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BOUNDING SCALAR CURVATURE AND DIAMETER ALONG THE KÄHLER RICCI FLOW (AFTER PERELMAN)

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Abstract In this short note we present a result of Perelman with detailed proof. The result states that if $g(t)$ is the Kähler Ricci flow on a compact, Kähler manifold M with $c_1(M) > 0$, the scalar curvature and diameter of $(M, g(t))$ stay uniformly bounded along the flow, for $t \in [0, \infty)$. We learned about this result and its proof from Grigori Perelman when he was visiting MIT in the spring of 2003. This may be helpful to people studying the Kähler Ricci flow.

Keywords: scalar curvature; diameter; Kähler Ricci flow

AMS 2000 *Mathematics subject classification:* Primary 53C44

1. Introduction

We will consider a Kähler Ricci flow

$$\frac{\mathrm{d}}{\mathrm{d}t} g_{i\bar{j}} = g_{i\bar{j}} - R_{i\bar{j}} = \partial_i \partial_{\bar{j}} u \tag{1.1}$$

on a compact, Kähler manifold M, with $c_1(M) > 0$, of an arbitrary complex dimension n. Cao proved in [**1**] that (1.1) has a solution for all time t. One of the most important questions regarding the Kähler Ricci flow is whether it develops singularities at infinity, that is whether the curvature of $g(t)$ blows up as $t \to \infty$. This question was only answered in the case in which the curvature operator or bisectional curvature is non-negative (see [**2**–**4**]). In 2003, Perelman made a surprising claim that the scalar curvature of $g(t)$ does not blow up as $t \to \infty$. He also showed us a sketched proof. This result of Perelman strengthens the belief that the Kähler Ricci flow converges to a Kähler Ricci soliton as t tends to infinity, at least outside a subvariety of complex codimension 2.

The goal of this paper is to give a detailed proof of Perelman's bound on a scalar curvature and a diameter.

Theorem 1.1 (Perelman). *Let $g(t)$ be a Kähler Ricci flow (1.1) on a compact, Kähler manifold M of complex dimension n, with $c_1(M) > 0$. There exists a uniform constant C, depending only on the initial metric so that*

- $|R(g(t))| \leqslant C$,
- $\operatorname{diam}(M, g(t)) \leqslant C$,
- $|u|_{C^1} \leqslant C$,

where the C^1-norm is with respect to the evolving metric $g(t)$ and u is normalized so that $(2\pi)^{-n}\int_M \mathrm{e}^{-u}\,\mathrm{d}V_{g(t)} = 1$.

The outline of the main steps of the proof of Theorem 1.1 is as follows.

(1) Getting a uniform lower bound on Ricci potential $u(t)$.

(2) Bounding $|\nabla u(t)|$ and a scalar curvature $R(t)$ by C^0-norm of Ricci potential $u(t)$. This can be achieved by considering the evolution equations for $|\nabla u|^2/(u + 2B)$ and $-\Delta u/(u + 2B)$, where B is a uniform constant such that $u + B > 0$, whose existence is guaranteed by step (1).

(3) Step (2) tells us that $\sqrt{u + 2B}$ is uniformly Lipschitz bounded and that it is enough to bound $\operatorname{diam}(M, g(t))$ in order to have uniform bounds on $|u(t)|_{C^1}$ and scalar curvature $R(t)$.

(4) To show that the diameters are uniformly bounded along the flow, we will argue by contradiction. We will assume that the diameters are unbounded as we approach infinity. Using that, we will show that the integral of the scalar curvature over some large annulus is bounded by CV, where C is a uniform constant and V is a volume of a slightly larger annulus than the one we started with. We can find such an annulus at every time t in the sequence of times for which the diameters go to infinity. By choosing similar cut-off functions as in the proof of Perelman's non-collapsing theorem in [**7**] we will show that we get a contradiction if the diameters are unbounded as we approach infinity.

The organization of the paper is as follows. In § 2 we will give the proof of Theorem 1.1. In § 3 we will discuss the convergence of the normalized Kähler Ricci flow, using Perelman's results.

2. The Ricci potential $u(t)$

In this section we will show that there is a uniform lower bound on $u(x, t)$. We will also show that it is enough to bound diameters of $(M, g(t))$ in order to have Theorem 1.1.

By taking the trace of (1.1) we get $\Delta u = n - R$. Let $\phi(t)$ be a metric potential, that is,

$$g_{i\bar{j}}(t) = g_{i\bar{j}}(0) + \partial_i\partial_{\bar{j}}\phi.$$

Then we can take $u(t) = (\mathrm{d}/\mathrm{d}t)\phi(t)$. Normalize so that

$$\int_M \mathrm{e}^{-u} = (2\pi)^n.$$

Define

$$\mu(g,\tau) = \inf_{\{f \mid \int_M \mathrm{e}^{-f}(4\pi\tau)^{-n}=1\}} (4\pi\tau)^{-n} \int_M \mathrm{e}^{-f}\{2\tau(R + |\nabla f|^2) + f - 2n\}\,\mathrm{d}V$$

to be Perelman's functional for $g(t)$ as in [**7**]. Perelman has proved that $\mu(g,\tau)$ is achieved. Take $f = u$ and $\tau = \frac{1}{2}$. Then by monotonicity of $\mu(g(t))$ along the Kähler Ricci flow,

$$\begin{aligned} A = \mu(g(0), \tfrac{1}{2}) &\leqslant \mu(g(t), \tfrac{1}{2}) \\ &\leqslant \int_M (2\pi)^{-n}\mathrm{e}^{-u}(R + |\nabla u|^2 + u - 2n) \\ &= \int_M (2\pi)^{-n}\mathrm{e}^{-u}(-\Delta u + |\nabla u|^2 - 2n + u) \\ &= \int_M (2\pi)^{-n}\Delta \mathrm{e}^{-u} - 2n + (2\pi)^{-n}\int_M \mathrm{e}^{-u}u \\ &= -2n + (2\pi)^{-n}\int_M \mathrm{e}^{-u}u. \end{aligned} \tag{2.1}$$

We have just proved the following lemma.

Lemma 2.1. *There is a uniform constant $C_1 = C_1(A)$ such that $\int \mathrm{e}^{-u}u \geqslant C_1$.*

Define $a = -(2\pi)^{-n}\int_M u\mathrm{e}^{-u}\,\mathrm{d}V$. In the following claim we will prove a lower bound on a.

Claim 2.2. *There is a uniform constant $C_2 > 0$ such that $a \geqslant -C_2$.*

Proof. Let $u_- = \min\{u, 0\}$ and $u_+ = \max\{0, u\}$. Then we have

$$\begin{aligned} a &= -(2\pi)^{-n}\int_M u\mathrm{e}^{-u}\,\mathrm{d}V = -(2\pi)^{-n}\int_M u_-\mathrm{e}^{-u_-}\,\mathrm{d}V - (2\pi)^{-n}\int_M u_+\mathrm{e}^{-u_+}\,\mathrm{d}V \\ &\geqslant -(2\pi)^{-n}\int_M u_+\mathrm{e}^{-u_+}\,\mathrm{d}V \geqslant -C_2 \end{aligned}$$

for some constant $C_2 \geqslant 0$, since $f(x) = x\mathrm{e}^{-x}$ is a bounded function for $x \geqslant 0$. □

Remark 2.3. It is well known that the scalar curvature is uniformly bounded from below along the flow. We may assume that $R > 0$.

Function $u(t)$ satisfies

$$\begin{aligned} \partial_i\partial_{\bar{j}}u_t &= g_{i\bar{j}} - R_{i\bar{j}} + \frac{\mathrm{d}}{\mathrm{d}t}\partial_i\partial_{\bar{j}}\ln\det(g_{i\bar{j}}(0) + \partial_i\partial_{\bar{j}}\phi) \\ &= \partial_i\partial_{\bar{j}}(u + \Delta u), \end{aligned}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}u = \Delta u + u + a, \tag{2.2}$$

where we can choose $a = -\int u\mathrm{e}^{-u}(2\pi)^{-n} \leqslant C$, uniformly bounded from above by the previous lemma.

Lemma 2.4. *Function $u(t)$ is uniformly bounded from below.*

Proof. If the Ricci potential u is very negative for some time t_0, say $u(t_0) \leqslant -2(n+C_1)$, from (2.2), by Lemma 2.1 and Remark 2.3 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}u = n - R + u + a \leqslant n + C_1 + u < 0, \tag{2.3}$$

at $t = t_0$, which implies that $u(t)$ stays very negative for $t \geqslant t_0$. If for some y_0 we have $u(y_0) \ll 0$ at some time t_0, $u(y) \ll 0$ for all y in some neighbourhood U of y_0, at time t_0. By (2.3), $u(y) \ll 0$ continues to hold in U, for all $t \geqslant t_0$. Then $(\mathrm{d}/\mathrm{d}t)u \leqslant C + u$ implies

$$u(t)(z) \leqslant \mathrm{e}^{t-t_0}(C + u(t_0)) \leqslant -\tilde{C}\mathrm{e}^{t}, \tag{2.4}$$

for $t \geqslant t_0$, for all $z \in U$, where $\tilde{C}$ depends on t_0. Then $\dot{\phi} = u$ yields

$$\phi(t)(z) \leqslant \phi(t_0)(z) - \tilde{C}\mathrm{e}^{t-t_0} \leqslant -C_1\mathrm{e}^{t} \tag{2.5}$$

for big enough t and all $z \in U$. On the other hand, $\int_M \mathrm{e}^{-u(t)} = (2\pi)^n$, which tells us that $u(t)$ cannot be very negative everywhere on M, that is, there is a uniform constant C_2 such that $u(x'_t, t) = \max_M u(t) \geqslant -C_2$. Since $\dot{\phi}(t) = u(t)$, from (2.2) we get

$$u(x'_t, t) - \phi(x'_t, t) \leqslant \max_M (u(\cdot\,, 0) - \phi(\cdot\,, 0)) + \tilde{C}t,$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t) - \phi(t)) = n - R + a \leqslant \tilde{C}$$

by Lemma 2.1 and Remark 2.3. This implies that

$$\max_M \phi(t) \geqslant -C_3 - \tilde{C}t \tag{2.6}$$

for a uniform constant C_3.

By taking the trace of $g(t) = g(0) + \mathrm{i}\partial\bar{\partial}\phi(t)$ at time $t = 0$, we get

$$-\Delta_0\phi(t) = -\operatorname{tr}_{g(0)} g(t) + n \leqslant n.$$

Consider a fixed metric $g(0)$. By Green's formula applied to $\phi(t)$ we have

$$\begin{aligned}
\phi(x_t, t) &= \frac{1}{\mathrm{Vol}_0(M)} \int_M \phi(y, t)\,\mathrm{d}V_0 - \frac{1}{\mathrm{Vol}(M)} \int_M \Delta_0\phi(y, t) G_0(x_t, y)\,\mathrm{d}V_0 \\
&\leqslant \frac{\mathrm{Vol}_0(M\backslash U)}{\mathrm{Vol}_0(M)} \sup_M \phi(\cdot\,, t) + \frac{\mathrm{Vol}_0(U)}{\mathrm{Vol}_0(M)} \int_U \phi(y, t)\,\mathrm{d}V_0 + C \\
&\leqslant \frac{\mathrm{Vol}_0(M\backslash U)}{\mathrm{Vol}_0(M)} \sup_M \phi(\cdot\,, t) - C_4\mathrm{e}^{t} + C
\end{aligned}$$

for $t \geqslant t_0$, where $\phi(x_t, t) = \max_M \phi(y, t)$ and G_0 is Green's function associated with metric $g(0)$ (recall that $\int_M G_0(x_t, y)\,\mathrm{d}V_0(y) = \text{const.}$). Since

$$\frac{\mathrm{Vol}_0(M \backslash U)}{\mathrm{Vol}_0(M)} < 1,$$

we get

$$\max_M \phi(\cdot\,, t) \leqslant -C_5 \mathrm{e}^t + C_6$$

for some uniform constants C_5, C_6. All constants C_2, C_3, C_4, C_5, C_6 are independent of $t \geqslant t_0$ and they all depend on t_0. This together with (2.6) yields a contradiction for big values of t. Therefore, there exists a uniform lower bound on $u(t)$, that is, there is some constant B such that $u(x, t) \geqslant B$ for all $(x, t) \in M \times [0, \infty)$. □

Standard computation gives the following evolution equations for Δu and $|\nabla u|^2$:

$$\Box(\Delta u) = \frac{\mathrm{d}}{\mathrm{d}t}\Delta u - \Delta^2 u = -|\nabla\bar{\nabla}u|^2 + \Delta u, \tag{2.7}$$

$$\Box(|\nabla u|^2) = \frac{\mathrm{d}}{\mathrm{d}t}|\nabla u|^2 - \Delta|\nabla u|^2 = -|\nabla\nabla u|^2 - |\nabla\bar{\nabla}u|^2 + |\nabla u|^2. \tag{2.8}$$

Proposition 2.5. *There is a uniform constant C such that*

$$|\nabla u|^2 \leqslant C(u + C), \tag{2.9}$$

$$R \leqslant C(u + C). \tag{2.10}$$

Proof. We will first prove an estimate (2.9) which we will need in the proof of Lemma 3.3. By Lemma 2.4 we may assume that $u(x, t) > -B$. The proof resembles the arguments in [**9**] and [**6**]. If $H = |\nabla u|^2/(u + 2B)$, by (2.7) and (2.8) we get

$$\Box H = \frac{-|\nabla\bar{\nabla}u|^2 - |\nabla\nabla u|^2}{u + 2B} + \frac{|\nabla u|^2(2B - a)}{(u + 2B)^2} + \frac{2\,\mathrm{Re}(\bar{\nabla}u \cdot \nabla|\nabla u|^2)}{(u + 2B)^2} - \frac{2|\nabla u|^4}{(u + 2B)^3}. \tag{2.11}$$

We can write

$$\frac{\mathrm{Re}(2\bar{\nabla}u \cdot \nabla|\nabla u|^2)}{(u + 2B)^2} - \frac{2|\nabla u|^4}{(u + 2B)^3} = (2 - \epsilon)\frac{\mathrm{Re}(\bar{\nabla}u \cdot \nabla H)}{u + 2B} + \epsilon\frac{\mathrm{Re}(\bar{\nabla}u \cdot \nabla|\nabla u|^2)}{(u + 2B)^2} - \epsilon\frac{|\nabla u|^4}{(u + 2B)^3} \tag{2.12}$$

for some small $\epsilon > 0$. Since

$$\begin{aligned} |\nabla_{\bar{i}} u \nabla_i(\nabla_j u \nabla_{\bar{j}} u)| &= |\nabla_{\bar{i}} u \nabla_i \nabla_j u \nabla_{\bar{j}} u + \nabla_{\bar{i}} u \nabla_j u \nabla_i \nabla_{\bar{j}} u| \\ &\leqslant |\nabla u|^2(|\nabla\nabla u| + |\nabla\bar{\nabla}u|), \end{aligned}$$

by the Cauchy–Schwarz inequality,

$$\begin{aligned} \epsilon\frac{|\nabla u \cdot \nabla|\nabla u|^2|}{(u + 2B)^2} &\leqslant \epsilon\frac{|\nabla u|^2(|\nabla\nabla u| + |\nabla\bar{\nabla}u|)|}{(u + 2B)^{3/2}(u + 2B)^{1/2}} \\ &\leqslant \frac{\epsilon}{2}\frac{|\nabla u|^4}{(u + 2B)^3} + \frac{\epsilon(|\nabla\nabla u|^2 + |\nabla\bar{\nabla}u|^2)}{u + 2B}. \end{aligned} \tag{2.13}$$

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Choose ϵ small so that $2\epsilon < \frac{1}{2}$. Combining (2.11), (2.12) and (2.13) yields

$$\square H \leqslant \frac{|\nabla u|^2(2B-a)}{(u+2B)^2} + (2-\epsilon)\frac{\bar{\nabla} u \cdot \nabla H}{u+2B} - \frac{\epsilon}{2}\frac{|\nabla u|^4}{(u+2B)^3}. \tag{2.14}$$

At a point at which H achieves its maximum we have that ∇H vanishes and therefore by maximum principle, an estimate (2.14) reduces to

$$0 \leqslant \frac{\mathrm{d}}{\mathrm{d}t} H_{\max} \leqslant \frac{|\nabla u|^2}{(u+2B)^2}\biggl(2B - a - \frac{\epsilon}{2}\frac{|\nabla u|^2}{u+2B}\biggr). \tag{2.15}$$

If we assume that

$$|\nabla u|^2 \gg u + 2B, \tag{2.16}$$

then a term on the right-hand side of (2.15) becomes negative for large t, which is a contradiction and therefore we have (2.9).

Our next goal is to prove that $-\Delta u$ is bounded by $C(u+C)$, which yields (2.10), since $\Delta u = n - R$. Let $K = -\Delta u/(u+2B)$, where B is a uniform constant as above. Similar computation as before gives that

$$\square\biggl(-\frac{\Delta u}{u+2B}\biggr) = \frac{|\nabla\bar{\nabla} u|^2}{u+2B} + \frac{(-\Delta u)(2B-a)}{(u+2B)^2} + 2\frac{\bar{\nabla} u \cdot \nabla K}{u+2B}.$$

Take $b > 1$. Then

$$\square\biggl(\frac{-\Delta u + b|\nabla u|^2}{u+2B}\biggr) = \frac{-b|\nabla\nabla u|^2 - (b-1)|\nabla\bar{\nabla} u|^2}{u+2B} + \frac{(-\Delta u + b|\nabla u|^2)(2B-a)}{(u+2B)^2} + \frac{2\bar{\nabla} u \cdot \nabla((-\Delta u + b|\nabla u|^2)/(u+2B))}{u+2B}.$$

Let

$$G = \frac{-\Delta u + b|\nabla u|^2}{u+2B}$$

and, by the maximum principle,

$$\frac{\mathrm{d}}{\mathrm{d}t} G_{\max} \leqslant -(b-1)\frac{|\nabla\bar{\nabla} u|^2}{u+2B} + \frac{(-\Delta u + b|\nabla u|^2)(2B-a)}{(u+2B)^2}.$$

In local coordinates,

$$(\Delta u)^2 = \biggl(\sum_i u_{i\bar{i}}\biggr)^2 \leqslant n\sum_i u_{i\bar{i}}^2 = n|\nabla\bar{\nabla} u|^2,$$

and therefore

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} G_{\max} &\leqslant -(b-1)\frac{(\Delta u)^2}{n(u+2B)} + \frac{(-\Delta u + b|\nabla u|^2)(2B-a)}{(u+2B)^2} \\ &\leqslant \frac{(-\Delta u)}{u+2B}\biggl\{\frac{2B-a}{u+2B} - \frac{(-\Delta u)}{n}\biggr\} + \frac{b|\nabla u|^2(2B-a)}{(u+2B)^2}. \end{aligned} \tag{2.17}$$

By Lemmas 2.1 and 2.4 we may assume that $(2B-a)/(u+2B)$ is bounded from above by a uniform constant. We have also proved the estimate (2.9) on $|\nabla u|$. If

$$-\Delta u \gg u + 2B, \tag{2.18}$$

by (2.17) we would have that $(\mathrm{d}/\mathrm{d}t)G_{\max} < 0$ for big values of t. This would imply $-\Delta u(t) \leqslant C(u+2B)$, for some uniform constant C and all big values of t, which contradicts (2.18). Therefore, there exists a uniform constant C such that (2.10) holds. □

Proposition 2.6. *Let $g(t)$ be the Kähler Ricci flow as above. There exists a positive constant $C = C(A)$ such that for every $x \in M$, $\mathrm{Vol}(B_{g(t)}(x,1)) \geqslant C$, for any metric $g(t)$ satisfying $|R| \leqslant 1$ on $B_{g(t)}(x,1)$.*

Proof. Let $g(t)$ be as before, a solution to a normalized Kähler Ricci flow equation, and let $\tilde{g}(s)$ be a solution to the equation $(\mathrm{d}/\mathrm{d}s)\tilde{g}(s) = -2\,\mathrm{Ric}(\tilde{g}(s))$. Reparametrization between these two flows is given by $\tilde{g}(s) = (1-2s)g(t(s))$, where $t(s) = -\ln(1-2s)$. The first flow has a solution for $t \in [0,\infty)$ and the second one has a maximal solution for $s \in [0,\frac{1}{2})$. The scalar curvature rescales as

$$R(\tilde{g}(s)) = \frac{R(g(t(s)))}{1-2s} \leqslant \frac{1}{1-2s}.$$

The following improvement of Perelman's non-collapsing result (noticed by Perelman himself) that requires only a scalar curvature bound can be found in [**5**]. The result was communicated to Kleiner and Lott by Tian. It says that there is a universal constant $\kappa = \kappa(\tilde{g}(0)) > 0$, so that for an unnormalized Ricci flow $(\mathrm{d}/\mathrm{d}s)\tilde{g}(s) = -2\,\mathrm{Ric}(\tilde{g}(s))$, if $|R(\tilde{g}(s))| \leqslant (1/r^2)$ in a ball $B_{\tilde{g}(s)}(p,r)$, then $\mathrm{Vol}_{\tilde{g}(s)} B_{\tilde{g}(s)}(p,r) \geqslant \kappa r^{2n}$. The detailed arguments of the proof can be found in [**5**] and [**8**], but for the convenience of a reader we will include it here as well. We argue by contradiction, that is, assume there are sequences $p_k \in M$ and $t_k \to \infty$ so that $|R| \leqslant (C/r_k^2)$, but $\mathrm{Vol}(B_k)r_k^{-2n} \to 0$ as $k \to \infty$, where $B_k = B_{t_k}(p_k, r_k)$. Let $\tau = r_k^2$. Define

$$u_k(x) = \mathrm{e}^{C_k}\phi(r_k^{-1}\,\mathrm{dist}(x,p_k)) \tag{2.19}$$

at t_k, where ϕ is a smooth function on R, equal to 1 on $[0,\frac{1}{2}]$, decreasing on $[\frac{1}{2},1]$ and equal to 0 on $[1,\infty)$, and 'dist' is a distance computed at time t_k. C_k is a constant to make u satisfy the constraint

$$\begin{aligned}(4\pi)^n &= \mathrm{e}^{2C_k} r_k^{-2n} \int_{B(p_k,r_k)} \phi(r_k^{-1}\,\mathrm{dist}(x,p_k))^2\,\mathrm{d}V \\ &\leqslant \mathrm{e}^{2C_k} r_k^{-2n}\,\mathrm{Vol}(B_k).\end{aligned}$$

Since $r_k^{-2n}\,\mathrm{Vol}\,B_k \to 0$, this shows that $C_k \to +\infty$. Recall that Perelman's functional

$$\mathcal{W}(g,u,\tau) = (4\pi\tau)^{-n}\int_M (2\tau(Ru^2 + 4|\nabla u|^2) + u^2\ln^2 u - 2nu^2)\,\mathrm{d}V_g.$$

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We compute

$$\begin{aligned}\mathcal{W}(u_k) &= (4\pi)^{-n} r_k^{-2n} \mathrm{e}^{2C_k} \int_{B(p_k,r_k)} (4|\phi'(r_k^{-1}\operatorname{dist}(x,p_k))|^2 - 2\phi^2 \ln\phi)\,\mathrm{d}V \\ &\qquad + r_k^2 \int_{B(p_k,r_k)} Ru^2 (4\pi)^{-n} r_k^{-n}\,\mathrm{d}V - 2n - 2C_k \\ &\leqslant (4\pi)^{-n} r_k^{-2n} \mathrm{e}^{2C_k} \int_{B(p_k,r_k)} (4|\phi'|^2 - 2\phi^2\ln\phi)\,\mathrm{d}V + r_k^2 \max_{B_k} R - 2n - 2C_k.\end{aligned}$$

Let $V(r) = \operatorname{Vol}(B(p_k, r))$. The necessary ingredients of the argument are that

(a) $r_k^{-2n}\operatorname{Vol}(B(p_k, r_k)) \to 0$;

(b) $r_k^2 R$ is uniformly bounded above;

(c) $\operatorname{Vol}(B(p_k, r_k))/\operatorname{Vol}(B(p_k, r_k/2))$ is uniformly bounded above.

If $\operatorname{Vol}(B(p_k, r_k))/\operatorname{Vol}(B(p_k, r_k/2)) < 3^n$ for all k, then we are done. If not, then for a given k we have that $\operatorname{Vol}(B(p_k, r_k))/\operatorname{Vol}(B(p_k, r_k/2)) \geqslant 3^n$. Let $r_k' = r_k/2$. We have $(r_k')^{-2n}\operatorname{Vol}(B(p_k, r_k')) \leqslant r_k^{-2n}\operatorname{Vol}(B(p_k, r_k))$ and $(r_k')^2 R \leqslant C_1$ on $B(p_k, r_k')$. Replace r_k by r_k'. If $\operatorname{Vol}(B(p_k, r_k'))/\operatorname{Vol}(B(p_k, r_k'/2)) < 3^n$, then we stop. If not, then we repeat the process and replace r_k' by $r_k'/2$. At some point we will achieve that

$$\frac{\operatorname{Vol}(B(p_k, r_k''))}{\operatorname{Vol}(B(p_k, r_k''/2))} < 3^n,$$

where r_k'' is of the form $r_k/2^m$. In what follows we consider the new sequence $\{p_k, r_k''\}_{k=1}^{\infty}$, which we rename to $\{p_k, r_k\}_{k=1}^{\infty}$. Hence $V(r_k) - V(r_k/2) \leqslant C'V(r_k/2)$. Therefore,

$$\begin{aligned}\int_{B(p_k,r_k)} (4|\phi'|^2 - 2\phi^2\ln\phi)\,\mathrm{d}V &\leqslant C(V(r_k) - V(r_k/2)) \\ &\leqslant CV(r_k/2) \\ &\leqslant C\int_{B_k} \phi^2\,\mathrm{d}V.\end{aligned}$$

Plugging this into the previous estimate for $\mathcal{W}$ and using the constraint

$$(4\pi\tau_k)^{-n}\int_M u_k^2\,\mathrm{d}V_{t_k} = 1,$$

we get

$$\mathcal{W}(u_k) \leqslant C'' - 2C_k. \tag{2.20}$$

Since $C_k \to +\infty$ and $\mu(g(t_k), r_k^2) \leqslant \mathcal{W}(g(t_k), u_k, r_k^2)$, we conclude that $\mu(g(t_k), r_k^2) \to -\infty$. By condition (a) we have $A \leqslant \mu(g(t_k), r_k^2) \to -\infty$, which is impossible.

The previous argument implies that $\operatorname{Vol}_{\tilde{g}(s)} B_{\tilde{g}(s)}(x, \sqrt{1-2s}) \geqslant \kappa(1-2s)^n$, since $R(\tilde{g}(s)) \leqslant 1/(1-2s)$, which by rescaling implies $\operatorname{Vol} B(x, 1) \geqslant \kappa$ at metric $g(t)$, where κ is a constant depending only on the initial metric $g(0)$. □

Claim 2.7. *There is a uniform constant C such that*

$$\begin{aligned} u(y,t) &\leqslant C\,\mathrm{dist}_t^2(x,y) + C, \\ R(y,t) &\leqslant C\,\mathrm{dist}_t^2(x,y) + C, \\ |\nabla u| &\leqslant C\,\mathrm{dist}_t(x,y) + C, \end{aligned}$$

where $u(x,t) = \min_{y\in M} u(y,t)$.

Proof. By Lemma 2.4 we can assume that $u \geqslant \delta > 0$, since otherwise we can consider $u+2B+\delta$ instead of u. From (2.9) it follows that $\sqrt{u+2B}$ is uniformly Lipschitz bounded since $|\nabla(\sqrt{u+2B})| \leqslant C = C(\delta)$ and, therefore,

$$\begin{aligned} |\sqrt{u}(y,t) - \sqrt{u}(z,t)| &\leqslant \frac{|\nabla u|(p,t)}{2\sqrt{u}}\,\mathrm{dist}_t(y,z) \\ &\leqslant \bar{C}\,\mathrm{dist}_t(y,z), \end{aligned}$$

and, therefore,

$$\begin{aligned} u(y,t) &\leqslant (\tilde{C}\,\mathrm{dist}_t(y,z) + \sqrt{u}(x,t))^2 \\ &\leqslant C_1\,\mathrm{dist}_t^2(x,y) + C_1 u(x,t). \end{aligned}$$

Assume that $u(x,t) \geqslant K(t)$. Then $u(y,t) \geqslant K(t)$ for all $y \in M$ and we would have

$$(2\pi)^n = \int_M \mathrm{e}^{-u}\,\mathrm{d}V_t \leqslant \mathrm{e}^{-K(t)}\,\mathrm{Vol}(M) \to 0$$

if $K(t) \to \infty$, which is not possible. Therefore, $u(x,t) \leqslant K$, for a constant that does not depend on t, and finally

$$u(y,t) \leqslant C\,\mathrm{dist}_t^2(y,x) + \tilde{C} \tag{2.21}$$

for some uniform constants C and $\tilde{C}$. The other two estimates in the claim follow from (2.21) and Proposition 2.5. □

By Claim 2.7 it follows that if we manage to estimate the diameter, we will get uniform bounds on the scalar curvature and the C^1-norm of u.

3. A uniform upper bound on diameters

In this section we want to prove the following proposition which will finish the proof of Theorem 1.1.

Proposition 3.1. *There is a uniform constant C such that* $\mathrm{diam}(M, g(t)) \leqslant C$.

We argue by contradiction. Assume that the diameters are unbounded in time. Denote by $d_t(z) = \mathrm{dist}_t(x,z)$ where $u(x,t) = \min_{y\in M} u(y,t)$.

Let $B(k_1,k_2)=\{z:2^{k_1}\leqslant d_t(z)\leqslant 2^{k_2}\}$. Consider an annulus $B(k,k+1)$. By Claim 2.7 we have that $R\leqslant C2^{2k}$ on $B(k,k+1)$. The ball $B(k,k+1)$ contains 2^{2k-1} balls of radii $1/2^k$. By Claim 2.7 and Proposition 2.6 we have that at time t

$$\operatorname{Vol}(B(k,k+1))\geqslant\sum_i\operatorname{Vol}(B(x_i,2^{-k}))\geqslant 2^{2k-1}2^{-2kn}C, \tag{3.1}$$

where $\{x_i\}$ are the centres of 2^{2k-1} balls contained in $B(k,k+1)$.

Claim 3.2. *For every $\epsilon>0$ we can find $B(k_1,k_2)$ with $k_1<k_2$, such that if $\operatorname{diam}(M,g(t))$ is large enough, then*

(a) $\operatorname{Vol}(B(k_1,k_2))<\epsilon$ *and*

(b) $\operatorname{Vol}(B(k_1,k_2))\leqslant 2^{10n}\operatorname{Vol}(B(k_1+2,k_2-2))$.

Proof. Since $\operatorname{Vol}_t(M)$ is constant along the flow, it is uniformly bounded. If the diameter is sufficiently big, there is k_0 such that for all $k_2\geqslant k_1\geqslant k_0$, we have that $\operatorname{Vol}(B(k_1,k_2))<\epsilon$. If our estimate (b) did not hold, that is, if

$$\operatorname{Vol}(B(k_1,k_2))\geqslant 2^{10n}\operatorname{Vol}(B(k_1+2,k_2-2)),$$

we would consider $B(k_1+2,k_2-2)$ instead and ask whether (b) holds for that ball. Assume that for every p, at the pth step we are still not able to find our radii so that (a) and (b) are satisfied. In that case, at the pth step we would have

$$\operatorname{Vol}(B(k_1,k_2))\geqslant 2^{10np}\operatorname{Vol}(B(k_1+2p,k_2-2p)).$$

In particular, assume we have the above estimate at the pth step so that $k_1+2p+1\sim k_2-2p$, which is for

$$2p\sim\tfrac{1}{2}(k_2-k_2-1). \tag{$*$}$$

Take $k_1=k/2$ and $k_2=3k/2$ for $k\gg 1$. In that case $(*)$ becomes $p\sim k/4$, $k_1+2p\sim k$ and $k_2-2p\sim k+1$. Combining this with (3.1) yields

$$\epsilon>\operatorname{Vol}(B(k_1,k_2))\geqslant 2^{10nk/4}\operatorname{Vol}(B(k,k+1))\geqslant 2^{10nk/4}C2^{2k}2^{-2nk}.$$

This leads to a contradiction if we let $k\to\infty$. This finishes the proof of our claim. □

For every t for which the diameter of $(M,g(t))$ becomes very big, find k_1 and k_2 as in Claim 3.2. Then we have the following lemma.

Lemma 3.3. *There exist r_1, r_2 and a uniform constant C such that $2^{k_1}\leqslant r_1\leqslant 2^{k_1+1}$, $2^{k_2-1}\leqslant r_2\leqslant 2^{k_2}$ and*

$$\int_{B(r_1,r_2)}R\leqslant CV,$$

where $B(r_1,r_2)=\{z\in M:r_1\leqslant d_t(z)\leqslant r_2\}$ and $V=\operatorname{Vol}(B(k_1,k_2))$.

Proof. We will first prove the existence of r_1, such that $2^{k_1} \leqslant r_1 \leqslant 2^{k_1+1}$ and

$$\operatorname{Vol} S(r_1) \leqslant 2\frac{V}{2^{k_1}}, \tag{3.2}$$

where $S(r)$ is a metric sphere of radius r. We have that

$$\frac{\mathrm{d}}{\mathrm{d}r} \operatorname{Vol}(B(r)) = \operatorname{Vol} S(r). \tag{3.3}$$

Assume that for all $r \in [2^{k_1}, 2^{k_1+1}]$ we have $\operatorname{Vol}(S(r)) \geqslant 2(V/2^{k_1})$. Integrate (3.3) in r. Then

$$\begin{aligned}\operatorname{Vol}(B(k_1, k_1+1)) &= \int_{2^{k_1}}^{2^{k_1+1}} \operatorname{Vol}(S(r))\,\mathrm{d}r \\ &> 2\frac{V}{2^{k_1}} 2^{k_1} = 2V = 2\operatorname{Vol}(B(k_1, k_2)),\end{aligned}$$

which is not possible, since $k_2 \gg k_1$ by the proof of Claim 3.2. If for all $r \in [2^{k_2-1}, 2^{k_2}]$ we have that $\operatorname{Vol}(S(r)) \geqslant 2(V/2^{k_2})$, similarly as above we would get $\operatorname{Vol}(B(k_2-1, k_2)) > V = B(k_1, k_2)$, which is not possible. Therefore, there exists $r_2 \in [2^{k_2-1}, 2^{k_2}]$ such that

$$\operatorname{Vol} S(r_2) \leqslant 2\frac{V}{2^{k_2}}. \tag{3.4}$$

Estimates (3.2), (3.4) together with bounds on ∇u obtained in Claim 2.7 imply

$$\begin{aligned}\int_{B(r_1,r_2)} R &= \int_{B(r_1,r_2)} (R-n) + n\operatorname{Vol}(B(r_1,r_2)) \\ &= -\int_{B(r_1,r_2)} \Delta u + n\operatorname{Vol}(B(r_1,r_2)) \\ &\leqslant \int_{S(r_1)} |\nabla u| + \int_{S(r_2)} |\nabla u| \\ &\leqslant \frac{V}{2^{k_1}} C 2^{k_1+1} + \frac{V}{2^{k_2}} C 2^{k_2+1} \\ &= \tilde{C} V < \tilde{C}\epsilon.\end{aligned}$$

□

We can now finish the proof of Proposition 3.1.

Proof of Proposition 3.1. The proof of the proposition is similar to the proof of Perelman's non-collapsing theorem from [**7**]. Assume $\operatorname{diam}(M, g(t))$ is not uniformly bounded in t, that is, there exists a sequence $t_i \to \infty$ such that $\operatorname{diam}(M, g(t_i)) \to \infty$. Let $\epsilon_i \to 0$ be a sequence of positive numbers. By Claim 3.2 we can find sequences k_1^i and k_2^i such that

$$\operatorname{Vol}_{t_i} B_{t_i}(k_1^i, k_2^i) < \epsilon_i, \tag{3.5}$$

$$\operatorname{Vol}(B_{t_i}(k_1^i, k_2^i)) \leqslant 2^{10n} \operatorname{Vol}(B(k_1^i + 2, k_2^i - 2)). \tag{3.6}$$

For each i, find r_1^i and r_2^i as in Lemma 3.3. Let ϕ_i be a sequence of cut-off functions such that $\phi(z) = 1$ for $z \in [2^{k_1^i+2}, 2^{k_2^i-2}]$ and equal to zero for $z \in (-\infty, r_1^i] \cup [r_2^i, \infty)$. Let $u_i(x) = \mathrm{e}^{C_i}\phi_i(\mathrm{dist}_{t_i}(x, p_i))$ such that $(2\pi)^{-n}\int_M u_i^2 = 1$. This implies

$$\begin{aligned}(2\pi)^n &= \mathrm{e}^{2C_i}\int_M \phi_i^2\\ &\leqslant \mathrm{e}^{2C_i}\,\mathrm{Vol}_{t_i} B_{t_i}(k_1^i, k_2^i+1)\\ &\leqslant \mathrm{e}^{2C_i}\epsilon_i.\end{aligned}$$

Since $\epsilon_i \to 0$, this is possible only if $\lim_{i\to\infty} C_i = -\infty$. By Perelman's monotonicity formula,

$$\begin{aligned}A &\leqslant \mathcal{W}(g(t_i), u_i, \tfrac{1}{2})\\ &= (2\pi)^{-n}\mathrm{e}^{2C_i}\int_{B_{t_i}(r_1^i, r_2^i)} (4|\phi_i'(\mathrm{dist}_{t_i}(y))|^2 - 2\phi_i^2\ln\phi_i)\,\mathrm{d}V_{t_i}\\ &\qquad + (2\pi)^{-n}\int_{B_{t_i}(r_1^i, r_2^i)} Ru_i^2\,\mathrm{d}V_{t_i} - 2n - 2C_i. \qquad (3.7)\end{aligned}$$

First of all by Lemma 3.3 and (3.6) we have

$$\begin{aligned}\int_{B_{t_i}(r_1^i, r_2^i)} Ru_i^2 &\leqslant \mathrm{e}^{2C_i}\int_{B_{t_i}(r_1^i, r_2^i)} R\\ &\leqslant \tilde{C}\mathrm{e}^{2C_i}\,\mathrm{Vol}_{t_i} B_{t_i}(k_1^i, k_2^i)\\ &\leqslant \tilde{C}\mathrm{e}^{2C_i}2^{10n}\,\mathrm{Vol}_{t_i} B_{t_i}(k_1^i+2, k_2^i-2)\\ &\leqslant \tilde{C}2^{10n}\int_M u_i^2\,\mathrm{d}V_{t_i}\\ &= \tilde{C}2^{10n}(2\pi)^n.\end{aligned}$$

By (3.6) we also have

$$\begin{aligned}\mathrm{e}^{2C_i}\int_{B_{t_i}(r_1^i, r_2^i)} &(4|\phi_i'(\mathrm{dist}_{t_i}(y))|^2 - 2\phi_i^2\ln\phi_i)\,\mathrm{d}V_{t_i}\\ &\leqslant C\mathrm{e}^{2C_i}\,\mathrm{Vol}_{t_i} B_{t_i}(k_1^i, k_2^i)\\ &\leqslant \mathrm{e}^{2C_i}C2^{10n}\,\mathrm{Vol}_{t_i} B_{t_i}(k_1^i+2, k_2^i-2)\\ &\leqslant C2^{10n}\int_M u_i^2\\ &= C2^{10n}(2\pi)^n.\end{aligned}$$

By (3.7) we get

$$A \leqslant \bar{C} - 2C_i \to -\infty$$

as $i \to \infty$, and we get a contradiction. Therefore, there is a uniform bound on $(M, g(t))$, which gives us uniform bounds on scalar curvatures and $|u(y,t)|_{C^1}$. □

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