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A SPECTRAL MAPPING THEOREM FOR SOME REPRESENTATIONS OF COMPACT ABELIAN GROUPS

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Dedicated to Professor Chinami Watari on his sixtieth birthday

We show that if G is a compact abelian group and U is a weakly continuous representation of G by means of isometries on a Banach space X, then $\sigma(\pi(\mu))=\overline{\hat{\mu}(sp(U))}$ holds for each measure μ in $\operatorname{reg}(M(G))$, where $\pi(\mu)$ denotes the generalized convolution operator in $B(X)$ defined by $\pi(\mu)x=\int_G U(t)x\,d\mu(t)\,(x\in X)$, σ the usual spectrum in $B(X)$, $sp(U)$ the Arveson spectrum of U, $\hat{\mu}$ the Fourier–Stieltjes transform of μ and $\operatorname{reg}(M(G))$ the largest closed regular subalgebra of the convolution measure algebra $M(G)$ of G. $\operatorname{reg}(M(G))$ contains all the absolutely continuous measures and discrete measures.

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1. Introduction and main result

Let G be a locally compact abelian group and U a weakly continuous representation of G by means of isometries on a Banach space X, i.e., a map $U: G\to B(X)$ satisfying

(i) $U(s+t)=U(s)U(t)$ for all $s,t\in G$, $U(0)=I$,

(ii) $\|U(s)x\|=\|x\|$ for $s\in G$, $x\in X$,

(iii) $G\to X;\ s\to U(s)x$ is weakly continuous for each $x\in X$.

Then this representation induces a continuous algebra homomorphism π of the convolution algebra $M(G)$ into $B(X)$ and such a homomorphism is written by $\pi(\mu)=\int_G U(t)\,d\mu(t)$ (cf. [5]). Let $sp(U)$ be the Arveson spectrum of U defined by

$$sp(U)=\bigcap\{Z(f): f\in \operatorname{Ker}(\pi|L^1(G))\}.$$

Here $Z(f)$ denotes the set of zeros of the Fourier transform $\hat{f}$ of f. In this setting, Connes [4] proved that for every Dirac measure μ the spectral mapping theorem (SMT): $\sigma(\pi(\mu))=\overline{\hat{\mu}(sp(U))}$ holds, where σ denotes the usual spectrum in $B(X)$ and $\hat{\mu}$ denotes the Fourier–Stieltjes transform of μ. Furthermore D'Antoni, Longo and Zsido [2] proved the SMT for the class of measures whose continuous part belongs to $L^1(G)$, the group algebra of G. Also, Eschmeier [5] proved the SMT in the case that U is the translation group representation and $X=L^1(G)$ or $M(G)$ and the convolution operator induced by μ has the weak 2-SDP (see [5, Theorem 2]). Here $M(G)$ denotes the Banach algebra of all bounded regular complex Borel measures on G. Also, since $M(G)$ is a semisimple commutative Banach algebra with identity, it follows from Albrecht's theorem [1] that there exists a largest closed regular subalgebra of $M(G)$, which we denote by $\operatorname{reg}(M(G))$.

With this notation, our main theorem can be stated as follows:

Theorem. *If G is a compact abelian group and $\mu\in\operatorname{reg}(M(G))$, we have $\sigma(\pi(\mu))=\overline{\hat{\mu}(sp(U))}$.*

Remark. The group algebra $L^1(G)$ and the discrete measures $M_d(G)$ are regular Banach subalgebras of $M(G)$. Then $L^1(G)+M_d(G)\subset\operatorname{reg}(M(G))$. In general, $L^1(G)+M_d(G)\neq\operatorname{reg}(M(G))$. In fact, let us denote by $\operatorname{top}(G)$ the class of all locally compact group topologies on G which are equal to or stronger than the original topology on G and denote by $L^*(G)$ the closed subalgebra of $M(G)$ generated by $\{L^1(G,\tau):\tau\in\operatorname{top}(G)\}$ as in [6]. Then we have that $L^1(G)+M_d(G)\subset L^*(G)\subset\operatorname{reg}(M(G))$ and $L^1(G)+M_d(G)\neq L^*(G)$ in general (cf. [6], [12]). Thus our result contains the Connes–D'Antoni–Longo–Zsido spectral mapping theorem for the compact case.

2. Lemmas

We first present the following result obtained in [7], which plays an essential role in the proof of the main theorem, and we include its proof for completeness.

Lemma 1. *Let X be a commutative Banach algebra with identity and B a Banach subalgebra of X. If B is regular, then for any $b\in B$ the Gelfand transform of b as an element of X is continuous on the carrier space Φ_X of X in the hull–kernel topology.*

Proof. We can assume without loss of generality that B contains the identity of X. Then it is sufficient to show that the restriction map θ: $\Phi_X\to\Phi_B$; $\phi\to\phi|B$ is continuous in the hull–kernel topology. To do this let F be a closed subset of Φ_B in the hull–kernel topology. Then $\{\phi\in\Phi_X:\phi|\ker F=0\}=\theta^{-1}(F)$. Also, since $\ker F\subset\ker\theta^{-1}(F)$, it follows that $\operatorname{hul}(\ker\theta^{-1}(F))\subset\{\phi\in\Phi_X:\phi|\ker F=0\}$. Therefore $\theta^{-1}(F)$ is closed in the hull–kernel topology. In other words, θ is continuous in this topology. □

We will next state the definition of BSE-algebras introduced by the first author and Hatori [10]. Let A be a commutative Banach algebra without order and $M(A)$ the multiplier algebra of A. It is well-known that $T\in M(A)$ can be represented as a bounded continuous complex-valued function $\hat{T}$ on Φ_A such that $\widehat{Ta}(\phi)=\hat{T}(\phi)\hat{a}(\phi)$ for all $a\in A$ and $\phi\in\Phi_A$ (cf. [8]). Set $\hat{M}(A)=\{\hat{T}: T\in M(A)\}$. We also denote by A^* the dual space of A and $C_{\mathrm{BSE}}(\Phi_A)$ the set of all continuous complex-valued functions σ on Φ_A which satisfy the following condition: there exists a positive real number β such that for every finite sequence of complex numbers $c_1,\dots,c_n$ and elements $\phi_1,\dots,\phi_n$ of Φ_A, the inequality

$$\left|\sum_{i=1}^{n} c_i\sigma(\phi_i)\right|\leq\beta\left\|\sum_{i=1}^{n} c_i\phi_i\right\|_{A^*}$$

holds.

Definition. A commutative Banach algebra A without order is said to be BSE if it satisfies the condition $\hat{M}(A)=C_{\mathrm{BSE}}(\Phi_A)$.

By the Bochner–Schoenberg–Eberlein theorem, the group algebra of a locally compact abelian group is BSE (cf. [**10**]). The following results can be observed in [**10**].

Lemma 2 ([**10**, Theorem 4, (ii)]). *Let A be a commutative Banach algebra without order, A^{**} its second dual and $C^b(\Phi_A)$ the set of all bounded continuous complex-valued functions on Φ_A. Then $C_{\mathrm{BSE}}(\Phi_A)=C^b(\Phi_A)\cap(A^{**}|\Phi_A)$.*

When a closed ideal I of a commutative Banach algebra A is essential as a Banach A-module, that is, I equals the closed linear span of $\{ax: a\in A,\ x\in I\}$, we call I an essential ideal.

Lemma 3 ([**10**, Theorem 8, (i)]). *Let A be a* BSE*-algebra with discrete carrier space and I an essential closed ideal of A. Then $\hat{M}(A/I)=C_{\mathrm{BSE}}(\Phi_{A/I})$, i.e., A/I is* BSE, *where A/I denotes the quotient algebra of A defined by I.*

The following lemma also plays an essential role in the proof of the main theorem.

Lemma 4. *Let A be a* BSE*-algebra with discrete carrier space and I a closed ideal of A such that $I^{\sim}=\ker(\mathrm{hul}(I))$ is essential. Then every multiplier on $A/I^{\sim}$ can be lifted as a multiplier on A, that is, if $\nu\in M(A/I^{\sim})$ and η is the canonical map of A onto $A/I^{\sim}$, then there exists $\mu\in M(A)$ such that $\eta(\mu a)=\nu\eta(a)$ for all $a\in A$.*

Proof. Let $\nu\in M(A/I^{\sim})$ and let η be the canonical map of A onto $A/I^{\sim}$. Note that the algebra $A/I^{\sim}$ is semisimple. Then it is sufficient to show that there exists $\mu\in M(A)$ such that $(\eta(\mu a))^{\wedge}=(\nu\eta(a))^{\wedge}$ for all $a\in A$. Here $^{\wedge}$ denotes the Gelfand transform on $A/I^{\sim}$. We have

$$\hat{\nu}\in\hat{M}(A/I^{\sim})=C_{\mathrm{BSE}}(\Phi_{A/I^{\sim}})\qquad\text{(by Lemma 3)}$$

$$=(A/I^{\sim})^{**}|\Phi_{A/I^{\sim}}\qquad\text{(by Lemma 2),}$$

so that there exists $H\in(A/I^{\sim})^{**}$ with $\hat{\nu}=H|\Phi_{A/I^{\sim}}$. Then we can find an element $F\in A^{**}$ such that $\eta^{**}(F)=H$, since $\eta^{**}: A^{**}\to(A/I^{\sim})^{**}$ is a surjection. We further have by the BSE property of A and Lemma 2 that

$$\hat{M}(A)=C_{\mathrm{BSE}}(\Phi_A)=A^{**}|\Phi_A.$$

Therefore we can find an element $\mu\in M(A)$ such that $\hat{\mu}=F|\Phi_A$. Let $\phi\in\Phi_A$ be such that $\phi|I^{\sim}=0$ and ϕ' the canonical image of ϕ in $\Phi_{A/I^{\sim}}$. Then we have

$$\hat{v}(\phi') = H(\phi') = \langle \phi', \eta^{**}(F) \rangle = \langle \eta^{*}(\phi'), F \rangle = \langle \phi, F \rangle$$

and hence for any $a \in A$,

$$\begin{aligned}(v\eta(a))^{\wedge}(\phi') &= \hat{v}(\phi')(\eta(a))^{\wedge}(\phi') = \langle \phi, F \rangle \hat{a}(\phi) \\ &= \hat{\mu}(\phi)\hat{a}(\phi) = (\mu a)^{\wedge}(\phi) = \phi'(\eta(\mu a)) \\ &= (\eta(\mu a))^{\wedge}(\phi').\end{aligned}$$

Consequently $(\eta(\mu a))^{\wedge} = (v\eta(a))^{\wedge}$ for all $a \in A$. □

Lemma 5. *If* $\mu \in M(G)$, *then* $\sigma(\pi(\mu)) \subset \{\mu^{\vee}(\phi) : \phi \in \Phi_{M(G)}, \ \mathrm{Ker}\,\pi \subset \mathrm{Ker}\,\phi\}$. *Here* $\mu^{\vee}$ *denotes the Gelfand transform of* $\mu \in M(G)$.

Proof. Let $\mu \in M(G)$. Then we have that

$$\begin{aligned}\sigma_{M(G)/\mathrm{Ker}\,\pi}(\mu + \mathrm{Ker}\,\pi) &= (\mu + \mathrm{Ker}\,\pi)^{\vee}(\Phi_{M(G)/\mathrm{Ker}\,\pi}) \\ &= \{\mu^{\vee}(\phi) : \mathrm{Ker}\,\pi \subset \mathrm{Ker}\,\phi\}.\end{aligned}$$

Also, since $M(G)/\mathrm{Ker}\,\pi \cong \pi(M(G)) \subset B(X)$, it follows that

$$\begin{aligned}\sigma_{M(G)/\mathrm{Ker}\,\pi}(\mu + \mathrm{Ker}\,\pi) &= \sigma_{\pi(M(G))}(\pi(\mu)) \\ &\supset \sigma_{B(X)}(\pi(\mu)).\end{aligned}$$

Therefore the desired inclusion follows. □

The following result was proved by D'Antoni, Longo and Zsido [2].

Lemma 6 ([2, Lemma 1]). $\sigma(\pi(\mu)) \supset \overline{\hat{\mu}(sp(U))}$ *for all* $\mu \in M(G)$.

In the next section we will show our main theorem using these lemmas.

3. Proof of theorem

Since $\overline{\hat{\mu}(sp(U))} \subset \sigma(\pi(\mu))$ by Lemma 6, we have only to show the reverse inclusion. To do this, let $\alpha \in \sigma(\pi(\mu))$. Then by Lemma 5, there exists $\phi_0 \in \Phi_{M(G)}: \alpha = \mu^{\vee}(\phi_0)$ and $\mathrm{Ker}\,\pi \subset \mathrm{Ker}\,\phi_0$.

Let us consider the natural homomorphism T_π of $M(G)/\mathrm{Ker}\,\pi$ into $M(L^1(G)/I_\pi)$ defined by

$$T_\pi(v + \mathrm{Ker}\,\pi)(f + I_\pi) = v * f + I_\pi \quad (v \in M(G),\ f \in L^1(G)),$$

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where $I_\pi = \operatorname{Ker}(\pi | L^1(G))$.

Since G is compact, it follows from [**9**, Corollary 8.3.2] that $I_\pi^{\sim} = I_\pi$. Note also that $L^1(G)$ is a BSE-algebra with discrete carrier space and it has an approximate identity; hence I_π is an essential ideal of $L^1(G)$. Then Lemma 4 implies that T_π is surjective, since $M(G) \cong M(L^1(G))$. Furthermore, T_π is injective. In fact, let $v \in M(G)$ be such that $\pi(v * f) = 0$ for all $f \in L^1(G)$. Given $\varepsilon > 0$, $x \in X$ and $\xi \in X^*$, the dual space of X, choose a neighbourhood V of zero such that

$$|\langle U(t)x, (\pi(v))^* \xi\rangle - \langle x, (\pi(v))^* \xi\rangle| < \varepsilon \qquad (t \in V).$$

Furthermore, choose a non-negative real-valued function $u_V \in L^1(G)$ vanishing off V and satisfying $\int_G u_V(t)\,dt = 1$. Then we have

$$\begin{aligned}
|\langle \pi(v)x, \xi\rangle| &\leqq |\langle \pi(u_V)x, (\pi(v))^*\xi\rangle - \langle x, (\pi(v))^*\xi\rangle| + |\langle \pi(u_V)x, (\pi(v))^*\xi\rangle| \\
&\leqq \int_V |\langle U(t)x, (\pi(v))^*\xi\rangle - \langle x, (\pi(v))^*\xi\rangle| u_V(t)\,dt \\
&< \varepsilon.
\end{aligned}$$

Since ε is arbitrary, it follows that $\langle \pi(v)x, \xi\rangle = 0$ for all $x \in X$ and $\xi \in X^*$; hence $\pi(v) = 0$. In other words, T_π is injective.

Here we take the following convention: for each $\phi \in \Phi_{M(G)}$ such that $\operatorname{Ker}\pi \subset \operatorname{Ker}\phi$, ϕ' denotes the element of $\Phi_{M(G)/\operatorname{Ker}\pi}$ defined by $\phi'(v + \operatorname{Ker}\pi) = \phi(v)\,(v \in M(G))$.

Since T_π is an isomorphism of $M(G)/\operatorname{Ker}\pi$ onto $M(L^1(G)/I_\pi)$, there exists an element ψ_0 of $\Phi_{M(L^1(G)/I_\pi)}$ such that $\phi'_0 = (T_\pi)^*\psi_0$. So we can find a net $\{\psi_\lambda\}$ in $\Phi_{M(L^1(G)/I_\pi)}$ such that $\psi_\lambda | L^1(G)/I_\pi \neq 0$ for all λ and $hk\text{–}\lim \psi_\lambda = \psi_0$, where "$hk$–lim" denotes the hull–kernel limit. Furthermore, we can find a net $\{\phi_\lambda\}$ in $\Phi_{M(G)}$ such that $\operatorname{Ker}\pi \subset \operatorname{Ker}\phi_\lambda$ and $(T_\pi)^*\psi_\lambda = \phi'_\lambda$ for all λ. Set $\xi_\lambda = \phi_\lambda | L^1(G)$ for each λ. Then each $\xi_\lambda \neq 0$. In fact, choose a function $f_0 \in L^1(G)$ such that $\psi_\lambda(f_0 + I_\pi) \neq 0$. Then for each λ, we have

$$\phi_\lambda(f_0) = \phi'_\lambda(f_0 + I_\pi) = \langle T_\pi(f_0 + I_\pi), \psi_\lambda\rangle = \psi_\lambda(f_0 + I_\pi) \neq 0,$$

so that $\xi_\lambda \neq 0$. Thus each ξ_λ belongs to $\Phi_{L^1(G)}(\cong \hat{G}$, the dual group of $G)$ and hence must belong to $sp(U)$, since $I_\pi \subset \operatorname{Ker}\xi_\lambda$ and $sp(U)$ can be regarded as the hull of I_π in $\Phi_{L^1(G)}$.

Of course $(T_\pi)^* | \Phi_{M(L^1(G)/I_\pi)}$ is continuous on $\Phi_{M(L^1(G)/I_\pi)}$ in the hull–kernel topology and hence

$$hk\text{–}\lim \phi'_\lambda = hk\text{–}\lim (T_\pi)^*\psi_\lambda = (T_\pi)^*\psi_0 = \phi'_0.$$

Therefore we have from [**11**, Theorem 2.6.6] that $hk\text{–}\lim \phi_\lambda = \phi_0$. Since $\mu \in \operatorname{reg}(M(G))$, it follows from Lemma 1 that $\mu^\vee$ is continuous on $\Phi_{M(G)}$ in the hull–kernel topology, so that

$$\lim \hat{\mu}(\xi_\lambda) = \lim \mu^\vee(\phi_\lambda) = \mu^\vee(\phi_0) = \alpha.$$

Consequently we have that $\alpha \in \overline{\hat{\mu}(sp(U))}$ and the reverse inclusion is shown. □

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