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ANALYTICAL VALUATION OF VULNERABLE OPTIONS IN A DISCRETE-TIME FRAMEWORK

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In this paper, we present a pricing model for vulnerable options in discrete time. A Generalized Autoregressive Conditional Heteroscedasticity process is used to describe the variance of the underlying asset, which is correlated with the returns of the asset. As for counterparty default risk, we study it in a reduced form model and the proposed model allows for the correlation between the intensity of default and the variance of the underlying asset. In this framework, we derive a closed-form solution for vulnerable options and investigate quantitative impacts of counterparty default risk on option prices.

Keyword: mathematical finance

1. INTRODUCTION

In this paper, we investigate the pricing issue of vulnerable options under a discrete-time Generalized Autoregressive Conditional Heteroscedasticity (GARCH) model. Vulnerable options refer to European options with counterparty default risk, and counterparty default risk refers to the risk in a financial contract that one counterparty defaults prior to maturity and fails to make the payments in accordance with agreed terms. Vulnerable options are usually traded in the over-the-counter (OTC) markets, and OTC contracts are indeed exposed to counterparty default risk, since there are no marking to market and delivery guaranty mechanisms. To protect counterparties from each other's default, the clearing counterparty (CCP) (CCP is a process by which financial transactions are cleared by a single central counterparty.) was introduced as a new regulatory approach to financial stability (see, e.g., Duffie and Zhu [12]). The main goal of introducing CCP is to prevent counterparties from each other's default and a large number of the OTC markets has moved toward the CCP. However, the OTC derivatives remain a significant part of the world of global finance. (The statistics in the ISDA survey "OTC Derivatives Market Analysis, Year-End 2012," published in June 2013, show that the notional outstanding of OTC derivatives approximated US$632.6 trillion on December 31, 2012.) Additionally, counterparty default risk has received much more attention due to the 2007–2008 financial crisis, and has been one of the risks facing all the participants in the OTC markets. The most cited example is the largest insurance company in the world, American International Group (AIG). AIG provided insurance against bond defaults by selling credit default swaps (CDS). CDS are

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frequently traded in the OTC markets and are priced by taking into account the possible defaults of the seller of the contract. For instance, Arora, Gandhi, and Longstaff [2] show that default risk is priced in the CDS markets by examining an extensive data set of CDS transaction prices. Since counterparty default risk indeed exists, taking counterparty default risk into account is necessary when valuing OTC contracts.

There are two methods to capture credit default risk, structural models and reduced form models. Structural models spring from the pioneering work by Black and Scholes [5] and Merton [25], in which a default happens if the value of a firm's assets is lower than debt obligations at maturity. In the structural models, credit events are triggered by the movements of the firm's value relative to some credit-event-triggering threshold (or barrier). Structural approaches have been adopted to study corporate bonds in the literature, including Black and Cox [4], Leland [23], Leland and Toft [24] and Anderson and Sundaresan [1], and vulnerable options, for instance, Johnson and Stulz [19], Klein [21], Cao and Wei [7], and Tian et al. [27]. Based on the first work on vulnerable options by Johnson and Stulz [19], Klein [21] presents an improved method of pricing vulnerable Black–Scholes options and derives an analytical pricing formula of the options with correlated credit risk. Cao and Wei [7] investigate the case where the counterparty has two types of liabilities: a corporate bond and a short position in a call option. In a recent work, Tian et al. [27] incorporate jump processes to describe the dynamics of asset prices and investigate the impact of jump risk on vulnerable option prices. Wang [29] investigates the differences in the prices of vulnerable options with different counterparties. Without examining underlying causalities of default, reduced form models assume Poisson-type arrivals of defaults, with an intensity of default governed by exogenous state variables. Examples of reduced form models include Artzner and Delbaen [3], Jarrow, Lando, and Turnbull [17], Lando [22], and Duffie and Singleton [11]. Adopting reduced form models of credit risk, Hull and White [16], Jarrow and Turnbull [18] and Fard [13] also consider the pricing of vulnerable options. Hull and White [16] consider derivative securities with default risk using the reduced form models, by assuming the holder of a security could only recover a proportion of its no-default value in the event of a default by the counterparty. Fard [13] describes the dynamics of the underlying asset using a kernel-biased completely random jump-diffusion process, and adopts a mean-reverting Ornstein-Uhlenbeck process to capture the default intensity. Based on the Esscher transform, the author determines an equivalent martingale measure (EMM) and obtains a closed-form price for vulnerable options.

This paper attempts to evaluate vulnerable options in a discrete-time framework, by considering credit default risk in a reduced form model and adopting a GRACH process to describe the variance of the underlying asset. GARCH processes are applied to take into account the volatility clustering phenomenon by Bollerslev [6] and are used to describe the dynamics of the underlying asset for valuing options (see, e.g., Duan [10], Heston and Nandi [14], and Christoffersen et al. [9]). Following Heston and Nandi [14], we use a particular GARCH process to describe the variance process. In contrast to earlier studies, this paper has four main characteristics. First, this paper is the first one to consider vulnerable options in a GARCH reduced form model and the possible default before maturity is considered. Second, the proposed model captures stochastic nature of volatility. There are few papers focusing on stochastic volatility when pricing vulnerable options. To our best knowledge, Wang and Wang [28] and Yang, Lee, and Kim [30] are the only exceptions, where the authors adopt different kinds of continuous-time stochastic volatility models. Third, the proposed model captures stochastic nature of correlation between returns and volatility for the underlying asset and allows for the correlation between the intensity of default and the variance of the underlying asset. Lastly, the closed-form solution for vulnerable options is also derived.

The remainder of this paper is organized as follows. In Section 2, the proposed framework is presented and we derive an explicit pricing formula of vulnerable European options. Section 3 presents numerical results to illustrate vulnerable option prices. Finally, concluding remarks are contained in Section 4. The detailed proofs are shown in the appendix.

2. THE MODEL

In this section, we present the pricing framework and derive a closed-form solution for vulnerable European options. Our formulation incorporates time-varying variance for the underlying asset described by GARCH processes and captures counterparty credit risk in a reduced form model. Additionally, the proposed model allows for the correlation between the variance of the underlying asset and the default intensity.

Assume that the uncertainty of the economy is described by a probability space $(\Omega, \mathscr{F}, P)$, equipped with an information flow $\{\mathscr{F}_t\}_{t\geq 0}$, where P is a real-world probability measure. Suppose that the underlying asset price satisfies the following process under P,

$$\begin{cases} \ln S(t+1) = \ln S(t) + r + \left(l_s - \frac{1}{2}\right) h_s(t+1) + \sqrt{h_s(t+1)} Z_s(t+1), \\ h_s(t+1) = w_s + b_s h_s(t) + a_s \left(Z_s(t) - c_s \sqrt{h_s(t)}\right)^2, \end{cases} \tag{2.1}$$

where $S(t)$ represents the value of the underlying asset at the close of day t, r is the interest rate, and l_s denotes the market price of risk, with expected returns being $r + l_s h_s(t+1)$ during the interval of time $[t, t+1]$. Shocks to the returns are generated by a standard normal variable $Z_s(t+1)$, and the conditional variance $h_s(t+1)$ is known at the end of day t. The term $-(1/2)h_s(t+1)$ is a convexity adjustment introduced such that the conditional expectation of returns becomes

$$\mathbb{E}_t^P \left[\frac{S(t+1)}{S(t)} \right] = e^{r + l_s h_s(t+1)}.$$

This class of GARCH processes are proposed by Heston and Nandi [14] and are stationary with finite mean and variance if $b_s + a_s c_s^2 < 1$ (see Heston and Nandi [14] for detail). This process allows for the correlation between the returns and variance, that is,

$$\mathrm{Cov}_{t-1}(h_s(t+1), \ln S(t)) = -2 a_s c_s h_s(t).$$

Positive values for a_s and c_s imply a negative correlation between returns and variance. In addition, GARCH models have an obvious advantage that volatility or variance is observable from the history of asset prices, compared with the continuous-time stochastic volatility models. To be specific, due to the dynamics of the underlying asset in (2.1), we can observe disturbance processes $Z_m(t)$ from the market price of the underlying asset,

$$Z_s(t) = \frac{\ln\{[S(t)]/[S(t-1)]\} - r - (l_s - [1/2]) h_s(t)}{\sqrt{h_s(t)}},$$

which results in an observable conditional variance $h_s(t+1)$ as follows:

$$\begin{aligned} h_s(t+1) &= w_s + b_s h_s(t) + a_s \left(Z_s(t) - c_s \sqrt{h_s(t)}\right)^2 \\ &= w_s + b_s h_s(t) + a_s \left(\frac{\ln\{[S(t)]/[S(t-1)]\} - r - (l_s - [1/2]) h_s(t)}{\sqrt{h_s(t)}} - c_s \sqrt{h_s(t)} \right)^2. \end{aligned}$$

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The above model can capture the time-varying variance and correlation between the returns and the variance of the underlying asset. To derive option prices, we now determine an EMM. First, define the following conditional Radon-Nikodym derivative:

$$
\begin{aligned}
L(t+1) &:= \frac{dQ}{dP}\Big|_{\mathcal{F}_t} \\
&= \frac{\exp\left\{\theta_s\sqrt{h_s(t+1)}Z_s(t+1)\right\}}{\mathbb{E}_t^P\left[\exp\left\{\theta_s\sqrt{h_s(t+1)}Z_s(t+1)\right\}\right]},
\end{aligned}
\tag{2.2}
$$

where θ_s is a constant. The form of the above Radon–Nikodym derivative is motivated by the affine structure of the pricing kernel (see, e.g., Christoffersen, Jacobs, and Ornthanalai [8]). To determine an EMM Q, the martingale condition should hold. The following result gives a necessary and sufficient condition.

Proposition 2.1: *The martingale condition holds if and only if*

$$\theta_s = -l_s.$$

Moreover, $Z_s(t) + l_s\sqrt{h_s(t)}$ *is a standard normal variable under* Q*.*

Based on the above condition ensuring that Q is an EMM, the risk-neutral dynamics of the underlying asset is given as follows.

Proposition 2.2: *The underlying asset price* $S(t)$ *satisfies the following processes under* Q*,*

$$
\begin{cases}
\ln S(t+1) = \ln S(t) + r - \dfrac{1}{2}h_s(t+1) + \sqrt{h_s(t+1)}Z_s^*(t+1), \\
h_s(t+1) = w_s + b_s h_s(t) + a_s\left(Z_s^*(t) - (c_s + l_s)\sqrt{h_s(t)}\right)^2,
\end{cases}
\tag{2.3}
$$

where $Z_s^*(t+1) := Z_s(t+1) + l_s\sqrt{h_s(t+1)}$ *is a standard normal variable under* Q*.*

A direct application of Proposition 1 in Heston and Nandi [14] helps us obtain Propositions 2.1 and 2.2. (The detailed proofs are available from the author upon request.) To price vulnerable options, we describe counterparty default risk and investigate the payoff of vulnerable options in the following subsections.

2.1. Counterparty Default Risk

In this subsection, we consider counterparty default risk in a reduced form model, where the default event is governed by a specified intensity process. We model the random time of default τ as the first jump time of a doubly stochastic Poisson process (Cox process) with intensity $\lambda(t)$, which is the conditional mean arrival rate of default measured in events per day. Suppose that the default intensity is governed by the following process under Q:

$$\lambda(t+1) = w_\lambda + b_\lambda\lambda(t) + a_\lambda\left(Z_\lambda(t)\right)^2, \tag{2.4}$$

where $w_\lambda > 0$, $b_\lambda \geq 0$, $a_\lambda \geq 0$, and $Z_\lambda(t)$ is a standard normal variable, capturing a time-varying intensity. The intensity process is wide-sense stationary if and only if $a_\lambda + b_\lambda < 1$

(see Bollerslev [6] for more detail). Note that $\lambda(t+1)$ is the mean arrival rate of default in $(t,t+1]$ given survival up to time t, we have that

$$Q(\tau > t+1|\mathscr{F}_t) = E_t^Q\left[e^{-\lambda(t+1)}\right] = e^{-\lambda(t+1)}. \quad \textbf{(2.5)}$$

Additionally, we assume that $Z_s^*(t+1)$ and $Z_\lambda(t+1)$ have a correlation coefficient ρ. From the dynamics of the underlying asset in (2.3) and the intensity process in (2.4), one gets that the covariance of the variance of the underlying asset with the default intensity is given by

$$\begin{aligned}
&\mathrm{Cov}_{t-1}\Big(h_s(t+1),\lambda(t+1)\Big)\\
&= \mathrm{Cov}_{t-1}\left(w_s + b_s h_s(t) + a_s\Big(Z_s^*(t) - (c_s+l_s)\sqrt{h_s(t)}\Big)^2, w_\lambda + b_\lambda\lambda(t) + a_\lambda\Big(Z_\lambda(t)\Big)^2\right)\\
&= \mathrm{Cov}_{t-1}\left(a_s\Big(Z_s^*(t) - (c_s+l_s)\sqrt{h_s(t)}\Big)^2, a_\lambda\Big(Z_\lambda(t)\Big)^2\right)\\
&= a_s a_\lambda \mathrm{Cov}_{t-1}\left(\Big(Z_s^*(t) - (c_s+l_s)\sqrt{h_s(t)}\Big)^2, \Big(Z_\lambda(t)\Big)^2\right)\\
&= a_s a_\lambda \mathrm{Cov}_{t-1}\left(\Big(Z_s^*(t)\Big)^2 - 2(c_s+l_s)\sqrt{h_s(t)}Z_s^*(t), \Big(Z_\lambda(t)\Big)^2\right)\\
&= 2a_s a_\lambda \rho^2,
\end{aligned}$$

where we have used the fact that $h_s(t)$ and $\lambda(t)$ are known at time $t-1$. We observe that our specification captures positive correlation between the variance of the underlying asset and the default intensity with positive values of a_s and a_λ and a non-zero ρ. Intuitively, the variance of the underlying asset and the default intensity are both positively correlated with market risk.

2.2. Vulnerable Option Prices

Based on the framework described above, we derive an explicit formula of vulnerable call options in this subsection. To this end, we first focus on the payoff of vulnerable call options. Due to the possibility that the counterparty defaults, the payoff of vulnerable options depends on whether default events occur or not during the lifetime of the options. Consequently, the payoff of vulnerable options consists of two parts. If there is no default event before the maturity T, the payoff of vulnerable options is equal to the payoff on a vanilla European call option. Mathematically, this part of the payoff can be expressed as $I(\tau > T)(S(T)-K)^+$, where τ denotes the default time and $I(\tau > T)$ indicates the case when no default events occur before the maturity T. If default events occur during the lifetime of the options, only a proportion of its market value can be recovered. In this case, the payoff of vulnerable options equals $\alpha\mathbb{E}^Q[e^{-r(T-\tau)}(S(T)-K)^+|\mathscr{F}_\tau]$, where $\mathbb{E}^Q[e^{-r(T-\tau)}(S(T)-K)^+|\mathscr{F}_\tau]$ represents the value of vulnerable call options at time τ and α is the recover rate. Therefore, the value of a vulnerable call option, C^*, can be represented by,

$$\begin{aligned}C^* &= e^{-rT}\mathbb{E}^Q\Big[I(\tau > T)(S(T)-K)^+\Big]\\&\quad + \mathbb{E}^Q\Big[I(0\le\tau\le T)\alpha\, e^{-r\tau}\mathbb{E}^Q\Big[e^{-r(T-\tau)}(S(T)-K)^+|\mathscr{F}_\tau\Big]\Big]\\&= e^{-rT}\mathbb{E}^Q\Big[I(\tau > T)(S(T)-K)^+\Big] + \alpha\, e^{-rT}\mathbb{E}^Q\Big[I(0\le\tau\le T)(S(T)-K)^+\Big],\end{aligned}$$

where the expectation is taken under the EMM Q. Note that $I(0\le\tau\le T) = 1 - I(\tau > T)$, we can rewrite C^* as follows:

$$\begin{aligned}C^* &= e^{-rT}\mathbb{E}^Q\Big[I(\tau > T)(S(T)-K)^+\Big] + \alpha\, e^{-rT}\mathbb{E}^Q\Big[I(0\le\tau\le T)(S(T)-K)^+\Big]\\&= (1-\alpha)e^{-rT}\mathbb{E}^Q\Big[I(\tau > T)(S(T)-K)^+\Big] + \alpha\, e^{-rT}\mathbb{E}^Q\Big[(S(T)-K)^+\Big]\\&:= (1-\alpha)e^{-rT}\Big(I_1 - KI_2\Big) + \alpha\, e^{-rT}\Big(I_3 - KI_4\Big),\end{aligned} \tag{2.6}$$

where I_1–I_4 are given by

$$I_1 = \mathbb{E}^Q\Big[S(T)I(\tau > T, S(T)\ge K)\Big], \tag{2.7}$$

$$I_2 = \mathbb{E}^Q\Big[I(\tau > T, S(T)\ge K)\Big], \tag{2.8}$$

$$I_3 = \mathbb{E}^Q\Big[S(T)I(S(T)\ge K)\Big], \tag{2.9}$$

$$I_4 = \mathbb{E}^Q\Big[I(S(T)\ge K)\Big]. \tag{2.10}$$

The closed form solutions for I_1–I_4 yield an explicit formula of vulnerable option prices C^* in (2.6). To calculate I_1–I_4, now we turn to derive the closed-form solution for the moment generating function of $x(T) := \ln S(T)$ and $\sum_{s=1}^{T}\lambda(s)$, which is denoted by $f(0;T,\phi_1,\phi_2)$ as follows:

$$f(0;T,\phi_1,\phi_2) = \mathbb{E}^Q\Big[e^{\phi_1 x(T)+\phi_2\sum_{s=1}^{T}\lambda(s)}\Big].$$

Note that

$$f(0;T,i\phi_1,i\phi_2) = \mathbb{E}^Q\Big[e^{i\phi_1 x(T)+i\phi_2\sum_{s=1}^{T}\lambda(s)}\Big]$$

is the characteristic function of $x(T) := \ln S(T)$ and $\sum_{s=1}^{T}\lambda(s)$. By inverting the characteristic function, we can calculate probabilities and hence obtain the expressions of I_1–I_4 shown in Proposition 2.4. The following result gives the explicit expression of $f(t;T,\phi_1,\phi_2)$ defined by

$$f(t;T,\phi_1,\phi_2) = \mathbb{E}_t^Q\Big[e^{\phi_1 x(T)+\phi_2\sum_{s=1}^{T}\lambda(s)}\Big].$$

Proposition 2.3: *The conditional moment generating function of $x(T)$ and $\sum_{s=1}^{T}\lambda(s)$, with the notation $x(T)=\ln S(T)$, admits the following form:*

$$f(t;T,\phi_1,\phi_2)=\exp\left\{\phi_1 x(t)+\phi_2\sum_{s=1}^{t}\lambda(s)+A(t;T,\phi_1,\phi_2)+B_1(t;T,\phi_1,\phi_2)h_s(t+1)\right.$$
$$\left.+\,B_2(t;T,\phi_1,\phi_2)\lambda(t+1)\right\},$$

where $A(t;T,\phi_1,\phi_2)$, $B_1(t;T,\phi_1,\phi_2)$ and $B_2(t;T,\phi_1,\phi_2)$ are given by

$$\begin{aligned}A(t;T,\phi_1,\phi_2)&=\phi_1 r+A(t+1;T,\phi_1,\phi_2)+w_sB_1(t+1;T,\phi_1,\phi_2)+w_\lambda B_2(t+1;T,\phi_1,\phi_2)\\&\quad-\frac{1}{2}\ln\left(1-2a_\lambda B_2(t+1;T,\phi_1,\phi_2)(1-\rho^2)\right)\\&\quad-\frac{1}{2}\ln\left(1-2\left(a_sB_1(t+1;T,\phi_1,\phi_2)+\frac{a_\lambda B_2(t+1;T,\phi_1,\phi_2)\rho^2}{1-2a_\lambda B_2(t+1;T,\phi_1,\phi_2)(1-\rho^2)}\right)\right),\\B_1(t;T,\phi_1,\phi_2)&=b_sB_1(t+1;T,\phi_1,\phi_2)-\frac{1}{2}\phi_1+a_s(c_s+l_s)^2B_1(t+1;T,\phi_1,\phi_2)\\&\quad+\frac{\left(\phi_1-2a_s(c_s+l_s)B_1(t+1;T,\phi_1,\phi_2)\right)^2}{2\left(1-2\left(a_sB_1(t+1;T,\phi_1,\phi_2)+\frac{a_\lambda B_2(t+1;T,\phi_1,\phi_2)\rho^2}{1-2a_\lambda B_2(t+1;T,\phi_1,\phi_2)(1-\rho^2)}\right)\right)},\\B_2(t;T,\phi_1,\phi_2)&=b_\lambda B_2(t+1;T,\phi_1,\phi_2)+\phi_2,\end{aligned}$$

and these coefficients can be obtained recursively using the terminal conditions,

$$A(T;T,\phi_1,\phi_2)=B_1(T;T,\phi_1,\phi_2)=B_2(T;T,\phi_1,\phi_2)=0.$$

Proof: See the appendix. ■

It should be noted that $B_2(t;T,\phi_1,\phi_2)$ has no independence on the variable ϕ_1 and can be derived in a closed form, that is, $B_2(t;T,\phi_1,\phi_2)=(1-b_\lambda^{T-t})/(1-b_\lambda)\phi_2$. In addition, $A(t;T,\phi_1,\phi_2)$ and $B_1(t;T,\phi_1,\phi_2)$ are defined recursively using the terminal conditions $A(T;T,\phi_1,\phi_2)=B_1(T;T,\phi_1,\phi_2)=0$. Based on the expression of the characteristic function $f(0;T,i\phi_1,i\phi_2)$, we can derive the vulnerable option price in (2.6). Hence, the expressions for $A(t;T,i\phi_1,i\phi_2)$ and $B_1(t;T,i\phi_1,i\phi_2)$ appear in the closed form solution for vulnerable option prices.

Proposition 2.4: *The price of vulnerable options with strike price K and maturity T is given by*

$$C^*=e^{-rT}\left[(1-\alpha)*\left(\Pi_1(0;T)+\frac{1}{2}f(0;T,1,-1)-K\Pi_2(0;T)-\frac{1}{2}Kf(0;T,0,-1)\right)\right.$$
$$\left.+\,\alpha\left(\Pi_3(0;T)+\frac{1}{2}f(0;T,1,0)-K\Pi_4(0;T)-\frac{1}{2}K\right)\right].$$

where the closed form solution for $f(0;T,\phi_1,\phi_2)$ *is derived in Proposition* 2.3 *and* $\Pi_1(0;T)$, $\Pi_2(0;T)$, $\Pi_3(0;T)$, *and* $\Pi_4(0;T)$ *are given by*

$$\Pi_1(0;T) = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi_1 \ln K} f(0;T,i\phi_1+1,-1)}{i\phi_1}\right] d\phi_1,$$

$$\Pi_2(0;T) = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi_1 \ln K} f(0;T,i\phi_1,-1)}{i\phi_1}\right] d\phi_1,$$

$$\Pi_3(0;T) = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi_1 \ln K} f(0;T,i\phi_1+1,0)}{i\phi_1}\right] d\phi_1,$$

$$\Pi_4(0;T) = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi_1 \ln K} f(0;T,i\phi_1,0)}{i\phi_1}\right] d\phi_1.$$

PROOF: See the appendix. ■

When $\alpha = 1$ (there is no loss when default occurs), the pricing formula in Proposition 2.4 reduces to the vanilla European call option pricing formula in Heston and Nandi [14]. Based on the explicit expression of the moment generating function, we have obtained the closed-form solution for vulnerable options. In the next section, we present some numerical results to illustrate vulnerable option prices.

3. NUMERICAL RESULTS

In this section, we illustrate vulnerable option prices using the derived pricing formula. In order to observe the quantitative impact of counterparty default risk, we choose the Heston and Nandi [14] model as a reference model. In addition, we contrast the values of vulnerable options issued by different counterparties. To be specific, we suppose that there are three counterparties (denoted by A, B, and C) issuing options with the same underlying asset, and these counterparties have different default probabilities as listed in Table 1. The data in Table 1 is from Hull [15], representing average cumulative default rates for corporate bonds in different rating categories during years 1970–2009. To determine four parameters' values in the dynamics of the default intensity, we choose four default probabilities during a period of 1.0, 3.0, 5.0, and 7.0 years. By equating the default probabilities implied by the proposed model (As shown in the previous section, the default probability in the proposed model during years 0–T is $1 - f(0;T,0,-1)$, and the closed-form solution for $f(0;T,0,-1)$ is given in Proposition 2.3.) with those in Table 1, we obtain the parameter values in the default intensity dynamics. (Since the default probabilities implied by the proposed model are of the recursive form, it is not easy to show the uniqueness and existence of the parameter values. We have to investigate it numerically. If the uniqueness and existence cannot be guaranteed, we can alternatively estimate the parameters by minimizing the

TABLE 1. Average Cumulative Default Rates (%), 1970–2009.

Term (years)	1.0	3.0	5.0	7.0
Counterparty A	0.176	0.912	1.926	2.996
Counterparty B	1.166	5.583	10.397	14.318
Counterparty C	4.546	16.188	25.895	34.473

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distance between theoretical and empirical cumulative default rates. The author thanks the referee for pointing out this issue.) The parameter values for three counterparties are listed in Table 2 and the corresponding cumulative default probabilities are plotted in Figure 1, using the derived formula of $1 - f(0; T, 0, -1)$.

As a reference point of the numerical results, Table 2 also summarizes a base set of parameters for the underlying asset. We set $w_s = 2.101 \times 10^{-17}$, $a_s = 3.317 \times 10^{-6}$,

TABLE 2. Parameter Values in the Base Case.

The underlying asset parameters		
Initial price	$S(0) = 100$	
Market price of risk	$l_s = 2.231$	
Parameters governing variance processes	$w_s = 2.101 \times 10^{-17}$	$a_s = 3.317 \times 10^{-6}$
	$b_s = 9.012 \times 10^{-1}$	$c_s = 1.276 \times 10^{2}$
Initial variance	$h_s(0) = 0.01/365$	
The default intensity parameters (A)		
Initial intensity	$\lambda(0) = 4.826 \times 10^{-6}$	
Parameters governing default intensities	$w_\lambda = 1.121 \times 10^{-8}$	$a_\lambda = 3.318 \times 10^{-12}$
	$b_\lambda = 9.994 \times 10^{-1}$	
The default intensity parameters (B)		
Initial intensity	$\lambda(0) = 3.213 \times 10^{-5}$	
Parameters governing default intensities	$w_\lambda = 1.542 \times 10^{-7}$	$a_\lambda = 2.601 \times 10^{-11}$
	$b_\lambda = 9.977 \times 10^{-1}$	
The default intensity parameters (C)		
Initial intensity	$\lambda(0) = 1.274 \times 10^{-4}$	
Parameters governing default intensities	$w_\lambda = 8.637 \times 10^{-7}$	$a_\lambda = 2.743 \times 10^{-10}$
	$b_\lambda = 9.949 \times 10^{-1}$	
Other parameters		
Interest rate	$r = 0.05$	
Strike price	$K = 100$	
Maturity	$T = 2.0$	
Correlation coefficient	$\rho = 0.50$	
Recovery rate	$\alpha = 0.50$	

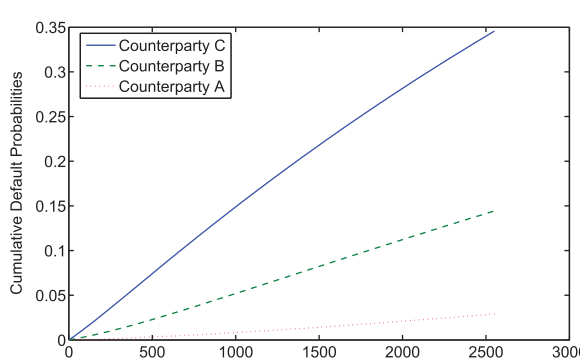

FIGURE 1. Cumulative default probabilities for Counterparties A, B, and C.

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$b_s = 9.012 \times 10^{-1}$, and $c_s = 1.276 \times 10^2$. The market price of risk is set to be $l_s = 2.231$. These parameters are borrowed from Christoffersen et al. [9], where the parameters are estimated using maximum likelihood and daily total returns from July 1, 1962 to December 31, 2001 on the Standard and Poor's 500 index. The annual initial variance is set to be 0.01, which means the instantaneous volatility is 0.10. As a base case, we suppose the vulnerable option is at the money and time to maturity is assumed to be 2 years. The variance of the underlying asset and the default intensity are correlated by setting $\rho = 0.5$. In the following, we investigate quantitative impacts of counterparty default risk on option prices with alternative maturities and spot-to-strike ratios.

Figure 2 depicts the values of call options with alternative maturities and several observations are in order. First, the values of the options without counterparty default risk are the largest and those of the options issued by Counterparty C are the smallest. Intuitively, once default events occur, option holders will suffer from the loss, reducing option values, and Counterparty C is mostly likely to default. Second, the default risk becomes more and more pronounced as the life of the option rises. Third, the values of the options issued by Counterparty A are very close to those without default risk, even when time to maturity increases to 5 years, while the gaps between option prices without default risk and those issued by Counterparty C increase quite quickly, as shown in Figure 3 more obviously. For instance, the distance between option prices with no default risk and those issued by Counterparty

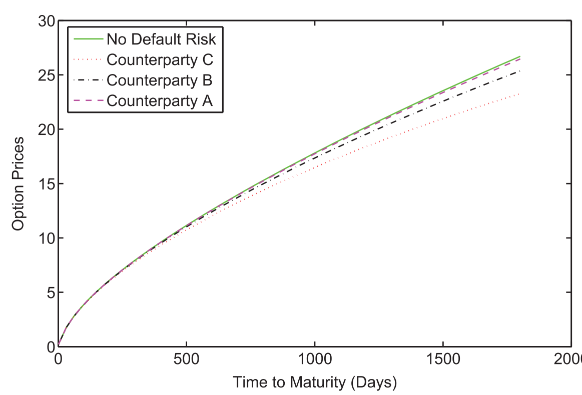

FIGURE 2. Option prices against time to maturity.

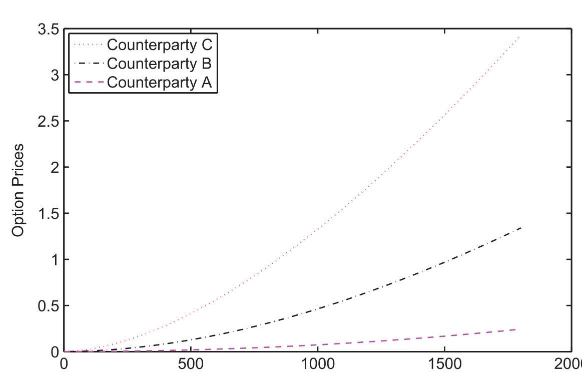

FIGURE 3. The Heston–Nandi price less than the proposed model price for call options with alternative maturities.

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C rises from 0.2416, corresponding to maturity $T = 1.0$ to 3.501, corresponding to maturity $T = 5.0$.

Figures 4 and 5 plot option prices varying with the spot to strike ratio. The impacts of counterparty default risk are more pronounced for deep-in-the-money options. For instance, the difference between the values of the options without default risk and those of the options issued by Counterparty A is only 0.0094 with spot-to-strike ratio being 0.8, and the difference increases slowly to 0.0841 in the case when spot-to-strike ratio is 1.20. In contrast, the differences between the values of the options without default risk and those of the options issued by Counterparty C are 0.1893 and 1.6912, respectively. Intuitively, once credit events occur, deep-in-the-money option holders will suffer from more potential credit losses.

Figure 6 presents option prices against recovery rate. Recovery rate only affects the payoff of the options when default happens, hence option prices in the Heston–Nandi model are not affected. Increasing the recovery rate from 0.40 to 0.60, values of the options issued by Counterparty A change from 14.33 to 14.34, while those corresponding to Counterparty C are 13.43 and 13.75, respectively. Hence, the more likely the counterparty is to default, the more sensitive option prices are to recovery rate.

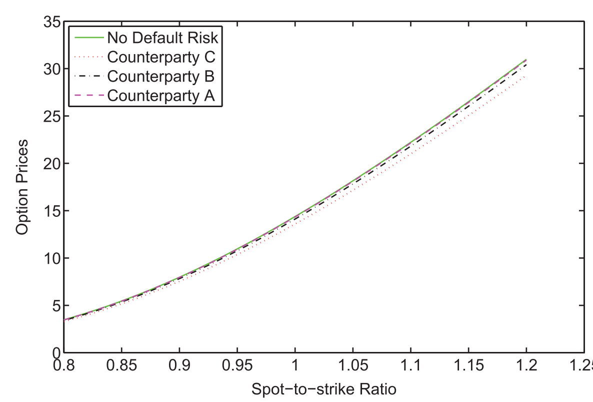

FIGURE 4. Option prices against spot-to-strike ratio.

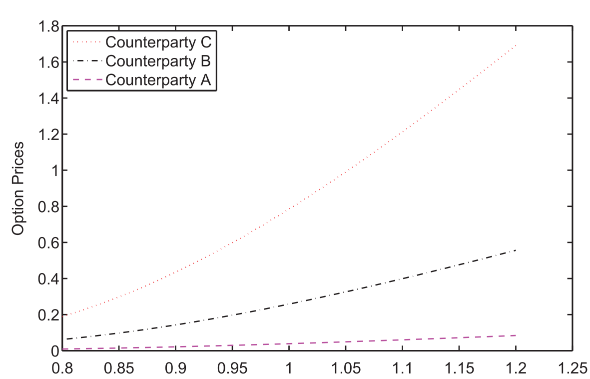

FIGURE 5. The Heston–Nandi price less than the proposed model price for call options with alternative pot-to-strike ratios.

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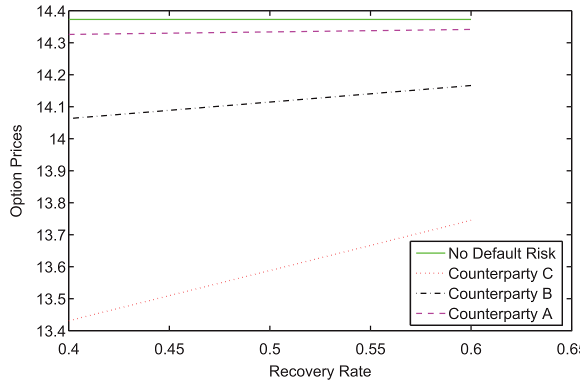

FIGURE 6. Option prices against recovery rate.

4. CONCLUSION

In this paper, we contribute to the literature through investigating vulnerable options in discrete-time models. The proposed model captures stochastic nature of variance, which is described by a GARCH process and correlated with the returns of the asset. The reduced form model is adopted to model counterparty default risk and the default intensity is time-varying stochastically. In addition, the correlation between the intensity of default and the variance of the underlying asset is allowed in the proposed model. Analytical formula of vulnerable options is obtained and numerical results are given to illustrate vulnerable option prices.

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APPENDIX

Proof of Proposition 2.3: Let $x(t) = \ln S(t)$. Denote by $f(t; T, \phi_1, \phi_2)$ the conditional moment generating function of $\sum_{s=1}^{T} \lambda(s)$ and $X(T)$, that is,

$$f(t; T, \phi_1, \phi_2) = \mathbb{E}_t^Q \left[e^{\phi_1 x(T) + \phi_2 \sum_{s=1}^{T} \lambda(s)} \right].$$

In the following, we show that the moment generating function has the log-linear form below:

$$f(t; T, \phi_1, \phi_2) = \exp \left\{ \phi_1 x(t) + \phi_2 \sum_{s=1}^{t} \lambda(s) + A(t; T, \phi_1, \phi_2) + B_1(t; T, \phi_1, \phi_2) h_s(t+1) \right. \\ \left. + B_2(t; T, \phi_1, \phi_2) \lambda(t+1) \right\}.$$

For convenience, we use the more parsimonious notations $f(t)$, $A(t)$, $B_1(t)$, and $B_2(t)$ to indicate $f(t; T, \phi_1, \phi_2)$, $A(t; T, \phi_1, \phi_2)$, $B_1(t; T, \phi_1, \phi_2)$ and $B_2(t; T, \phi_1, \phi_2)$.

At time T, $x(T)$ and $\sum_{s=1}^{T}\lambda(s)$ are known and it holds that $f(T)=\exp\{\phi_1 x(T)+\phi_2\sum_{s=1}^{T}\lambda(s)\}$, which in turn implies the following terminal conditions:

$$A(T)=B_1(T)=B_2(T)=0.$$

Applying the law of iterated expectations to $f(t)$ yields that

$$\begin{aligned}
f(t) &= \mathbb{E}_t^Q\Big[e^{\phi_1 x(T)+\phi_2\sum_{s=1}^{T}\lambda(s)}\Big]\\
&= \mathbb{E}_t^Q\Big[\mathbb{E}_{t+1}^Q\Big[e^{\phi_1 x(T)+\phi_2\sum_{s=1}^{T}\lambda(s)}\Big]\Big]\\
&= \mathbb{E}_t^Q\Big[f(t+1)\Big]\\
&= \mathbb{E}_t^Q\Bigg[\exp\Bigg\{\phi_1 x(t+1)+\phi_2\sum_{s=1}^{t+1}\lambda(s)+A(t+1)+B_1(t+1)h_s(t+2)\\
&\qquad +\ B_2(t+1)\lambda(t+2)\Bigg\}\Bigg].
\end{aligned}$$

Substituting the dynamics of $x(t+1)$, $h_s(t+2)$, and $\lambda(t+2)$, we have that

$$\begin{aligned}
f(t) = \mathbb{E}_t^Q\Bigg[\exp\Bigg\{&\phi_1 x(t)+\phi_1 r-\frac{1}{2}\phi_1 h_s(t+1)+\phi_1\sqrt{h_s(t+1)}Z_s^*(t+1)\\
&+\phi_2\sum_{s=1}^{t+1}\lambda(s)+A(t+1)\\
&+B_1(t+1)\Big(w_s+b_s h_s(t+1)+a_s(Z_s^*(t+1)-(c_s+l_s)\sqrt{h_s(t+1)})^2\Big)\\
&+\ B_2(t+1)\Big(w_\lambda+b_\lambda\lambda(t+1)+a_\lambda(Z_\lambda(t+1))^2\Big)\Bigg\}\Bigg].
\end{aligned}$$

Rearranging terms implies that

$$\begin{aligned}
f(t) &= \mathbb{E}_t^Q\Bigg[\exp\Bigg\{\phi_1 x(t)+\phi_1 r-\frac{1}{2}\phi_1 h_s(t+1)+\phi_2\sum_{s=1}^{t+1}\lambda(s)+A(t+1)\\
&\quad +\ B_1(t+1)\Big(w_s+b_s h_s(t+1)\Big)+B_2(t+1)\Big(w_\lambda+b_\lambda\lambda(t+1)\Big)+\Psi\Bigg\}\Bigg]\\
&= \mathbb{E}_t^Q\Bigg[\exp\Bigg\{\phi_1 x(t)+\phi_2\sum_{s=1}^{t}\lambda(s)+\phi_1 r+A(t+1)+w_s B_1(t+1)+w_\lambda B_2(t+1)\\
&\quad +\ \left(b_s B_1(t+1)-\frac{1}{2}\phi_1\right)h_s(t+1)+(b_\lambda B_2(t+1)+\phi_2)\lambda(t+1)+\Psi\Bigg\}\Bigg]\\
&= \exp\Bigg\{\phi_1 x(t)+\phi_2\sum_{s=1}^{t}\lambda(s)+\phi_1 r+A(t+1)+w_s B_1(t+1)+w_\lambda B_2(t+1)\\
&\quad +\left(b_s B_1(t+1)-\frac{1}{2}\phi_1\right)h_s(t+1)+(b_\lambda B_2(t+1)+\phi_2)\lambda(t+1)\Bigg\}\\
&\quad \times\mathbb{E}_t^Q\Big[\exp\{\Psi\}\Big],
\end{aligned}$$

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where Ψ has the following form:

$$\Psi = \phi_1 \sqrt{h_s(t+1)} Z_s^*(t+1) + a_s B_1(t+1)\Big(Z_s^*(t+1) - (c_s+\lambda_s)\sqrt{h_s(t+1)}\Big)^2 \\ + a_\lambda B_2(t+1)\Big(Z_\lambda(t+1)\Big)^2,$$

To derive $\mathbb{E}_t^Q\Big[\exp\{\Psi\}\Big]$, now we investigate $\mathbb{E}_t^Q\Big[\exp\Big\{a_\lambda B_2(t+1)\Big(Z_\lambda(t+1)\Big)^2\Big\}\Big|Z_s^*(t+1)\Big]$. Note that $Z_\lambda(t+1)$ and $Z_s^*(t+1)$ are two standard normal variables with a correlation coefficient ρ, which allows us to rewrite $Z_\lambda(t+1)$ as follows:

$$Z_\lambda(t+1) = \rho Z_s^*(t+1) + \sqrt{1-\rho^2}Z,$$

with Z being a standard normal variable, independent of $Z_s^*(t+1)$. Based on the above expression, one has that

$$\begin{aligned} &\mathbb{E}_t^Q\left[\exp\left\{a_\lambda B_2(t+1)\Big(Z_\lambda(t+1)\Big)^2\right\}\Big|Z_s^*(t+1)\right] \\ &= \mathbb{E}_t^Q\left[\exp\left\{a_\lambda B_2(t+1)\Big(\sqrt{1-\rho^2}Z + \rho Z_s^*(t+1)\Big)^2\right\}\Big|Z_s^*(t+1)\right] \\ &= \mathbb{E}_t^Q\left[\exp\left\{a_\lambda B_2(t+1)(1-\rho^2)\left(Z + \frac{\rho}{\sqrt{1-\rho^2}}Z_s^*(t+1)\right)^2\right\}\Big|Z_s^*(t+1)\right] \\ &= \exp\left\{-\frac{1}{2}\ln\Big(1-2a_\lambda B_2(t+1)(1-\rho^2)\Big) + \frac{a_\lambda B_2(t+1)\rho^2}{1-2a_\lambda B_2(t+1)(1-\rho^2)}\Big(Z_s^*(t+1)\Big)^2\right\}, \end{aligned}$$

where in the last equality we have used the fact

$$Ee^{a(Z+b)^2} = e^{-\frac{1}{2}\ln(1-2a)+\frac{ab^2}{1-2a}},$$

with Z being a standard normal variable.

In the following, we focus on $\mathbb{E}_t^Q\Big[\exp\{\Psi\}\Big]$, which in turn gives us the form of $f(t)$. Note that $Z_s^*(t+1)$ is a standard normal variable. We derive $\mathbb{E}_t^Q\Big[\exp\{\Psi_s\}\Big]$ as follows:

$$\begin{aligned} &\mathbb{E}_t^Q\left[\exp\left\{\phi_1\sqrt{h_s(t+1)}Z_s^*(t+1) + a_s B_1(t+1)\Big(Z_s^*(t+1) - (c_s+l_s)\sqrt{h_s(t+1)}\Big)^2\right.\right. \\ &\quad \left.\left. +\ a_\lambda B_2(t+1)\Big(Z_\lambda(t+1)\Big)^2\right\}\right] \\ &= \mathbb{E}_t^Q\Big[\mathbb{E}_t\Big[\exp\Big\{\phi_1\sqrt{h_s(t+1)}Z_s^*(t+1) + a_s B_1(t+1)\Big(Z_s^*(t+1) - (c_s+l_s)\sqrt{h_s(t+1)}\Big)^2 \\ &\quad + a_\lambda B_2(t+1)\Big(Z_\lambda(t+1)\Big)^2\Big\}\Big|Z_s^*(t+1)\Big]\Big] \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_t^Q \left[\exp\left\{ \phi_1 \sqrt{h_s(t+1)} Z_s^*(t+1) + a_s B_1(t+1) \Big(Z_s^*(t+1) - (c_s + l_s)\sqrt{h_s(t+1)} \Big)^2 \right\} \right. \\
&\quad \times \mathbb{E}_t \left[\exp\left\{ a_\lambda B_2(t+1) \Big(Z_\lambda(t+1) \Big)^2 \right\} \Big| Z_s^*(t+1) \right] \Bigg] \\
&= \mathbb{E}_t^Q \left[\exp\left\{ \phi_1 \sqrt{h_s(t+1)} Z_s^*(t+1) + a_s B_1(t+1) \Big(Z_s^*(t+1) - (c_s + l_s)\sqrt{h_s(t+1)} \Big)^2 \right\} \right. \\
&\quad \times \exp\left\{ -\frac{1}{2} \ln\Big(1 - 2a_\lambda B_2(t+1)(1-\rho^2)\Big) + \frac{a_\lambda B_2(t+1)\rho^2}{1 - 2a_\lambda B_2(t+1)(1-\rho^2)} \Big(Z_s^*(t+1) \Big)^2 \right\} \Bigg] \\
&= \mathbb{E}_t^Q \Bigg[\exp\Bigg\{ \phi_1 \sqrt{h_s(t+1)} Z_s^*(t+1) + a_s B_1(t+1) \Big(Z_s^*(t+1) - (c_s + l_s)\sqrt{h_s(t+1)} \Big)^2 \\
&\quad - \frac{1}{2} \ln\Big(1 - 2a_\lambda B_2(t+1)(1-\rho^2)\Big) + \frac{a_\lambda B_2(t+1)\rho^2}{1 - 2a_\lambda B_2(t+1)(1-\rho^2)} \Big(Z_s^*(t+1) \Big)^2 \Bigg\} \Bigg] \\
&:= \mathbb{E}_t^Q \left[\exp\left\{ \mu_1 \Big(Z_s^*(t+1) \Big)^2 + \mu_2 Z_s^*(t+1) + \mu_3 \right\} \right],
\end{aligned}
$$

where μ_1, μ_2, and μ_3 are defined by

$$
\begin{aligned}
\mu_1 &= a_s B_1(t+1) + \frac{a_\lambda B_2(t+1)\rho^2}{1 - 2a_\lambda B_2(t+1)(1-\rho^2)}, \\
\mu_2 &= \Big(\phi_1 - 2a_s(c_s + l_s) B_1(t+1) \Big) \sqrt{h_s(t+1)}, \\
\mu_3 &= a_s (c_s + l_s)^2 B_1(t+1) h_s(t+1) - \frac{1}{2} \ln\Big(1 - 2a_\lambda B_2(t+1)(1-\rho^2)\Big).
\end{aligned}
$$

Recall that $Z_s^*(t+1)$ is a standard normal variable, and one gets that

$$
\begin{aligned}
&\mathbb{E}_t^Q \left[\exp\left\{ \mu_1 \Big(Z_s^*(t+1) \Big)^2 + \mu_2 Z_s^*(t+1) + \mu_3 \right\} \right] \\
&\quad = \mathbb{E}_t^Q \left[\exp\left\{ \mu_1 \Big(Z_s^*(t+1) + \frac{\mu_2}{2\mu_1} \Big)^2 + \mu_3 - \frac{\mu_2^2}{4\mu_1} \right\} \right] \\
&\quad = \exp\left\{ \mu_3 - \frac{\mu_2^2}{4\mu_1} \right\} \mathbb{E}_t \left[\exp\left\{ \mu_1 \Big(Z_s^*(t+1) + \frac{\mu_2}{2\mu_1} \Big)^2 \right\} \right] \\
&\quad = \exp\left\{ \mu_3 - \frac{\mu_2^2}{4\mu_1} \right\} \times \exp\left\{ -\frac{1}{2} \ln(1 - 2\mu_1) + \frac{\mu_1}{1 - 2\mu_1} \left(\frac{\mu_2}{2\mu_1} \right)^2 \right\},
\end{aligned}
$$

where in the last equality we have also used the fact

$$
E e^{a(Z+b)^2} = e^{-\frac{1}{2}\ln(1-2a) + [ab^2/(1-2a)]},
$$

with Z being a standard normal variable.

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Substituting the expressions of μ_1, μ_2, and μ_3, and completing some algebra shows that

$$\begin{aligned}
&\mu_3 - \frac{\mu_2^2}{4\mu_1} - \frac{1}{2}\ln(1-2\mu_1) + \frac{\mu_1}{1-2\mu_1}\left(\frac{\mu_2}{2\mu_1}\right)^2 \\
&\quad = \mu_3 - \frac{1}{2}\ln(1-2\mu_1) + \frac{\mu_2^2}{2(1-2\mu_1)} \\
&\quad = a_s(c_s+l_s)^2 B_1(t+1)h_s(t+1) - \frac{1}{2}\ln\Big(1-2a_\lambda B_2(t+1)(1-\rho^2)\Big) \\
&\qquad - \frac{1}{2}\ln\left(1-2\Big(a_s B_1(t+1) + \frac{a_\lambda B_2(t+1)\rho^2}{1-2a_\lambda B_2(t+1)(1-\rho^2)}\Big)\right) \\
&\qquad + \frac{\Big(\phi_1 - 2a_s(c_s+l_s)B_1(t+1)\Big)^2}{2\left(1-2\left(a_s B_1(t+1) + \frac{a_\lambda B_2(t+1)\rho^2}{1-2a_\lambda B_2(t+1)(1-\rho^2)}\right)\right)} h_s(t+1) \\
&\quad = -\frac{1}{2}\ln\Big(1-2a_\lambda B_2(t+1)(1-\rho^2)\Big) - \frac{1}{2}\ln\left(1-2\Big(a_s B_1(t+1) + \frac{a_\lambda B_2(t+1)\rho^2}{1-2a_\lambda B_2(t+1)(1-\rho^2)}\Big)\right) \\
&\qquad + \left(a_s(c_s+l_s)^2 B_1(t+1) + \frac{\Big(\phi_1 - 2a_s(c_s+l_s)B_1(t+1)\Big)^2}{2\Big(1-2\Big(a_s B_1(t+1) + \frac{a_\lambda B_2(t+1)\rho^2}{1-2a_\lambda B_2(t+1)(1-\rho^2)}\Big)\Big)}\right) h_s(t+1).
\end{aligned}$$

Therefore, we have obtained that

$$\begin{aligned}
\mathbb{E}_t^Q\Big[\exp\{\Psi\}\Big] = \exp\Bigg\{ &-\frac{1}{2}\ln\Big(1-2a_\lambda B_2(t+1)(1-\rho^2)\Big) \\
&- \frac{1}{2}\ln\left(1-2\Big(a_s B_1(t+1) + \frac{a_\lambda B_2(t+1)\rho^2}{1-2a_\lambda B_2(t+1)(1-\rho^2)}\Big)\right) \\
&+ \Bigg(a_s(c_s+l_s)^2 B_1(t+1) \\
&+ \frac{\Big(\phi_1 - 2a_s(c_s+l_s)B_1(t+1)\Big)^2}{2\Big(1-2\Big(a_s B_1(t+1) + \frac{a_\lambda B_2(t+1)\rho^2}{1-2a_\lambda B_2(t+1)(1-\rho^2)}\Big)\Big)}\Bigg) h_s(t+1)\Bigg\}.
\end{aligned}$$

Hence, $A(t)$, $B_1(t)$, and $B_2(t)$ are given by

$$\begin{aligned}
A(t) &= \phi_1 r + A(t+1) + w_s B_1(t+1) + w_\lambda B_2(t+1) \\
&\quad - \frac{1}{2}\ln\Big(1-2a_\lambda B_2(t+1)(1-\rho^2)\Big) \\
&\quad - \frac{1}{2}\ln\left(1-2\left(a_s B_1(t+1) + \frac{a_\lambda B_2(t+1)\rho^2}{1-2a_\lambda B_2(t+1)(1-\rho^2)}\right)\right), \\
B_1(t) &= b_s B_1(t+1) - \frac{1}{2}\phi_1 + a_s(c_s+l_s)^2 B_1(t+1) + \frac{\Big(\phi_1 - 2a_s(c_s+l_s)B_1(t+1)\Big)^2}{2\Big(1-2\Big(a_s B_1(t+1)\frac{a_\lambda B_2(t+1)\rho^2}{1-2a_\lambda B_2(t+1)(1-\rho^2)}\Big)\Big)}, \\
B_2(t) &= b_\lambda B_2(t+1) + \phi_2.
\end{aligned}$$

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Thus, we have proved the moment generating function has the log-linear form,

$$f(t;T,\phi_1,\phi_2) = \exp\left\{\phi_1 x(t) + \phi_2 \sum_{s=1}^{t} \lambda(s) + A(t;T,\phi_1,\phi_2) + B_1(t;T,\phi_1,\phi_2)h_s(t+1) + B_2(t;T,\phi_1,\phi_2)\lambda(t+1)\right\}.$$

These coefficients can be obtained recursively using the terminal conditions,

$$A(T;T,\phi_1,\phi_2) = B_1(T;T,\phi_1,\phi_2) = B_2(T;T,\phi_1,\phi_2) = 0. \qquad \blacksquare$$

Proof of Proposition 2.4: Recall the moment generating function of $x(T)$ and $\sum_{s=1}^{T}\lambda(s)$ in Proposition 2.3, with the notations $x(t) = \ln S(t)$,

$$f(0;T,\phi_1,\phi_2) = \mathbb{E}^Q\left[e^{\phi_1 x(T) + \phi_2 \sum_{s=1}^{T}\lambda(s)}\right]. \tag{A.1}$$

To deal with the term I_1 in (2.7), we define a new probability measure Q_1 as follows:

$$Q_1(A) := \frac{E^Q\left[I(A)S(T)I(\tau > T)\right]}{E^Q\left[S(T)I(\tau > T)\right]},$$

for any events $A \in \mathscr{F}_T$ and $I(\cdot)$ is an indicator function. Based on the definition of Q_1, we have the characteristic function of $x(T)$ under Q_1,

$$\begin{aligned} f_1(0;T,i\phi_1) &= E^{Q_1}\left[e^{i\phi_1 X(T)}\right] \\ &= \frac{E^Q\left[e^{i\phi_1 X(T)}S(T)I(\tau > T)\right]}{E^Q\left[S(T)I(\tau > T)\right]} \\ &= \frac{E^Q\left[e^{(i\phi_1+1)X(T)}I(\tau > T)\right]}{E^Q\left[S(T)I(\tau > T)\right]} \\ &= \frac{E^Q\left[e^{(i\phi_1+1)X(T)}e^{-\sum_{s=1}^{T}\lambda(s)}\right]}{E^Q\left[e^{X(T)-\sum_{s=1}^{T}\lambda(s)}\right]} \\ &= \frac{f(0;T,i\phi_1+1,-1)}{f(0;T,1,-1)}, \end{aligned}$$

where we have used the definition of $f(0;T,\phi_1,\phi_2)$ in (A.1).

Standard probability theory (see, e.g., Kendall and Stuart [20], and Shephard [26]) implies the distribution function $F_1(x(T);x)$ corresponding to the characteristic function $f_1(0;T,i\phi_1)$,

$$F_1(x(T);x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big[\frac{e^{-i\phi_1 x} f_1(0;T,i\phi_1)}{i\phi_1}\Big] d\phi_1,$$

where Re[ ] denotes the real part of a complex number. Hence, we have that

$$\begin{aligned} Q_1(x(T) \geq \ln K) &= 1 - F_1(x(T);\ln K) \\ &= \frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big[\frac{e^{-i\phi_1 \ln K} f_1(0;T,i\phi_1)}{i\phi_1}\Big] d\phi_1 + \frac{1}{2}. \end{aligned}$$

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Recall the definition of the probability measure Q_1, we have that

$$Q_1(x(T) \geq \ln K) = \frac{E^Q\Big[I(x(T) \geq \ln K)S(T)I(\tau > T)\Big]}{E^Q\Big[S(T)I(\tau > T)\Big]},$$

which in turn implies

$$\begin{aligned}
&E^Q\Big[I(x(T) \geq \ln K)S(T)I(\tau > T)\Big] \\
&= Q_1(x(T) \geq \ln K) * E^Q\Big[S(T)I(\tau > T)\Big] \\
&= \left(\frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big[\frac{e^{-i\phi_1 \ln K} f_1(0;T,i\phi_1)}{i\phi_1}\Big]d\phi_1 + \frac{1}{2}\right) f(0;T,1,-1) \\
&= \frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big[\frac{e^{-i\phi_1 \ln K} f(0;T,i\phi_1+1,-1)}{i\phi_1}\Big]d\phi_1 + \frac{1}{2}f(0;T,1,\,-1) \\
&:= \Pi_1(0;T) + \frac{1}{2}f(0;T,1,-1),
\end{aligned}$$

Rewrite I_1 in (2.7) and we obtain

$$\begin{aligned}
I_1 &= \mathbb{E}^Q\Big[S(T)I(\tau > T, S(T) \geq K)\Big] \\
&= \mathbb{E}^Q\Big[I(S(T) \geq K)S(T)I(\tau > T)\Big] \\
&= \Pi_1(0;T) + \frac{1}{2}f(0;T,1,-1),
\end{aligned}$$

where $\Pi_1(0;T) = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big[\frac{e^{-i\phi_1 \ln K} f(0;T,i\phi_1+1,-1)}{i\phi_1}\Big]d\phi_1$.

As for I_2, we can derive it similarly. Define a new probability measure Q_2 as follows:

$$Q_2(A) := \frac{E^Q\Big[I(A)I(\tau > T)\Big]}{E^Q\Big[I(\tau > T)\Big]},$$

for any events $A \in \mathscr{F}_T$ and $I(\cdot)$ is an indicator function. Analogously, the characteristic function of $x(T)$ under Q_2 is given by

$$\begin{aligned}
f_2(0;T,i\phi_1) &= E^{Q_2}\Big[e^{i\phi_1 X(T)}\Big] \\
&= \frac{E^Q\Big[e^{i\phi_1 X(T)}I(\tau > T)\Big]}{E^Q\Big[I(\tau > T)\Big]} \\
&= \frac{E^Q\Big[e^{i\phi_1 X(T)}e^{-\sum_{s=1}^T \lambda(s)}\Big]}{E^Q\Big[e^{-\sum_{s=1}^T \lambda(s)}\Big]} \\
&= \frac{f(0;T,i\phi_1,-1)}{f(0;T,0,-1)}.
\end{aligned}$$

Thanks to standard probability theory (see, e.g., Kendall and Stuart [20]), the distribution function $F_2(x(T);x)$ corresponding to the characteristic function $f_2(0;T,i\phi_1)$ is

$$F_2(x(T);x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big[\frac{e^{-i\phi_1 x} f_2(0;T,i\phi_1)}{i\phi_1}\Big]d\phi_1,$$

where Re[ ] denotes the real part of a complex number. Hence, we have that

$$\begin{aligned} Q_2(x(T) \geq \ln K) &= 1 - F_2(x(T); \ln K) \\ &= \frac{1}{\pi} \int_0^\infty \mathrm{Re}\Big[\frac{e^{-i\phi_1 \ln K} f_2(0; T, i\phi_1)}{i\phi_1}\Big] d\phi_1 + \frac{1}{2}. \end{aligned}$$

Therefore, we obtain the explicit expression for I_2 as follows:

$$\begin{aligned} I_2 &= \mathbb{E}^Q\Big[I(\tau > T, S(T) \geq K)\Big] \\ &= Q_2(x(T) \geq \ln K) * Q(\tau > T) \\ &= Q_2(x(T) \geq \ln K) * E^Q\Big[e^{-\sum_{s=1}^T \lambda(s)}\Big] \\ &= Q_2(x(T) \geq \ln K) * f(0; T, 0, -1) \\ &:= \Pi_2(0; T) + \frac{1}{2} f(0; T, 0, -1), \end{aligned}$$

where

$$\Pi_2(0; T) = \frac{1}{\pi} \int_0^\infty \mathrm{Re}\Big[\frac{e^{-i\phi_1 \ln K} f(0; T, i\phi_1, -1)}{i\phi_1}\Big] d\phi_1.$$

To derive I_3, we define a new probability measure Q_3,

$$Q_3(A) := \frac{E^Q\Big[I(A)S(T)\Big]}{E^Q\Big[S(T)\Big]},$$

and the characteristic function of $x(T)$ under Q_3 is given by

$$\begin{aligned} f_3(0; T, i\phi_1) &= E^{Q_3}\Big[e^{i\phi_1 X(T)}\Big] \\ &= \frac{E^Q\Big[e^{i\phi_1 X(T)} S(T)\Big]}{E^Q\Big[S(T)\Big]} \\ &= \frac{f(0; T, i\phi_1 + 1, 0)}{f(0; T, 1, 0)}. \end{aligned}$$

Similarly, one gets that

$$Q_3(x(T) \geq \ln K) = \frac{1}{\pi} \int_0^\infty \mathrm{Re}\Big[\frac{e^{-i\phi_1 \ln K} f_3(0; T, i\phi_1)}{i\phi_1}\Big] d\phi_1 + \frac{1}{2}$$

and

$$\begin{aligned} I_3 = \mathbb{E}^Q[S_T I(S_T \geq K)] &= Q_3(x(T) \geq \ln K) * \mathbb{E}^Q[S_T] \\ &= Q_3(x(T) \geq \ln K) * f(0; T, 1, 0) \\ &:= \Pi_3(0; T) + \frac{1}{2} f(0; T, 1, 0), \end{aligned}$$

where

$$\Pi_3(0; T) = \frac{1}{\pi} \int_0^\infty \mathrm{Re}\Big[\frac{e^{-i\phi_1 \ln K} f(0; T, i\phi_1 + 1, 0)}{i\phi_1}\Big] d\phi_1.$$

Analogously, standard probability theory implies that

$$\begin{aligned}\Pi_4(0;T) &:= Q(x(T) \geq \ln K) \\ &= 1 - Q(x(T) \leq \ln K) \\ &= \frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big[\frac{e^{-i\phi_1 \ln K} f(0;T,i\phi_1,0)}{i\phi_1}\Big] d\phi_1 + \frac{1}{2} \\ &:= \Pi_4(0;T) + \frac{1}{2}.\end{aligned}$$

Therefore, the price of vulnerable options is given by

$$\begin{aligned}C^* = e^{-rT}\Big[&(1-\alpha) * \Big(\Pi_1(0;T) + \frac{1}{2}f(0;T,1,-1) - K\Pi_2(0;T) - \frac{1}{2}Kf(0;T,0,-1)\Big) \\ &+ \alpha\Big(\Pi_3(0;T) + \frac{1}{2}f(0;T,1,0) - K\Pi_4(0;T) - \frac{1}{2}K\Big)\Big].\end{aligned}$$

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