Pattern equivariant functions, deformations and equivalence of tiling spaces

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Abstract. We re-investigate the theory of deformations of tilings using P-equivariant cohomology. In particular, we relate the notion of asymptotically negligible shape functions introduced by Clark and Sadun to weakly P-equivariant forms. We then investigate more closely the relation between deformations of patterns and homeomorphism or topological conjugacy of pattern spaces.

1. *Introduction*

The study of aperiodic systems in physics or geometry has led to the definition of cohomology groups associated with aperiodic tilings or point sets of $\mathbb{R}^n$ (we use here the word pattern to mean either of them). In physics some elements of these groups are related (via K-theory and cyclic cohomology) to topologically quantized transport properties; see [**KR06**] for a recent overview. In geometry, Sadun, Williams and Clark have given an interpretation of the cohomology group (with values in $\mathbb{R}^n$) in terms of deformation theory of tilings [**SW03, CS06**]. In short, an (admissible) 1-cocycle defines a deformation of a tiling by redefining the shape of its tiles. If the cocycle is a coboundary then the deformed tiling is locally derivable from the original one. A deformation alters the properties of the dynamical system associated with the tiling, except if the new tiling is topologically conjugate to the old one, a notion which is, however, strictly weaker than being mutually locally derivable. To capture also topological conjugacy in cohomological terms, Clark and Sadun introduced the concept of asymptotically negligible cocycles and showed that such cocycles yield deformations which are topologically conjugate.

Clark and Sadun used a formulation of tiling cohomology which is based on the Anderson–Putnam–Gähler construction. This construction furnishes a system of finite CW-complexes so that one can make use of their cellular cohomology. Our aim here is to provide a formulation of deformation theory in terms of pattern equivariant functions making use of de Rham cohomology. From this point of view a deformation of a pattern

is given by a pattern equivariant 1-form. This allows us to formulate the somewhat ad hoc notion of asymptotically negligible cocycles in terms of natural analytic properties of pattern equivariant functions (Theorem 3.4). In fact, as was introduced in [**KP06**], there are naturally strongly and weakly pattern equivariant forms on $\mathbb{R}^n$ and we find that the theory of deformation modulo topological conjugacy of a pattern P is related to the mixed quotient

$$Z^1_{s-P}(\mathbb{R}^n, \mathbb{R}^n)/B^1_{w-P}(\mathbb{R}^n, \mathbb{R}^n) \cap Z^1_{s-P}(\mathbb{R}^n, \mathbb{R}^n),$$

where $Z^1_{s-P}(\mathbb{R}^n, \mathbb{R}^n)$ denotes the $\mathbb{R}^n$-valued strongly P-equivariant closed 1-forms and $B^1_{w-P}(\mathbb{R}^n, \mathbb{R}^n)$ the $\mathbb{R}^n$-valued weakly P-equivariant exact 1-forms. A study of the mixed quotient for certain classes of tilings is under investigation; for substitution tilings the results of [**CS06**] point out that this is worthwhile.

As has already been observed by Sadun and Williams, deformations give rise to homeomorphisms between tiling spaces. We review this result in the framework of P-equivariant functions. We find that homeomorphisms coming from deformations preserve the canonical transversals and show that this property characterizes such homeomorphisms: any homeomorphism between two pattern spaces preserving the canonical transversals and identifying the two patterns comes from a deformation (Corollary 4.11).

We further investigate deformations which give (topologically) conjugate patterns. Our Theorem 4.13 can be seen as the analogue of Clark and Sadun's result about asymptotically negligible cocycles: deformations differing from the identity by a weakly P-equivariant 1-form lead to conjugate patterns. More precisely, we find that they give rise to a second homeomorphism which also identifies P with its deformed pattern, but does not preserve the canonical transversal. Instead it commutes with the $\mathbb{R}^n$ actions. Under an additional assumption on the deformation, which we call boundedness, a converse can be obtained: bounded deformations which yield (pointed) conjugate patterns differ from the identity by a weakly P-equivariant 1-form (Theorem 4.15).

Finally, we present a detailed analysis of the question of invertibility of deformations (which somehow is hidden in the notion of admissibility in [**CS06**]). We show that there exists a neighbourhood of the identity deformation which contains only invertible deformations.

2. Preliminaries

2.1. *Pattern spaces and their dynamical systems.*　The objects we are interested in and which we want somehow to compare are closed subsets of Euclidean space, mostly of finite local complexity. These will mostly be uniformly discrete sets, but for comparison with [**CS06**] we consider also tilings; for shortness we call them patterns. A tiling can either be viewed as a covering of Euclidean space by compact sets, which we assume for simplicity to be polyhedral and touch face to face, or as a closed subset, namely the set given by the boundary points of its tiles. The latter point of view is the one taken for the definition of the distance.

Let P, Q be two closed subsets of $\mathbb{R}^n$. Define their distance to be

$$D(P, Q) = \inf\left\{\epsilon > 0 \,\middle|\, d_H\left(B_{1/\epsilon}[P] \cup \partial B\left(0, \frac{1}{\epsilon}\right), B_{1/\epsilon}[Q] \cup \partial B\left(0, \frac{1}{\epsilon}\right)\right) \leq \epsilon\right\}.$$

Here $B(x, r)$ is the (open) ball of radius r around x, $\partial B(x, r)$ denotes its boundary,

$$B_r[P] = B(0, r) \cap P,$$

and d_H is the Hausdorff distance. We call $B_r[P]$, $x \in \mathbb{R}^n$, the r-patch of P around zero. Again, for tilings we define their r-patches viewing them as the closed subset given by the boundary points of their tiles.

For a closed subset $P \subset \mathbb{R}^n$ the completion of $\mathrm{orb}(P) := \{P - x \mid x \in \mathbb{R}^n\}$, the $\mathbb{R}^n$-orbit of P, is called the continuous hull of P, written Ω_P,

$$\Omega_P := \overline{\mathrm{orb}(P)}^D.$$

Ω_P is a compact space which carries a continuous action of $\mathbb{R}^n$ induced by the translation action on P. We denote the action of x by $\omega \mapsto \omega - x$. This defines the (continuous) dynamical system $(\Omega_P, \mathbb{R}^n)$ of P.

P is uniformly discrete if its points have a minimal distance, i.e. if

$$r_{\min}(P) = \inf\{|x - y| : x \neq y \in P\}$$

is strictly positive. If $h \in \mathbb{R}^n$ and $|h| < r_{\min}/2$ then $D(P, P - h) \leq |h|$.

P is relatively dense if it does not have arbitrarily large holes, i.e. there exists r such that, for all $x \in \mathbb{R}^n$, $B_r[P - x]$ contains at least one point. A uniformly discrete and relatively dense set is called a Delone set.

If P is uniformly discrete we consider also its discrete hull

$$\Xi_P := \overline{\{P - p \mid p \in P\}}^D.$$

It is a transversal for the $\mathbb{R}^n$ action and also referred to as the canonical transversal of P. If T is a tiling we associate to it the discrete set T^0 containing the vertices of its tiles. Most of the statements below about discrete sets P apply to tilings when we take $P = T^0$. For instance, we define the transversal of a tiling T to be $\Xi_T := \overline{\{T - p \mid p \in T^0\}}^D$ †.

P (or T) has finite local complexity (with respect to the translation group) if for any given r there are only finitely many distinct r-patches $B_r[P - x]$ with $x \in P$ ($x \in T^0$).

Finite local complexity has strong implications on the structure of the hull. First, all elements of the hull of a discrete pattern P with finite local complexity can be viewed as uniformly discrete subsets of $\mathbb{R}^n$ (including, if P is not relatively dense, the empty set). Second, such a hull carries an equivalent but somewhat simpler metric, namely

$$D_t(Q, Q') = \inf_{r > 0}\{r^{-1} \mid B_r[Q] \cong_{r^{-1}} B_r[Q']\},$$

where we write $B_r[Q] \cong_{r^{-1}} B_r(Q')$ for there exists $x, x' \in B(0, 1/2r)$ with $B_r[Q - x] = B_r[Q' - x']$. Restricted to the canonical transversal, the metric is even equivalent to

$$D_0(Q, Q') = \inf_{r > 0}\{r^{-1} \mid B_r[Q] = B_r[Q']\}.$$

Third, the canonical transversal Ξ_P is totally disconnected. Finally, at least if P is also relatively dense, the hull is a foliated space whose leaves are locally homeomorphic to $\mathbb{R}^n$.

† Another transversal of the hull of a tiling would be obtained if one replaced T^0 by a set of punctures; that is, a set containing a point from each tile. The results would be similar.

To see this let $U_\epsilon(Q)$ be the open ϵ-ball around $Q \in \Omega_P$ with respect to the metric D_t and $T_{Q,\epsilon,V} = \{Q' - v \mid B_{1/\epsilon}[Q'] = B_{1/\epsilon}[Q], \ v \in V\}$ where $V \subset \mathbb{R}^n$. Then for any small enough ϵ one finds $\epsilon_1, \epsilon_2 > 0$ such that $U_{\epsilon_1}(Q) \subset T_{Q,\epsilon,B(0,\epsilon)} \subset U_{\epsilon_2}(Q)$. If P is relatively dense then translates $U_\epsilon(Q) - x$ with $Q \in \Xi_P$ and $|x| < r_{\max}(P)$ generate the topology and hence the family of sets $T_{Q,\epsilon,B(x,\epsilon)}, Q \in \Xi_P, |x| < r_{\max}(P), \epsilon < r_{\min}(P)/2$ generates the topology as well. Now $T_{Q,\epsilon,B(x,\epsilon)}$ is the homeomorphic image of $T_{Q,\epsilon,\{0\}} \times B(x, \epsilon)$ under the map $(Q', v) \mapsto Q' - v$. We thus have charts which are Cartesian products of totally disconnected sets (clopen subsets of Ξ_P) with open balls of $\mathbb{R}^n$. These are the foliation charts. This has two consequences which will be important further down: first, the path-connected components of Ω_P are the orbits under the translation action; second, the pre-image under the map $\mathbb{R}^n \to \mathrm{orb}(P)$: $x \mapsto P - x$ of the path-connected component of P in $\mathrm{orb}(P) \cap U_\epsilon(P)$ lies in $B(0, \epsilon)$.

2.2. *Notions of equivalence.*

There are a variety of equivalence relations between patterns. Some of them are expressed directly between the patterns, as, for instance, mutual local derivability.

Definition 2.1. [**BSJ91**] Let P and Q be subsets of $\mathbb{R}^n$. Q is locally derivable from P if there exists an $R > 0$ such that, for all $x, y \in \mathbb{R}^n$,

$$B_R[P - x] = B_R[P - y] \Longrightarrow \{0\} \cap (Q - x) = \{0\} \cap (Q - y).$$

We say that R is a derivability range of the local derivation.

Equivalently, Q is locally derivable from P if and only if for all $r > 0$ there exists r' such that, for all $x, y \in \mathbb{R}^n$,

$$B_{r'}[P - x] = B_{r'}[P - y] \Longrightarrow B_r[Q - x] = B_r[Q - y].$$

This has the following interpretation which explains the name: one can construct the r-patch of Q around x from the r'-patch of P around x.

Other notions of equivalence refer to the spaces and dynamical systems defined by the patterns. For instance, we could regard patterns as equivalent if they have:
(1)   homeomorphic hulls; or
(2)   topological conjugate (continuous) dynamical systems.
Relation (2) implies relation (1) (trivially). In fact, we will find a stronger relation associated with a deformation, namely an invertible deformation P' of P (we explain the notion further down) leads to a homeomorphism between the hulls which preserves the canonical transversal and identifies P with P'. Likewise we find a relation stronger than (2), namely we say that P' is pointed conjugate to P if there is a topological conjugacy between the (continuous) dynamical systems which maps P to P'. We will also say that P' is pointed semi-conjugate to P if there is a continuous surjection $\Omega_P \to \Omega_{P'}$ which commutes with the action of $\mathbb{R}^n$ and maps P to P'. Note that pointed conjugacy implies homeomorphic hulls but the conjugacy will not, in general, preserve the canonical transversal.

2.3. *P-equivariant functions.* Topological spaces can equivalently be described by commutative C^*-algebras. Here these algebras consist of P-equivariant functions. For that we need to assume that our subset P has finite local complexity.

Roughly speaking, a function f on $\mathbb{R}^n$ is P-equivariant with range R if it satisfies the following property: if the patterns in P surrounding two points x and y are equal in a ball of radius R (after translating by $-x$ and $-y$, respectively), then f must take the same values at x and y. More precisely, we have the following.

Definition 2.2. [**Ke03**] Let f be a function defined on $\mathbb{R}^n$. We say that f is P-equivariant with range R if, for $x, y \in \mathbb{R}^n$,

$$B_R[P - x] = B_R[P - y] \Longrightarrow f(x) = f(y).$$

f is strongly P-equivariant if it is P-equivariant with finite range. We also call a function f defined on a subset $Q \subset \mathbb{R}^n$ which is locally derivable from P strongly P-equivariant if it satisfies the above conditions for $x, y \in Q$.

Note that Q is locally derivable from P if and only if the indicator function on Q is strongly P-equivariant.

A priori our definition does not require a strongly P-equivariant function to satisfy any further regularity conditions. By adding such conditions we arrive naturally at the concept of weakly P-equivariant functions. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and

$$s_k(f) = \sup\{|D^\alpha f(x)| : x \in \mathbb{R}^n, |\alpha| \le k\},$$

$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, where $D^\alpha f = ((\partial^{\alpha_1}/\partial^{\alpha_1} x_1) \cdots (\partial^{\alpha_n}/\partial^{\alpha_n} x_n)) f$, and $| \cdot |$ denotes any norm on $\mathbb{R}^m$ but $|\alpha| = \sum_i^n \alpha_i$.

Definition 2.3. We denote by $C_{s-P}^k(\mathbb{R}^n, \mathbb{R}^m)$, $k \in \mathbb{N} \cup \{\infty\}$ the space of strongly P-equivariant C^k-functions over $\mathbb{R}^n$ with values in $\mathbb{R}^m$. We say that a C^k-function f is weakly P-equivariant if, for all $\epsilon > 0$, there exists a strongly P-equivariant C^k-function f_ϵ such that $s_k(f - f_\epsilon) < \epsilon$, if k is finite, and, for all $l \in \mathbb{N}$, $s_l(f - f_\epsilon) < \epsilon$, if $k = \infty$. The space of weakly P-equivariant C^k-functions over $\mathbb{R}^n$ with values in $\mathbb{R}^m$ is denoted by $C_{w-P}^k(\mathbb{R}^n, \mathbb{R}^m)$.

Since strongly P-equivariant functions are bounded, $C_{w-P}^k(\mathbb{R}^n, \mathbb{R}^m)$ is the closure of $C_{s-P}^k(\mathbb{R}^n, \mathbb{R}^m)$ in the space of functions $f : \mathbb{R}^n \to \mathbb{R}^m$ which have bounded continuous derivatives up to order k with respect to the topology induced by s_k, if k is finite, or by all s_l, $l \in \mathbb{N}$, if $k = \infty$. Smooth strongly or weakly P-equivariant forms can then be identified with elements of $C_{s-P}^\infty(\mathbb{R}^n, \Lambda \mathbb{R}^{n*})$ or $C_{w-P}^\infty(\mathbb{R}^n, \Lambda \mathbb{R}^{n*})$, respectively. As usual, $df = \sum_i (\partial/\partial x_i) f \, dx_i$, and we view $\{dx_i \mid i = 1, \ldots, n\}$ as a base for $\mathbb{R}^{n*}$.

Complex-valued continuous weakly P-equivariant functions† form a C^*-algebra. The map $f \mapsto f_P$, $f_P(x) := f(P - x)$ induces an algebra isomorphism between $C(\Omega_P)$ and $C_{w-P}^0(\mathbb{R}^n, \mathbb{C})$. The compactness of Ω_P is reflected in the following lemma whose simple proof is left to the reader.

† These were also simply called P-equivariant functions over $\mathbb{R}^n$.

LEMMA 2.4. *Let P be a uniformly discrete set with finite local complexity. If $f \in C^0_{w-P}(\mathbb{R}^n, \mathbb{R}^m)$ then its image is compact. If even $f \in C^0_{s-P}(\mathbb{R}^n, \mathbb{R}^m)$ then its restriction to P takes only finitely many values.*

We now present some general results about P-equivariant functions.

LEMMA 2.5. *Let P be a uniformly discrete set of finite local complexity. Let φ and ψ be two differentiable functions which coincide on P. Suppose that $d\varphi$ and $d\psi$ are strongly P-equivariant. Then $\varphi - \psi$ is strongly P-equivariant.*

Proof. Let $\eta = \varphi - \psi$. Hence η vanishes on P and $d\eta$ is P-equivariant, say with range R. Let $r = R + r_{\min}$. Let $B_r[P - x] = B_r[P - y]$. Choose $p \in P$ such that $p - x \in B_{r_{\min}}[P - x]$. Let $q = p - x + y$. Then $q - y = p - x \in B_{r_{\min}}[P - y]$ and, by P-equivariance of $d\eta$,

$$\eta(x) - \eta(p) = \int_p^x d\eta = \int_q^y d\eta = \eta(y) - \eta(q).$$

Since $\eta(p) = \eta(q)$, η is strongly P-equivariant.   □

Let $f : P \to \mathbb{R}^n$ be a function. For $h = P - P := \{p - q \mid p, q \in P\}$, we define $\Delta_h f : P \cap (P - h) \to \mathbb{R}^n$ by

$$\Delta_h f(x) = f(x + h) - f(x).$$

LEMMA 2.6. *Let $\psi : P \to \mathbb{R}^m$ be a function on a uniformly discrete set P with finite local complexity.*

(1) *If ψ is strongly P-equivariant there exists a smooth strongly P-equivariant function $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ which restricts on P to ψ.*

(2) *If, for all $\epsilon > 0$, there exists a strongly P-equivariant function $\psi_\epsilon : P \to \mathbb{R}^m$ such that, for all $p \in P$, $|\psi(p) - \psi_\epsilon(p)| < \epsilon$, then there exists a smooth weakly P-equivariant function $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ which restricts on P to ψ.*

(3) *Suppose that P is, moreover, relatively dense. If, for all $h \in P - P$, the function $\Delta_h \psi : P \cap (P - h) \to \mathbb{R}^n$ is strongly P-equivariant then there exists a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ which restricts on P to ψ and has strongly P-equivariant differential $d\varphi$.*

Proof. Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a smooth positive function with support contained in $B(0, r)$, $r \leq r_{\min}/2$. For the first two parts of the lemma we consider

$$\varphi(x) = \sum_{p \in P} \rho(x - p)\psi(p)$$

and suppose that $\rho(0) = 1$ so that $\varphi(p) = \psi(p)$ for $p \in P$. If ψ is P-equivariant with range R then φ is smooth and P-equivariant with range $R + r_{\min}/2$.

Suppose that ψ satisfies the conditions of the second part. Then, by the first part, $\varphi_\epsilon(x) = \sum_{p \in P} \rho(x - p)\psi_\epsilon(p)$ defines a smooth strongly P-equivariant function. Now

$$|D^\alpha(\varphi(x) - \varphi_\epsilon(x))| \leq \|\psi - \psi_\epsilon\|_\infty \left| \sum_{p \in p} D^\alpha \rho(x - p) \right| \leq \epsilon \|D^\alpha \rho\|_\infty$$

and hence φ_ϵ tends to φ in the Fréchet topology when ϵ tends to zero.

For the third part consider the Voronoi domains V_p for $p \in P$. Let A be an upper bound for their diameter; it is finite by the relative density of P. Define for all x in the interior $\text{int}(V_p)$ of the domain

$$\tilde{\phi}(x) := \psi(p). \qquad (2.1)$$

Then $\tilde{\phi}$ is defined almost everywhere and

$$\varphi := \rho * \tilde{\phi}$$

is a smooth function where we suppose that $\int \rho = 1$ so that φ coincides with ψ on P provided r is small enough. Now suppose that for all $h = P - P$ with $|h| \le A$, $\Delta_h \psi$ is P-equivariant with range $R > 3A$, and hence, if $B_R(P - p) = B_R(P - q)$ for some $p, q \in P \cap P - h$, then, for almost all $x \in B(0, R)$, $\tilde{\phi}(x + p + h) - \tilde{\phi}(x + p) = \tilde{\phi}(x + h + q) - \tilde{\phi}(x + q)$. This implies that the equation $\tilde{\phi}(x + p) - \psi(p) = \tilde{\phi}(x + q) - \psi(q)$, which by (2.1) holds for $x \in V_p - p$, extends to $x \in B(0, 2A)$. It follows that $\tilde{\phi}(x + p + h) - \psi(p) = \tilde{\phi}(x + h + q) - \psi(q)$ for $x \in B(0, 2A)$ and hence

$$(d\rho * \tilde{\phi})(x + p + h) = (d\rho * (\tilde{\phi} - \psi(q) + \psi(p)))(x + q + h) = (d\rho * \tilde{\phi})(x + q + h)$$

where we have used that $d\rho * (\psi(q) - \psi(p)) = 0$. Since $d\varphi(x) = (d\rho * \tilde{\phi})(x)$ the statement follows. $\qquad \square$

We finally provide a condition under which a differentiable function with strongly P-equivariant differential is weakly P-equivariant.

PROPOSITION 2.7. *Let $P \in \mathbb{R}^n$ be a Delone set of finite local complexity and $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a C^k-function, $1 \le k \in \mathbb{N} \cup \{\infty\}$, with strongly P-equivariant differential. Then φ is weakly P-equivariant if and only if for all ϵ there exists a finite partition of P into subsets $P_1, \dots, P_N$ which are locally derivable from P and such that for all $i \in \{1, \dots, N\}$ for all $p, q \in P_i$: $|\varphi(p) - \varphi(q)| < \epsilon$.*

Proof. Suppose that φ is weakly P-equivariant. Choose ϵ and a strongly P-equivariant function φ_ϵ such that $\|\varphi - \varphi_\epsilon\|_\infty < \epsilon$. In particular, $\varphi_\epsilon(P)$ is a finite set, let us say $\{x_1, \dots, x_N\}$. Define $P_i = \varphi_\epsilon^{-1}(x_i) \cap P$ which is certainly locally derivable from P. The P_i partition P and by construction for all $p, q \in P_i$ we have $|\varphi(p) - \varphi(q)| < 2\epsilon$.

We prove the converse: given δ choose a finite partition $P_1, \dots, P_N$ of sets which are locally derivable from P and such that, for all $i \in \{1, \dots, N\}$ for all $p, q \in P_i$, $|\varphi(p) - \varphi(q)| < \delta$. Note that this and the strong P-equivariance of $d\varphi$ imply that φ has bounded derivatives $D^\alpha \varphi$, $|\alpha| \le k$. Choose a point p_i in each P_i. Consider the Voronoi domains V_p of p in P. For x in the interior of V_p and $p \in P_i$ let $\sigma_\delta(x)$ be given by $\sigma_\delta(x) = \varphi(p) - \varphi(p_i)$. Then σ_δ is defined almost everywhere and is constant on the interior of the Voronoi domains. Since the P_i are locally derivable from P, $\varphi - \sigma_\delta$ is strongly P-equivariant. It is also δ-close to φ where defined and the aim is to smoothen it out.

Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a smooth positive function with support contained in $B(0, 1)$ and $\int \rho = 1$. Let $\rho_\delta(x) = \delta^{-n} \rho(x/\delta)$. Define $\varphi_{\delta_1, \delta_2} = \rho_{\delta_1} * (\varphi - \sigma_{\delta_2})$ which is by construction a smooth strongly P-equivariant function. We claim that for all $\epsilon > 0$ and $l \le k$ there exists δ_1, δ_2 such that $s_l(|\varphi - \varphi_{\delta_1, \delta_2}|) < \epsilon$. This then proves that we can approximate φ in the relevant topology by strongly P-equivariant functions.

To prove the claim we fix ϵ and $l \leq k$. We have

$$\|\rho_{\delta_1} * \sigma_{\delta_2}\|_\alpha \leq \|\sigma_{\delta_2}\|_\infty \|D^\alpha \rho_{\delta_1}\|_1 \leq \delta_2 \delta_1^{-|\alpha|} \|D^\alpha \rho\|_1.$$

Hence

$$s_l(|\varphi - \varphi_{\delta_1, \delta_2}|) \leq s_l(|\varphi - \rho_{\delta_1} * \varphi|) + \sum_{|\alpha| \leq l} \delta_2 \delta_1^{-|\alpha|} \|D^\alpha \rho\|_1.$$

Since $D^\alpha \varphi$ is bounded continuous for $|\alpha| \leq k$ we can find δ_1 such that $s_l(|\varphi - \rho_{\delta_1} * \varphi|) < \epsilon/2$ and then δ_2 such that $\sum_{|\alpha| \leq l} \delta_2 \delta_1^{-|\alpha|} \|D^\alpha \rho\|_1 < \epsilon/2$. □

2.4. *Cohomology groups for patterns and tilings.*

Let $\mathcal{P}$ be a uniformly discrete subset or a tiling, of finite local complexity. Associated with $\mathcal{P}$ is a cohomology group $H(\mathcal{P}, A)$, the pattern cohomology, with coefficients in an abelian group A. It can be described in a variety of equivalent ways as follows.

2.4.1. *Cech-cohomology.*

$H(\mathcal{P}, A)$ is the Cech-cohomology $\check{H}(\Omega_\mathcal{P}, A)$ of the continuous hull.

2.4.2. *Cellular cohomology.*

Let T be a polyhedral tiling, that is a tiling whose tiles are polyhedra and match face to face. We can view T as an infinite CW-complex for $\mathbb{R}^n$ choosing an orientation of its cells such that translational congruent cells have the same orientation. We also assume that T has finite local complexity. The Anderson–Putnam–Gähler construction expresses $\Omega_\mathcal{P}$ as an inverse limit of CW-complexes. We recall the construction: the k-neighbourhood of a tile t in a polyhedral tiling T is the patch of all tiles of T which meet the $(k-1)$-neighbourhood of t. Here the 0-neighbourhood of t is simply t itself. A k-collared tile is a tile labelled with its k-neighbourhood. A k-collared prototile is a translational congruence class of a k-collared tile. Since T has finite local complexity there are only finitely many k-collared prototiles. The CW-complex Γ^k is the complex obtained from the disjoint union of all k-collared prototiles upon identifying any two faces f_1, f_2 of two k-collared prototiles p_1, p_2 if one finds in T two tiles t_1, t_2 whose k-neighbourhood corresponds to the two k-collared prototiles p_1, p_2 such that the faces of t_1 and t_2 which correspond to f_1, f_2 agree in T. There are surjective maps $\alpha_k : \Gamma^{k+1} \to \Gamma^k$ associating to a point in a $k+1$-collared prototile the same point in the k-collared prototile obtained by reducing the labelling of the $(k+1)$-neighbourhood to the k-neighbourhood. Then one has the following result [**G, Sa03**]: Ω_T is the inverse limit of the chain

$$\cdots \xrightarrow{\alpha_{k+1}} \Gamma^{k+1} \xrightarrow{\alpha_k} \Gamma^k \cdots \xrightarrow{\alpha_0} \Gamma^0.$$

Moreover, we have maps $\pi_k : T \to \Gamma^k$ associating to a point x in a tile of T the point at the same position in the k-collared prototile to which it corresponds.

As a consequence, $\check{H}(\Omega_\mathcal{P}, A)$ is the direct limit of the chain

$$H(\Gamma^0, A) \xrightarrow{\alpha_0^*} \cdots H(\Gamma^k, A) \xrightarrow{\alpha_k^*} H(\Gamma^{k+1}, A) \cdots.$$

Here $H(\Gamma^k, A)$ is the cohomology of CW-complexes. In particular, there are homeomorphisms $I_k : H(\Gamma^k, A) \to \check{H}(\Omega_\mathcal{P}, A)$ satisfying $I_k = I_{k+1} \circ \alpha_k^*$.

2.4.3. *P-equivariant cohomology.* The strongly P-equivariant forms on $\mathbb{R}^n$ form a sub-complex of the de Rham complex for $\mathbb{R}^n$. The cohomology of this complex is called the strongly P-equivariant cohomology (of $\mathbb{R}^n$) and is denoted by $H_{s-\mathcal{P}}(\mathbb{R}^n, \mathbb{R})$. It has been shown in [**KP06**] that $H_{s-\mathcal{P}}(\mathbb{R}^n, \mathbb{R})$ is isomorphic to $\check{H}(\Omega_{\mathcal{P}}, \mathbb{R})$. This isomorphism has been revisited in [**Sadun**] where the description of Ω_P as an inverse limit was used. We provide an explicit description of this isomorphism in degree one.

Let T be a polyhedral tiling of finite local complexity and set $P = T^0$. Recall the quotient map $\pi_k : T \to \Gamma^k$. Let e be a 1-cell in Γ^k. Then $\pi_k^{-1}(e)$ is a union of (oriented) edges of T. Suppose that $d\varphi$ is P-equivariant with range r and that r is small compared with k (i.e. the tubular r-neighbourhood of any tile in T is smaller than its k-neighbourhood). Then $\int_{\tilde{e}} d\varphi$ is independent of the choice of $\tilde{e} \in \pi_k^{-1}(e)$, since two such pre-images agree on their k-neighbourhood. Denoting by $Z_{r-P}^1(\mathbb{R}^n, \mathbb{R}^n)$ the closed P-equivariant 1-forms of range r we may thus define $J_r^k : Z_{r-P}^1(\mathbb{R}^n, \mathbb{R}^n) \to Z^1(\Gamma^k, \mathbb{R}^n)$ by

$$J_r^k(d\varphi)(e) = \int_{\tilde{e}} d\varphi = \varphi(t(\tilde{e})) - \varphi(s(\tilde{e})), \tag{2.2}$$

where $s(\tilde{e})$ and $t(\tilde{e})$ are the vertices at which $\tilde{e}$ starts and terminates. It is clear that $J_r^k(d\varphi)$ is closed, i.e. vanishes on cycles. $J_r^k(d\varphi)$ is exact if and only if also $\varphi(s(\tilde{e}))$ is independent of the choice of the pre-image for e which means that φ is strongly P-equivariant (with a range which is large compared to k but finite). It follows that $\Theta : H_{s-P}^1(\mathbb{R}^n, \mathbb{R}^n) \to \check{H}^1(\Omega_P, \mathbb{R}^n)$

$$\Theta[d\varphi] = I_k[J_r^k(d\varphi)]$$

is well defined. It does not depend on the precise choice for r and k as long as $d\varphi$ is P-equivariant with range r and k large compared to r. Using this flexibility one easily sees that Θ is additive and hence defines a group homeomorphism.

We show that Θ is surjective. Let $f \in Z^1(\Gamma^k, \mathbb{R}^n)$, choose $x_0 \in T^0$ and define $\psi : T^0 \to \mathbb{R}^n$ by

$$\psi(t(\tilde{e})) - \psi(s(\tilde{e})) = f(\pi_k(\tilde{e})), \quad \psi(x_0) = 0, \tag{2.3}$$

for all (oriented) edges $\tilde{e}$ of T. Let $h \in \mathbb{R}^n$ be the vector corresponding to the edge $\tilde{e}$. Then $\Delta_h \psi(x) = \Delta_h \psi(y)$ for all $x, y \in T^0 \cap (T^0 - h)$ which have the same $(k+1)$-neighbourhood. It follows that $\Delta_h \psi$ is P-equivariant with a range large compared to $k+1$. By part (3) of Lemma 2.6 there exists a smooth function φ which coincides with ψ on P and has strongly P-equivariant differential. It follows that, for large enough r and l, $J_r^l(d\varphi) = f \circ \alpha_{l-1} \cdots \alpha_k$ and hence $\Theta([d\varphi]) = I_k([f])$. Hence Θ is surjective. To see that Θ is injective we note that, by (2.2), $J_r^k(d\varphi) = 0$ whenever φ is constant on the vertices of the tiling. From Lemma 2.5 it follows then that φ is strongly P-equivariant, and hence $[d\varphi] = 0$.

Also the weakly P-equivariant forms on $\mathbb{R}^n$ form a sub-complex of the de Rham complex for $\mathbb{R}^n$ and hence a cohomology group which is called the weakly P-equivariant cohomology of $\mathbb{R}^n$. We will not consider weakly P-equivariant cohomology in this article, but one of our conclusions will be that the mixed quotient

$$Z_{s-P}^1(\mathbb{R}^n, \mathbb{R}^n)/B_{w-P}^1(\mathbb{R}^n, \mathbb{R}^n) \cap Z_{s-P}^1(\mathbb{R}^n, \mathbb{R}^n)$$

is a parameter space of deformations of P modulo bounded deformations which are pointed topological semi-conjugate to it.

3.   The first cohomology and deformations

3.1.   The approach of Williams, Sadun and Clark.

Let T be a polyhedral tiling of finite local complexity with its system of Anderson–Putnam–Gähler complexes Γ^k, $k \geq 0$.

A shape function is an $\mathbb{R}^n$-valued 1-cocycle on Γ^k, i.e. a function $f : \Gamma^k \to \mathbb{R}^n$ such that $\delta f(w) := f(\delta w) = 0$ for any 2-cell w in Γ^k (δw is the chain corresponding to the boundary of w). With a little luck a shape function defines a new tiling (up to translation) which is obtained from T by changing the shape of the prototiles. A 1-cell e of Γ^k is an (oriented) edge of a k-collared prototile. $f(e)$ is a vector in $\mathbb{R}^n$ and hence defines an oriented straight line segment up to translation which is going to be an edge of a new prototile. Replacing each edge e of a k-collared prototile by a line segment parallel to $f(e)$, the condition $\delta f = 0$ guarantees that these line segments can be matched together at their boundary points with the same combinatorics as in the prototile. One thus obtains a new 1-skeleton. *A priori* there is no reason that one can complete the new 1-skeleton to a tile by adding in higher-dimensional faces in such a way that the combinatorial structure is that of the prototile. But if this is the case, we can construct from T, tile by tile, a whole new tiling up to an overall translation, and hence a new tiling space which we denote by $(\Omega_T)_f$. It is somewhat clear that all this works if the new edges differ only slightly from the old edges. It then makes sense to speak of a deformation of T.

A question which we will investigate in more detail than has been done in [**CS06**] is whether the above process is invertible, i.e. whether one can obtain T back from the new tiling by a similar procedure. Certainly, if the new tiling would be periodic whereas the old one was not, the process could not be inverted. One may argue that invertibility should be no problem if the new edges differ little from the old edges. This is, however, only obvious if k is fixed, since the larger k the more shapes are redefined by the shape function.

Let us call shape functions that allow for all this admissible.

THEOREM 3.1. [**SW03, CS06**] *Let $f : \Gamma^0 \to \mathbb{R}^n$ be a cocycle which is admissible. Then Ω_T is homeomorphic to $(\Omega_T)_f$.*

THEOREM 3.2. [**CS06**] *Let $f, g : \Gamma^0 \to \mathbb{R}^n$ be two cocycles which are admissible. If $I_0(f) = I_0(g)$ then $(\Omega_T)_f$ is mutually locally derivable with $(\Omega_T)_g$.*

Here the relation of being mutually locally derivable is extended to hulls of tilings by saying that two such hulls are mutually locally derivable if there exists a topological conjugacy between them which maps a tiling to a mutually locally derivable tiling with a uniform derivability range.

Clark and Sadun introduce the notion of asymptotically negligible elements in $H^1(\Omega_T, \mathbb{R}^n)$. Let us suppose that r is larger than the maximal diameter of a prototile. Then an ordered pair of vertices (x_1, x_2) of T is called a recurrence of size r if $B_r[T - x_1] = B_r[T - x_2]$ but $B_{r+\delta}[T - x_1] \neq B_{r+\delta}[T - x_2]$ for $\delta > 0$. If k is small enough so that the $(k + 1)$-neighbourhood of any tile is contained in the r-tubular neighbourhood of the tile, then a path between x_1 and x_2 along edges defines a loop in the 1-skeleton of Γ^k, different

paths leading to homologous loops. An element $\eta \in H^1(\Gamma^k, \mathbb{R}^n)$ can be evaluated on (the cycle defined by) a loop in Γ^k. It is said to be asymptotically negligible if for all ϵ there exists an r_ϵ such that, when η is evaluated on a loop in the 1-skeleton of Γ^k which is defined by a recurrence of size larger than r_ϵ, then the result is smaller than ϵ in norm. The images under the I_k of asymptotically negligible elements in $H^1(\Gamma^k, \mathbb{R}^n)$ are by definition the asymptotic elements of $H^1(\Omega_T, \mathbb{R}^n)$.

THEOREM 3.3. **[CS06]** *Let* $f, g : \Gamma^0 \to \mathbb{R}^n$ *be two cocycles which are admissible. If* $I_0(f) - I_0(g)$ *is asymptotically negligible then* $((\Omega_T)_f, \mathbb{R}^n)$ *is topologically conjugate to* $((\Omega_T)_g, \mathbb{R}^n)$.

It is rather clear that the three theorems extend to shape functions on Γ^k. Note also that if $f : \Gamma^k \to \mathbb{R}^n$ is a shape function then $f \circ \alpha_k : \Gamma^{k+1} \to \mathbb{R}^n$ is also a shape function and it leads to the same deformation.

3.2. *Negligible elements versus weakly P-equivariant 1-forms.* The aim is to reinterpret negligible elements as weakly P-equivariant 1-forms and to interpret Theorem 3.3 in terms of a mixed quotient.

THEOREM 3.4. *An element* $\eta \in \check{H}^1(\Omega_P, \mathbb{R}^n)$ *is asymptotically negligible if and only if* $\Theta^{-1}(\eta)$ *belongs to* $B^1_{w-P}(\mathbb{R}^n, \mathbb{R}^n) \cap Z^1_{s-P}(\mathbb{R}^n, \mathbb{R}^n)/B^1_{s-P}(\mathbb{R}^n, \mathbb{R}^n)$.

Proof. Suppose that $d\varphi \in B^1_{w-P}(\mathbb{R}^n, \mathbb{R}^n) \cap Z^1_{s-P}(\mathbb{R}^n, \mathbb{R}^n)$. Hence for all $\epsilon > 0$ there exists R_ϵ such that $\varphi = \varphi_\epsilon + \psi_\epsilon$ where φ_ϵ is P-equivariant with range R_ϵ and $\|\psi_\epsilon\|_\infty < \epsilon$. Also $d\varphi$ is P-equivariant with some finite range which we may suppose to be smaller than R_ϵ. Let k be large compared to R_ϵ and r_ϵ be large compared to $k + 1$. Finally, let (x_1, x_2) be a recurrence of size larger than or equal to r_ϵ. Denote by $[x_1, x_2]$ the cycle it defines in Γ^k. Then we have

$$J^k_{R_\epsilon}(d\varphi)([x_1, x_2]) = \varphi_\epsilon(x_2) - \varphi_\epsilon(x_1) + \psi_\epsilon(x_2) - \psi_\epsilon(x_1).$$

The first difference drops out since $[x_1, x_2]$ is a loop in Γ^k and so $|J^k_{R_\epsilon}(d\varphi)([x_1, x_2])| < 2\epsilon$. This shows that $\Theta([d\varphi])$ is asymptotically negligible.

For the converse suppose that $f \in Z^1(\Gamma^k, \mathbb{R}^n)$ such that $[f]$ is asymptotically negligible. As explained in §2.4.3, $\Theta^{-1}([f])$ is represented by the differential of a smooth strongly P-equivariant function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ which extends the function $\psi : T^0 \to \mathbb{R}^n$ defined as in (2.3). Let $\epsilon > 0$ and r_ϵ such that f applied to a loop defined by a recurrence (x_1, x_2) of size larger than or equal to r_ϵ is smaller in norm than ϵ. By the finite local complexity of T there exist finitely many points $p_1, \dots, p_f \in T^0$ such that the sets $T^0_i := \{y \in T^0 \mid B_{r_\epsilon}[T - y] = B_{r_\epsilon}[T - p_i]\}$ are pairwise disjoint, locally derivable from T^0, and their union exhausts T^0. Furthermore, the elements of $\bigcup_i T^0_i \times T^0_i$ are precisely the recurrences of size larger than or equal to r_ϵ. Hence for all $x_1, x_2 \in T^0_i$ we have $|\varphi(x_1) - \varphi(x_2)| = |f([x_1, x_2])| < \epsilon$. From Proposition 2.7 it follows therefore that φ is weakly P-equivariant. $\square$

4. Deformations and P-equivariant functions

The aim of this section is to present a theory of deformations which is formulated entirely in the framework of P-equivariant functions. We will recover the results of Theorems 3.1–3.3 (see Theorems 4.8, 4.13 and Corollary 4.20) but are able to go further as we obtain refined information about the homeomorphisms involved. As a consequence, we are able to decide whether two pattern spaces are homeomorphic (with homeomorphism preserving the abstract transversal) by means of pattern equivariant functions, and to a large extent this turns out to be the case for topological conjugacy as well. Finally, we investigate the question of invertibility of deformations.

Our first aim is to interpret the elements entering in the first P-equivariant cohomology with values in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is contractible, any closed 1-form on $\mathbb{R}^n$, P-equivariant or not, can be written as $d\varphi$ for a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$. The first P-equivariant cohomology is therefore related to functions, but the restriction that their differential is P-equivariant does not imply that they are themselves P-equivariant which is what makes the cohomology non-trivial. For instance, $\varphi = \mathrm{id}$ is never P-equivariant, but $d\,\mathrm{id}$ is.

One may view Ω_P as the compactification of $\mathbb{R}^n$ defined by $P \in \mathbb{R}^n$. It is then natural to ask how a function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ affects the compactification. In other words, how does $\Omega_{\varphi(P)}$ compare to Ω_P? Furthermore, one may ask how the pattern dynamical systems compare.

We will often need to make the following assumption about φ.

Hypothesis 4.1. For all $r > 0$ there exists $r' > 0$ such that for all $x \in \mathbb{R}^n$

$$B(\varphi(x), r) \cap \varphi(P) \subset \varphi(B(x, r') \cap P).$$

This assumption is, for instance, verified if

$$\text{for all } r > 0 \text{ there exists } r' > 0 \text{ for all } x \in \mathbb{R}^n : \varphi^{-1}(B(\varphi(x), r)) \subset B(x, r'). \qquad (4.1)$$

Note that if φ satisfies (4.1) and η is bounded then also $\varphi - \eta$ satisfies (4.1). We say that $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a bi-Lipschitz map if there exists $\lambda > 1$ (the Lipschitz constant) such that

$$\lambda^{-1}|x - y| \leq |\varphi(x) - \varphi(y)| \leq \lambda|x - y|.$$

A bi-Lipschitz map is a homeomorphism which satisfies (4.1) with $r' = \lambda r$. In fact, it is injective and continuous, and hence has open image by Brouwer's theorem. Moreover, its image is also closed since a pre-image of a Cauchy sequence is a Cauchy sequence, and hence it is surjective. A further advantage of a bi-Lipschitz map is that it maps Delone sets to Delone sets. We are particularly interested in the case $\varphi = \mathrm{id} + \eta$ where η is differentiable and $\|d\eta\|_\infty < 1$ (or $\varphi = g + \eta$ with $g \in GL(n, \mathbb{R})$ and $\|d\eta\|_\infty < (\|g^{-1}\|_\infty)^{-1}$). Such a map φ is a bi-Lipschitz map with $\lambda^{-1} = 1 - \|d\eta\|_\infty$.

LEMMA 4.2. *Let P be a uniformly discrete set of finite local complexity. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a function, $g \in GL(n, \mathbb{R})$ and $\varphi - g$ be P-equivariant with range R. Then we have, for all $r > 0$ and $x, y \in \mathbb{R}^n$,*

$$B_{R+r}[P - x] = B_{R+r}[P - y] \Longrightarrow$$
$$\text{for all } h \in B(0, r) : \varphi(x + h) - g(x) = \varphi(y + h) - g(y).$$

In particular,

$$B_{R+r}[P - x] = B_{R+r}[P - y] \implies B_r(\varphi(P) - g(x)) = B_r(\varphi(P) - g(y))$$

and $\varphi(P)$ is locally derivable from $g(P)$.

Proof. By assumption, $B_{R+r}[P - x] = B_{R+r}[P - y]$ implies that, for all $h \in B(0, r)$, $\varphi(x + h) - g(x + h) = \varphi(y + h) - g(y + h)$. The statement follows therefore from the linearity of g. The last statement is trivial for $g = \mathrm{id}$. Now if $\varphi - g$ is strongly P-equivariant then $\varphi \circ g^{-1} - \mathrm{id}$ is strongly $g(P)$-equivariant. This implies the last statement for general g. □

Note that $d(\varphi - g)$ is strongly P-equivariant whenever $d\varphi$ is strongly P-equivariant.

LEMMA 4.3. *Let P be a uniformly discrete set of finite local complexity and let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function. If $d\varphi$ is P-equivariant with range R then, for all $x, y \in \mathbb{R}^n$,*

$$B_{R+r}[P - x] = B_{R+r}[P - y] \implies$$
$$\text{for all } h \in B(0, r) : \varphi(x + h) - \varphi(x) = \varphi(y + h) - \varphi(y).$$

If, moreover, φ satisfies Hypothesis 4.1, then for all $r > 0$ there exists r' such that

$$B_{r'}[P - x] = B_{r'}[P - y] \implies B_r(\varphi(P) - \varphi(x)) = B_r(\varphi(P) - \varphi(y)). \quad (4.2)$$

Proof. Let $B_{R+r}[P - x] = B_{R+r}[P - y]$ and $h \in B_r$. As $d\varphi$ is P-equivariant with range R we have $d\varphi(x + h) = d\varphi(y + h)$. In particular, $\int_x^{x+h} d\varphi = \int_y^{y+h} d\varphi$ (we can integrate along a straight line). As $\varphi(x + h) - \varphi(x) = \int_x^{x+h} d\varphi$ the first statement holds true.

Suppose that $B_{R+r'}[P - x] = B_{R+r'}[P - y]$. According to the first statement this implies that

$$\varphi(B(x, r') \cap P) - \varphi(x) = \varphi(B(y, r') \cap P) - \varphi(y). \quad (4.3)$$

Now let r be given and choose r' according to Hypothesis 4.1. Then, for any $z \in \mathbb{R}^n$, $(\varphi(B(z, r') \cap P) - \varphi(z)) \cap B(0, r) = (\varphi(B(z, r') \cap P) \cap B(\varphi(z), r)) - \varphi(z) = (\varphi(P) \cap B(\varphi(z), r)) - \varphi(z)$. The latter is by definition $B_r[\varphi(P) - \varphi(z)]$ and so (4.3) implies that $B_r[\varphi(P) - \varphi(x)] = B_r[\varphi(P) - \varphi(y)]$. □

COROLLARY 4.4. *Let P be a uniformly discrete set of finite local complexity. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function which fulfils Hypothesis 4.1. If $d\varphi$ is strongly P-equivariant then $\varphi(P)$ has finite local complexity.*

Proof. Finite local complexity of P means that there are, for any given r', only finitely many different $B(0, r') \cap (P - x)$, $x \in P$. Thus we conclude from Lemma 4.3 that there are also only finitely many different $B(0, r) \cap (\varphi(P) - z)$, $z \in \varphi(P)$. □

There is no reason for $\varphi(P)$ to have finite local complexity if we require merely weak P-equivariance of $d(\varphi - g)$ or $d\varphi$.

How can we interpret (4.2)? Clearly it allows one to construct the r-patch of $\varphi(P)$ around $\varphi(x)$ from the r'-patch of P around x, but this involves the passage from $\varphi(x)$ to x

which is *a priori* not local (i.e. not determined by a patch of P). Nevertheless, if we forget about the position of the patches in $\mathbb{R}^n$ and consider their translational congruence classes we obtain from (4.2) a local map: to each translational congruence class of an r-patch of P is associated a unique translational congruence class of an r'-patch of $\varphi(P)$. In one dimension this gives a rewriting rule. In higher dimensions one obtains a local derivation in the sense of [**Ke97**]. If $\varphi = \mathrm{id} + \eta$ with small $\|d\eta\|_\infty$ and P is the set of vertices of a polyhedral tiling the local map defined by (4.2) can be interpreted as a deformation of the tiling induced by changing the length of the edges of its tiles [**SW03**]; see §3.1. In view of this and the results in §2.4.3 we define the following.

Definition 4.5. Let P be a Delone set in $\mathbb{R}^n$. A Delone set $P' \subset \mathbb{R}^n$ is a deformation of P if there exists a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ satisfying Hypothesis 4.1 and having a strongly P-equivariant differential such that $P' = \varphi(P)$.

We have a lot of freedom to choose φ. In fact, since the joint kernel of the maps J_r^k from §2.4.3 is

$$N_{s-P}^1(\mathbb{R}^n, \mathbb{R}^n) = \{d\eta \in B_{s-P}^1(\mathbb{R}^n, \mathbb{R}^n) \mid \eta \text{ is constant on } P\},$$

the elements of $Z_{s-P}^1(\mathbb{R}^n, \mathbb{R}^n)/N_{s-P}^1(\mathbb{R}^n, \mathbb{R}^n)$ parametrize the deformations of P (or of a tiling T with $P = T^0$).

We end this section with the following important definition, which is inspired from the above lemma.

Definition 4.6. Let P be a uniformly discrete set of finite local complexity.

(1)   Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^0-function such that $\varphi - g$ is weakly P-equivariant. We define $\Phi_g : \mathrm{orb}(P) \to \Omega_{\varphi(P)}$,

$$\Phi_g(P - x) := \varphi(P) - g(x).$$

(2)   Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function such that $d\varphi$ is strongly P-equivariant. We define $\Phi_\varphi : \mathrm{orb}(P) \to \Omega_{\varphi(P)}$,

$$\Phi_\varphi(P - x) := \varphi(P) - (\varphi(x) - \varphi(0)).$$

That these maps are well defined is the issue of the following lemma.

LEMMA 4.7. *We assume the notation and conditions of the last definition. In the first case $P - x = P$ implies that $\varphi(P) - g(x) = \varphi(P)$. In the second case $P - x = P$ implies that $\varphi(P) - \varphi(x) = \varphi(P) - \varphi(0)$.*

Proof. Suppose that $\varphi - g$ is strongly P-equivariant. By Lemma 4.2, for all $h \in P$, $\varphi(-x + h) - g(-x) = \varphi(h)$ which implies that $\varphi(P) = \varphi(P) - g(x)$. If $\varphi - g$ is merely weakly P-equivariant we can find for all $\epsilon > 0$ an approximation φ_ϵ of φ such that $\varphi_\epsilon - g$ is strongly P-equivariant and $\|\varphi_\epsilon - \varphi\|_\infty < \epsilon$. The above then implies that $d_h(\varphi(P) - g(x), \varphi(P)) < 2\epsilon$.

Now suppose that $d\varphi$ is strongly P-equivariant. By Lemma 4.3, for all $h \in P$, $\varphi(x + h) - \varphi(x) = \varphi(h) - \varphi(0)$. This implies that $\varphi(P + x) = \varphi(P) + \varphi(x) - \varphi(0)$ and hence the statement. □

In the following we shall see that Φ_g and Φ_φ can be extended to maps between Ω_P and $\Omega_{\varphi(P)}$, the second only in the strongly P-equivariant case. This is the case once we can show that they are uniformly continuous, since a uniformly continuous map between (not necessarily complete) metric spaces has a unique continuous extension to the completions.

4.1. *Closed strongly P-equivariant 1-forms and deformations.* We investigate more closely P-equivariant 1-forms, that is P-equivariant differentials, and how they affect the tiling spaces and dynamical systems. We start by the observation that a strongly P-equivariant 1-form gives rise to a continuous map between hulls of Delone sets preserving the canonical transversal, an observation which is slightly stronger than Theorem 3.1.

THEOREM 4.8. *Let P be a Delone set of finite local complexity. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map which fulfils Hypothesis 4.1 and whose differential $d\varphi$ is strongly P-equivariant. Then Φ_φ is a uniformly continuous map. It thus extends to a continuous map $\Phi_\varphi : \Omega_P \to \Omega_{\varphi(P)}$. If $\varphi(0) = 0$ then $\Phi_\varphi(\Xi_P) = \Xi_{\varphi(P)}$. If φ is surjective then Φ_φ is surjective.*

Proof. Choose $r > 0$ and then $r' \geq r$ according to Lemma 4.3. Let $D(P - x, P - y) < 1/r'$. Since P has finite local complexity we can find $x', y' \in B(0, 1/2r')$ such that $B_{r'}[P - x - x'] = B_{r'}[P - y - y']$. By Lemma 4.3 we then have $B_r[\varphi(P) - \varphi(x + x')] = B_r[\varphi(P) - \varphi(y + y')]$ and so $D(\varphi(P) - \varphi(x + x'), \varphi(P) - \varphi(y + y')) < 1/r$. Since P has finite local complexity $d\varphi$ is bounded and $|\varphi(x + x') - \varphi(x)| \leq \|d\varphi\|_\infty |x'|$. If r is sufficiently large then $D(\varphi(P) - \varphi(x + x'), \varphi(P) - \varphi(x)) \leq \|d\varphi\|_\infty |x'|$. Hence the triangle inequality gives

$$D(\varphi(P) - \varphi(x), \varphi(P) - \varphi(y)) \leq \frac{1}{r} + \frac{\|d\varphi\|_\infty}{2r'} + \frac{\|d\varphi\|_\infty}{2r'}.$$

As this tends to 0 for r tending to infinity, we conclude that Φ_φ is uniformly continuous.

That Φ_φ preserves the canonical transversal provided $\varphi(0) = 0$ follows immediately from the construction of Φ_φ. Since $\Phi_\varphi(\Xi_P)$ is closed and contains $\{\varphi(P) - \varphi(p) \mid p \in P\}$ we must have $\Phi_\varphi(\Xi_P) = \Xi_{\varphi(P)}$.

Finally, if φ is surjective then $\mathrm{orb}(P') = \Phi_\varphi(\mathrm{orb}(P))$, which is dense and hence implies surjectivity of Φ_φ. $\square$

Note that since $d\varphi$ is strongly P-equivariant $\varphi(P)$ has finite local complexity, but it need not be a Delone set.

We are now interested in the converse direction: given a continuous map between hulls of Delone sets, can we associate a strongly P-equivariant 1-form to it?

Let P, P' be aperiodic Delone sets of finite local complexity and $\Phi : \Omega_P \to \Omega_{P'}$ a continuous map satisfying $\Phi(P) = P'$. Since the orbits under the translation action are the path-connected components Φ must preserve orbits. Hence for any x there exists a x' such that $\Phi(P - x) = P' - x'$. Since P' is aperiodic x' is uniquely determined by x. Thus there exists a family of functions $\varphi_{P-y} : \mathbb{R}^n \to \mathbb{R}^n$, $y \in \mathbb{R}^n$ such that

$$\Phi(P - y - x) = P' - y - \varphi_{P-y}(x), \quad \varphi_{P-y}(0) = 0. \tag{4.4}$$

LEMMA 4.9. *For all $y \in \mathbb{R}^n$, $\varphi_{P-y} : \mathbb{R}^n \to \mathbb{R}^n$ is uniformly continuous. Furthermore, for all $x \in \mathbb{R}^n$, $\mathrm{orb}(P) \ni \omega \mapsto \varphi_\omega(x) \in \mathbb{R}^n$ is uniformly continuous.*

Proof. To show that φ_P is uniformly continuous choose $0 < \epsilon < r_{\min}(P')/3$, where $r_{\min}(P')$ is the minimal distance in P'. By the uniform continuity of Φ there exists a δ such that, for all $x_0 \in \mathbb{R}^n$, $|v| < \delta$ implies that

$$D(\Phi(P - x_0), \Phi(P - x_0 - v)) < \epsilon. \qquad (4.5)$$

By the remarks at the end of §2.1 the pre-image under $w \mapsto \Phi(P - x_0) - w$ of the path-connected component of $\Phi(P - x_0)$ in $U_\epsilon(\Phi(P - x_0))$ is contained in $B(0, \epsilon)$. Therefore, $\varphi_{P-x_0}(B(0, \delta)) \subset B(0, r_{\min}(P')/3)$.

Furthermore, (4.5) implies that either $|\varphi_P(x_0 + v) - \varphi_P(x_0)| \leq \epsilon$ or $|\varphi_P(x_0 + v) - \varphi_P(x_0)| \geq r_{\min}(P') - \epsilon$. But the second possibility cannot hold since $|\varphi_P(x_0 + v) - \varphi_P(x_0)| = |\varphi_{P-x_0}(v)| \leq r_{\min}(P')/3$. This proves uniform continuity of φ_P. That of φ_{P-y} follows in the same way.

To show that $\omega \mapsto \varphi_\omega(y)$ is uniformly continuous on the orbit of P we choose $\epsilon > 0$, which is small compared with $|y|^{-1}$. Note that $B_r[\omega] = B_r[\omega']$ implies that $B_{r-|x|}[\omega - x] = B_{r-|x|}[\omega' - x]$ provided that $r > |x|$. Using this property, the (uniform) continuity of Φ and boundedness of $\varphi_{\omega'}$, we can find $\delta > 0$ such that $D(\omega, \omega') < \delta$ implies that, for all $y' \in B(0, |y|)$, $D(\Phi(\omega - y'), \Phi(\omega' - y')) < \epsilon$ and $D(\Phi(\omega) - \varphi_{\omega'}(y'), \Phi(\omega') - \varphi_{\omega'}(y')) < \epsilon$ and therefore

$$D(\Phi(\omega) - \varphi_{\omega'}(y'), \Phi(\omega) - \varphi_\omega(y')) < 2\epsilon.$$

This implies that $|\varphi_\omega(y') - \varphi_{\omega'}(y')|$ is either smaller than 2ϵ or larger than $r_{\min}(P') - 2\epsilon$. From the continuity of φ_ω and the condition $\varphi_\omega(0) = 0$ it follows therefore that $|\varphi_\omega(y) - \varphi_{\omega'}(y)| < 2\epsilon$ (for small enough ϵ). $\qquad \square$

We now require, in addition, that Φ preserves the canonical transversal.

THEOREM 4.10. *Let P, P' be aperiodic Delone sets of finite local complexity. Let $\Phi : \Omega_P \to \Omega_{P'}$ be a continuous map satisfying $\Phi(\Xi_P) \subset \Xi_{P'}$ and $\Phi(P) = P'$. Then there exists a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with strongly P-equivariant differential which coincides with φ_P (as defined by (4.4)) on P and satisfies $\varphi(P) \subset P'$.*

Proof. We can sharpen the last part of the proof of Lemma 4.9 provided $w, w' \in \Xi_P$ and $y = P - P$. Let $\epsilon > 0$, $M > 0$ and $p, q \in P$. By the reasoning of the proof of Lemma 4.9 there exists $R > M$ such that $B_R[P - p] = B_R[P - q]$ implies that, for all $y \in B_M[P - p]$, $D(\Phi(P - p - y), \Phi(P - q - y)) < \epsilon$ and $|\varphi_{P-p}(y) - \varphi_{P-q}(y)| < \epsilon$. Moreover, $\Phi(P - p - y)$ and $\Phi(P - q - y)$ lie in $\Xi_{P'}$ and hence the finite local complexity of P' allows us to replace D by D_0 to obtain $B_{1/\epsilon}[\Phi(P - p - y)] = B_{1/\epsilon}[\Phi(P - q - y)]$. Hence $|\varphi_{P-p}(y) - \varphi_{P-q}(y)| < \epsilon$ forces $|\varphi_{P-p}(y) - \varphi_{P-q}(y)|$ to vanish. A simple calculation shows that $\varphi_{P-p}(y) = \varphi_P(p + y) - \varphi_P(p)$. Hence, for all $h \in P - P$, $\Delta_h \varphi_P : P \cap (P - h) \to \mathbb{R}^n$ is strongly P-equivariant. Applying Lemma 2.6(3) we obtain a smooth function φ with strongly P-equivariant differential which coincides with φ_P on P.

Finally, if $p \in P$ then $\Phi(P - p) \in \Xi_{P'}$ and hence $0 \in P' - \varphi(p)$. This shows that $\varphi(P) \subset P'$. $\square$

We cannot expect that under the hypothesis of the theorem $\varphi(P)$ is all of P'. For instance, if P' is obtained from P by adding points in a locally derivable way then $\varphi = \mathrm{id}$ would be the resulting map but P would be a proper subset of P'.

COROLLARY 4.11. *Let P, P' be aperiodic Delone sets of finite local complexity. Let $\Phi : \Omega_P \to \Omega_{P'}$ be a homeomorphism which maps Ξ_P onto $\Xi_{P'}$ and P to P'. Then the map φ constructed in the theorem satisfies Hypothesis 4.1 and $P' = \varphi(P)$.*

Proof. Since Φ is a homeomorphism, φ_P is also a homeomorphism and its inverse φ_P^{-1} is the map constructed in (4.4) from $\Phi^{-1} : \Omega_{P'} \to \Omega_P$; we denote it here by $\varphi_{P'}$. By Lemma 4.9, it is uniformly continuous and hence we can find at least one $r > 0$ for which there exists $r' > 0$, for all $y \in \mathbb{R}^n$, where $\varphi_{P'}(B(y, r)) \subset B(\varphi_{P'}(y), r')$. Our aim is to show that this implies that for all $r > 0$, there exists $r' > 0$, for all $y \in \mathbb{R}^n$, where $\varphi_{P'}(B(y, r)) \subset B(\varphi_{P'}(y), r')$ and hence that φ_P satisfies (4.1). To see this, let $0 \in A \subset B(0, r)$ be a finite set such that $B(0, \frac{3}{2}r) \subset \bigcup_{a \in A} B(a, r)$. Then the above implies that

$$\varphi_{P'}\left(B\left(y, \frac{3}{2}r\right)\right) \subset \bigcup_{a \in A} B(\varphi_{P'}(y + a), r').$$

By Lemma 4.9 the maps $y \mapsto \varphi_{P'}(y + a) - \varphi_{P'}(y)$ are weakly P'-equivariant and hence bounded. If we denote a common bound for their norm by M then $\bigcup_{a \in A} B(\varphi_{P'}(y + a), r') \subset B(\varphi_{P'}(y), r' + M)$. This shows that φ_P satisfies (4.1). Since φ_P and φ are uniformly continuous (φ as $d\varphi$ is bounded), $\varphi_P - \varphi$ is bounded and hence φ satisfies Hypothesis 4.1.

We have $\Phi(\mathrm{orb}(P)) = \mathrm{orb}(P')$ and $\Phi^{-1}(\Xi_{P'}) = \Xi_P$. Now if $y \in P'$ then the first identity implies that there exists a x such that $P' - \varphi_P(x) = P' - y$, and hence $\varphi_P(x) - y$ lies in the periodicity lattice of P'. Using $\varphi(0) = 0$ and continuity we can arrange for $\varphi_P(x) = y$. Now the second identity implies that $P - x \in \Xi_P$ and hence $x \in P$. $\square$

Remark 4.12. The map φ constructed in Theorem 4.10 may differ substantially from φ_P away from P, but it defines the same deformation.

A refinement of Lemma 2.6(3) allows us to construct maps which are arbitrarily close, namely for all $\epsilon > 0$ there exists a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with strongly P-equivariant differential such that $\|\varphi_P - \varphi\|_\infty < \epsilon$ and φ coincides with φ_P on P.

This can be seen as follows. Fix $\delta > 0$. Uniform continuity of $\omega \mapsto \varphi_\omega(x)$ is equivalent to weak P-equivariance of $y \mapsto \varphi_{P-y}(x) = \Delta_x \varphi_P(y)$ ($x \in \mathbb{R}^n$ fixed)[†]. Hence we can find a partition $\{P_i\}_i$ of P such that for all $x \in B(0, 3A)$ and $p, q \in P_i$ we have $|\Delta_x \varphi_P(p) - \Delta_x \varphi_P(q)| < \delta$. Furthermore, taking if necessary a refinement of that partition, the strong P-equivariance of $\Delta_h \varphi_P : P \cap (P - h) \to \mathbb{R}^n$ allows us to assume that, for $h \in P - P$, $|h| \leq A$, $\Delta_h \varphi_P$ takes the same value on all points of P_i. Choose $p_i \in P_i$ and set, for $p \in P_i$ and $x \in \mathrm{int} V_p - p$,

$$\tilde{\phi}(x + p) = \psi(x + p_i) - \psi(p_i) + \psi(p). \tag{4.6}$$

[†] We extend the definition of $\Delta_x \varphi_P(y) = \varphi_P(y + x) - \varphi_P(y)$ to $x, y \in \mathbb{R}^n$.

Now we proceed exactly as in the proof of Lemma 2.6(3) thereby obtaining a smooth function $\varphi = \rho * \tilde{\phi}$ which has strongly P-equivariant differential. If we choose δ small enough then $\|\varphi - \varphi_P\|_\infty$ is smaller than any given ϵ. The only drawback is that $\varphi - \varphi_P$ does not vanish on P. But the restriction of $\varphi - \varphi_P$ to P is strongly P-equivariant. We may therefore use Lemma 2.6(3) to add a smooth strongly P-equivariant function to φ such that the result coincides with φ_P on P. This correction is of order ϵ in the $\| \cdot \|_\infty$-norm and so we are done.

4.2. *Exact P-equivariant 1-forms.* Exact P-equivariant 1-forms are of the form df where f is P-equivariant. This implies that f is bounded. Such maps are not good candidates to compare P with $f(P)$ as the latter could not even be uniformly discrete. It is more fruitful to consider exact 1-forms of the form $d(\varphi - \mathrm{id})$ and then compare P with $\varphi(P)$.

THEOREM 4.13. *Let P be a Delone set of finite local complexity, $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^0-function and $g \in GL(n, \mathbb{R})$. Suppose that $\varphi - g$ is weakly P-equivariant. Then Φ_g is a uniformly continuous map which satisfies $\Phi_g(\omega - x) = \Phi_g(\omega) - g(x)$. In particular, Φ_{id} extends to a topological semi-conjugacy between $(\Omega_P, \mathbb{R}^n)$ and $(\Omega_{\varphi(P)}, \mathbb{R}^n)$ mapping P to P'. If $\varphi - g$ is even strongly P-equivariant then $\varphi(P)$ is a Delone set.*

Proof. We consider first the case that $\varphi - g$ is P-equivariant with range R. Clearly $g(P)$ is a Delone set. Since $\varphi(P) = g(P) + (\varphi - g)(P)$ and $(\varphi - g)(P)$ is finite by Lemma 2.4, also $\varphi(P)$ is a Delone set. For the second statement choose $r > 0$ and let $r' = R + r$. Let $D(P - x, P - y) < 1/r'$. Since P has finite local complexity we can find $x', y' \in B(0, 1/2r')$ such that $B_{r'}[P - x - x'] = B_{r'}[P - y - y']$. By Lemma 4.2 we then have $B_r[\varphi(P) - g(x + x')] = B_r[\varphi(P) - g(y + y')]$ and so $D(\varphi(P) - g(x + x'), \varphi(P) - g(y + y')) < 1/r$. If r is sufficiently large then $D(\varphi(P) - g(x + x'), \varphi(P) - g(x)) \leq |g(x')| \leq \|g\|_\infty / 2r'$. Hence the triangle inequality gives

$$D(P - g(x), P - g(y)) \leq \frac{1}{r} + \frac{\|g\|_\infty}{R + r}.$$

Since this tends to zero when r tends to infinity we conclude that Φ_g is uniformly continuous.

Now suppose that $\varphi - g$ is merely weakly P-equivariant and hence for any $\epsilon > 0$ there exists a strongly P-equivariant C^0-function φ_ϵ such that $\|\varphi - \varphi_\epsilon\|_\infty < \epsilon$. Now

$$D(\Phi_g(P - x), \Phi_{\epsilon g}(P - x)) \leq d_H(\varphi(P), \varphi_\epsilon(P)) \leq \epsilon$$

and a further application of the triangle inequality implies that Φ_g is uniformly continuous.

The surjectivity of g implies that the image of Φ_g contains the orbit of $\varphi(P)$ and hence that Φ_g is surjective. The relation $\Phi_g(\omega - x) = \Phi_g(\omega) - g(x)$ is clear for $\omega = P - y$ and follows for general ω from the continuity of Φ_g. $\square$

We now investigate conditions which lead to statements in the converse direction: assuming that P' is a deformation of P such that the dynamical systems are pointed topological conjugate, is there a weakly P-equivariant map φ such that $\varphi(P) = P'$? Clark and Sadun obtained related results for substitution tilings [**CS06**].

Consider two patterns P and P' of finite local complexity and suppose that $\Psi : \Omega_P \to \Omega_{P'}$ is a semi-conjugacy mapping P to P'. Since it commutes with the $\mathbb{R}^n$ actions, the maps constructed in Lemma 4.9 are all equal to the identity. Moreover, since Ψ has to be the continuous extension of $P - x \mapsto P' - x$ we must have

$$\Psi = \Phi_{\mathrm{id}}.$$

Of course, the semi-conjugacy is not assumed to preserve the canonical transversals. If we assume in addition that P' is the image of P under a C^1-function φ satisfying Hypothesis 4.1 and having strongly P-equivariant differential, then we obtain from Theorem 4.8 a (in general different) map $\Phi_\varphi : \Omega_P \to \Omega_{P'}$ which preserves the canonical transversal but does not commute with the $\mathbb{R}^n$ actions. Our main question is, therefore, under which additional conditions does uniform continuity of Φ_{id} and Φ_φ imply that $\varphi - \mathrm{id}$ is weakly P-equivariant?

A first observation to make is that boundedness of $\varphi - \mathrm{id}$ is necessary, as weakly P-equivariant functions are bounded. We call a deformation bounded if $\varphi - \mathrm{id}$ is bounded. We show below that boundedness of $\varphi - \mathrm{id}$ is sufficient, provided P' is aperiodic and uniformly discrete.

LEMMA 4.14. *Let $P \subset \mathbb{R}^n$ be an aperiodic, uniformly discrete set. For all $M > 0$ there exists $r > 0$ such that, for all $h \in \mathbb{R}^n$, $|h| \le M$, the following holds: if $B_r[P] = B_r[P - h]$ then $h = 0$.*

Proof. Let us suppose the contrary. Then there exist $M > 0$, a sequence $(r_n)_n$ tending to $+\infty$, and points $h_n \in \mathbb{R}^n$, $|h_n| \le M$ such that $B_{r_n}[P] = B_{r_n}[P - h_n]$ but $h_n \ne 0$. Since $\{h \in \mathbb{R}^n : |h| \le M\}$ is compact we can assume, perhaps after taking a sub-sequence, that $(h_n)_n$ converges, let us say to h. This limit satisfies $h \ne 0$ since $B_{r_n}[P] = B_{r_n}[P - h_n]$ for large enough r_n implies that either $h_n = 0$ or $|h_n| \ge r_{\min}(P)$. It follows that $(P - h_n)_n$ converges to $P - h$ and that $P - h = P$. This contradicts the assumption that P is aperiodic. $\square$

THEOREM 4.15. *Let P' be an aperiodic and uniformly discrete deformation of a Delone set P of finite local complexity. Thus, in particular, $P' = \varphi(P)$ for some smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with strongly P-equivariant differential and satisfying Hypothesis 4.1. Suppose that P' is, moreover, pointed topologically semi-conjugate to P and hence Φ_{id} is uniformly continuous. If $\varphi - \mathrm{id}$ is bounded then it is weakly P-equivariant.*

Proof. Without loss of generality, we may assume that $\varphi(0) = 0$. Then $\Phi_\varphi : \Omega_P \to \Omega_{P'}$, which is uniformly continuous by Theorem 4.8, is given by

$$\Phi_\varphi(P - x) = \Phi_{\mathrm{id}}(P - x) + \eta(x),$$

where $\eta = \varphi - \mathrm{id}$. By definition of the metric, uniform continuity of Φ_φ and Φ_{id} implies that for any $\epsilon > 0$ there exists $\delta > 0$ such that, for all $x, y \in \mathbb{R}^n$,

$$B_{\delta^{-1}}[P - x] = B_{\delta^{-1}}[P - y]$$
$$\implies \begin{cases} B_{1/\epsilon}[\Phi_\varphi(P - x)] \cong_\epsilon B_{1/\epsilon}[\Phi_\varphi(P - y)] \\ B_{1/\epsilon}[\Phi_\varphi(P - x) + \eta(x)] \cong_\epsilon B_{1/\epsilon}[\Phi_\varphi(P - y) + \eta(y)]. \end{cases}$$

Suppose that $\epsilon^{-1} > 2(\|\eta\|_\infty + 1)$. Let $p \in P$. Then we have, for all $x \in \mathbb{R}^n$,

$$B_{\delta^{-1}}[P - x] = B_{\delta^{-1}}[P - p] \Longrightarrow \text{there exists } x' \in \mathbb{R}^n, |x'| < 2\epsilon :$$
$$B_{\frac{1}{2}\epsilon^{-1}}[\Phi_\varphi(P - p)] = B_{\frac{1}{2}\epsilon^{-1}}[\Phi_\varphi(P - p) + \eta(x) - \eta(p) - x'].$$

Since η is bounded we can apply Lemma 4.14 to $\Phi_\varphi(P - p)$ with $h = \eta(x) - \eta(p) - x'$ to conclude that $|\eta(x) - \eta(p)| < 2\epsilon$. By the finite local complexity of P there exists a finite subset $\{p_1, \ldots, p_k\} \subset P$ such that the sets $\{p \in P \mid B_{\delta^{-1}}[P - p] = B_{\delta^{-1}}[P - p_i]\}$ partition P. They are clearly locally derivable from P. Hence we can apply Proposition 2.7 to obtain the statement. $\qquad\square$

The following example shows that the hypothesis of boundedness of $\varphi - \mathrm{id}$ is not implied by the other conditions of the theorem. The Penrose tilings can be defined by means of a substitution. This is a locally defined decomposition γ of the tiles into smaller tiles followed by a rescaling by a factor of τ, the golden ratio. If T is a Penrose tiling then $\gamma(T)$ and T are mutually locally derivable, since the substitution is recognizable. In particular, T and $\gamma(T)$ are pointed topologically conjugate and so are their vertex sets T^v and $\gamma(T)^v$. Now a few Penrose tilings have the property that they are invariant under the fourth iterate of the substitution. Hence for those, $\gamma^4(T)^v = \tau^{-4}T^v$, and thus $\gamma^4(T)^v$ is a deformation of T^v given by the map $\varphi = \tau^{-4}\mathrm{id}$. Clearly $\varphi - \mathrm{id}$ is unbounded.

4.3. *Deforming back.*

Recall that according to Definition 4.5 a deformation of a pattern P is determined by a function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ whose differential is strongly P-equivariant. The deformed pattern is then $\varphi(P)$. This does not imply that P is a deformation of $\varphi(P)$, i.e. that there exists a function $\psi : \mathbb{R}^n \to \mathbb{R}^n$ whose differential is strongly $\varphi(P)$-equivariant such that $\psi(\varphi(P)) = P$. For instance, $\varphi(P)$ could be periodic whereas P was not†. Thus deformations are not always invertible. Although this is not analysed in [**CS06**], it is quite clear that, for fixed k (cf. §2.4.2), there exists a neighbourhood of the identity shape function such that shape functions in that neighbourhood are admissible and define deformations which can be inverted. It is not clear *a priori*, however, that one can obtain a lower bound on the size of this neighbourhood which is uniform in k. We will in this section effectively establish such a lower bound (Theorem 4.18) and thereby guarantee that small deformations are invertible, i.e. can be deformed back.

Given a uniformly discrete set P with minimal distance between points $r_{\min}$ we define $A_P : [r_{\min}, \infty) \to \mathbb{R}^+$,

$$A_P(r) = \inf\left\{|h - h'| : h, h' \in \bigcup_{x \in P} B_r[P - x], h \neq h'\right\}.$$

$A_P(r)$ is a monotonically decreasing function which is bounded from above by $r_{\min}$. Moreover, if P has finite local complexity then $A_P(r)$ is strictly positive since then $\bigcup_{x \in P} B_r[P - x]$ is a finite set.

For the following lemma and its corollary we do not require any P-equivariance property of φ or $d\varphi$. In particular, $\varphi(P)$ will usually not have finite local complexity.

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† Definition 4.5 does not allow for additional markings of the points of $\varphi(P)$ which could render the deformation invertible.

LEMMA 4.16. *Let P be a uniformly discrete set of finite local complexity. For all $r > 0$ there exists $1 > \epsilon > 0$ such that, for all differentiable functions $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ which satisfy $\|d(\varphi - \mathrm{id})\|_\infty < \epsilon$, the following holds: for all $x, y \in P$*

$$B_{r'}[\varphi(P) - \varphi(x)] = B_{r'}[\varphi(P) - \varphi(y)] \Longrightarrow$$
$$\begin{cases} B_r[P - x] = B_r[P - y] & \text{and} \\ \text{for all } h \in B_r[P - x] : \varphi(x + h) - \varphi(x) = \varphi(y + h) - \varphi(y) \end{cases}$$

where $r' = r(1 - \epsilon)^{-1}$.

Proof. The statement is trivial for $r < r_{\min}$, the minimal distance in P, so we assume that $r \geq r_{\min}$. Define

$$\epsilon = \sup_{0 < t < 1} \{2t(1 - t)^{-2}r \leq A_P((1 - t)^{-2}r)\}.$$

Suppose that $\|d(\varphi - \mathrm{id})\|_\infty < \epsilon$. Let $r' = r(1 - \epsilon)^{-1}$ and $S_{r'}(x) := \{h \in P - x \mid \varphi(x + h) - \varphi(x) \in B_{r'}[\varphi(P) - \varphi(x)]\}$. Since φ is bi-Lipschitz with constant $\lambda = (1 - \epsilon)^{-1}$ we have

$$B_{(1-\epsilon)r'}[P - x] \subset S_{r'}(x) \subset B_{(1-\epsilon)^{-1}r'}[P - x]. \tag{4.7}$$

Now let $B_{r'}[\varphi(P) - \varphi(x)] = B_{r'}[\varphi(P) - \varphi(y)]$. This implies that there exists a bijection $\beta : S_{r'}(x) \to S_{r'}(y)$ such that

$$\varphi(y + \beta(h)) - \varphi(y) = \varphi(x + h) - \varphi(x).$$

It follows that $\beta(h) - h = \eta(x + h) - \eta(x) - \eta(y + \beta(h)) - \eta(y)$ with $\eta = \varphi - \mathrm{id}$ and hence

$$|\beta(h) - h| \leq (|h| + |\beta(h)|)\|d(\varphi - \mathrm{id})\|_\infty.$$

By the right inclusion of (4.7),

$$(|h| + |\beta(h)|)\|d(\varphi - \mathrm{id})\|_\infty < 2(1 - \epsilon)^{-1}r'\epsilon = A_P((1 - \epsilon)^{-1}r').$$

Since $\beta(h), h \in \bigcup_{x \in P} B_r[P - x]$, the definition of A_P implies that $h = \beta(h)$ and so $S_{r'}(x) = S_{r'}(y)$. From the left inclusion of (4.7), it follows that $B_r[P - x] = B_r[P - y]$ and, since $h = \beta(h)$, for all $h \in B_r[P - x]$, $\varphi(x + h) - \varphi(x) = \varphi(y + h) - \varphi(y)$. □

COROLLARY 4.17. *Let P be a Delone set of finite local complexity. There exists $1 > \epsilon > 0$ such that, for all differentiable functions $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ which satisfy $\|d(\varphi - \mathrm{id})\|_\infty < \epsilon$, the following holds: for all $r > 0$ there exists $r' > 0$ for all $x, y \in P$*

$$B_{r'}[\varphi(P) - \varphi(x)] = B_{r'}[\varphi(P) - \varphi(y)] \Longrightarrow B_r[P - x] = B_r[P - y]. \tag{4.8}$$

Proof. Choose $\delta > 0$ and $r_0 > 0$ such that, for all $r \geq r_0$ and $x \in P$, the following holds:

$$B_{r+\delta}[P - x] \subset \bigcup_{h \in B_r[P-x]} B_r[P - x - h] + h. \tag{4.9}$$

This is possible since P is relatively dense. Choose ϵ such that the statement of Lemma 4.16 is satisfied with $r = r_0$ and some $r' = r'_0$. For $x \in P$, let

$$V(x) := \bigcup_{h \in B_r[P-x]} B_{r'_0}[\varphi(P) - \varphi(x + h)] + \varphi(x + h) - \varphi(x).$$

$V(x)$ is a subset of $\varphi(P) - \varphi(x)$. Then

$$B(\varphi(x + h) - \varphi(x), r_0') \cap V(x) = B_{r_0'}[\varphi(P) - \varphi(x + h)] + \varphi(x + h) - \varphi(x). \quad (4.10)$$

Now let $V(x) = V(y)$, $x, y \in P$. This implies, in particular, that $B_{r_0'}[\varphi(P) - \varphi(x)] = B_{r_0'}[\varphi(P) - \varphi(y)]$ from which we conclude from Lemma 4.16 that $B_{r_0}[P - x] = B_{r_0}[P - y]$ and, for all $h \in B_{r_0}[P - x]$, $\varphi(x + h) - \varphi(x) = \varphi(y + h) - \varphi(y)$. The latter implies with (4.10) that, for all $h \in B_{r_0}[P - x]$, $B_{r_0'}[\varphi(P) - \varphi(x + h)] = B_{r_0'}[\varphi(P) - \varphi(y + h)]$ and hence by Lemma 4.16 that, for all $h \in B_{r_0}[P - x]$, $B_{r_0}[P - x - h] = B_{r_0}[P - y - h]$ holds and, for all $h' \in B_{r_0}[P - x - h]$, $\varphi(x + h + h') - \varphi(x + h) = \varphi(y + h + h') - \varphi(y + h)$. In particular, we find that, for all $h'' \in B_{r_0}[P - x - h] + h$,

$$\begin{aligned}
\varphi(x + h'') - \varphi(x) &= \varphi(x + h + (h'' - h)) - \varphi(x + h) + \varphi(x + h) - \varphi(x) \\
&= \varphi(y + h + (h'' - h)) - \varphi(y + h) + \varphi(y + h) - \varphi(y) \\
&= \varphi(y + h'') - \varphi(y),
\end{aligned}$$

as $h'' - h \in B_{r_0}[P - x - h]$. Thus $V(x) = V(y)$, $x, y \in P$ implies that $B_{r_1}[P - x] = B_{r_1}[P - y]$ and, for all $h \in B_{r_1}[P - x]$, $\varphi(x + h) - \varphi(x) = \varphi(y + h) - \varphi(y)$ where $r_1 = r_0 + \delta$. Since $V(x)$ is finite there exists an r_1' such that $V(x) \subset B(0, r_1')$. It follows that, for $x, y \in P$, $B_{r_1'}[\varphi(P) - \varphi(x)] = B_{r_1'}[\varphi(P) - \varphi(y)]$ implies that $V(x) = V(y)$. We have thus proven that the statement of Lemma 4.16 holds with the same ϵ but $r = r_1$ and $r' = r_1'$. Iterating this argument yields the statement of the corollary. $\quad\square$

Note that in the statement of the corollary the points x and y were taken from P and so the corollary does not imply that $d\varphi^{-1}$ is strongly $\varphi(P)$-equivariant†. To obtain the statement for arbitrary $x, y \in \mathbb{R}^n$ we need to require that $d\varphi$ is strongly P-equivariant.

THEOREM 4.18. *Let P be a Delone set of finite local complexity. There exists $1 > \epsilon > 0$ such that, for all differentiable functions $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ whose differential satisfies $\|d(\varphi - \mathrm{id})\|_\infty < \epsilon$ and is strongly P-equivariant the following holds: for all $r > 0$ there exists $r' > 0$ for all $x, y \in \mathbb{R}^n$*

$$B_{r'}[\varphi(P) - \varphi(x)] = B_{r'}[\varphi(P) - \varphi(y)] \Longrightarrow B_r[P - x] = B_r[P - y]. \quad (4.11)$$

Proof. Given P and ϵ according to Corollary 4.17 let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-function whose differential $d\varphi$ is P-equivariant with range R and satisfies $\|d(\varphi - \mathrm{id})\|_\infty < \epsilon$. Let $r \geq \max\{\lambda r_{\max}, R\}$ where $r_{\max} = \inf\{r \mid \text{for all } x \in \mathbb{R}^n : B_r[P - x] \neq \emptyset\}$, $\lambda = (1 - \epsilon)^{-1}$. Let $r' = r'(r + \lambda r_{\max})$ according to Corollary 4.17.

Suppose that $B_{r' + \lambda r_{\max}}[\varphi(P) - \varphi(x)] = B_{r' + \lambda r_{\max}}[\varphi(P) - \varphi(y)]$. Choose $h \in \varphi(P) - \varphi(x)$ such that $|h| \leq \lambda r_{\max}$. Let $x' = \varphi^{-1}(\varphi(x) + h)$ and $y' = \varphi^{-1}(\varphi(y) + h)$. Then $\varphi(x')$, $\varphi(y') \in \varphi(P)$ and $B_{r'}[\varphi(P) - \varphi(x')] = B_{r'}[\varphi(P) - \varphi(y')]$. From Corollary 4.17 we conclude that $B_{r + \lambda r_{\max}}[P - x'] = B_{r + \lambda r_{\max}}[P - y']$. Since $d\varphi$ is P-equivariant with range r the latter implies that $\varphi(x) - \varphi(x') = \varphi(y' + x - x') - \varphi(y')$ which, in turn, is equivalent to $\varphi(y) = \varphi(y' + x - x')$. Since φ is invertible we get $y - y' = x - x'$ and hence $B_r[P - x] = B_r[P - y]$. $\quad\square$

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† The argument breaks down if one allows arbitrary $x \in \mathbb{R}^n$, as replacing in the definition of A_P the union over $x \in P$ by a union over $x \in \mathbb{R}^n$ would yield $A_P(s) = 0$.

COROLLARY 4.19. *Let P be a Delone set of finite local complexity. There exists $1 > \epsilon > 0$ such that for all differentiable functions $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ whose differential satisfies $\|d(\varphi - \mathrm{id})\|_\infty < \epsilon$ and is strongly P-equivariant we also have that $d\varphi^{-1}$ is strongly $\varphi(P)$-equivariant.*

Proof. Given P there exists ϵ according to the last theorem. Suppose that $\|d(\varphi - \mathrm{id})\|_\infty < \epsilon$ and that $d\varphi$ is P-equivariant with range r. Let r' correspond to the one needed in the last theorem. Let $B_{r'}(\varphi(P) - \varphi(x)) = B_{r'}(\varphi(P) - \varphi(y))$. By the last theorem this implies that $B_r(P - x) = B_r[P - y]$. Hence $d\varphi(x) = d\varphi(y)$. Since

$$(d\varphi^{-1})(\varphi(x)) = (d\varphi(x))^{-1}$$

(the inverse in the matrix sense) we see that $d\varphi^{-1}$ is P-equivariant with range r'. □

We can now answer the question as to what it means if two closed forms differ by an exact form, at least for forms which are close to $d\mathrm{id}$.

COROLLARY 4.20. *Let P be a Delone set of finite local complexity. There exists $1 > \epsilon > 0$ such that if $d\varphi$ and $d\psi$ are strongly P-equivariant, $\|d(\varphi - \mathrm{id})\|_\infty < \epsilon$, and furthermore if $\varphi - \psi$ is weakly P-equivariant (or even strongly P-equivariant) then $(\psi - \varphi) \circ \varphi^{-1}$ is weakly $\varphi(P)$-equivariant (or even strongly $\varphi(P)$-equivariant). In particular, $\psi \circ \varphi^{-1}$ induces a topological semi-conjugacy between $(\Omega_{\varphi(P)}, \mathbb{R}^n)$ and $(\Omega_{\psi(P)}, \mathbb{R}^n)$ (or even $\psi(P)$ is locally derivable from $\varphi(P)$).*

Proof. Suppose that $\varphi - \psi$ is P-equivariant with range r. By Theorem 4.18 we find r' such that $B_{r'}[\varphi(P) - x] = B_{r'}[\varphi(P) - y]$ implies that $B_r[P - \varphi^{-1}(x)] = B_r[P - \varphi^{-1}(x)]$. Since $\eta(x) = (\varphi - \psi)(\varphi^{-1}(x))$ the latter implies that $\eta(x) = \eta(y)$. Hence η is P-equivariant with range r' and so the result follows directly from Lemma 4.2 (the φ, g and P there correspond to $\mathrm{id} - \eta$, id and $\varphi(P)$ here). □

4.4. *Locally deriving back.* When are $\varphi(P)$ and P mutually locally derivable? Suppose that $\varphi - \mathrm{id}$ is strongly P-equivariant so that, in particular, $\varphi(P)$ is locally derivable from P. The last corollary gives an answer to the question of under which condition can P be locally derived back from $\varphi(P)$, namely this is the case if $\|d(\varphi - \mathrm{id})\|_\infty < \epsilon$ for some ϵ whose size could be estimated using the details of Corollary 4.17.

4.5. *Cohomological interpretation.* We interpret our results in cohomological terms.

Let $B_{\|\cdot\|_\infty}(\alpha, \epsilon)$ be the open ϵ-ball around $\alpha \in Z^1_{s-P}(\mathbb{R}^n, \mathbb{R}^n)$ with respect to the $\|\cdot\|_\infty$-norm. A good measure for the size of the deformation defined by φ is given by $\inf\{\|d\varphi + d\eta\|_\infty : d\eta \in N^1_{s-P}(\mathbb{R}^n, \mathbb{R}^n)\}$. It is scale invariant and corresponds to the maximal relative change of distance between points resulting from the deformation.

Theorem 4.8 and Corollaries 4.19 and 4.20 allow for the following interpretation. For all Delone sets of finite local complexity there exists an ϵ such that the elements of $Z^1_{s-P}(\mathbb{R}^n, \mathbb{R}^n) \cap B_{\|\cdot\|_\infty}(d\mathrm{id}, \epsilon)$ define invertible deformations of P. If, moreover, two such elements differ by an element of $B^1_{w-P}(\mathbb{R}^n, \mathbb{R}^n)$ (or even $B^1_{s-P}(\mathbb{R}^n, \mathbb{R}^n)$) then they define pointed topological conjugate (or even mutually locally derivable) deformations.

Theorems 4.13 and 4.15 say that the elements near the class of $d$id in the mixed quotient $Z^1_{s-P}(\mathbb{R}^n, \mathbb{R}^n)/B^1_{w-P}(\mathbb{R}^n, \mathbb{R}^n) \cap Z^1_{s-P}(\mathbb{R}^n, \mathbb{R}^n)$ parametrize small deformations modulo bounded deformations which are in the same pointed conjugacy class. Here we say that a deformation defined by a map φ is bounded if $\varphi - $ id is bounded, a condition which implies that the deformation is homotopic to the original pattern in the Hausdorff metric. Note that a bounded deformation has finite size but the converse need not be true.

On the level of pattern spaces and their associated dynamical systems the following picture has emerged (Theorem 4.10). An element $d\varphi$ near $d$id of $Z^1_{s-P}(\mathbb{R}^n, \mathbb{R}^n)$ defines a homeomorphism from the hull of P to that of the deformation it defines which restricts to a homeomorphism between the canonical transversals and maps P to its deformation. If $d(\varphi - \mathrm{id})$ lies even in $B^1_{w-P}(\mathbb{R}^n, \mathbb{R}^n) \cap Z^1_{s-P}(\mathbb{R}^n, \mathbb{R}^n)$ then it defines a second homeomorphism, this time a pointed conjugacy between the continuous dynamical systems. The two homeomorphisms need not coincide.

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