(W_r)_{r\in\mathbb{N}}$ is the cuspidal decomposition of $(a_n)_{n\in\mathbb{N}}$, we write

$$\xi = [a_0, a_1, \dots]_{\mathbb{D}} = [W_0, W_1, \dots]_{\mathbb{D}}. \tag{2.14}$$

REMARK 2.2. If $W_{r-1} := (a_{n(r-1)}, \dots, a_{n(r)-1})$ and $W_r := (a_{n(r)}, \dots, a_{n(r+1)-1})$ are two consecutive cuspidal words in the cuspidal decomposition of a sequence $(a_n)_{n\in\mathbb{N}}$ satisfying Condition (2.9), then the word $(a_{n(r)-1}, a_{n(r)}, \dots, a_{n(r+1)-1})$ can be cuspidal.

https://doi.org/10.1017/S0004972720001483 Published online by Cambridge University Press

3. The main theorem

The tools in Section 2 induce a boundary expansion on $\mathbb{R}$. Let $\Gamma_0 < \Gamma$ be the free subgroup and $\Omega_{\mathbb{D}} \subset \mathbb{D}$ the ideal polygon in Section 2.2. Then $\mathcal{P}_{\Gamma_0} = \Gamma_0(\Omega_{\mathbb{D}} \cap \partial\mathbb{D})$ by Theorem 4.2.5 in [4]. Since Γ_0 has finite index in Γ, the two groups have the same set of parabolic fixed points, that is,

$$\mathcal{P}_{\Gamma} = \Gamma_0(\Omega_{\mathbb{D}} \cap \partial\mathbb{D}). \tag{3.1}$$

3.1. Geometric length of cuspidal words. Fix the list $\mathcal{S} = (A_1, \ldots, A_p)$ as in (1.5). Let $\Omega_{\mathbb{H}} := \varphi^{-1}(\Omega_{\mathbb{D}}) \subset \mathbb{H}$ be the pre-image of $\Omega_{\mathbb{D}}$ under the map in (2.1). Any vertex ξ of $\Omega_{\mathbb{D}}$ corresponds to a unique vertex $\zeta = \varphi^{-1}(\xi)$ of $\Omega_{\mathbb{H}}$. For any such vertex ζ, consider $B \in \Gamma$ and $k \in \{1, \ldots, p\}$ with

$$\zeta = BA_k \cdot \infty. \tag{3.2}$$

Any side s_a of $\Omega_{\mathbb{D}}$ corresponds to a unique side $e_a := \varphi^{-1}(s_a)$ of $\Omega_{\mathbb{H}}$, where $a \in \mathcal{A}$.

If $BA_k \cdot \infty = B'A_j \cdot \infty$, then $j = k$. Moreover, $B' = BP$, where $P \in \Gamma$ is parabolic fixing $A_k \cdot \infty$ (recall that in any Fuchsian group Γ with cusps, if $G \in \Gamma$ satisfies $G \cdot \zeta = \zeta$ for some $\zeta \in \mathcal{P}_{\Gamma}$, then G is parabolic). Hence, the map $z \mapsto A_k^{-1}PA_k(z)$ is a horizontal translation in $\mathbb{H}$. If s and s' are geodesics in $\mathbb{D}$ having ξ as common end point, then their pre-images in $\mathbb{H}$ under $\varphi \circ B \circ A_k$ are parallel vertical half-lines whose distance does not depend on the choice of B in (3.2). This gives a well-defined positive real number

$$\Delta(s, s', \xi) := |\mathrm{Re}(A_k^{-1}B^{-1}\varphi^{-1}(s)) - \mathrm{Re}(A_k^{-1}B^{-1}\varphi^{-1}(s'))|.$$

Fix a cuspidal word $W = (a_0, \ldots, a_n)$ and the vertex ξ_W of $\Omega_{\mathbb{D}}$ associated to W in Section 2.4. For $n \geq 1$, (2.13) implies that the geodesics $s_{a_0}, F_{a_0}(s_{a_1}), \ldots, F_{a_0,\ldots,a_{n-1}}(s_{a_n})$ all have ξ_W as common end point (see Figure 3). Define the *geometric length* $|W| \geq 0$ of W as

$$|W| := \begin{cases} \Delta(s_{a_0}, F_{a_0,\ldots,a_{n-1}}(s_{a_n}), \xi_W) & \text{if } n \geq 1, \\ 0 & \text{if } n = 0. \end{cases} \tag{3.3}$$

For $a \in \mathcal{A}$, set $G_a = \varphi^{-1} \circ F_a \circ \varphi$. Set $G_{a_0,\ldots,a_n} := G_{a_0} \circ \cdots \circ G_{a_n}$ for any word $(a_0, \ldots, a_n)$ and $G_{W_0,\ldots,W_r} = G_{a_0,\ldots,a_n}$ if $(a_0, \ldots, a_n) = W_0 * \cdots * W_r$. Define the interval

$$[a_0, \ldots, a_n]_{\mathbb{H}} := \varphi^{-1}([a_0, \ldots, a_n]_{\mathbb{D}}) = G_{a_0,\ldots,a_n}(\partial\mathbb{H} \setminus [\widehat{a_n}]_{\mathbb{H}}).$$

Set $[a_0, a_1, \ldots]_{\mathbb{H}} := \varphi^{-1}([a_0, a_1, \ldots]_{\mathbb{D}})$, that is, encode $\alpha \in \mathbb{R}$ by the same cutting sequence as $\varphi(\alpha) \in \mathbb{D}$. If $(a_n)_{n\in\mathbb{N}}$ has cuspidal decomposition $(W_r)_{r\in\mathbb{N}}$, (2.14) becomes

$$\alpha = [W_0, W_1, \ldots]_{\mathbb{H}} := [a_0, a_1, \ldots]_{\mathbb{H}}. \tag{3.4}$$

For $r \in \mathbb{N}$, let W_r be the rth cuspidal word. Set $\zeta_{W_r} := \varphi^{-1}(\xi_{W_r})$. The convergents of α are

$$\zeta_r := G_{W_0,\ldots,W_{r-1}} \cdot \zeta_{W_r}, \quad r \in \mathbb{N}. \tag{3.5}$$

For $k = 1, \ldots, p$, let $\mu_k > 0$ be such that the primitive parabolic element $P_k \in A_k\Gamma A_k^{-1}$ fixing ∞ acts by $P_k(z) = z + \mu_k$. Set $\mu := \max\{\mu_1, \ldots, \mu_p\}$.

https://doi.org/10.1017/S0004972720001483 Published online by Cambridge University Press

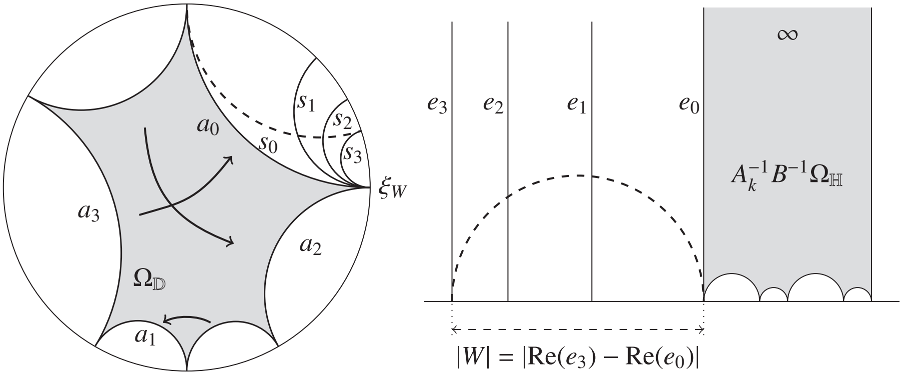

FIGURE 3. Geometric length $|W|$ of a right cuspidal word $W = (a_0, a_1, a_2, a_3)$. The arrows inside $\Omega_{\mathbb{D}}$ represent the action of $F_{a_0}, F_{a_1}, F_{a_2}$. The arcs $s_0 := s_{a_0}$, $s_1 := F_{a_0}(s_{a_1})$, $s_2 := F_{a_0,a_1}(s_{a_2})$, $s_3 := F_{a_0,a_1,a_2}(s_{a_3})$ share the common vertex ξ_W, which is sent to ∞ under the map $A_k^{-1}B^{-1}\varphi^{-1}$. Thus, the arcs s_0, s_1, s_2, s_3 in $\mathbb{D}$ are sent to parallel vertical arcs $e_i := \varphi^{-1}(s_i)$ in $\mathbb{H}$.

THEOREM 3.1 (Main theorem). *For any $r \in \mathbb{N}$ with $|W_r| > 0$,*

$$\frac{1}{|W_r| + 2\mu} \leq D(G_{W_0,\dots,W_{r-1}} \cdot \zeta_{W_r})^2 \cdot |\alpha - G_{W_0,\dots,W_{r-1}} \cdot \zeta_{W_r}| \leq \frac{1}{|W_r|}. \tag{3.6}$$

Moreover, there exists $\epsilon_0 > 0$, depending only on $\Omega_{\mathbb{D}}$ and $\mathcal{S}$, such that for any $G \in \Gamma$ and $k = 1, \dots, p$ with $D(G \cdot z_k) \neq 0$, the condition

$$D(G \cdot z_k)^2 \cdot |\alpha - G \cdot z_k| < \epsilon_0$$

implies that there exists some $r \in \mathbb{N}$ such that

$$G \cdot z_k = G_{W_0,\dots,W_{r-1}} \cdot \zeta_{W_r}, \quad \text{where } |W_r| > 0. \tag{3.7}$$

REMARK 3.2. Equation (3.6) holds for any choice of $\mathcal{S}$ as in (1.5), and this follows because geometric length and denominators satisfy a form of equivariance under the choice of $\mathcal{S}$. Equation (3.7) shows that, for any choice of the subgroup Γ_0, all good enough approximations of a given α belong to the sequence of its convergents.

3.2. Reduced form of parabolic fixed points. Fix $G \cdot z_k \in \mathcal{P}_\Gamma$. Recall (3.1) and write elements of Γ_0 in the generators $\{G_a : a \in \mathcal{A}\}$. There exist a unique admissible word $b_0, \dots, b_m$ and a vertex ζ of $\Omega_{\mathbb{H}}$ which is not an end point of $e_{\widehat{b_m}}$ such that

$$G \cdot z_k = G_{b_0,\dots,b_m} \cdot \zeta.$$

The representation above is called the *reduced form* of the parabolic fixed point $G \cdot z_k$. In the next lemmas (Lemmas 3.3 and 3.4), let $(b_0, \dots, b_m)$ be a nontrivial admissible word and let ζ_0 be a vertex of $\Omega_{\mathbb{H}}$ which is not an end point of $e_{\widehat{b_m}}$, so that $G_{b_0,\dots,b_m} \cdot \zeta_0$ is a parabolic fixed point written in its reduced form and different from ∞.

https://doi.org/10.1017/S0004972720001483 Published online by Cambridge University Press

LEMMA 3.3. *There exists a constant $\kappa_1 > 0$, depending only on $\Omega_{\mathbb{H}}$, such that*

$$|\zeta_0 - G^{-1}_{b_0,\dots,b_m} \cdot \infty| \geq \kappa_1,$$

that is, the vertex ζ_0 and the pole of $G_{b_0,\dots,b_m}$ stay at distance uniformly bounded from below.

PROOF. By (2.10), $G_{b_0,\dots,b_m}(\mathbb{R} \setminus [\widehat{b_m}]_{\mathbb{H}}) = [b_0, \dots, b_m]_{\mathbb{H}}$. Since ∞ does not belong to the interior of $[b_0, \dots, b_m]_{\mathbb{H}}$, the pole of $G_{b_0,\dots,b_m}$ belongs to the closure of $[\widehat{b_m}]_{\mathbb{H}}$. The lemma follows because ζ_0 is a vertex of $\Omega_{\mathbb{H}}$ different from the end points of $e_{\widehat{b_m}}$. □

LEMMA 3.4. *There exists a constant $\kappa_2 > 0$, depending only on $\Omega_{\mathbb{H}}$ and $\mathcal{S}$, such that the following statements hold.*

(1) *If ζ_1 is a vertex of $\Omega_{\mathbb{H}}$ different from ζ_0, then*

$$D(G_{b_0,\dots,b_m} \cdot \zeta_0) \geq \kappa_2 \cdot D(G_{b_0,\dots,b_m} \cdot \zeta_1).$$

(2) *If b_{m+1} satisfies $b_{m+1} \neq \widehat{b_m}$ and ζ_2 is a vertex of $\Omega_{\mathbb{H}}$ with $G_{b_{m+1}} \cdot \zeta_2 \neq \zeta_0$, then*

$$D(G_{b_0,\dots,b_m} \cdot \zeta_0) \geq \kappa_2 \cdot D(G_{b_0,\dots,b_m,b_{m+1}} \cdot \zeta_2).$$

PROOF. Part (1). Set $G := G_{b_0,\dots,b_m}$, $\zeta := G \cdot \zeta_0$ and $\zeta' := G \cdot \zeta_1$. If $\zeta' = \infty$, then the statement is trivially true. If $D(G \cdot \zeta_1) \neq 0$, let $\zeta_0 = B_0 A_k \cdot \infty$ and $\zeta_1 = B_1 A_j \cdot \infty$ as in (3.2). Referring to (1.3), let c, d be the entries of G. Let a_0, c_0 and a_1, c_1 be the entries of $B_0 A_k$ and $B_1 A_j$, respectively. We prove an upper bound for

$$\frac{D(G_{b_0,\dots,b_m} \cdot \zeta_1)}{D(G_{b_0,\dots,b_m} \cdot \zeta_0)} = \left|\frac{ca_1 + dc_1}{ca_0 + dc_0}\right|.$$

We cannot have $c_0 = c_1 = 0$ because $\zeta_0 \neq \zeta_1$ and in particular ζ_0, ζ_1 cannot both be equal to ∞. Moreover, $G \cdot \zeta_0, G \cdot \zeta_1$ are both different from ∞; thus, the condition $c = 0$ implies that $c_0, c_1 \neq 0$. Hence, (1) follows for $c = 0$ because the ratio above equals $|c_1/c_0|$, which varies in a finite set of values and is therefore bounded from above. If $c, c_0, c_1 \neq 0$, then

$$\left|\frac{ca_1 + dc_1}{ca_0 + dc_0}\right| = \left|\frac{c_1}{c_0}\right| \cdot \left|\frac{(a_1/c_1) - (-d/c)}{(a_0/c_0) - (-d/c)}\right| = \left|\frac{c_1}{c_0}\right| \cdot \left|\frac{\zeta_1 - (G^{-1} \cdot \infty)}{\zeta_0 - (G^{-1} \cdot \infty)}\right|.$$

In this case (1) follows because $|c_1/c_0|$ is bounded from above, and Lemma 3.3 gives a lower bound for the denominator of the second factor (the numerator is not bounded, but as it increases the ratio converges to 1). If $c, c_0 \neq 0$ and $c_1 = 0$, then Lemma 3.3 gives

$$\left|\frac{ca_1 + dc_1}{ca_0 + dc_0}\right| = \left|\frac{a_1}{c_0}\right| \cdot \left|\frac{1}{(a_0/c_0) - (-d/c)}\right| = \left|\frac{a_1}{c_0}\right| \cdot \left|\frac{1}{\zeta_0 - (G^{-1} \cdot \infty)}\right| \leq \left|\frac{a_1}{c_0 \cdot \kappa_1}\right|$$

https://doi.org/10.1017/S0004972720001483 Published online by Cambridge University Press

and (1) follows on observing that a_1/c_0 varies in a finite set of values. Finally, if we have $c, c_1 \neq 0$ and $c_0 = 0$, then

$$\left|\frac{ca_1 + dc_1}{ca_0 + dc_0}\right| = \left|\frac{a_1}{a_0} - (-d/c)\frac{c_1}{a_0}\right| \leq \left|\frac{a_1}{a_0}\right| + |G^{-1} \cdot \infty|\left|\frac{c_1}{a_0}\right|.$$

In this case $\zeta_0 = \infty$, which is not an end point of $[\widehat{b_m}]$. Thus, $[\widehat{b_m}]$ is contained in the compact interval of $\mathbb{R}$ delimited by the two parallel vertical segments of $\Omega_{\mathbb{H}}$. Hence, $|G^{-1} \cdot \infty|$ is uniformly bounded because the pole $G^{-1} \cdot \infty$ belongs to the closure of $[\widehat{b_m}]$ (see proof of Lemma 3.3). Part (1) follows in this case too and the proof is complete.

Part (2) follows similarly, replacing ζ_1 by $\zeta_* := G_{b_{m+1}} \cdot \zeta_2$ and observing that, since $G_{b_{m+1}}$ varies in the finite set $\{G_a : a \in \mathcal{A}\}$, then also the entries of $X \in \mathrm{SL}(2, \mathbb{R})$ with $G_{b_{m+1}} \cdot \zeta_2 = X \cdot \infty$ vary in a finite set. Moreover, $\zeta_0 \neq \zeta_*$ and so $G \cdot \zeta_0 \neq G \cdot \zeta_*$. □

3.3. Proof of Theorem 3.1. By a standard separation property of parabolic fixed points (see [6, Section A]), there exists a constant $S_0 > 0$, depending only on Γ and $\mathcal{S}$, such that for any $G \cdot z_i$ and $F \cdot z_j$ in $\mathcal{P}_\Gamma$ with $G \cdot z_i \neq F \cdot z_j$,

$$|G \cdot z_i - F \cdot z_j| \geq \frac{S_0}{D(G \cdot z_i)D(F \cdot z_j)}. \tag{3.8}$$

Let $\alpha = [a_0, a_1, \ldots]_{\mathbb{H}} = [W_0, W_1, \ldots]_{\mathbb{H}}$ be the expansion of $\alpha \in \mathbb{R}$ as in (3.4).

3.3.1. Proof of (3.6). Fix $r \in \mathbb{N}$ with $|W_r| > 0$. Take $k \in \{1, \ldots, p\}$ and $B \in \Gamma$ as in (3.2), that is, $\zeta_{W_r} = BA_k \cdot \infty$. As in Figure 4, let $T > 0$ be such that the horoball

$$B_T := G_{W_0,\ldots,W_{r-1}} BA_k(\{z \in \mathbb{H} : \mathrm{Im}(z) > T\})$$

is tangent at $G_{W_0,\ldots,W_{r-1}} \cdot \zeta_{W_r}$ with radius $\rho(B_T) = |\alpha - G_{W_0,\ldots,W_{r-1}} \cdot \zeta_{W_r}|$. Equations (1.6) and (1.7) give

$$D(G_{W_0,\ldots,W_{r-1}} \cdot \zeta_{W_r})^2 \cdot |\alpha - G_{W_0,\ldots,W_{r-1}} \cdot \zeta_{W_r}| = c^2(G_{W_0,\ldots,W_{r-1}} BA_k) \cdot \frac{\mathrm{Diam}(B_T)}{2} = \frac{1}{2T}.$$

The geodesic in $\mathbb{H}$ with end points $(G_{W_0,\ldots,W_{r-1}} BA_k)^{-1} \cdot \infty$ and $(G_{W_0,\ldots,W_{r-1}} BA_k)^{-1} \cdot \alpha$ is tangent to $\{z \in \mathbb{H} : \mathrm{Im}(z) > T\}$. Equation (3.6) follows because (3.3) gives

$$|W_r| \leq 2T \leq |W_r| + 2\mu.$$

3.3.2. Proof of (3.7). Referring to Section 3.2, let ζ_0 be the vertex of $\Omega_{\mathbb{H}}$ and $(b_0, \ldots, b_m)$ be the admissible word such that the reduced form of the parabolic fixed point $G \cdot z_k$ is

$$G \cdot z_k = G_{b_0,\ldots,b_m} \cdot \zeta_0,$$

where ζ_0 is not an end point of $e_{\widehat{b_m}}$ whenever $(b_0, \ldots, b_m)$ is not the empty word. Assume that $D(G \cdot z_k)^2 |\alpha - G \cdot z_k| < \epsilon_0$, where the constant $\epsilon_0 > 0$ will be determined later.

https://doi.org/10.1017/S0004972720001483 Published online by Cambridge University Press

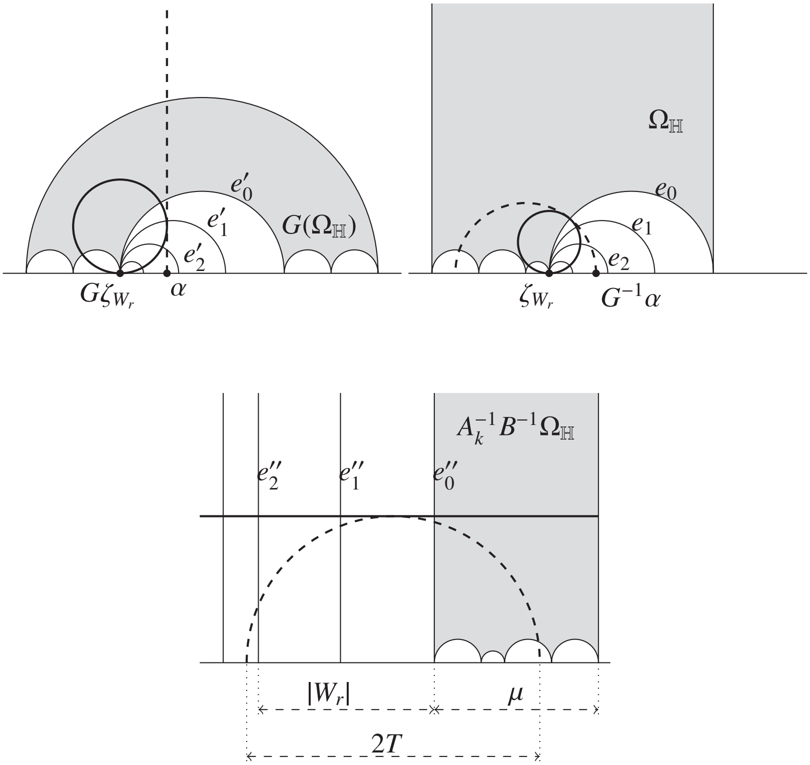

FIGURE 4. The rth cuspidal word $W_r = (a_0, a_1, a_2)$ of α is the first cuspidal word of $G^{-1} \cdot \alpha$, where $G = G_{W_0,\dots,W_{r-1}}$. The vertex ζ_{W_r} of $\Omega_{\mathbb{H}}$ is common to the arcs $e_0 = e_{a_0}$, $e_1 := G_{a_0} e_{a_1}$ and $e_2 := G_{a_0 a_1} e_{a_2}$. The arcs $e_i' = Ge_i$ share the vertex $G\zeta_{W_r}$. The point ζ_{W_r} is sent to ∞ and the arcs e_0, e_1, e_2 are sent to the parallel vertical arcs e_0'', e_1'', e_2''. We have $|W_r| = |\mathrm{Re}(e_2'') - \mathrm{Re}(e_0'')|$.

Step (0). Assume that $(b_0, \dots, b_m)$ is the empty word, so that $\zeta_0 = G \cdot z_k \neq \infty$. Consider the extra assumption $|W_0| > 0$ and $\zeta_0 = \zeta_{W_0}$ on pairs (α, ζ_0), where $\zeta_{W_0} = \varphi^{-1}(\xi_{W_0})$ and ξ_{W_0} is the vertex of $\Omega_{\mathbb{D}}$ associated to W_0 as in Section 2.4. Define $\epsilon_0 > 0$ by

$$\epsilon_0 := \inf_{(\alpha,\zeta_0)} D(\zeta_0)^2 \cdot |\alpha - \zeta_0|,$$

where the infimum is taken over all pairs (α, ζ_0) not satisfying the extra assumption. With such ϵ_0, the statement follows whenever $(b_0, \dots, b_m)$ is the empty word.

Step (1). Now assume that $(b_0, \dots, b_m)$ is not the empty word. Then $G \cdot z_k$ is an interior point of $[b_0, \dots, b_m]_{\mathbb{H}}$. Let ζ_1, ζ_2 be the end points of $[\widehat{b_m}]$, which are vertices of $\Omega_{\mathbb{H}}$ different from ζ_0. By (2.10), the end points of $[b_0, \dots, b_m]_{\mathbb{H}}$ are $\zeta_i' := G_{b_0,\dots,b_m} \cdot \zeta_i$ for $i = 1, 2$. Let $N \geq -1$ be maximal with $a_n = b_n$ for any $n = 0, \dots, N$, where the last condition is empty for $N = -1$, and where $N \leq m$. Observe that the condition

https://doi.org/10.1017/S0004972720001483 Published online by Cambridge University Press

$N \leq m-1$ implies that $\alpha \notin [b_0, \ldots, b_m]_{\mathbb{H}}$ and therefore

$$|\alpha - G \cdot z_k| \geq \min_{i=1,2} |\zeta_i' - G \cdot z_k| = \min_{i=1,2} |G_{b_0,\ldots,b_m} \cdot \zeta_i - G_{b_0,\ldots,b_m} \cdot \zeta_0|$$
$$\geq \frac{S_0}{D(G_{b_0,\ldots,b_m} \cdot \zeta_0)} \cdot \min_{i=1,2} \frac{1}{D(G_{b_0,\ldots,b_m} \cdot \zeta_i)} \geq \frac{S_0 \kappa_2}{D(G_{b_0,\ldots,b_m} \cdot \zeta_0)^2},$$

where the third inequality follows from Lemma 3.4(1) and the second from (3.8). Therefore $N = m$, provided that $\epsilon_0 < \kappa_2 S_0$.

We have proved that $[a_0, \ldots, a_m]_{\mathbb{H}} = [b_0, \ldots, b_m]_{\mathbb{H}}$. Moreover, $G \cdot z_k$ does not belong to the interior of $[a_0, \ldots, a_m, a_{m+1}]_{\mathbb{H}}$, since the latter is a subinterval of $[b_0, \ldots, b_m]_{\mathbb{H}}$ delimited by the image under $G_{b_0,\ldots,b_m}$ of two consecutive vertices of $\Omega_{\mathbb{H}}$. The same argument as in the first part of Step (1), which is left to the reader, shows that $G \cdot z_k$ is an end point of $[a_0, \ldots, a_m, a_{m+1}]_{\mathbb{H}}$.

Step (2). We show that $G \cdot z_k = G_{b_0,\ldots,b_m} \cdot \zeta_0$ is an end point of $[a_0, \ldots, a_{m+2}]_{\mathbb{H}}$. Otherwise, $G \cdot z_k$ does not belong to the closure of $[a_0, \ldots, a_{m+2}]_{\mathbb{H}}$. Since $\alpha \in [a_0, \ldots, a_{m+2}]_{\mathbb{H}}$,

$$|\alpha - G \cdot z_k| \geq |G_{b_0,\ldots,b_m,a_{m+1}} \cdot \zeta_3 - G_{b_0,\ldots,b_m} \cdot \zeta_0|$$
$$\geq \frac{S_0}{D(G_{b_0,\ldots,b_m} \cdot \zeta_0) D(G_{b_0,\ldots,b_m,a_{m+1}} \cdot \zeta_3)} \geq \frac{S_0 \kappa_2}{D(G_{b_0,\ldots,b_m} \cdot \zeta_0)^2},$$

where $G_{b_0,\ldots,b_m,a_{m+1}} \cdot \zeta_3$ is the end point of $[a_0, \ldots, a_{m+2}]_{\mathbb{H}}$ which is closest to $G \cdot z_k$ and where ζ_3 is a vertex of $\Omega_{\mathbb{H}}$ which is not an end point of $e_{\widehat{a_{m+1}}}$. Here, we use (3.8) and Lemma 3.4(2). The inequality is absurd because of the condition $\epsilon_0 < \kappa_2 S_0$.

Step (3). Let r be minimal such that $(a_0, \ldots, a_m)$ is an initial factor of $W_0 * \cdots * W_{r-1}$. If $(a_0, \ldots, a_{m+2})$ is also an initial factor of $W_0 * \cdots * W_{r-1}$, then $G_{W_0,\ldots,W_{r-1}} \cdot \xi_{W_{r-1}}$ is a common end point of the intervals $[a_0, \ldots, a_m]_{\mathbb{H}}$, $[a_0, \ldots, a_{m+1}]_{\mathbb{H}}$ and $[a_0, \ldots, a_{m+2}]_{\mathbb{H}}$, according to (2.13). Without loss of generality,

$$G_{W_0,\ldots,W_{r-1}} \cdot \xi_{W_{r-1}} = \inf[a_0, \ldots, a_m]_{\mathbb{H}} = \inf[a_0, \ldots, a_{m+1}]_{\mathbb{H}} = \inf[a_0, \ldots, a_{m+2}]_{\mathbb{H}}.$$

The common end point is not $G \cdot z_k$, which belongs to the interior of $[a_0, \ldots, a_m]_{\mathbb{H}}$. Thus, Step (1) implies that $G \cdot z_k = \sup[a_0, \ldots, a_{m+1}]_{\mathbb{H}}$, which gives a contradiction because $G \cdot z_k$ is an end point of $[a_0, \ldots, a_{m+2}]_{\mathbb{H}}$ by Step (2). Hence, $W_0 * \cdots * W_{r-1}$ is equal either to $(a_0, \ldots, a_m)$ or to $(a_0, \ldots, a_{m+1})$. Moreover, (a_{m+1}, a_{m+2}) is a cuspidal word because $[a_0, \ldots, a_{m+1}]_{\mathbb{H}}$ and $[a_0, \ldots, a_{m+2}]_{\mathbb{H}}$ share the end point $G \cdot z_k$.

If $W_0 * \cdots * W_{r-1} = (a_0, \ldots, a_m)$, the word (a_{m+1}, a_{m+2}) is an initial factor of W_r, that is, $|W_r| > 0$ and $\zeta_0 = \zeta_{W_r}$.

If $W_0 * \cdots * W_{r-1} = (a_0, \ldots, a_{m+1})$, the word $W' := (a_{m+1}) * W_r$ is also cuspidal (this is allowed by Remark 2.2). If $|W_r| = 0$, that is, $W_r = (a_{m+2})$, then $G \cdot z_k$ does not belong to the closure of $[a_0, \ldots, a_{m+3}]_{\mathbb{H}}$ and we reach a contradiction by

$$|\alpha - G \cdot z_k| \geq |G_{b_0,\ldots,b_m} \cdot \zeta_0 - G_{b_0,\ldots,b_m,a_{m+1},a_{m+2}} \cdot \zeta_3| \geq \frac{S_0 \kappa_2}{D(G_{b_0,\ldots,b_m} \cdot \zeta_0)^2},$$

https://doi.org/10.1017/S0004972720001483 Published online by Cambridge University Press

where ζ_3 is a vertex of $\Omega_{\mathbb{H}}$ and $G_{b_0,\dots,b_m,a_{m+1},a_{m+2}} \cdot \zeta_3$ is the end point of $[a_0,\dots,a_{m+3}]_{\mathbb{H}}$ which is closest to $G \cdot z_k$. In the last inequality we reason as in Step (2), replacing κ_2 by a smaller constant and extending Part (2) of Lemma 3.4 one more step, in order to compare $D(G_{b_0,\dots,b_m} \cdot \zeta_0)$ and $D(G_{b_0,\dots,b_m,a_{m+1},a_{m+2}} \cdot \zeta_3)$. Since W' is cuspidal with $|W'| > 0$, we have $\zeta_0 = \zeta_{W'}$. But we also have $\zeta_{W'} = G_{a_{m+1}} \cdot \zeta_{W_r}$, which implies that

$$G_{b_0,\dots,b_m} \cdot \zeta_0 = G_{a_0,\dots,a_m} \cdot G_{a_{m+1}} \cdot \zeta_{W_r} = G_{W_0,\dots,W_{r-1}} \cdot \zeta_{W_r}.$$

In both cases (3.7) follows. The proof of Theorem 3.1 is complete.

Acknowledgements

The author is grateful to M. Artigiani and C. Ulcigrai. The author is also grateful to the anonymous referee for helpful remarks and suggestions.

References

[1] M. Artigiani, L. Marchese and C. Ulcigrai, ‘The Lagrange spectrum of a Veech surface has a Hall ray’, *Groups Geom. Dyn.* **10** (2016), 1287–1337.

[2] M. Artigiani, L. Marchese and C. Ulcigrai, ‘Persistent Hall rays for Lagrange spectra at cusps of Riemann surfaces’, *Ergod. Th. & Dynam. Sys.* **40**(8) (2020), 2017–2072.

[3] R. Bowen and C. Series, ‘Markov maps associated with fuchsian groups’, *Publ. Math. Inst. Hautes Études Sci.* **50** (1979), 153–170.

[4] S. Katok, *Fuchsian Groups*, Chicago Lectures in Mathematics (University of Chicago Press, Chicago, IL, 1992).

[5] J. Lehner, ‘Diophantine approximation on Hecke groups’, *Glasg. Math. J.* **27** (1985), 117–127.

[6] L. Marchese, ‘Transfer operators and dimension of bad sets for non-uniform Fuchsian lattices’, Preprint, 2018, arXiv:1812.11921.

[7] H. Nakada, ‘On the Lenstra constant associated to the Rosen continued fractions’, *J. Eur. Math. Soc.* **12** (2010), 55–70.

[8] S. J. Patterson, ‘Diophantine approximation in Fuchsian groups’, *Philos. Trans. Roy. Soc. Lond. Ser. A* **282**(1309) (1976), 527–563.

[9] D. Rosen, ‘A class of continued fractions associated to certain properly discontinuous groups’, *Duke Math. J.* **21** (1954), 549–562.

[10] D. Rosen and T. Schmidt, ‘Hecke groups and continued fractions’, *Bull. Aust. Math. Soc.* **46** (1992), 459–474.

[11] T. Schmidt and M. Sheingorn, ‘Riemann surfaces have Hall rays at each cusp’, *Illinois J. Math.* **41**(3) (1997), 378–397.

[12] P. Tukia, ‘On discrete groups of the unit disk and their isomorphisms’, *Ann. Acad. Sci. Fenn. Ser. A I* **504** (1972), 1–45.

LUCA MARCHESE, Dipartimento di Matematica,
Università di Bologna, Piazza di Porta San Donato 5,
40126 Bologna, Italy
e-mail: luca.marchese4@unibo.it