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STOCHASTIC COMPARISONS ON RESIDUAL LIFE AND INACTIVITY TIME OF SERIES AND PARALLEL SYSTEMS

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Inactivity time is a reliability conception dual to the residual life. In this article, we establish some stochastic comparisons on inactivity time and the residual life of series and parallel systems, respectively. Some applications are presented as well.

1. INTRODUCTION

Series and parallel systems are two familiar reliability structures: A series system functions if and only if each of its components functions, whereas a parallel system functions if and only if at least one of its components functions. In practical situations, one often meets two basic systems: a system composed of used units and a used system. Li and Zhang [5] have proved that the life of a parallel or series system composed of used i.i.d. elements is stochastically larger than that of a used parallel or series system; similar results were also derived for the inactivity time. Recently, Pellerey and Petakos [6] obtained a more general conclusion, which asserts that the life of a coherent system composed of used elements is stochastically larger than that of a used coherent system. In this article, we establish some stochastic comparison results on their inactivity time and residual life for parallel or series system, respectively. The results of Li and Zhang [5] are improved in the sense that the residual life of a parallel (series) system composed of i.i.d. used elements is larger (smaller) than that of a used parallel (series) system of i.i.d. elements in likelihood ratio order. The

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hazard rate order is proved to be valid when the concerned systems are of independent but not identical elements. Some related applications are presented as well.

Throughout this article, the terms *increasing* and *decreasing* mean nondecreasing and nonincreasing, respectively. All random variables under consideration are restricted to nonnegative cases.

For convenience, let us first recall some key definitions and well-known notions which will be used. $a/0$ is taken to be equal to ∞ whenever $a \geq 0$.

Let X and Y be two random variables with absolutely continuous cumulative distribution functions $F(x)$ and $G(x)$, respectively, and probability density functions $f(x)$ and $g(x)$, respectively. Denote their survival functions by $\bar{F}(x) = 1 - F(x)$ and $\bar{G} = 1 - G$, respectively.

Definition 1:

1. *X is said to be smaller than Y in stochastic dominance order (denoted by $X \leq_{\text{st}} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all x.*
2. *X is said to be smaller than Y in hazard rate order (denoted by $X \leq_{\text{hr}} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x.*
3. *X is said to be smaller than Y in reversed hazard rate order (denoted by $X \leq_{\text{rh}} Y$) if $G(x)/F(x)$ is increasing in x.*
4. *X is said to be smaller than Y in the likelihood ratio order (denoted by $X \leq_{\text{lr}} Y$) if $g(x)/f(x)$ is increasing in x.*

For ease of reference, relations among these orderings are presented as follows (see, e.g., Shaked and Shanthikumar [8]):

$$\begin{array}{ccc} X \leq_{lr} Y & \Rightarrow & X \leq_{\text{hr}} Y \\ \Downarrow & & \Downarrow \\ X \leq_{\text{rh}} Y & \Rightarrow & X \leq_{\text{st}} Y. \end{array}$$

2. MAIN RESULTS

Assume X and Y, two component lifetimes, to be mutually independent random variables; the lifetime of a parallel system composed of X and Y can be expressed as $\max\{X, Y\}$ and the lifetime of a series system composed of X and Y can be expressed as $\min\{X, Y\}$. The residual life (Ross [7]) and the inactivity time (IT) (Chandra and Roy [3], Block, Savits, and Singh [2]) of the used components with age $t \geq 0$ are respectively defined as

$$X_t = (X - t \mid X > t), \qquad X_{(t)} = (t - X \mid X \leq t); \tag{1}$$

their survival functions can be represented as

$$P(X_t > x) = \bar{F}(x+t)/\bar{F}(t), \qquad P(X_{(t)} > x) = F(t-x)/F(t), \tag{2}$$

where $\bar{F}$ and $\bar{G}$ are survival functions of X and Y, respectively.

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Assume that X and Y are mutually independent; it is obvious that

$$\max\{X_t, Y_t\}, \qquad \min\{X_t, Y_t\}$$

give respective random lives of the parallel and series systems of used components and their residual lives are

$$(\max\{X, Y\})_t, \qquad (\min\{X, Y\})_t.$$

The expressions

$$\max\{X_{(t)}, Y_{(t)}\}, \qquad \min\{X_{(t)}, Y_{(t)}\}$$

present the maximum and the minimum of two inactivity times, respectively, and their respective inactivity times are

$$(\max\{X, Y\})_{(t)}, \qquad (\min\{X, Y\})_{(t)}.$$

It is obvious that

$$P((\min\{X, Y\})_t > x) = \frac{\bar{F}(x+t)\bar{G}(x+t)}{\bar{F}(t)\bar{G}(t)},$$
$$P(\min\{X_t, Y_t\} > x) = \frac{\bar{F}(x+t)\bar{G}(x+t)}{\bar{F}(t)\bar{G}(t)},$$

so

$$(\min\{X, Y\})_t \stackrel{\text{st}}{=} \min\{X_t, Y_t\}, \qquad t \geq 0. \tag{3}$$

Now, we present our main results.

THEOREM 1: *Assume that X and Y are i.i.d., then, for all $t \geq 0$,*

$$(\max\{X, Y\})_t \leq_{\text{lr}} \max\{X_t, Y_t\}. \tag{4}$$

PROOF: Since

$$P((\max\{X, Y\})_t \leq x) = \frac{F^2(x+t) - F^2(t)}{1 - F^2(t)}, \qquad x \geq 0,$$
$$P(\max\{X_t, Y_t\} \leq x) = \left(\frac{F(x+t) - F(t)}{1 - F(t)}\right)^2, \qquad x \geq 0,$$

their probability density functions are respectively

$$\frac{d}{dx}\left(\frac{F^2(x+t) - F^2(t)}{1 - F^2(t)}\right) = \frac{1}{1 - F^2(t)}\left(\frac{d}{dx} F^2(x+t)\right)$$
$$= \frac{2F(x+t)f(x+t)}{1 - F^2(t)}$$

and

$$\frac{d}{dx}\left(\frac{F(x+t)-F(t)}{1-F(t)}\right)^2 = \frac{1}{(1-F(t))^2}\frac{d}{dx}(F(x+t)-F(t))^2$$
$$= \frac{2(F(x+t)-F(t))f(x+t)}{(1-F(t))^2}.$$

Note that

$$\frac{2(F(x+t)-F(t))f(x+t)}{(1-F(t))^2}\left[\frac{2F(x+t)f(x+t)}{1-F^2(t)}\right]^{-1}$$
$$= \frac{F(x+t)-F(t)}{(1-F(t))^2}\frac{1-F^2(t)}{F(x+t)}$$
$$= \frac{1+F(t)}{1-F(t)}\left(1-\frac{F(t)}{F(x+t)}\right)$$

is increasing in x. Thus, the desired relation in (4) is valid. ■

THEOREM 2: *Assume that X and Y are i.i.d.; then, for all $t \geq 0$,*

$$(\max\{X,Y\})_{(t)} \leq_{\text{lr}} \max\{X_{(t)},Y_{(t)}\}, \qquad (\min\{X,Y\})_{(t)} \geq_{\text{lr}} \min\{X_{(t)},Y_{(t)}\}. \tag{5}$$

PROOF: Note that

$$P((\max\{X,Y\})_{(t)} \leq x) = 1-\frac{F^2(t-x)}{F^2(t)}, \qquad t \geq x \geq 0,$$
$$P(\max\{X_{(t)},Y_{(t)}\} \leq x) = \left(1-\frac{F(t-x)}{F(t)}\right)^2, \qquad t \geq x \geq 0,$$

and their probability density functions are respectively

$$\frac{d}{dx}\left(1-\frac{F^2(t-x)}{F^2(t)}\right) = \frac{2F(t-x)f(t-x)}{F^2(t)},$$
$$\frac{d}{dx}\left(1-\frac{F(t-x)}{F(t)}\right)^2 = \frac{2(F(t)-F(t-x))f(t-x)}{F^2(t)}.$$

Because

$$\frac{2(F(t)-F(t-x))f(t-x)}{F^2(t)}\left[\frac{2F(t-x)f(t-x)}{F^2(t)}\right]^{-1} = \frac{F(t)}{F(t-x)}-1$$

is obviously increasing in x, this yields the first relation in (5).

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Now, we turn to the second one:

$$P((\min\{X,Y\})_{(t)} \le x) = 1 - \frac{1-\bar{F}^2(t-x)}{1-\bar{F}^2(t)}, \qquad t \ge x \ge 0,$$

$$P(\min\{X_{(t)},Y_{(t)}\} \le x) = 1 - \frac{F^2(t-x)}{F^2(t)}, \qquad t \ge x \ge 0.$$

Their respective probability density functions are

$$\frac{d}{dx}\left(1 - \frac{1-\bar{F}^2(t-x)}{1-\bar{F}^2(t)}\right) = \frac{2\bar{F}(t-x)f(t-x)}{1-\bar{F}^2(t)},$$

$$\frac{d}{dx}\left(1 - \frac{F^2(t-x)}{F^2(t)}\right) = \frac{2F(t-x)f(t-x)}{F^2(t)}.$$

Since

$$\begin{aligned}
&\frac{2\bar{F}(t-x)f(t-x)}{1-\bar{F}^2(t)}\left[\frac{2F(t-x)f(t-x)}{F^2(t)}\right]^{-1} \\
&= \frac{F^2(t)}{1-\bar{F}^2(t)}\,\frac{\bar{F}(t-x)}{F(t-x)} \\
&= \frac{F^2(t)}{1-\bar{F}^2(t)}\left(\frac{1}{F(t-x)} - 1\right)
\end{aligned}$$

is increasing in x, this yields the second relation in (5). ■

In the following, we will consider the case that X and Y are assumed to be only independent.

THEOREM 3: *Assume that X and Y are independent (not necessarily identical); then, for all $t \ge 0$,*

$$(\max\{X,Y\})_t \le_{\text{hr}} \max\{X_t, Y_t\}. \qquad \textbf{(6)}$$

PROOF: Since

$$P((\max\{X,Y\})_t > x) = \frac{1-F(x+t)G(x+t)}{1-F(t)G(t)}, \qquad x \ge 0,$$

$$P(\max\{X_t,Y_t\} > x) = 1 - \left(1 - \frac{\bar{F}(x+t)}{\bar{F}(t)}\right)\left(1 - \frac{\bar{G}(x+t)}{\bar{G}(t)}\right), \qquad x \ge 0,$$

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it can be verified that

$$\left[1-\left(1-\frac{\bar F(x+t)}{\bar F(t)}\right)\left(1-\frac{\bar G(x+t)}{\bar G(t)}\right)\right]\left[\frac{1-F(x+t)G(x+t)}{1-F(t)G(t)}\right]^{-1}$$
$$=\frac{1-F(t)G(t)}{\bar F(t)\bar G(t)}\,\frac{\bar F(t)\bar G(x+t)+\bar G(t)\bar F(x+t)-\bar F(x+t)\bar G(x+t)}{1-F(x+t)G(x+t)}$$
$$=\frac{1-F(t)G(t)}{\bar F(t)\bar G(t)}\left(1-\frac{F(t)\bar G(x+t)+G(t)\bar F(x+t)}{1-F(x+t)G(x+t)}\right)$$

is increasing in $x \geq 0$, and so we obtain the inequality in (6). ■

THEOREM 4: *Assume that X and Y are independent (not necessarily identical); then, for all* $t \geq 0$,

$$(\max\{X,Y\})_{(t)} \leq_{\mathrm{hr}} \max\{X_{(t)},Y_{(t)}\}, \qquad (\min\{X,Y\})_{(t)} \geq_{\mathrm{hr}} \min\{X_{(t)},Y_{(t)}\}. \tag{7}$$

PROOF: The survival functions of the concerned random variables are respectively

$$P((\max\{X,Y\})_{(t)} > x) = \frac{F(t-x)G(t-x)}{F(t)G(t)}, \qquad t \geq x \geq 0,$$

$$P(\max\{X_{(t)},Y_{(t)}\} > x) = \frac{F(t-x)}{F(t)} + \frac{G(t-x)}{G(t)} - \frac{F(t-x)G(t-x)}{F(t)G(t)}, \qquad t \geq x \geq 0.$$

Note that

$$\left(\frac{F(t-x)}{F(t)}+\frac{G(t-x)}{G(t)}-\frac{F(t-x)G(t-x)}{F(t)G(t)}\right)\left[\frac{F(t-x)G(t-x)}{F(t)G(t)}\right]^{-1}$$
$$=\frac{G(t)F(t-x)+G(t-x)F(t)-F(t-x)G(t-x)}{F(t-x)G(t-x)}$$
$$=\frac{G(t)}{G(t-x)}+\frac{F(t)}{F(t-x)}-1$$

is obviously increasing in x; thus, the first inequality in (7) is obtained.

Since

$$P((\min\{X,Y\})_{(t)} > x) = \frac{1-\bar F(t-x)\bar G(t-x)}{1-\bar F(t)\bar G(t)},$$

$$P(\min\{X_{(t)},Y_{(t)}\} > x) = \frac{F(t-x)G(t-x)}{F(t)G(t)},$$

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and

$$\begin{aligned}
&\frac{1-\bar F(t-x)\bar G(t-x)}{1-\bar F(t)\bar G(t)}\left[\frac{F(t-x)G(t-x)}{F(t)G(t)}\right]^{-1}\\
&\quad=\frac{1-\bar F(t-x)\bar G(t-x)}{1-\bar F(t)\bar G(t)}\;\frac{F(t)G(t)}{F(t-x)G(t-x)}\\
&\quad=\frac{F(t)G(t)}{1-\bar F(t)\bar G(t)}\;\frac{1-(1-F(t-x))(1-G(t-x))}{F(t-x)G(t-x)}\\
&\quad=\frac{F(t)G(t)}{1-\bar F(t)\bar G(t)}\left(\frac{1}{F(t-x)}+\frac{1}{G(t-x)}-1\right),
\end{aligned}$$

it is increasing in x. The second relation in (7) is reduced also. ■

Remark: In fact, Theorems 1–4 are available for parallel and series systems composed of n components.

3. SOME APPLICATIONS

In this section, we present some interesting application results of our main theorems in Section 2. They are closely related to some aging conceptions and arranged into three separate parts. For convenience, we first present these aging conceptions as follows.

DEFINITION 2:

1. *An absolutely continuous random variable X has a* Polya frequency of order 2 *(PF2) if it has a log concave density function.*
2. *A random life X with distribution function F is said to be of* increasing failure rate *(IFR) (*decreasing failure rate *(DFR)) if its hazard rate function* $\lambda(x) = f(x)/\bar F(x)$ *is increasing (decreasing) on its interval of support.*
3. *A random life X is said to be of* decreasing reversed hazard rate *(DRHR) if its reversed hazard rate* $\bar\lambda(x) = f(x)/F(x)$ *is decreasing in time* $t \geq 0$.

For more details on PF2 and IFR, refer to Barlow and Proschan [1]; for details on DRHR, see Chandra and Roy [3]. It should be pointed out that

$$X \text{ is PF2} \Rightarrow X \text{ is IFR} \Rightarrow X \text{ is DRHR}.$$

Application 1: Theorem 1.C.22 of Shaked and Shanthikumar [8] stresses that, for a nonnegative random variable X,

$$X \text{ is PF2} \quad \text{if and only if } X \geq_{\text{lr}} X_t, \quad \text{for all } t \geq 0. \tag{8}$$

Suppose X and Y are mutually independent and identical PF2 random lives; then,

$$X \geq_{\text{lr}} X_t, \qquad Y \geq_{\text{lr}} Y_t, \qquad t \geq 0.$$

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By Theorem 1.C.9 of Shaked and Shanthikumar [8], we have, for all $t \geq 0$,

$$\min\{X, Y\} \geq_{\text{lr}} \min\{X_t, Y_t\}$$

and

$$\max\{X, Y\} \geq_{\text{lr}} \max\{X_t, Y_t\}.$$

In combination with (3) and Theorem 2, it holds that, for all $t \geq 0$,

$$\min\{X, Y\} \geq_{\text{lr}} (\min\{X, Y\})_t$$

and

$$\max\{X, Y\} \geq_{\text{lr}} (\max\{X, Y\})_t.$$

By (8) again, PF2 is preserved under both the formation of parallel systems and that of series systems with i.i.d. components.

Application 2: One characterization of IFR (DFR) [8, Thm. 1.B.19] is given as follows:

$$X \text{ is IFR (DFR)} \quad \text{if and only if } X \geq_{\text{hr}} (\leq_{\text{hr}})\, X_t, \qquad \text{for all } t \geq 0. \qquad \textbf{(9)}$$

Suppose X and Y are mutually independent IFR (DFR) random lives; then,

$$X \geq_{\text{hr}} (\leq_{\text{hr}})\, X_t, \qquad Y \geq_{\text{hr}} (\leq_{\text{hr}})\, Y_t, \qquad t \geq 0.$$

By Theorem 1.B.3 of Shaked and Shanthikumar [8], we have

$$(\min\{X, Y\}) \geq_{\text{hr}} (\leq_{\text{hr}}) \min\{X_t, Y_t\}, \qquad t \geq 0.$$

In combination with (3), it holds that

$$\min\{X, Y\} \geq_{\text{hr}} (\leq_{\text{hr}})\, (\min\{X, Y\})_t, \qquad t \geq 0.$$

By (9) again, IFR and DFR are both preserved under the formation of series systems of independent components which are not necessarily identical.

Theorem 1.B.4 of Shaked and Shanthikumar [8] asserts the following: If $(X_i, Y_i)(i = 1, \ldots, n)$ are mutually independent and $X_i(Y_i)$ $(i = 1, \ldots, n)$ are identical, then $X_i \geq_{\text{hr}} Y_j$ for every pair of (i, j) implies that $X_{(r)} \geq_{\text{hr}} Y_{(r)}$ for all $r = 1, \ldots, n$. In a completely similar manner, it can be shown, through Theorem 3, that IFR and DFR are both preserved under the formation of parallel systems of i.i.d. components.

In fact, Grosh [4] has obtained an analytical proof for the preservation property of IFR under the formation of parallel systems of i.i.d. components. It should be pointed out here that the preservation of IFR under parallel systems is not satisfied when the components are not identically distributed. This is shown in the well-known counterexample [1].

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Application 3: It can be shown that X is DRHR if and only if its inactivity time $X_{(t)}$ is increasing in $t \geq 0$. Shaked and Shanthikumar [8] presented one of their characterizations in Theorem 1.B.32(ii) as follows:

$$X \text{ is DRHR} \quad \text{if and only if } X \geq_{\mathrm{rh}} X_t, \qquad \text{for all } t \geq 0. \tag{10}$$

Suppose X and Y are i.i.d. DRHR random lives; then,

$$X \geq_{\mathrm{rh}} X_t, \qquad Y \geq_{\mathrm{rh}} Y_t, \qquad t \geq 0.$$

By Theorem 1.B.23 of Shaked and Shanthikumar [8], we have

$$\max\{X, Y\} \geq_{\mathrm{rh}} \max\{X_t, Y_t\}, \qquad t \geq 0.$$

Note that the likelihood ratio order implies the reversed hazard rate order; in combination with Theorem 1, it holds that

$$\max\{X, Y\} \geq_{\mathrm{rh}} (\max\{X, Y\})_t, \qquad t \geq 0.$$

By (10) again, DRHR is preserved under the formation of parallel systems of i.i.d. components.

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