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AN EXISTENTIAL $\emptyset$-DEFINITION OF $\mathbb{F}_q[[t]]$ IN $\mathbb{F}_q((t))$

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Abstract. We show that the valuation ring $\mathbb{F}_q[[t]]$ in the local field $\mathbb{F}_q((t))$ is existentially definable in the language of rings with no parameters. The method is to use the definition of the henselian topology following the work of Prestel-Ziegler to give an $\exists$-$\mathbb{F}_q$-definable bounded neighbouhood of 0. Then we "tweak" this set by subtracting, taking roots, and applying Hensel's Lemma in order to find an $\exists$-$\mathbb{F}_q$-definable subset of $\mathbb{F}_q[[t]]$ which contains $t\mathbb{F}_q[[t]]$. Finally, we use the fact that $\mathbb{F}_q$ is defined by the formula $x^q - x = 0$ to extend the definition to the whole of $\mathbb{F}_q[[t]]$ and to rid the definition of parameters.

Several extensions of the theorem are obtained, notably an $\exists$-$\emptyset$-definition of the valuation ring of a nontrivial valuation with divisible value group.

§1. **Introduction.** This paper deals with questions of definability in power series fields. Unless stated otherwise, all definitions will be in the language $\mathcal{L}_{\text{ring}}$ of rings. Let $q = p^k$ be a power of a prime and let $\mathbb{F}_q((t))$ be the field of formal power series over the finite field $\mathbb{F}_q$; sometimes this is called the field of Laurent series over $\mathbb{F}_q$. The ring $\mathbb{F}_q[[t]]$ of formal power series with nonnegative exponents is the valuation ring of the t-adic valuation on $\mathbb{F}_q((t))$.

A predicate is said to be $\exists$-C-definable, for a subset C of the field, if it is definable by an existential formula with parameters from C. In particular, it is $\exists$-$\emptyset$-definable if it is defined by an existential formula which uses no parameters.

In section 2 of this paper we prove the following theorem.

THEOREM 1.1. *$\mathbb{F}_q[[t]]$ is $\exists$-$\emptyset$-definable in $\mathbb{F}_q((t))$.*

This result fits into a long history of definitions of valuation rings in valued fields. In the particular case of power series fields, a lot is already known.

If $K = \mathbb{C}$ or $\mathbb{Q}_p$, then $K[[t]]$ is not $\exists$-K-definable in $K((t))$. For the proof of this, see Observation A.1 in the appendix.

In the field $\mathbb{Q}_p$ the valuation ring $\mathbb{Z}_p$ is $\exists$-$\emptyset$-definable by the formula $\exists y\ 1 + x^l p = y^l$, for any prime $l \neq p$. This formula is not, however, uniform in p. Analogies between $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ naturally suggest the following "folkloric" definition: $\mathbb{F}_q[[t]]$ is defined in $\mathbb{F}_q((t))$ by the formula $\exists y\ 1 + x^l t = y^l$, whenever l is a prime not equal to p.

Other definitions are also well-known. One example is an $\exists\forall\exists\forall$-definition with no parameters due to Ax, from [1], which applies to all power series fields.

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FACT 1.2 (Implicit in [1]). *Let F be any field. Then $F[[t]]$ is $\exists\forall\exists\forall$-$\emptyset$-definable in $F((t))$.*

Another definition, in even greater generality, which uses no parameters is due to the second author and is from [4]. However, this definition is not existential.

FACT 1.3 (Lemma 3.6, [4]). *Let F be any field and suppose that $\mathcal{O}$ is an henselian rank 1 valuation ring on F with a nondivisible value group. Then $\mathcal{O}$ is $\emptyset$-definable.*

Recent work of Cluckers-Derakhshan-Leenknegt-Macintyre on the uniformity of definitions of valuation rings in henselian valued fields includes the following theorem. They use certain expansions of the ring language by predicates: the Macintyre predicate P_2 is interpreted as the set of squares and P_2^{AS} is interpreted as the image of the polynomial $y^2 + y$.

FACT 1.4 (Theorems 2 and 3, [2]). *There is an existential formula in $\mathcal{L}_{\text{ring}} \cup \{P_2^{AS}\}$ which defines the valuation ring in all henselian valued fields with finite or pseudo-finite residue field. Furthermore, if the residue field is not of characteristic 2 then this formula is equivalent to an existential $\mathcal{L}_{\text{ring}} \cup \{P_2\}$-formula.*

One consequence of Theorem 1.1 is in the study of definability in $\mathbb{F}_q((t))$: it reduces questions of existential definability in the language of valued fields (for example $\mathcal{L}_{\text{ring}}$ expanded with a unary prediate for the valuation ring) to existential definability in $\mathcal{L}_{\text{ring}}$ without needing more parameters.

It is famously unknown whether or not the theory of $\mathbb{F}_p((t))$ is decidable, whereas $\mathbb{Q}_p$ is decidable by the work of Ax-Kochen and Ershov. In [3] Denef and Schoutens prove that Hilbert's 10th problem has a positive solution in $\mathbb{F}_q[[t]]$ (in the language $\mathcal{L}_{\text{ring}} \cup \{t\}$ of discrete valuation rings) on the assumption of Resolution of Singularities in characteristic p. As a consequence of Theorem 1.1, we prove in Corollary 3.4 that Hilbert's 10th problem in $\mathcal{L}_{\text{ring}}$ has a solution over $\mathbb{F}_q((t))$ if and only if it has a solution over $\mathbb{F}_q[[t]]$. Of course, the analogous result for the language $\mathcal{L}_{\text{ring}} \cup \{t\}$ follows from the "folkloric" definition above.

As an imperfect field, $\mathbb{F}_p((t))$ cannot be model complete in the language of rings; however, it is still unknown whether it is model complete in a relatively "nice" expansion of that language, for example some analogy of the Macintyre language (see [5]) suitable for positive characteristic.

§2. The $\exists$-$\emptyset$-definition of $\mathbb{F}_q[[t]]$ in $\mathbb{F}_q((t))$.

2.1. Spheres and balls in valued fields. We briefly make a few definitions and notational conventions. Let $(K, \mathcal{O})$ be a valued field, let v be the corresponding valuation, and let vK denote the value group.

DEFINITION 2.1. For $n \in vK$, we let

1. $S(n) := v^{-1}(\{n\})$ be the set of elements of value n,
2. $B(n) := v^{-1}((n, \infty])$ be the *open ball* of radius n around 0, and
3. $\bar{B}(n) := v^{-1}([n, \infty])$ be the *closed ball* of radius n around 0.

We let $\sqcup$ denote a disjoint union.

LEMMA 2.2. *Let $n \in vK$. Then*

1. $B(n) \subseteq S(n) - S(n)$,
2. $\bar{B}(n) = S(n) \sqcup B(n)$, *and*
3. $\bar{B}(n) - \bar{B}(n) = \bar{B}(n)$.

PROOF.

1. Let $x \in B(n)$ and let $y \in S(n)$. Then $v(y) = n < v(x)$, so that $v(x - y) = n$ (by an elementary consequence of the ultrametric inequality) and $x - y \in S(n)$. Thus $x = x - y + y \in S(n) - S(n)$.
2. Let $x \in \bar{B}(n)$. Then either $v(x) = n$ or $v(x) > n$.
3. Let $x, y \in \bar{B}(n)$. By the ultrametric inequality $v(x - y) \geq n$. Thus $x - y \in \bar{B}(n)$. ⊣

2.2. An $\exists$-definable filter base for the neighbourhood filter of zero.

DEFINITION 2.3 (Section 7, [7]). Let K be any field. We say that K is *t-henselian* if there is a field topology $\mathcal{T}$ on K induced by an absolute value or a valuation with the property that, for each $n \in \mathbb{N}$, there exists $U \in \mathcal{T}$ such that $0 \in U$ and such that each $f \in \{x^{n+1} + x^n + u_{n-1}x^{n-1} + \cdots + u_0 \mid u_i \in U\}$ has a root in K. In this case, $\mathcal{T}$ is said to be a *t-henselian topology*.

We say that a given field topology is definable if there is a base of the filter of neighbourhoods around zero which forms a definable family. The following lemma shows that in a t-henselian field there is a base for the filter of neighbourhoods of zero which forms an existentially definable family. It is due to Prestel (from [6]), and corrects an earlier result of Prestel-Ziegler (from [7]).

Let $D := D_x$ denote the formal derivative with respect to the variable x.

LEMMA 2.4 (Proof of Lemma, [6]). *Suppose that K is t-henselian and not separably closed. Let $f \in K[x]$ be a separable irreducible polynomial without a zero in K. Let $a \in K \setminus Z(Df)$ be any element which is not a zero of the formal derivative of f. Let $U_{f,a} := \{\frac{1}{f(x)} - \frac{1}{f(a)} \mid x \in K\}$. Then $\mathcal{U} := \{c \cdot U_{f,a} | c \in K^\times\}$ is a base for the filter of open neighbourhoods around zero in the (unique) t-henselian topology.*

We prove a simple consequence of the Lemma.

PROPOSITION 2.5. *Let K be t-henselian and suppose that $C \subseteq K$ is a relatively algebraically closed subfield of K which is not separably closed. There exists $V \subseteq K$ which is an $\exists$-C-definable bounded neighbourhood of 0 in the t-henselian topology.*

PROOF. We choose $f \in C[x]$ to be nonlinear, irreducible, and separable. Let $n := \deg(f)$; thus $\deg(Df) \leq n - 1$. If $|C| > n - 1$ then we may choose $a \in C \setminus Z(Df)$. On the other hand, if C is a finite field, then C allows separable extensions of degree 2. So we may choose f to be of degree 2; whence Df is of degree ≤ 1 and again there exists $a \in C$ which is not a root of Df. Let $V := U_{f,a}$. Clearly V is $\exists$-C-definable. As discussed in Lemma 2.4, V is a bounded neighbourhood of 0. ⊣

2.3. An $\exists$-F-definable set between $\mathcal{O}$ and $\mathcal{M}$ in $F((t))$. Now let $K := F((t))$ be the field of formal power series over a field F. Let v be the t-adic valuation, let $\mathcal{O} := F[[t]]$ be the valuation ring of v, let $\mathcal{M} := t\mathcal{O}$ be its maximal ideal, and let

$vK = \mathbb{Z}$ be its value group. Note that $(K, \mathcal{O})$ is henselian. Let $C \subseteq K$ be any subset. Let $\mathcal{P} := S(1)$ be the set of elements of value 1; thus $\mathcal{P}$ is the set of uniformisers.

In the following proposition we show how to "tweak" a definable bounded neighbourhood of 0 until we obtain a subset of $\mathcal{O}$ containing $\mathcal{M}$, in such a way as to preserve definability.

PROPOSITION 2.6. *Suppose that $V \subseteq K$ is an $\exists$-C-definable bounded neighbourhood of* 0.

1. *There exists $W \subseteq K$ which is bounded, $\exists$-C-definable, and is such that $\mathcal{P} \subseteq W$.*
2. *There exists $X \subseteq K$ which is bounded, $\exists$-C definable, and is such that $\mathcal{M} \subseteq X$.*
3. *There exists $Y \subseteq K$ which is bounded by $\mathcal{O}$, $\exists$-C-definable, and is such that $\mathcal{M} \subseteq Y$.*

PROOF.

1. V is a neighbourhood of 0. Let $n \in \mathbb{Z}$ be such that $B(n) \subseteq V$. Without loss of generality, we suppose that $n \geq 0$. Choose any $m > n$; then $\mathcal{P}^m \subseteq S(m) \subseteq B(n) \subseteq V$. Let $\phi(x)$ be the formula expressing $x^m \in V$, and let $W := \phi(K)$ be the set defined by ϕ in K. Note that W is $\exists$-C-definable, and $\mathcal{P} \subseteq W$.

   It remains to show that W is bounded. Since V is bounded, there exists $l \in \mathbb{Z}$ such that $V \subseteq B(l)$. Let $l' := \min\{l, -1\}$ and let $b \notin B(l')$. Since $vb \leq l' \leq -1 < 0$, we have that $vb^m = mvb \leq vb \leq l' \leq l$. Thus $b^m \notin V$ and
   $$\left(x^m \in V \implies x \in B(l')\right).$$
   So $W \subseteq B(l')$.
2. Let $W' := W \cup \{0\}$ and set $X := W - W'$. Clearly X is bounded and $\exists$-C-definable. By Lemma 2.2, we see that $B(1) \subseteq S(1) - S(1) = \mathcal{P} - \mathcal{P} \subseteq W - W \subseteq X$. Also $\mathcal{P} \subseteq W - 0 \subseteq X$. Thus $\mathcal{M} = \bar{B}(1) = \mathcal{P} \sqcup B(1) \subseteq X$.
3. X is bounded but contains $\mathcal{M}$, so one may choose $h \in \mathbb{N}$ such that $X \subseteq B(-h)$. Let $\psi(x)$ be the formula expressing $x^h \in X$, and set $Y := \psi(K) - \psi(K)$. Observe that Y is $\exists$-C-definable. It remains to show that Y is bounded by $\mathcal{O}$ and that $\mathcal{M} \subseteq Y$.

   If $va \leq -1$ then $va^h = hva \leq -h$. Thus if $va \leq -1$, then $a^h \notin B(-1) \supseteq X$ and $a \notin \psi(K)$. Therefore, $\psi(K) \subseteq \mathcal{O}$. By Lemma 2.2, $Y = \psi(K) - \psi(K) \subseteq \mathcal{O} - \mathcal{O} = \mathcal{O}$.

   Since $\mathcal{P}^h \subseteq S(h)$ (where $\mathcal{P}^h$ is the set of h-th powers of elements of $\mathcal{P}$) and $S(h) \subseteq \mathcal{M} \subseteq X$; we have that $\mathcal{P} \subseteq \psi(K)$. Thus $\mathcal{P} - \mathcal{P} \subseteq \psi(K) - \psi(K)$. By Lemma 2.2, $B(1) \subseteq \mathcal{P} - \mathcal{P}$; thus $B(1) \subseteq \psi(K) - \psi(K)$. Since $0^h = 0 \in \mathcal{M} \subseteq X$, $0 \in \psi(K)$ and $\mathcal{P} - 0 \subseteq \psi(K) - \psi(K)$. By another application of Lemma 2.2, this means that $\mathcal{M} = \mathcal{P} \sqcup B(1) \subseteq \psi(K) - \psi(K) = Y$, as required. ⊣

2.4. The $\exists$-$\emptyset$-definition of $\mathbb{F}_q[[t]]$ in $\mathbb{F}_q((t))$. Finally, we consider the special case where F is the finite field $\mathbb{F}_q$ for q a prime power. Thus now we have $K = \mathbb{F}_q((t))$, $\mathcal{O} = \mathbb{F}_q[[t]]$, and $\mathcal{M} = t\mathbb{F}_q[[t]]$.

LEMMA 2.7. *There exists an $\exists$-$\mathbb{F}_q$-definable bounded neighbourhood of* 0.

PROOF. $\mathbb{F}_q \subseteq K$ is relatively algebraically closed in K and is not separably closed. By Proposition 2.5 there exists V with the required properties. ⊣

PROPOSITION 2.8. *$\mathcal{O}$ is $\exists$-$\mathbb{F}_q$-definable in K.*

PROOF. We combine Lemma 2.7 and Proposition 2.6 to obtain an $\exists$-$\mathbb{F}_q$-definable set Y which contains $\mathcal{M}$ and is bounded by $\mathcal{O}$. Note that $\mathbb{F}_q$ is the set of zeros of the polynomial $x^q - x$ in K. Let χ be the formula $\exists y(y^q - y = 0 \wedge x \in y + Y)$. This is obviously an $\exists$-$\mathbb{F}_q$-formula. Since $\mathcal{O} = \mathbb{F}_q + \mathcal{M}$ and $\mathcal{M} \subseteq Y \subseteq \mathcal{O}$, it is clear that $\chi(K) = \mathcal{O}$. ⊣

We will improve Proposition 2.8 by removing the parameters. In the definition of the set $U_{f,a}$ we used a and the coefficients of f as parameters. All of these come from $\mathbb{F}_q$, but not necessarily from $\mathbb{F}_p$. Although elements of $\mathbb{F}_q$ are not closed terms, they are algebraic over $\mathbb{F}_p$. We use this algebraicity and a few simple tricks to find an existential formula with no parameters which defines $\mathcal{O}$.

LEMMA 2.9. *There exists an $\exists$-$\emptyset$-definable bounded neighbourhood of* 0.

PROOF. We seek a polynomial $f \in \mathbb{F}_p[x]$ which is irreducible in $\mathbb{F}_q[x]$ and is such that not all elements of $\mathbb{F}_q$ are roots of Df, i.e., $x^q - x \nmid Df$.

Write $q = p^k$ and let l be the least prime not dividing k. We claim that $l \leq 1 + k$. To see this, let P be the set of primes dividing k and consider $1 + \prod_{p \in P} p \leq 1 + k$. Then Euclid's famous argument shows that there is a prime $l \notin P$ (thus not dividing k) which does divide $1 + \prod_{p \in P} p$.

As a consequence $l \leq p^k = q$. Let $f \in \mathbb{F}_p[x]$ be an irreducible polynomial of degree l. Since $l \nmid k$, f is still irreducible in $\mathbb{F}_q[x]$. Furthermore, Df is of degree $\leq l - 1 < q$. Thus it cannot be the case that every element of $\mathbb{F}_q$ is a zero of Df. For any $a \in \mathbb{F}_q$ which is not a zero of Df, $U_{f,a} = \{\frac{1}{f(x)} - \frac{1}{f(a)} \mid x \in K\}$ is an $\exists$-$\mathbb{F}_q$-definable bounded neighbourhood of 0. We note that the only parameter in this definition not from $\mathbb{F}_p$ is a.

The union of finitely many bounded neighbourhoods of 0 is also a bounded neighbourhood of 0. Thus the formula ζ, which is defined to be

$$\exists y\ (y^q - y = 0 \wedge \neg Df(y) = 0 \wedge x \in U_{f,y}),$$

is an $\exists$-$\mathbb{F}_p$-formula which defines the union

$$V := \bigcup \{U_{f,a} \mid a \in \mathbb{F}_q, Df(a) \neq 0\}.$$

Finally note that each element of $\mathbb{F}_p$ is the image of a closed term; thus each remaining parameter can be replaced by a closed term and we are left with an $\exists$-$\emptyset$-definition of V. ⊣

REMARK 2.10. Here is an alternative method to find an irreducible separable polynomial $f \in \mathbb{F}_p[x]$ and an element $a \in \mathbb{F}_p$ which is not a root of Df.

Let l be a prime such that $p \nmid l \nmid k$. Let $g \in \mathbb{F}_p[x]$ be any monic irreducible polynomial of degree l. Since $l \nmid k$, g is still irreducible over $\mathbb{F}_q$. Let α be a root of g in a field extension. Either the coefficient of x^{l-1} in g is nonzero; or else we consider $h := g(x-1)$, which is the minimal polynomial of $\alpha + 1$. The coefficient of x^{l-1} in h is then $l \neq 0$. Thus we may assume that the x^{l-1} term in g is nonzero. The polynomial $f := x^l g(1/x)$ is the minimal polynomial of $1/\alpha$ and has nonzero linear term. Therefore, $Df(0) \neq 0$. Thus $U_{f,0}$ is an $\exists$-$\mathbb{F}_p$-definable bounded neighbourhood of 0. As before, elements of $\mathbb{F}_p$ are closed terms, so we may remove all parameters from the definition.

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Finally, we prove Theorem 1.1.

THEOREM 1.1. *$\mathcal{O}$ is $\exists$-$\emptyset$-definable in K.*

PROOF. From Lemma 2.9 we obtain an $\exists$-$\emptyset$-definable bounded neighbourhood of 0. Using again Proposition 2.6, we obtain an $\exists$-$\emptyset$-definable set Y which contains $\mathcal{M}$ and is bounded by $\mathcal{O}$. As before, we let χ be the formula

$$\exists y\ (y^q - y = 0 \wedge x \in y + Y).$$

This is an $\exists$-formula with no parameters and it defines $\mathcal{O}$. ⊣

Nevertheless the formula still depends on $\mathbb{F}_q$ in several ways: our choices of m and h in Proposition 2.6 and our choice of f in Theorem 1.1 depend on $\mathbb{F}_q$. The number q also appears directly in several of the formulas. All these factors tell us that χ is highly nonuniform in q. In fact, in the recent paper of Cluckers, Derakhshan, Leenknegt, and Macintyre ([2]) it is shown that no definition exists which is uniform in p or in k (where $q = p^k$).

REMARK 2.11. With a little more effort we can be more explicit about the formula χ. Suppose for the moment that $K = \mathbb{F}_p((t))$. Let $\wp := x^p - x$ and let $f := \wp - 1$. Observe that $\wp - 1$ is separable and irreducible in $K[x]$ and $Df(1) = D(\wp)(1) = -1 \neq 0$. Denote $\mathbf{x} = (x_1, \ldots, x_4)$ and $\mathbf{y} = (y_1, \ldots, y_4)$. Working back through the formulas and rearranging, we find that $\mathbb{F}_p[[t]]$ is defined by

$$\exists ab\mathbf{xy} \begin{pmatrix} \wp(x - a + b) = 0 \ \wedge \ a^h = x_1 - x_2 \ \wedge \\ b^h = x_3 - x_4 \ \wedge \bigwedge_{i=1}^{4} f(y_i)(x_i^m - 1) - 1 = 0 \end{pmatrix},$$

where $h, m \in \mathbb{N}$ are chosen as in the proof of Proposition 2.6.

§3. Extensions of the result.

3.1. The field $\bigcup_{n\in\mathbb{N}} \mathbb{F}_q((t^{1/n}))$ of Puiseux series. Let $K^{\mathrm{Px}} := \bigcup_{n\in\mathbb{N}} \mathbb{F}_q((t^{1/n}))$ denote the field of Puiseux series over $\mathbb{F}_q$, where $(t^{1/n})_{n\in\mathbb{N}}$ is a compatible system of n-th roots of t (for $n \in \mathbb{N}$). Note that K^{Px} can be formally defined as a direct limit. Let $\mathcal{O}^{\mathrm{Px}} := \bigcup_{n\in\mathbb{N}} \mathbb{F}_q[[t^{1/n}]]$ denote the valuation ring of the t-adic valuation. Note that the value group is $\mathbb{Q}$.

The following theorem is the first example of an $\exists$-$\emptyset$-definition of a nontrivial valuation ring with divisible value group.

THEOREM 3.1. *$\mathcal{O}^{\mathrm{Px}}$ is $\exists$-$\emptyset$-definable in K^{Px}.*

PROOF. By Theorem 1.1, we may let χ be an $\exists$-formula (with no parameters) which defines $\mathcal{O}$ in K. In each field $\mathbb{F}_q((t^{1/n}))$ the formula χ defines the valuation ring $\mathbb{F}_q[[t^{1/n}]]$ since each of these fields is isomorphic to $\mathbb{F}_q((t))$. In the union, χ defines the union of the valuation rings (in any union of structures an existential formula defines the unions of sets that it defines in each of the structures). Thus χ defines $\mathcal{O}^{\mathrm{Px}} = \bigcup_{n\in\mathbb{N}} \mathbb{F}_q[[t^{1/n}]]$, as required. ⊣

3.2. The perfect hull $\mathbb{F}_q((t))^{\mathrm{perf}}$. Let $K^{\mathrm{perf}} := \bigcup_{n\in\mathbb{N}} \mathbb{F}_q((t^{p^{-n}}))$ be the *perfect hull* of $\mathbb{F}_q((t))$; this is also formally defined as a direct limit. Now we use Theorem 1.1 to existentially define the valuation ring $\mathcal{O}^{\mathrm{perf}} := \bigcup_{n\in\mathbb{N}} \mathbb{F}_q[[t^{p^{-n}}]]$ in K^{perf}.

THEOREM 3.2. *$\mathcal{O}^{\mathrm{perf}}$ is $\exists$-$\emptyset$-definable in K^{perf}.*

PROOF. The proof is almost identical to the proof of Theorem 3.1. ⊣

3.3. Consequences for $\exists$-definability in $\mathcal{L}_{\text{val}}$. We return to the field $\mathbb{F}_q((t))$ and its valuation ring $\mathbb{F}_q[[t]]$. Let $\mathcal{L}_{\text{val}} := \mathcal{L}_{\text{ring}} \cup \{O\}$ be the language of valued fields, so that $(\mathbb{F}_q((t)), \mathbb{F}_q[[t]])$ is an $\mathcal{L}_{\text{val}}$-structure. The most important consequence of Theorem 1.1 is that questions of existential definability in $\mathcal{L}_{\text{val}}$ reduce to questions of existential definability in $\mathcal{L}_{\text{ring}}$, in the field $\mathbb{F}_q((t))$.

Let $C \subseteq \mathbb{F}_q((t))$ be any subfield of parameters.

PROPOSITION 3.3. *Let $\phi \in \mathcal{L}_{\text{val}}$ be an existential formula with parameters in C. Then there exists an existential formula $\psi \in \mathcal{L}_{\text{ring}}$ with parameters in C such that ϕ and ψ are equivalent modulo the theory of $\mathbb{F}_q((t))$.*

PROOF. Recall that we denote $K := \mathbb{F}_q((t))$, $\mathcal{O} := \mathbb{F}_q[[t]]$, and $\mathcal{M} := t\mathbb{F}_q[[t]]$. Let $\mathbf{b} = (b_i)_{i<q}$ be some indexing of the field $\mathbb{F}_q$ such that $b_0 = 0$. Let ρ be a quantifier-free formula in free variables $\mathbf{y} = (y_i)_{i<q}$ expressing the quantifier-free type of $\mathbf{b}$. We let π be the formula

$$\exists \mathbf{y} \left(x \in O \wedge \rho(\mathbf{y}) \wedge \bigwedge_{0<i<q} y_i + x \in O^{\times} \right).$$

We claim that π existentially defines $\mathcal{M}$. Let $a \in \mathcal{O}$. Then $a \in \mathcal{M}$ if and only if, for each $b \in \mathbb{F}_q^{\times}$, $a + b \in \mathcal{O} \setminus \mathcal{M} = \mathcal{O}^{\times}$; that is if and only if $K \models \pi(a)$. Thus π is an $\exists$-$\emptyset$-definition for $\mathcal{M}$. Consequently, $K \setminus \mathcal{O} = (\mathcal{M} \setminus \{0\})^{-1}$ is $\exists$-$\emptyset$-definable; and so $\mathcal{O}$ is $\forall$-$\emptyset$-definable.

Since $\mathcal{O}$ is both $\forall$-$\emptyset$-definable and $\exists$-$\emptyset$-definable, we may convert any $\exists$-C-formula ϕ of $\mathcal{L}_{\text{val}}$ into an $\exists$-C-formula ψ of $\mathcal{L}_{\text{ring}}$. ⊣

COROLLARY 3.4. *Hilbert's* 10*th problem has a solution over* $\mathbb{F}_q((t))$ *if and only if it does so over* $\mathbb{F}_q[[t]]$, *in any language which expands the language of rings by adding constants from* $\mathbb{F}_q[[t]]$.

PROOF. Let ϕ be a quantifier-free formula with $\mathbf{x}$ the tuple of free-variables. Suppose that Hilbert's 10th problem (H10) has a solution over $\mathbb{F}_q((t))$. In order to decide the existential sentence $\exists \mathbf{x}\, \phi(\mathbf{x})$ in $\mathbb{F}_q[[t]]$ we apply our algorithm for $\mathbb{F}_q((t))$ to the sentence

$$\exists \mathbf{x} \left(\phi(\mathbf{x}) \wedge \bigwedge_{x \in \mathbf{x}} \chi(x) \right),$$

where χ denotes the existential formula defining $\mathbb{F}_q[[t]]$ in $\mathbb{F}_q((t))$.

Conversely, suppose that H10 has a solution over $\mathbb{F}_q[[t]]$. By standard equivalences in the theory of fields we may assume that ϕ is the formula $f = 0$ for some polynomial $f \in \mathbb{F}_p[\mathbf{x}]$.

We need to find a quantifier-free formula which is realised in $\mathbb{F}_q[[t]]$ if and only if f has a zero in $\mathbb{F}_q((t))$. We adopt the convention that tuples are allowed to be empty, so the empty tuple is a subtuple of any tuple. For a variable $x \in \mathbf{x}$ we let d_x denote the degree of f in x; and for any subtuple $\mathbf{x}' \subseteq \mathbf{x}$ we let $\mathbf{x}'' := (\mathbf{x} \setminus \mathbf{x}') \cup \{x^{-1} | x \in \mathbf{x}'\}$ be a new tuple formed from $\mathbf{x}$ by inverting the elements of $\mathbf{x}'$. Then we set $f_{\mathbf{x}'} := f(\mathbf{x}'') \prod_{x \in \mathbf{x}'} x^{d_x}$. Importantly, $f_{\mathbf{x}'}$ is a polynomial. Note that if $\mathbf{x}'$ is empty, then $f_{\mathbf{x}'} = f$. Finally we let ϕ' be the formula

$$\bigvee_{\mathbf{x}' \subseteq \mathbf{x}} \left(f_{\mathbf{x}'} = 0 \wedge \bigwedge_{x \in \mathbf{x}'} \neg x = 0 \right).$$

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Then $\mathbb{F}_q((t)) \models \exists \mathbf{x}\ f(\mathbf{x}) = 0$ if and only if $\mathbb{F}_q[[t]] \models \exists \mathbf{x}\ \phi'(\mathbf{x})$. Therefore, in order to decide $\exists \mathbf{x}\ \phi(\mathbf{x})$ in $\mathbb{F}_q((t))$ we apply our algorithm for $\mathbb{F}_q[[t]]$ to the existential sentence $\exists \mathbf{x}\ \phi'(\mathbf{x})$. ⊣

Appendix A. The fields $\mathbb{C}((t))$ and $\mathbb{Q}_p((t))$. The following observation is well-known, but we give the proof here for completeness.

Observation A.1. *$K[[t]]$ is not $\exists$-K-definable in $K((t))$ for $K = \mathbb{Q}_p, \mathbb{C}$.*

Proof. Let $K((t))^{\mathrm{Px}} := \bigcup_{n\in\mathbb{N}} K((t^{1/n}))$ be the field of Puiseux series and let $K[[t]]^{\mathrm{Px}} := \bigcup_{n\in\mathbb{N}} K[[t^{1/n}]]$. If $K[[t]]$ is $\exists$-K-definable in $K((t))$ then $K[[t]]^{\mathrm{Px}}$ is $\exists$-K-definable in $K((t))^{\mathrm{Px}}$ by the same formula. If $K = \mathbb{C}$ then, by Puiseux's Theorem, $\mathbb{C}((t))^{\mathrm{Px}}$ is algebraically closed and thus no infinite co-infinite subsets are definable. In particular, $\mathbb{C}[[t]]^{\mathrm{Px}}$ is not definable.

Now let $K = \mathbb{Q}_p$ and let ϕ be an existential formula (with parameters). Suppose that ϕ defines $\mathbb{Q}_p[[t]]$ in $\mathbb{Q}_p((t))$; then in $\mathbb{Q}_p((t))^{\mathrm{Px}}$ the formula ϕ defines $\mathbb{Q}_p[[t]]^{\mathrm{Px}}$, which is a proper subring. Note also that $\mathbb{Q}_p$ is contained in this definable set. The field $\mathbb{Q}_p((t))^{\mathrm{Px}}$ is p-adically closed, thus $\mathbb{Q}_p \preceq \mathbb{Q}_p((t))^{\mathrm{Px}}$. Thus ϕ defines $\mathbb{Q}_p$ in $\mathbb{Q}_p$, which is *not* a proper subset. This is a contradiction because $\mathbb{Q}_p$ is an elementary substructure of $\mathbb{Q}_p((t))^{\mathrm{Px}}$. ⊣

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