J. Austral. Math. Soc. (Series A) 39 (1985), 1–20

INVEX FUNCTIONS AND DUALITY

B. D. CRAVEN and B. M. GLOVER

(Received 25 January 1983; revised 18 June 1983)

Communicated by B. Mond

Abstract

For both differentiable and nondifferentiable functions defined in abstract spaces we characterize the generalized convex property, here called cone-invexity, in terms of Lagrange multipliers. Several classes of such functions are given. In addition an extended Kuhn-Tucker type optimality condition and a duality result are obtained for quasidifferentiable programming problems.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 90 C 30.

1. Introduction

The Kuhn-Tucker conditions for a constrained minimization problem become also sufficient for a (global) minimum if the functions are assumed to be convex, or to satisfy certain generalized convex properties [14]. Hanson [10] showed that a minimum was implied when convexity was replaced by a much weaker condition, called *invex* by Craven [4], [5]. For the problem,

$$\text{Minimize}\, f_0(x) \quad \text{subject to } -g(x) \in S,$$

where S is a closed convex cone, the vector $f = (f_0, g)$ is required to have a certain property, here called *cone-invex*, in relation to the cone $\mathbf{R}_+ \times S$. Some conditions necessary, or sufficient, for cone-invex were given in Craven [5]; see also Hanson and Mond [12]. However, it would be useful to characterize some recognizable classes of cone-invex functions.

The present paper (a) represents several classes of cone-invex functions, (b) characterizes the cone-invex property, for differentiable functions, in terms of Lagrange multipliers (Theorems 2 and 3), using Motzkin's (or Gale's) alternative

https://doi.org/10.1017/S1446788700022126 Published online by Cambridge University Press

theorem; (c) extends some of these results to a class of nondifferentiable functions, namely quasidifferentiable functions [16]. In this final section we shall also establish a Kuhn-Tucker type optimality condition and a duality theorem for cone-invex programs with a quasidifferentiable objective function. Several examples are given to illustrate the results.

2. Definitions and symbols

Consider the constrained minimization problem:

$$(P) \qquad \underset{x \in X_0}{\text{Minimize}}\, f_0(x) \quad \text{subject to } -g(x) \in S,$$

in which f_0: $X_0 \to \mathbf{R}$ and g: $X_0 \to Y$ are (Fréchet) differentiable functions, X and Y are normed spaces, $X_0 \subset X$ is an open set, and $S \subset Y$ is a closed convex cone. Suppose that (P) attains a local minimum at $x = a$. If a suitable constraint qualification is also assumed, then the Kuhn-Tucker conditions hold:

$$(KT) \qquad (\exists \lambda \in S^*)\, f_0'(a) + \lambda g'(a) = 0, \qquad \lambda g(a) = 0, -g(a) \in S.$$

Here $S^* = \{v \in Y'$: $(\forall y \in S) vy \geqslant 0\}$, in which Y' denotes the (topological) dual space of Y, and vy denotes the evaluation of the functional v at $y \in Y$. (In finite dimensions, vy may be expressed as $v^T y$, in terms of column vectors. Note that x and y here do not generally relate to the spaces X and Y.)

The Kuhn-Tucker conditions are valid under weaker differentiability assumptions on f_0 and g, in particular, if the functions are *linearly Gâteaux differentiable* at the point a, ([3]). Now let $f = (f_0, g)$: $X_0 \to \mathbf{R} \times Y$; $K = \mathbf{R}_+ \times S$, where $\mathbf{R}_+ = [0, \infty)$; $r = (1, \lambda)$; $S_0 = \{\alpha(y + g(a))$: $\alpha \in \mathbf{R}_+,\ y \in S\}$, $K_0 = \mathbf{R}_+ \times S_0$. Then $K_0^* = \mathbf{R}_+ \times S_0^*$, where $S_0^* = \{\lambda \in S^*$: $\lambda g(a) = 0\}$. Then $r = (r_0, \lambda)$ with $r_0 \in \mathbf{R}_+$, $\lambda \in S_0^*$. Now (KT) holds if and only if there holds:

$$(KT+) \quad (\exists r \in K_0^*, r_0 > 0) rf'(a) = 0, \qquad rf(a) = f_0(a);\ -g(a) \in S.$$

Let $KT(P)$ denote the set of a X_0 such that $(KT+)$ holds, for some $r \in K_0^*$. Let $Z = \mathbf{R} \times Y$. Denote by (D_1) the formal *Wolfe dual* of the problem (P), namely

$$(D_1) \quad \underset{u \in X_0, \lambda \in Y'}{\text{Maximize}}\, f_0(u) + \lambda g(u) \quad \text{subject to } \lambda \in S^*, f_0'(u) + \lambda g'(u) = 0.$$

Let $E = \{x \in X_0$: $-g(x) \in S\}$, the feasible set of (P); denote by W the set of $u \in X_0$, such that (u, λ) is feasible for (D_1), for some $\lambda \in S^*$. The formal *Lagrangean dual* of the problem (P) is the problem

$$(D_2) \qquad \underset{\lambda \in S^*}{\text{Maximize}}\, \phi(\lambda), \quad \text{where } \phi(\lambda) = \inf\{f_0(x) + \lambda g(x): x \in X_0\}.$$

Note that weak duality ([3]) holds automatically for (P) and (D_2), that is $f_0(x) \geqslant \phi(\lambda)$ whenever x is feasible for (P) and λ for (D_2).

A function $h\colon X_0 \to Y$ is *S-convex* if, whenever $0 < \alpha < 1$ and $x, y \in X_0$,

$$\alpha h(x) + (1-\alpha)h(y) - h(\alpha x + (1-\alpha)y) \in S;$$

h is *locally S-convex* at $a \in X_0$ if this inclusion holds whenever $x, y \in U$, where U is a neighborhood of a in X_0. If the function h is linearly Gâteaux differentiable then h is S-convex if and only if, for each $x, y \in X_0$,

$$h(x) - h(y) - h'(y)(x-y) \in S. \tag{1}$$

The function h is *S-sublinear* if h is S-convex and positively homogeneous of degree one (that is, $h(\alpha x) = \alpha h(x)$, $\forall \alpha \geqslant 0$). If $Y = \mathbf{R}$, $S = \mathbf{R}_+$ we shall denote the *subdifferential* of a convex function h at $a \in X_0$ by $\partial h(a)$, where

$$\partial h(a) = \{v \in X' \colon v(x-a) \leqslant h(x) - h(a), \text{ for all } x \in X_0\}.$$

If h is continuous at a then $\partial h(a)$ is a non-empty weak* compact convex subset of X' ([17]); by (1) if h is linearly Gâteaux differentiable at a then $\partial h(a) = \{h'(a)\}$.

Following [5], a function $f\colon X_0 \to Z$ is called *K_0-invex*, with respect to a function $\eta\colon X_0 \times X_0 \to X$, if, for each $x, u \in X_0$,

$$f(x) - f(u) - f'(u)\eta(x,u) \in K_0. \tag{2}$$

(This property may be called *cone-invex* when the cone K_0 is fixed.) The function f is called *K_0-invex at u on $E \subset X_0$* if (2) holds for given $u \in X$, and for each $x \in E$. We are assuming f is linearly Gâteaux differentiable.

Define the following (possibly empty) set, contingent on a set $D \subset X$,

$$\operatorname{aint} D = \{x \in D \colon (\forall z \in X, z \neq 0)\,(\exists \delta > 0)\, x + \delta z \in D\}.$$

If D is convex, $x + \alpha\delta z \in D$ also when $0 < \alpha < 1$, so aint D equals the algebraic interior of D, as usually defined. If the cone S has non-empty (topological) interior int S, then $\varnothing \neq \operatorname{int} S \subset \operatorname{aint} S$. Define the *polar* sets of sets $V \subset X$ and $A \subset X'$ as

$$V^0 = \{w \in X' \colon (\forall x \in V)\, w(x) \geqslant -1\};$$

$$A^0 = \{x \in X \colon (\forall w \in A)\, w(x) \geqslant -1\}.$$

We shall also require in section V the following (not necessarily linear) concept of differentiability. A function $h\colon X_0 \to Y$ is *directionally differentiable* at $a \in X_0$ if the limit

$$h'(a, x) = \lim_{\alpha \downarrow 0} \alpha^{-1}(h(a + \alpha x) - h(a))$$

exists for each $x \in X$, in the strong topology of Y. If $Y = \mathbf{R}$ and h is a convex functional then, for each $a \in X_0$, $h'(a, \cdot)$ exists and is sublinear ([17]).

For a function $h: X_0 \to \mathbf{R}$, the *level sets* of h ([24]) are the sets

$$L_h(\alpha) = \{x \in X_0 : h(x) \leqslant \alpha\}, \qquad (\alpha \in \mathbf{R}),$$

and the *effective domain* of $L_h(\cdot)$ is the set $G_h = \{\alpha \in \mathbf{R}: L_h(\alpha) \neq \varnothing\}$. This point-to-set mapping L_h is called *lower semi-continuous* (LSC) at $\alpha \in G_h$ if $x \in L_h(\alpha)$, $G_h \supset (\alpha_i) \to \alpha$ imply the existence of an integer k and a sequence (x_i) such that $x_i \in L_h(\alpha_i)$ $(i = k, k+1, \ldots)$ and $x_i \to x$. The point-to-set mapping L_h is *strictly lower semi-continuous* (SLSC) at $\alpha \in G_h$ if $x \in L_h(\alpha)$, $G_h \supset (\alpha_i) \to \alpha$ imply the existence of an integer k, a sequence (x_i), and $b(x) > 0$ such that $x_i \in L_h(\alpha_i - b(x)\|x_i - x\|)$, $(i = k, k+1, \ldots)$, and $x_i \to x$.

The range of a function f is denoted by ran f; the nullspace by $N(f)$. For a continuous linear function $B: X \to Y$ we will denote by $B^T: Y' \to X'$ the transpose operator of B, defined by $B^T(w) = w \circ B$, for each $w \in Y'$. The constraint $-g(x) \in S$ of (P) is called *locally solvable* at $a \in X_0$ if $-g(a) \in S$ and, whenever $d \in X$ satisfies the linearized inclusion $-g(a) - g'(a)d \in S$, there exists a local solution $x = a + \alpha d + o(\alpha)$ to $-g(x) \in S$, valid for all sufficiently small $\alpha > 0$. We are assuming g is linearly Gâteaux differentiable at a, however if g is merely directionally differentiable at a then we can replace the linearized inclusion by $-g(a) - g'(a, d) \in S$. In particular, the constraint $-g(x) \in S$ is locally solvable at $a \in X_0$ if ([3]) g is continuously Fréchet differentiable and the set $g(a) + \operatorname{ran}(g'(a)) + S$ contains a neighborhood of 0 in Y.

For a set $A \subset X$, we shall denote the *closure* of A by $\bar{A}$. We shall assume throughout that the dual space X' (or Y') is endowed with the weak* topology (see [3]), thus for a set $V \subset X'$, $\bar{V}$ represents the weak* closure of V. The results in Section 4 do not depend on the dimensions of the spaces, and would extend readily to locally convex spaces (for example, space of distributions).

3. Classes of cone-invex functions

In this section we illustrate the broad nature of cone-invexity by presenting several classes of such functions, and some simple concrete examples.

(I) Each cone-convex function is invex, by (1) with $\eta(x, a) = x - a$.

(II) Let $q: X \to Y$ and $\varphi: X \to X$ be Hadamard differentiable with q S-convex and φ surjective ($\varphi(X) = X$). Assume further that either (a) $(\forall a \in X)\operatorname{ran}(\varphi'(a)) = X$, or (b) $(\forall x, a \in X)$ $[\operatorname{ran}(\varphi'(a))]^* \subset N(\varphi(x) - \varphi(a))$, and $[\varphi'(a), \varphi(x) - \varphi(a)]^T(X')$ is (weak*) closed. (In (b), we consider $\varphi(x) - \varphi(a) \in X''$, the second dual of X; note also that (a) implies (b).) Then the function $g = q \circ \varphi$ is S-invex on X. For hypothesis (a), this follows from [4]. For hypothesis (b), let $A = \varphi'(a)$

https://doi.org/10.1017/S1446788700022126 Published online by Cambridge University Press

and let $c = \varphi(x) - \varphi(a)$, given $a, x \in X_0$. Then

$$[\operatorname{ran}(A)]^* \subset N(c) \Leftrightarrow \{(\forall\lambda \in X')\lambda A = 0 \Rightarrow \lambda(c) \geqslant 0\}$$

since $[\operatorname{ran}(A)]^* = N(A^T)$

$\Leftrightarrow (0, -1) \notin [A, c]^T(X')$ since $[A, c]^T(X')$ is closed

$\Leftrightarrow (\exists\eta \equiv \eta(x, a) \in X) A\eta = c$

by Theorem 7 (below) and the Remark following it

$\Rightarrow (q \circ \varphi)'(a)\eta(x, a) = q'(\varphi(a))\varphi'(a)\eta(x, a) = q(\varphi(a))(\varphi(x) - \varphi(a))$

$= q(\varphi(x)) - q(\varphi(a)) - s$ for some $s \in S$, since q is S-convex

$\Rightarrow q \circ \varphi$ is S-invex at a.

Example 1. Let $X = \mathbf{R}^2$, $Y = \mathbf{R}$, $S = \mathbf{R}_+$, $q(x, y) = 3x^2 - 2xy + 2y^2$ and $\varphi(x, y) = (x - ax^3, y + by^3)$, where $a, b > 0$. Then $g = q \circ \varphi$ is invex on X but not convex.

(III) Let α: $X_0 \to Y$ be S-convex; let β: $X_0 \to \mathbf{R}$ satisfy $\beta(X_0) \subset \mathbf{R}_+\backslash\{0\}$; let α and β be Fréchet (or linearly Gâteaux) differentiable. Assume either (a) β is convex and $\alpha(X_0) \subset -S$; or (b) β is affine; or (c) β is concave and $\alpha(X_0) \subset S$. Then the function $g(\cdot) = \alpha(\cdot)/\beta(\cdot)$ is S-invex on X_0 with kernel η defined by

$$\eta(x, a) = (\beta(a)/\beta(x))(x - a).$$

The spaces X and Y here may have any dimensions, finite or infinte.

Proof. Let $x, a \in X_0$. Then, with the stated η,

$$g(x) - g(a) - g'(a)\eta(x, a) = \frac{1}{\beta(x)}\left\{\alpha(x) - \frac{\alpha(a)\beta(x)}{\beta(a)} - \frac{\beta(a)\alpha'(a) - \alpha(a)\beta'(a)}{\beta(a)}(x - a)\right\}$$

(since $\beta(x) > 0$)

$$= [\beta(x)]^{-1}\{\alpha(a) + \alpha'(a)(x - a) - [\alpha(a)/\beta(a)][\beta(a) + \beta'(a)(x - a)] - \alpha'(a)(x - a) + [\alpha(a)/\beta(a)]\beta'(a)(x - a) + s\}$$

for some $s \in S$, because $\alpha(x) - [\alpha(a)(x - a)] \in S$ since α is S-convex, and

$$-\alpha(a)\beta(x) + \alpha(a)[\beta(a) + \beta'(a)(x - a)] \in S$$

assuming either (a) or (b) or (c). Hence $g(x) - g(a) - g'(a)\eta(x, a) = [\beta(x)]^{-1}s \in S$.

https://doi.org/10.1017/S1446788700022126 Published online by Cambridge University Press

EXAMPLE 2. Let $X = Y = \mathbf{R}^2$, $S = \mathbf{R}^2_+$ (the nonnegative orthant in $\mathbf{R}^2$), $\beta(x, y) = 1 - x$, $\alpha(x, y) = (x^2 + y^2, x + 2y)$. Then $g(\cdot) = \alpha(\cdot)/\beta(\cdot)$ is $\mathbf{R}^2_+$-invex on $\{(x, y): x < 1\}$. In this case g is not $\mathbf{R}^2_+$-convex since the Hessian matrix of $(x + 2y)/(1 - x)$ is never positive semidefinite.

EXAMPLE 3. Let $X = \mathbf{R}^n$; let $\alpha(x) = (\alpha_1(x), \alpha_2(x), \ldots, \alpha_m(x)) \in \mathbf{R}^m$ where each componant α_i is convex on an open domain $X_0 \subset \mathbf{R}^n$; let $\beta(x) = c + d^T x$, with constant $c \in \mathbf{R}$ and $d \in \mathbf{R}^n$; assume that $\beta(x) > 0$ when $x \in X_0$; let $S = \mathbf{R}^m_+$. Then $g(\cdot) = \alpha(\cdot)/\beta(\cdot)$ is $\mathbf{R}^m_+$-invex on X_0, but not generally $\mathbf{R}^m_+$-convex.

EXAMPLE 4. Let X, X_0 and β be as in Example 3; let $I = [0, 1]$. Let h: $X_0{}^* \times I \to \mathbf{R}$ be any function such that $h(\cdot, t)$ is convex on X_0 for each $t \in I$ and $h(x, \cdot)$ is continuous on I for each $x \in X_0$. Let $C(I)$ denote the space of continuous functions from I into $\mathbf{R}$; let $C_+(I)$ denote the convex cone of nonnegative functions in $C(I)$. Define the function α: $X_0 \to C(I)$ by $(\forall x \in X_0, \forall t \in I)\alpha(x)(t) = h(x, t)$. Then α is $C_+(I)$ convex on X_0, and $g(\cdot) = \alpha(\cdot)/\beta(\cdot)$ is $C_+(I)$-invex on X_0.

REMARK. In (III), η depends on β alone, not on α. For a fixed affine function β: $X_0 \to \mathbf{R}_+\backslash\{0\}$, the convex cone $\{g(\cdot) = \alpha(\cdot)/\beta(\cdot)$: α is S convex on $X_0\}$ consists of functions, all of which are S-invex with the same kernel η. A similar statement holds when the hypothesis on β is replaced by another of the hypothesis in (III). (Unfortunately, there is no obvious extension to a cone of invex functions $\alpha(\cdot)/\beta(\cdot)$ with both α and β varying over appropriate classes of functions.) Similarly, if φ: $X \to X$ is surjective and $\operatorname{ran}(\varphi'(a)) = X$ for each $a \in X$, then the convex cone $\{g = q \circ \varphi$: q: $X \to Y$ is S-convex$\}$ consists of S-invex functions with the same kernel η, using (II). The result in (III) remains valid when α and β are merely directionally differentiable, provided that the definition of cone-invexity is suitably extended, as given below in (11).

(IV) For real-valued functions (that is, $Y = \mathbf{R}$) it will be shown (in section (IV) that every pseudoconvex function ([14]) is invex. The converse is *not* valid.

(V) Let g: $X_0 \to Y$ be a linearly Gâteaux differentiable function and suppose that there exists a point $\bar{x} \in X_0$ such that

$$g'(a)\bar{x} \in -\operatorname{int} S. \tag{3}$$

This assumes that $\operatorname{int} S \neq \varnothing$. We shall show that g is S-invex at a on X_0.

PROOF. Since $\operatorname{int} S \neq \varnothing$, there is a weak* compact convex set $B \subset Y'$ such that $0 \notin B$ and $S^* = \operatorname{cone} B$ (B is called a *base* for S^*, see [6]). Thus (3) can be expressed equivalently as $(\forall \lambda \in B)\lambda g'(a)\bar{x} < 0$. Now, since B is weak* compact,

$$(\exists \theta \in \mathbf{R})(\forall \lambda \in B)\ \lambda g'(a)\bar{x} \leqslant \theta < 0. \tag{4}$$

For each $x \in X_0$, let $b_x = \inf\{\lambda[g(x) - g(a)]: \lambda \in B\}$; clearly $b_x > -\infty$ since B is weak* compact and convex. By (4), since $\theta < 0$ and $b_x > -\infty$, there exists a sufficiently large positive number γ_x such that $b_x \geqslant \gamma_x\theta$ (thus γ_x is defined for each $x \in X_0$). The invex kernel is now defined by $\eta(x, a) = \gamma_x\bar{x}$. Thus

$$(\forall x \in X_0)(\forall\lambda \in B)\, \lambda g'(a)\eta(x, a) \leqslant \gamma_x\theta \leqslant b_x \leqslant \lambda[g(x) - g(a)].$$

Hence g is S-invex at a on X_0.

This is a slightly extended version of a result by Hanson and Mond [12] in finite dimensions. Note that (3) is a version of the well-known Slater constraint qualification for a program such as (P).

EXAMPLE 5. Let $X = Y = \mathbf{R}^2$, and $S = \mathbf{R}^2_+$, then the function $g(x, y) = (y - x^2 - 1, x + y^2 - 2)$ is $\mathbf{R}^2_+$-invex at the point $a = (0,0)$ on $\mathbf{R}^2$, since

$$g'(0,0)(-1,-1) = (-1,-1) \in -\operatorname{int}\mathbf{R}^2_+.$$

Clearly g is not $\mathbf{R}^2_+$-convex at $(0,0)$.

4. Differentiable functions

In this section we shall consider invexity in mathematical programming problems involving linearly Gâteaux differentiable functions.

THEOREM 1. (a) (*Hanson* [10], *Craven* [5]) *Let $a \in KT(P)$; let f be K_0-invex at a with respect to η on $E = \{x \in X_0: -g(x) \in S\}$. Then (P) attains a global minimum at the point a.*

(b) *Let f_0 be invex on X_0, then $a \in X_0$ is a (global) minimum of f_0 over X_0 if and only if $f_0'(a) = 0$.*

PROOF. (a) Let $x \in E$. Then, since $\lambda g(x) \leqslant 0$ and $\lambda g(a) = 0$ for each $\lambda \in S_0^*$,

$$f_0(x) - f_0(a) \geqslant rf(x) - rf(a) \geqslant rf'(a)\eta(x, a) = 0,$$

since $a \in KT(P)$. Here $r = (1, \lambda)$, where λ is the Lagrange multiplier associated to a.

(b) Since X_0 is an open set we need only establish sufficiency, which follows immediately by (2) with $f_0'(a) = 0$.

REMARK 1. If E is replaced by $E \cap U$, where U is a neighborhood of a in X_0, then (P) attains a local minimum at a in (a). As a consequence of Theorem 2 (to follow) we will establish the converse to part (b) above (see Remark 3), thus if every stationary point of f_0 is a minimum then f_0 is invex on X_0.

THEOREM 2. *Let $a \in KT(P)$; define K_0 from K and $g(a)$; assume the regularity hypotheses, that the convex cone $J_x = [f(x) - f(a), f'(a)]^T(K_0^*)$ is weak* closed for each $x \in E$, and that $g'(a)u \in -\text{aint}\, S$ for some $u \in X$. Then f is K_0-invex on E at a, for some η, if and only if*

$$(5) \qquad (\forall x \in E) f_0(x) + \lambda g(x) \geqslant f_0(a) + \lambda g(a),$$

where λ is any Lagrange multiplier satisfying (KT). Also, if f is K_0-invex on W and J_x is weak closed for each $x \in W$ then (D_1) reaches a maximum at $(u, v) = (a, \lambda)$.*

PROOF. Let $-g(x) \in S$ and $x \neq a$. Let $M = f'(a)$ and $c = f(x) - f(a)$. Then,

$$(6) \quad (\exists \eta \in X) f(x) - f(a) - f'(a)\eta \in K_0$$

$$\Leftrightarrow \quad (\exists \mu \in X, \exists t \in \mathbf{R}) ct - M\mu \in K_0,\ t \in \text{int}\, \mathbf{R}_+ \text{ (by substituting } \eta = \mu/t)$$

$$\Leftrightarrow \quad (\exists (t, \mu) \in \mathbf{R} \times X)[c, M]\begin{bmatrix} t \\ \mu \end{bmatrix} \in K_0,\ [0, 1]\begin{bmatrix} t \\ \mu \end{bmatrix} \in \text{int}\, \mathbf{R}_+$$

$$\Leftrightarrow \quad (\nexists, q) p[1, 0] + q[c, M] = 0,\ q \in K_0^*,\ 0 \neq p \in \mathbf{R}_+$$

(by Motzkin's alternative theorem, see [3, p. 32], since the cone $J_x = [c, M]^T(K^*)$ is weak* closed)·

$$\Leftrightarrow \quad (\nexists r \in K_0^*) r(c) = -1,\ rM = 0 \quad (\text{substituting } r = p^{-1}q)$$

$$(7) \quad \Leftrightarrow \quad [(r \in K_0^*, rM = 0) \Rightarrow r(c) \geqslant 0]$$

$$(8) \quad \Leftrightarrow \quad [(r \in K_0^*, r_0 > 0, rM = 0) \Rightarrow r(c) \geqslant 0]$$

(since if $0 \neq r \in K_0^*$, $r_0 = 0$, $rM = 0$ then $0 = \lambda g'(a)u < 0$, from $0 \neq \lambda \in S^*$ and $g'(a)u \in -\text{aint}\, S$; the case $r = 0$ is trivial. Note $r = (r_0, \lambda)$.).

$$\Leftrightarrow \quad f_0(x) + \lambda g(x) \geqslant f_0(a) + \lambda g(a)$$

(since $(1, \lambda)M = 0$ for any Lagrange multiplier λ satisfies (KT)).

Finally if f is K_0-invex on W then, using the above characterization,

$$f_0(z) + \bar{\lambda} g(z) \geqslant f_0(y) + \bar{\lambda} g(y),$$

for all $z, y \in W$ with $(y, \bar{\lambda})$ feasible for (D_1). Since $a \in KT(P)$, (a, λ) is feasible for (D_1), thus $f_0(a) = f_0(a) + \lambda g(a) \geqslant f_0(a) + \bar{\lambda} g(a) \geqslant f_0(y) + \bar{\lambda} g(y)$, for each $y \in W$. Hence (a, λ) is optimal for (D_1). It is easily shown that if f is K_0-invex on $W \cup E$ then duality holds between (P) and (D_1), (see [10]).

REMARK 2. The proof that (5) characterizes K_0-invexity at a in Theorem 2 does not require the assumption that $\lambda g(a) = 0$. If it is assumed then (5) is equivalent to $f_0(x) - f_0(a) \geqslant -\lambda g(x) \geqslant 0$, and thus to something a little stronger than a minimum of (P) at the point a. A corresponding result for local minimization follows if x is restricted to $E \cap U$, where U is a neighborhood of a.

If the cone J_x is not assumed weak* closed, then the Kuhn-Tucker conditions (KT) may be replaced by the doubly asymptotic Kuhn-Tucker conditions (see [25], [18], [7])

$$(AKT) \qquad (\exists\, \mathrm{net}(r_\alpha) \subset K^*) r_\alpha f'(a) \to 0, \qquad r_\alpha f(a) \to f_0(a), a \in E,$$

where the net (r_α) need not converge. Denote by $AKT(P)$ the set of points a at which (AKT) holds for (P). Then the result of applying Motzkin's theorem in the proof of Theorem 2 is replaced by

$$(\nexists\, \mathrm{net}(r_\alpha) \subset K_0^*) r_\alpha(c) \to -1, \qquad r_\alpha M \to 0.$$

Hence with $a \in AKT(P)$ and $g'(a)u \in -\mathrm{aint}\, S$ for some $u \in X$, f is K_0-invex at a on E if and only if

$$f_0(x) + \lambda_\alpha g(x) \geqslant f_0(a) + \lambda_\alpha g(a)$$

holds eventually, that is for all $\alpha \geqslant \bar{\alpha}$ (some index).

We now establish Theorem 2 under alternative regularity assumptions and characterize K_0-invexity using the Lagrangean dual, (D_2).

Theorem 3. *Let $a \in KT(P)$; define K_0 from K and $g(a)$; assume that J_x is weak* closed for each $x \in E$. In addition suppose that one of the following is satisfied*:

(a) *g is S-convex at a.*

(b) $(\forall \lambda \in S_0^* \setminus \{0\})(\exists x = x(\lambda) \in X)\lambda g'(a)x < 0$.

Then f is K_0-invex at a on E if and only if (5) holds for each $x \in E$ and each Lagrange multiplier λ. Furthermore, if J_x is weak closed for each $x \in X_0$, then f is K_0-invex at a on X_0 if and only if the Lagrangean dual (D_2) reaches a maximum at λ (the Lagrange multiplier satisfying (KT) at a) with $\varphi(\lambda) = f_0(a)$.*

Proof. By inspection of the proof of Theorem 2 we need only establish (8) $\Rightarrow$ (7) and the result will follow. Hence assume $r \in K_0^*$, $r_0 = 0$ and $rM = 0$. Thus, letting $r = (r_0, \lambda)$, $\lambda g'(a) = 0$ with $\lambda \in S_0^*$. If (a) holds then, by (1), $\lambda g(x) \geqslant \lambda g(a)$, $(\forall x \in E)$ and consequently $r(c) \geqslant 0$ as required. If (b) holds then the result follows as in Theorem 2, since $\lambda g'(a) \neq 0$, for each $\lambda \in S_0^* \setminus \{0\}$.

Finally, suppose (D_2) reaches a maximum at λ with $\phi(\lambda) = f_0(a)$ and $a \in KT(P)$. Then, by weak duality, we have

$$f_0(x) + \lambda g(x) \geqslant \phi(\lambda) = f_0(a) = f_0(a) + \lambda g(a), \qquad \forall x \in X_0.$$

Thus by the above f is K_0-invex at a on X_0. Conversely, if f is K_0-invex at a on X_0 then $\phi(\lambda) = f_0(a) + \lambda g(a) = f_0(a)$ using (5).

Remark 3. (i) From the proof of Theorem 2 (in particular since (6) $\Leftrightarrow$ (7)) we have the following equivalent condition for cone-invexity, namely f is K-invex at a

on a set $D \subset X$ if and only if

$$(9) \qquad [(r \in K^*, rf'(a) = 0) \Rightarrow rf(x) \geqslant rf(a), \forall x \in D],$$

(we are assuming J_x is weak* closed for each $x \in D$). For real-valued functions the condition (9) gives the following: f is invex on D if and only if every stationary point of f in D is a (global) minimum. Functions satisfying this latter condition have been extensively studied by Zang, Choo and Avriel [24] (see also [22], [23]). Using the characterization (9) we easily obtain

$$f \text{ is } K\text{-invex at } a \text{ on } D \Leftrightarrow rf \text{ is invex at } a \text{ on } D, \quad \text{for all } r \in K^*.$$

Note that we do not need to specify that η be the same for all $r \in K^*$, this follows since (9) is independent of η. Now, coupling this result with the work in [24] we obtain the following technical characterization of cone-invexity:

$$f \text{ is } K\text{-invex on an open set } D \subset \mathbf{R}^n \text{ if and only if}$$
$$(\forall r \in K^*) L_{rf}(\cdot) \text{ is strictly lower semi-continuous on } G_{rf}.$$

(ii) Under suitable regularity assumptions [3], the Fritz John conditions

$$(FJ) \quad (\exists \lambda \in S^*, \exists \tau \geqslant 0, (\tau, \lambda) \neq (0,0)) \tau f_0'(a) + \lambda g'(a) = 0, \qquad \lambda g(a) = 0,$$

are necessary for optimality at $a \in E$; equivalently,

$$(FJ+) \qquad (\exists r \in K_0^*, r \neq 0) \, rf'(a) = 0.$$

Hence, using (9) above, it follows that f is K_0-invex at a on E if and only if either, $(FJ+)$ is *not* satisfied at $a \in E$ or, the corresponding Lagrangean function $L(r, x) = rf(x)$ (for $r \in \mathbf{R} \times Y'$) attains a minimum at a over E. This result assumes that J_x is weak* closed for each $x \in E$, but does not require the other regularity conditions of Theorems 2 and 3. It is possible to consider Fritz John type conditions in an asymptotic form (see [7]) which would be applicable when J_x is not necessarily closed. The conditions (FJ) are known to be satisfied when the cone S has non-empty (topological) interior ([3]).

(iii) The weak* closure assumption on the convex cone J_x is satisfied under either of the following assumptions:

(a) K_0 is a polyhedral cone, (in particular if $K = \mathbf{R}^n_+$).

(b) $[f(x) - f(a), f'(a)](\mathbf{R}_+ \times X) + K_0 = \mathbf{R} \times Y$, for each $x \in E$.

In part (b) we need the additional assumption that X and Y are complete, for the details see Nieuwenhuis [15], or Glover [7, Lemma 3]. Other sufficient conditions are given in Zalinescu [20] and Holmes [13].

(iv) In Section 3 it was claimed that every pseudoconvex function is invex, this now follows easily from part (i) above since every stationary point of a pseudoconvex function is a (global) minimum. A related result was given in [24, Theorem 2.3] where it was shown that for a pseudoconvex function, $f \colon X \to \mathbf{R}$, $L_f(\cdot)$ is SLSC on G_f; which is equivalent to invexity by part (i) above.

(v) In this section we have characterized cone-invexity at Kuhn-Tucker points using the Motzkin alternative theorem; for finite systems of differentiable functions on $\mathbf{R}^n$, a similar approach was suggested by Hanson [10] using Gale's alternative theorem.

5. Nondifferentiable functions

In this section we shall discuss cone-invexity for a class of nondifferentiable functions. We use the concept of quasidifferentiability to show that under cone-invex hypotheses the generalized Kuhn-Tucker conditions of Glover [7] are sufficient for optimality.

DEFINITION. A function $g\colon X_0 \to Y$ is *S^*-quasidifferentiable* at $a \in X_0$ if g is directionally differentiable at a and, for each $\lambda \in S^*$, there is a non-empty weak* compact convex set $\tilde{\partial}(\lambda g)(a)$ such that

$$(10) \qquad g'(a, x) = \sup\{w(x) \colon w \in \tilde{\partial}(\lambda g)(a)\}.$$

Clearly if g is S^*-quasidifferentiable at a then $\lambda g'(a, \cdot\,)$ is a continuous sublinear functional for each $\lambda \in S^*$. Hence $\tilde{\partial}(\lambda g)(a)$ coincides with $\partial(\lambda g)'(a, 0)$ that is the subdifferential of $\lambda g'(a, \cdot\,)$ at 0 in the sense of convex analysis (see [17]). If g is S-convex at a then $\tilde{\partial}(\lambda g)(a) = \partial(\lambda g)(a)$; for convenience we shall omit the $\sim$ in the sequel.

Clearly every linearly Gâteaux differentiable function is S^*-quasidifferentiable with $\partial(\lambda g)(a) = \{\lambda g'(a)\}$. For more general classes of nondifferentiable functions which are quasidifferentiable see Pshenichnyi [16], Craven and Mond [6], Clarke [2], and Borwein [1].

Let $g\colon X_0 \to Y$ be directionally differentiable at $a \in X_0$, then g will be called *S-invex at a on* a set $D \subset X_0$ if, for each $x \in D$, there is a $\eta(x, a) \in X$ with

$$(11) \qquad g(x) - g(a) - g'(a, \eta(x, a)) \in S.$$

THEOREM 4 (*Sufficient Kuhn-Tucker Theorem*). *Consider problem* (P) *with* $a \in E$. *Let* f_0 *be quasidifferentiable at* a *and* g *S^*-quasidifferentiable at* a. *Further suppose that* f *is* K-*invex at* a *on* E *and that the generalized Kuhn-Tucker conditions*

$$(12) \qquad 0 \in (\partial f_0(a) \times \{0\}) + \overline{\bigcup_{\lambda \in S^*} (\partial(\lambda g)(a) \times \{\lambda g(a)\})}$$

are satisfied. Then a *is optimal for* (P).

https://doi.org/10.1017/S1446788700022126 Published online by Cambridge University Press

PROOF. It is easily seen that (12) is equivalent to the existence of $w \in \partial f_0(a)$ and nets $(\lambda_\alpha) \subset S^*$, $(w_\alpha) \subset X'$ with $w_\alpha \in \partial(\lambda_\alpha g)(a)$ for all α, such that

$$w + w_\alpha \to 0, \qquad \lambda_\alpha g(a) \to 0. \tag{13}$$

Let $x \in E$, then

$$\begin{aligned} f_0(x) - f_0(a) &\geqslant f_0'(a, \eta), \quad \text{by invexity} \\ &\geqslant w(\eta), \quad \text{since } w \in \partial f_0(a) \\ &= \lim_\alpha \left[-w_\alpha(\eta)\right], \quad \text{by (13)} \\ &\geqslant \liminf_\alpha \left[-\lambda_\alpha g'(a, \eta)\right], \quad \text{since } w_\alpha \in \partial(\lambda_\alpha g)(a), \forall \alpha \\ &\geqslant \liminf_\alpha (-\lambda_\alpha(g(x) - g(a))), \quad \text{by invexity} \\ &\geqslant \liminf_\alpha \lambda_\alpha g(a), \quad \text{as } x \in E \text{ and } \lambda_\alpha \in S^* \\ &= 0, \quad \text{by (13)}. \end{aligned}$$

Thus $f_0(x) - f_0(a) \geqslant 0$, for all $x \in E$ and so a is optimal.

REMARK 4. Theorem 4 generalizes the result of Hanson [10] and Craven [5] given in Theorem 1(a). The condition (12) has been shown to be necessary for optimality by Glover [7, Theorem 4] under the quasidifferentiability assumptions of Theorem 4 and the additional hypotheses that f_0 is arc-wise directionally differentiable at a ([6]) and g is locally solvable at a. In the special case of Theorem 4 in which f_0 and g are linearly Gâteaux differentiable at a it is easily shown that (12) is equivalent to (AKT).

We shall now consider an alternative characterization of optimality for invex programs under stronger hypotheses.

THEOREM 5. *For problem* (P) *let* $a \in E$; *let* f_0 *be quasidifferentiable at a and let g be linearly Gâteaux differentiable at a. Furthermore assume* ran$([g'(a), g(a)])$ *is closed, X and Y are complete, and g is locally solvable at a. Then a necessary condition for a to be a minimum of* (P) *is that*

$$(\exists v \in \overline{Q})\ 0 \in \partial f_0(a) + v g'(a), \qquad v g(a) = 0, \tag{14}$$

where

$$Q = S^* - N\left([g'(a), g(a)]^T\right). \tag{15}$$

If f is K-invex at a on E then (14) *is sufficient for optimality at a.*

PROOF. (Necessity) Let $a \in E$ be optimal for (P). Then by Craven and Mond [6], using the local solvability hypothesis, there is no solution $(\alpha, x) \in \mathbf{R} \times X$ to

$$f_0'(a, x) < 0, \qquad \alpha g(a) + g'(a)x \in -S. \tag{16}$$

https://doi.org/10.1017/S1446788700022126 Published online by Cambridge University Press

Let $A = [g'(a), g(a)]$, then (16) is equivalent to

$$A(\alpha, x) \in -S \Rightarrow f_0'(a, x) \geqslant 0. \tag{17}$$

Thus, by the separation theorem, ([16]), (17) is equivalent to

$$0 \in (\partial f_0(a) \times \{0\}) - [A^{-1}(-S)]^*. \tag{18}$$

By Theorem 1 in [8], $[A^{-1}(-S)]^* = A^T(\overline{Q})$ with Q given by (15). Thus (14) and (18) are equivalent as required.

(Sufficiency) Suppose (14) is satisfied at $a \in E$ and f is K-invex at a on E. Let $x \in E$. By (14) there are nets $(\lambda_\alpha) \subset S^*$, $(w_\alpha) \subset N(A^T)$ with $v = \lim_\alpha(\lambda_\alpha - w_\alpha)$.

Now,

$$\begin{aligned} f_0(x) - f_0(a) &\geqslant f_0'(a, \eta), \\ &\geqslant -vg'(a)\eta, \quad \text{by (14)} \\ &= \lim_\alpha [-(\lambda_\alpha - w_\alpha) g'(a)\eta] \\ &= \lim_\alpha [-\lambda_\alpha g'(a)\eta], \quad \text{since } w_\alpha \in N(A^T), \forall \alpha \\ &\geqslant \liminf_\alpha [-\lambda_\alpha (g(x) - g(a))], \quad \text{by invexity} \\ &\geqslant \liminf_\alpha [\lambda_\alpha g(a)], \quad \text{since } x \in E \\ &= vg(a), \quad \text{as } w_\alpha g(a) = 0, \forall \alpha \\ &= 0, \quad \text{by (14)}. \end{aligned}$$

Thus a is optimal for (P).

REMARK 5. Theorem 5 generalizes the results in [8]. If $Y = \mathbf{R}^n$ then the closed range condition is automatically satisfied. This result provides a non-asymptotic Kuhn-Tucker condition even if the usual 'closed cone' condition is not satisfied.

Consider the following program related to (P).

$$(D_3) \qquad \begin{array}{ll} \text{Maximize} & f_0(u) + vg(u) \\ \text{subject to} & v \in \overline{Q(u)}, 0 \in \partial f_0(u) + vg'(u) \end{array}$$

where f_0 is quasidifferentiable, g is linearly Gâteaux differentiable and $Q(u) = S^* - N([g'(u), g(u)^T])$. Let $W_1 = \{u \in X\colon (u, v) \text{ is feasible for } (D_3) \text{ for some } v \in \overline{Q(u)}\}$.

THEOREM 6. *Let f be K-invex on $W_1 \cup E$ then weak duality holds for (P) and (D_3). Let $a \in E$ be optimal for (P) and let* (14) *be satisfied for some $v \in \overline{Q}$, then (D_3) reaches a maximum at (a, v) with* $\text{Min}(P) = \text{Max}(D_3)$. *Thus (D_3) is a dual program to (P).*

https://doi.org/10.1017/S1446788700022126 Published online by Cambridge University Press

PROOF. Let $x \in E$ and let $(u, \bar{v})$ be feasible for (D_3). Then,

$$\begin{aligned} f_0(x) - f_0(u) - \bar{v}g(u) &\geqslant f_0'(u, \eta) - \bar{v}g(u) \\ &= -\bar{v}g'(u)\eta - \lim_\alpha (\lambda_\alpha - w_\alpha)g(u) \\ &\qquad \text{where we choose } (\lambda_\alpha) \text{ and } (w_\alpha) \text{ as in Theorem 5} \\ &= -\bar{v}g'(u)\eta + \lim_\alpha [-\lambda_\alpha g(u)], \quad \text{as } w_\alpha g(u) = 0 \\ &\geqslant -\bar{v}g'(u)\eta + \liminf_\alpha [\lambda_\alpha(g(x) - g(u))], \quad \text{as } x \in E \\ &\geqslant -\bar{v}g'(u)\eta + \liminf_\alpha \lambda_\alpha g'(u), \quad \text{by invexity} \\ &= -\bar{v}g'(u)\eta + \bar{v}g'(u)\eta, \quad \text{as } w_\alpha g'(u) = 0, \forall\alpha \\ &= 0. \end{aligned}$$

Thus weak duality holds for (P) and (D_3).

Let $a \in E$ be optimal for (P). Now by assumption there is a $v \in \bar{Q} = \overline{Q(a)}$ such that (14) holds. Thus (a, v) is feasible for (D_3). Hence, by weak duality,

$$f_0(a) + vg(a) = f_0(a) \geqslant f_0(u) + v\bar{g}(u)$$

for all $(u, \bar{v})$ feasible for (D_3). Thus (a, v) is optimal for (D_3) and $\mathrm{Min}(P) = f_0(a) = \mathrm{Max}(D_3)$.

In order to establish a version of Theorem 2 for quasidifferentiable functions we require the following theorem of the alternative. We no longer require the completeness assumptions on X and Y.

THEOREM 7. *Let h: $X \to Y$ be S-sublinear and weakly continuous. Let $z \in Y$. Then exactly one of the following is satisfied*:

(i) $(\exists x \in X)\ -h(x) + z \in S$.

(ii) $$(0,1) \in \overline{\bigcup_{\lambda \in S^*} (\partial(\lambda h)(0) \times \{\lambda(z)\})}.$$

PROOF. [Not (ii) $\Rightarrow$ (i)]. For convenience let $B = \bigcup_{\lambda \in S^*}(\partial(\lambda h)(0) \times \{\lambda(z)\})$. Clearly B is a convex cone. Now suppose $(0, -1) \notin B$. Thus, by the separation theorem ([3, p. 23]), $\exists(\hat{x}, \beta) \in X \times \mathbf{R}$ such that

$$\begin{aligned} -\beta &> \sup\{\bar{w}(\hat{x}, \beta) : \bar{w} \in \bar{B}\} \\ &= \sup\{\bar{w}(\hat{x}, \beta) : \bar{w} \in B\} \\ &\geqslant \sup\{w(\hat{x}) : w \in \partial(\lambda h)(0)\} + \beta\lambda(z), \quad \text{for any } \lambda \in S^* \\ &= \lambda h(\hat{x}) + \beta\lambda(z), \quad \text{by continuity and sublinearity of } \lambda h. \end{aligned} \tag{19}$$

Also as $0 \in S^*$, $-\beta > 0$. Let $\gamma = -\beta$. Then, for any $\lambda \in S^*$,

$$\lambda h(\hat{x}) - \lambda(\gamma z) < \gamma \Leftrightarrow \lambda(h(\bar{x}) - z) < 1, \quad \text{where } \bar{x} = \hat{x}/\gamma$$

$$\Rightarrow -h(\bar{x}) + z \in (S^*)^0 = S, \quad \text{as } S \text{ is a closed convex cone}$$

$$\Rightarrow \text{(i) is satisfied by } \bar{x}.$$

[(i) $\Rightarrow$ Not (ii)]. Suppose $-h(x) + z \in S$ for some $x \in X$; and suppose, if possible, that $(0, -1) \in \bar{B}$. Hence there are nets $(\lambda_\alpha) \subset S^*$ and $(w_\alpha) \subset X'$ such that $w_\alpha \in \partial(\lambda_\alpha h)(0)$, $\forall \alpha$, and $w_\alpha \to 0$, $\lambda_\alpha(z) \to -1$. Thus $w_\alpha(x) \to 0$. Now, for each α, $0 \geqslant \lambda_\alpha(h(x) - z) \geqslant w_\alpha(x) - \lambda_\alpha(z) \to 1$. Thus we have a contradiction, hence $(0, -1) \notin \bar{B}$ and (ii) is not satisfied.

Remark 6. Vercher [19] (see also Goberna *et al.* [9]) has established a result similar to Theorem 7 for arbitrary systems of sublinear functions defined on $\mathbf{R}^n$. It is possible to weaken the continuity requirement in Theorem 7 to λh lower semi-continuous for each $\lambda \in S^*$ (the proof is identical since (19) remains valid using [21, Theorem 1]). Consider the special case of Theorem 7 in which $h = C$, a continuous linear function. Then (ii) becomes

$$(0, -1) \in \overline{[C, z]^T(S^*)}. \tag{20}$$

If the convex cone $[C, z]^T(S^*)$ is weak* closed then (20) becomes

$$(\exists \lambda \in S^*) \lambda C = 0, \qquad \lambda(z) = -1. \tag{21}$$

Thus the first section of proof in Theorem 2 has established this 'linear' version of Gale's alternative theorem.

Consider the generalized Kuhn-Tucker conditions given in Theorem 4. If the convex cone $\bigcup_{\lambda \in S^*}(\partial(\lambda g)(a) \times \{\lambda g(a)\})$ is weak* closed then (12) is equivalent to

$$(GKT) \qquad (\exists \lambda \in S^*)\ 0 \in \partial f_0(a) + \partial(\lambda g)(a), \qquad \lambda g(a) = 0.$$

Let $GKT(P)$ denote the set of $a \in E$ such that (GKT) holds for some λ.

Theorem 8. *Let $a \in GKT(P)$; at a, let f_0 be quasidifferentiable and let g be S_0^*-quasidifferentiable. For each $x \in E$, assume that the convex cone*

$$J'_x = \bigcup_{r \in K_0^*} (\partial(rf)(a) \times \{r(f(x) - f(a))\})$$

is weak closed. Further assume that one of the following conditions is satisfied:*

(i) *g is S-convex at a.*

(ii) $(\exists u \in X)\ g'(a, u) \in -\text{aint } S$.

Then f is K_0-invex at a on E if and only if $f_0(x) + \lambda g(x) \geqslant f_0(a) + \lambda g(a)$, where λ is any Lagrange multiplier satisfying (GKT) at a, for all $x \in E$. Also f is K_0-invex on X_0 at a if and only if the Lagrangean dual (D_2) reaches a maximum at (a, λ) with $\phi(\lambda) = f_0(a)$.

PROOF. Let $x \in E$, $x \neq a$. Then

f is K_0-invex at a on E

$$\Leftrightarrow \quad (\exists \eta \in X) f(x) - f(a) - f'(a, \eta) \in K_0$$
$$\Leftrightarrow \quad (0, -1) \notin \bar{J}'_x = J'_x, \quad \text{by Theorem 7}$$
(since f is K_0-quasidifferentiable at a it follows that $f'(a, \cdot)$ is K_0-sublinear and $rf'(a, \cdot)$ is continuous $(r \in K_0^*)$)
$$(22) \quad \Leftrightarrow \quad [(0, \gamma) \in J'_x \Rightarrow \gamma \geqslant 0]$$
$$\Leftrightarrow \quad [r \in K_0^*, r_0 > 0, 0 \in \partial(rf)(a) \Rightarrow rf(x) \geqslant rf(a)]$$
(since (i) and (ii) will ensure that the case $r_0 = 0$ is satisfied as in the proof of Theorem 2)
$$\Leftrightarrow \quad f_0(x) + \lambda g(x) \geqslant f_0(a) + \lambda g(a)$$
(where λ is any multiplier satisfying (GKT) at a.

Now since $a \in GKT(P)$, $\exists \lambda \in S_0^*$ with $0 \in \partial f_0(a) + \partial(\lambda g)(a)$. Also as $f_0'(a, \cdot)$ and $\lambda g'(a, \cdot)$ are continuous convex functions we have $\partial(f_0 + \lambda g)(a) = \partial f_0(a) + \partial(\lambda g)(a)$. Thus $0 \in \partial(rf)(a)$ where $r = (1, \lambda)$). The final result then follows as in Theorem 2.

REMARK 7. If we remove the closed cone assumption on J'_x then (22) becomes $[(0, \gamma) \in \bar{J}'_x \Rightarrow \gamma \geqslant 0]$ which is equivalent to

$$\left[(r_\alpha) \subset K_0^*, w_\alpha \in \partial(r_\alpha f)(a), w_\alpha \to 0, r_\alpha(f(x) - f(a)) \to \gamma \Rightarrow \gamma \geqslant 0\right].$$

This is the analogue of the asymptotic conditions discussed following Theorem 2.

We can consider generalized Fritz John conditions (under suitable regularity and quasidifferentiability assumptions (see [7]) for problems (P) to attain a minimum at $a \in E$; namely

(GFJ)
$$(\exists \lambda \in S^*, \exists \tau \geqslant 0, (\tau, \lambda) \neq (0,0))\, 0 \in \tau \partial f_0(a) + \partial(\lambda g)(a), \qquad \lambda g(a) = 0.$$

Equivalently, since $f_0'(a, \cdot)$ and $\lambda g'(a, \cdot)$ are continuous, we have

$$(GFJ+) \qquad (\exists r \in K_0^*, r \neq 0)\, 0 \in \partial(rf)(a).$$

Thus, analogously to Remark 3, part (ii), f is K_0-invex at a on E if and only if either $(GFJ+)$ is *not* satisfied at $a \in E$, or, the corresponding Lagrangean function attains a minimum at a over E. This result follows easily from (22); we need only assume J'_x is weak* closed for each $x \in E$.

6. Examples

(i)(P_1)

$$\begin{aligned} &\text{Minimize} \quad f_0(x, y) = x^3 + y^3 \\ &\text{subject to} \quad g_1(x, y) = x^2 + y^2 - 4 \leqslant 0 \\ &\qquad\qquad\quad g_2(x, y) = y - x + 2 \leqslant 0. \end{aligned}$$

Let $a = (0, -2) \in E$. It is easily shown that a is a Kuhn-Tucker point for (P_1) with (unique) Lagrange multiplier $\lambda = (3, 0)$. Let $(x, y) \in E$, then it is easily shown that $y = \mu x - 2$, for some $\mu \in [0, 1]$; and $x \geqslant 0$. Thus,

$$\begin{aligned} f_0(x, y) + 3g_1(x, y) &= x^2 + y^3 + 3x^2 + 3y^2 - 12 \\ &= (1 + \mu^3)x^3 + 3(1 - \mu^2)x^2 - 8 \\ &\geqslant -8 = f_0(0, -2) + 3g_1(0, -2). \end{aligned}$$

Thus, since the constraints of (P_1) are convex, we have by Theorem 3 that $f = (f_0, g_1, g_2)$ is $\mathbf{R}^3_+$-invex on E at a. Hence, by Theorem 1(a), a is a minimum of (P_1).

We can actually define a suitable function η as follows:

Let $\eta(x, y) = (\eta_1(x, y), \eta_2(x, y))$, for $(x, y) \in E$, where

$$\begin{aligned} \eta_1(x, y) &= \max\{-g_1(x, y)/4 \colon (x, y) \in E\} - \min\{g_2(x, y) \colon (x, y) \in E\}, \\ \eta_2(x, y) &= -g_1(x, y)/4. \end{aligned}$$

We do not have duality between (P_1) and (D_1) in this case; we will show that if f is not $\mathbf{R}^3_+$-invex on $W \cup E$. Let $\alpha > 0$ and consider the point $(0, -\alpha)$.

$$\begin{aligned} f_0'(0, -\alpha) + \lambda_1 g_1'(0, -\alpha) + \lambda_2 g_2'(0, -\alpha) &= (-\lambda_2, 3\alpha^2 - 2\alpha\lambda_1 + \lambda_2) = 0 \\ &\Leftrightarrow \lambda_2 = 0, \lambda_1 = 3\alpha/2, \quad (\text{thus } (0, -\alpha) \in W, \forall \alpha > 0). \end{aligned}$$

Now,

$$\begin{aligned} f_0(0, -\alpha) + (3\alpha/2)g_1(0, -\alpha) &= -\alpha^3 + (3\alpha/2)(\alpha^2 - 4) \\ &= \tfrac{1}{2}\alpha(\alpha^2 - 12) \\ &> -8, \quad \text{for all } \alpha \in [0, 2) \\ &= f_0(0, -2) + 3g_1(0, -2). \end{aligned}$$

Thus a is not a maximum of (D_1) and f is *not* $\mathbf{R}^3_+$-invex on $W \cup E$. It should be noted that f is not $\mathbf{R}^3_+$-convex at a.

(ii) (Hanson and Mond [12])

$$(P_2) \qquad \begin{aligned} \text{Minimize} \quad & f_0(x, y) = -2y^3 - 6x^2 + 3y^2 + 6x + 6y - 7 \\ \text{subject to} \quad & g_1(x, y) = -3x^4 + y^2 - 3x - 3y + 2 \leqslant 0 \\ & g_2(x, y) = 2x^4 + 2x^2 - y^2 + 1 \leqslant 0 \\ & g_3(x, y) = 2xy - 6x - 1 \leqslant 0. \end{aligned}$$

In [12] it was shown that $f = (f_0, g_1, g_2, g_3)$ is K_0-invex on E at $a = (0, 1)$ by constructing a suitable function η. We shall apply Theorem 2. At the point a only g_1 and g_2 are binding constraints, thus $S = \mathbf{R}^3_+$, $S_0 = \mathbf{R}^2_+ \times \mathbf{R}$, $S_0^* = \mathbf{R}^2_+ \times \{0\}$ and $K_0 = \mathbf{R}^3_+ \times \mathbf{R}$. Theorem 2 is applicable since, for $g = (g_1, g_2, g_3)$ we have

$$g'(0,1)(1,1) = (-3, -2, -4) \in -\text{int}\, S.$$

Clearly a is a Kuhn-Tucker point with (unique) Lagrange multiplier $\lambda = (2, 3, 0)$. It is easily shown that $f_0(x, y) + 2g_1(x, y) + 3y_2(x, y) = -2$, for all $(x, y) \in E$. Thus f is K_0-invex at a on E, and consequently a is a minimum of (P_2).

(iii) (Craven [5])

$$(P_3) \qquad \begin{aligned} \text{Minimize} \quad & f_0(x, y) = \tfrac{1}{3}x^3 - y^2 \\ \text{subject to} \quad & g_1(x, y) = \tfrac{1}{2}x^2 + y^2 - 1 \leqslant 0 \\ & g_2(x, y) = x^2 + (y-1)^2 - \rho^2 \leqslant 0 \end{aligned}$$

(for $\rho > 0$ (to be specified) sufficiently small). For this problem the point $a = (0, 1)$ is a Kuhn-Tucker point with (unique) Lagrange multiplier $\lambda = (1, 0)$. Now, $f_0(x, y) + g_1(x, y) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 1 \geqslant -1 = f_0(0, 1) + g_1(0, 1)$, for all $(x, y) \in \mathbf{R}^2$ with $x \geqslant -3/2$, (this determines ρ so that $(x, y) \in E \Rightarrow x \geqslant -3/2$). Thus $f = (f_0, g_1, g_2)$ is $\mathbf{R}^2_+ \times \mathbf{R}$-invex at a on E and consequently a is a minimum of (P_3). Note that $g = (g_1, g_2)$ is $\mathbf{R}^2_+$-convex so that Theorem 3 is applicable.

(iv) In [11] Hanson and Mond defined the following class of generalized convex functions, to extend the concept of invexity. Let $\psi\colon X \to \mathbf{R}$ be a differentiable function. Then ψ is in this class over a set $C \subset X$ if, for each $x, a \in C$, there is a sublinear functional $F_{x,a}\colon X' \to \mathbf{R}$ such that

$$\psi(x) - \psi(a) \geqslant F_{x,a}(\psi'(a)). \tag{23}$$

They claimed this extended the idea of invexity to a wider class of function. We shall show that if ψ satisfies (23) then ψ is actually invex on C. The proof follows immediately from part (i) of Remark 3. For if $\psi'(a) = 0$ for some $a \in C$, then $F_{x,a}(\psi'(a)) = 0$ (by sublinearity) for all $x \in C$, thus $\psi(x) \geqslant \psi(a)$, and consequently every stationary point of ψ in C is a minimum. Hence ψ is invex on C.

Now suppose $\psi = \psi_i$ satisfies (23) for $i = 1, \ldots, n$. Let $\beta_i \geqslant 0$ $(\forall i)$ and $\beta = (\beta_1, \ldots, \beta_n)$ define $\Phi(\beta, \cdot\,) = \sum \beta_i \psi_i(\cdot)$. Thus

$$\begin{aligned} F_{x,a}(\Phi'(\beta, a)) &= F_{x,a}\left(\sum \beta_i \psi_i'(a)\right) \\ &\leqslant \sum \beta_i (\psi_i(x) - \psi_i(a)) \\ &= \Phi(\beta, x) - \Phi(\beta, a), \quad \text{for all } x \in C, \beta \in \mathbf{R}^n_+. \end{aligned}$$

Hence if $\Phi'(\beta, a) = 0$ then $\Phi(\beta, x) \geqslant \Phi(\beta, a)$, for all $x \in C$. Thus, by (9) and part (ii) in Remark 3, $\Psi = (\psi_1, \ldots, \psi_n)$ is $\mathbf{R}^n_+$-invex on C. Thus $F_{x,a}$ can be assumed linear and (23) is equivalent to invexity.

Hanson and Mond [11] also defined another class of generalized invex functions from (23) (in a manner analogous to the definition of pseudoconvex functions from convex functions); namely a differentiable function ψ is in this new class over $C \subset X$ if, for each $x, a \in C$, there is a sublinear functional $F_{x,a}$: $X' \to \mathbf{R}$ such that

$$[F_{x,a}(\psi'(a)) \geqslant 0 \Rightarrow \psi(x) \geqslant \psi(a)]. \tag{24}$$

It now follows immediately, as above, that if ψ satisfies (24) then ψ is invex on C, since every stationary point is a (global) minimum.

Example 4 in Section 3 shows that these invex concepts are also applicable in infinite dimensions.

Acknowledgement

The authors thank two referees for their comments, which have improved the presentation of the paper.

References

[1] J. M. Borwein, 'Fractional programming without differentiability', *Math. Programming* **11** (1976), 283–290.

[2] F. H. Clarke, 'Generalized gradients of Lipschitz functionals', *Adv. in Math.* **40** (1981), 52–67.

[3] B. D. Craven, *Mathematical programming and control theory* (Chapman and Hall, London, 1978).

[4] B. D. Craven, 'Duality for generalized convex fractional programs', *Generalized concavity in optimization and economics*, edited by S. Schaible and W. T. Ziemba, pp. 473–490 (Academic Press, New York), 1981.

[5] B. D. Craven, 'Invex functions and constrained local minima', *Bull. Austral. Math. Soc.* **24** (1981), 357–366.

[6] B. D. Craven and B. Mond, 'Lagrangean conditions for quasidifferentiable optimization', *Survey of mathematical programming*, Volume 1, Akadémiai Kiadó, Budapest, 1979, edited by A. Prékopa, pp. 177–192, (proceedings of the 9th International Programming Symposium, Budapest, 1976).

[7] B. M. Glover, 'A generalized Farkas lemma with applications to quasidifferentiable programming', *Z. Oper. Res.* **26** (1982), 125–141.
[8] B. M. Glover, 'Differentiable programming in Banach spaces', *Math. Operationsforsch. Statist. Ser. Optimization* **14** (1983), 499–508.
[9] M. A. Goberna, M. A. Lopez, J. Pastor and E. Vercher, 'An approach to semi-infinite programming via consequence relations', Conference paper, Intern. Symposium Semi-infinite Programming and Applications, University of Texas, Austin, September, 1981.
[10] M. A. Hanson, 'On sufficiency of the Kuhn-Tucker conditions', *J. Math. Anal. Appl.* **80** (1981), 545–550.
[11] M. A. Hanson and B. Mond, 'Further generalizations of convexity in mathematical programming', *J. Inform. Optim. Sci.* **3** (1982), 25–32.
[12] M. A. Hanson and B. Mond, *Necessary and sufficient conditions for global optimality in constrained optimization*, FSU Statistics Report M-600, Florida State University, Tallahassee, November, 1981.
[13] R. B. Holmes, *Geometric functional analysis and its applications* (Springer-Verlag, New York, 1975).
[14] O. L. Mangasarian, *Nonlinear programming* (McGraw-Hill, New York, 1969).
[15] J. W. Nieuwenhus, 'Another application of Guignard's generalized Kuhn-Tucker conditions', *J. Optim. Theory Appl.* **30** (1980), 117–125.
[16] B. N. Pshenichnyi, *Necessary conditions for an extremum* (Marcel Dekker, New York, 1971).
[17] R. T. Rockafellar, *Conjugate duality and optimization* (SIAM Regional Conference Series in Applied Mathematics 16, Philadelphia, 1974).
[18] L. I. Trudzik, 'Asymptotic Kuhn-Tucker conditions in abstract spaces', *Numer. Funct. Anal. Optim.* **4** (1982), 355–369.
[19] E. Vercher, *Teoremas de alternativa para sistemas infinitos aplicación a la programación y juegos semi-infinitos* (Tesis Doctoral, Universidad de Valencia, Spain, 1981).
[20] C. Zalinescu, 'A generalization of the Farkas lemma and applications to convex programming', *J. Math. Analy. Appl.* **66** (1978), 651–678.
[21] C. Zalinescu, 'On an abstract control problem', *Numer. Funct. Anal. Appl.* **2** (1980), 531–542.
[22] I. Zang and M. Avriel, 'On functions whose local minima are global', *J. Optim. Theory Appl.* **16** (1975), 183–190.
[23] I. Zang, E. U. Choo and M. Avriel, A note on functions whose local minima are global, *J. Optim. Theory Appl.* **18** (1976), 555–559.
[24] I. Zang, E. U. Choo and M. Avriel, 'On functions whose stationary points are global minima', *J. Optim. Theory Appl.* **22** (1977), 195–207.
[25] S. Zlobec, 'Extensions of asymptotic Kuhn-Tucker conditions in mathematical programming', *SIAM J. Appl. Math.* **21** (1971), 448–460.

Department of Mathematics
University of Melbourne
Parkville, Victoria 3052
Australia