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CONTINUITY OF ATTRACTORS AND INVARIANT MEASURES FOR ITERATED FUNCTION SYSTEMS

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ABSTRACT. We prove the "folklore" results that both the attractor A and invariant measure μ of an N-map Iterated Function System (IFS) vary continuously with variations in the contractive IFS maps as well as the probabilities. This represents a generalization of Barnsley's result showing the continuity of *attractors* with respect to variations of a parameter appearing in the IFS maps. Some applications are presented, including approximations of attractors and invariant measures of nonlinear IFS, as well as some novel approximations of Julia sets for quadratic complex maps.

1. **Introduction.** In this paper we derive some results regarding the continuity of attractors and invariant measures for N-map Iterated Function Systems (IFS) with respect to changes in the contractive maps as well as the associated probabilities. Barnsley [1, Section 3.11] has already shown that the attractor A of a contractive IFS varies continuously with respect to parameters in the IFS maps. There are obvious applications when one considers both *animation*, where images (IFS attractors) are deformed and possibly translated in time in an apparently continuous manner and *rendering*, where the shading of an image is manipulated. The results reported here represent a generalization of Barnsley's results. A primary motivation is in the inverse problem, where a local "fine tuning" of the IFS maps and probabilities is performed to optimize the approximation of sets or measures with IFS attractors or invariant measures.

The layout of the paper is as follows. Section 2 provides a glossary of the notation employed in this paper, followed by a very brief review of "traditional" IFS. In Section 3, a basic continuity property of fixed points of contractive maps is first shown. The continuity results for IFS attractors and invariant measures then follow. In Section 4 are presented some applications, including the approximation of attractors and measures of nonlinear IFS, as well as the approximation of Julia sets of quadratic maps with attractors of IFS composed of nonanalytic maps.

2. **Glossary of notation and brief review of** IFS. In this paper, the following notation will be employed:

(X,d) a compact metric space. (In applications, where X is the "base" space of the IFS, X is typically a compact subset of $\mathbf{R}^n$, *e.g.* $[0,1]$, $[0,1]^2$. In this case, the norm on X will be denoted as $\|x\|$, *i.e.* $\|x-y\| = d(x,y)$.)

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D diam$(X) = \sup_{x,y\in X} d(x,y)$.

$C(X)$ $= \{f: X \longrightarrow \mathbf{R}\}$, the space of continuous real-valued functions on X.

Lip(X) $= \{f: X \longrightarrow \mathbf{R} \mid |f(x) - f(y)| \le d(x,y), x, y \in X\}$.

Con(X) $= \{w: X \longrightarrow X \mid d(w(x), w(y)) \le s d(x,y), 0 \le s < 1, \forall x, y \in X\}$: the set of contraction maps on X. We shall refer to s as the *contractivity factor* of w.

$[\mathrm{Con}(X)]^N$ $= \{\mathbf{w} = (w_1, w_2, \ldots, w_N) \mid w_i \in \mathrm{Con}(X), i = 1, 2, \ldots N\}$, the set of N-map contractive IFS on X.

$\mathbf{H}^N$ $= \{\mathbf{p} = (p_1, p_2, \ldots, p_N) \mid p_i \ge 0, \sum_{i=1}^N p_i = 1\}$, the set of probability vectors for the N-map IFS on X.

$\mathcal{S}_{\mathrm{IFS}}^N(X)$ $= [\mathrm{Con}(X)]^N \times \mathbf{H}^N$, the topological space of N-map contractive IFS on X with associated probabilities.

$\mathcal{H}(X)$ the set of non-empty compact subsets of X.

h Hausdorff metric on $\mathcal{H}(X)$: Let the distance between a point $x \in X$ and a set $A \in \mathcal{H}(X)$ be given by

$$d(x, A) = \inf_{y\in A} d(x,y).$$

Then for $A, B \in \mathcal{H}(X)$,

$$h(A,B) = \max\Big[\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\Big].$$

$(\mathcal{H}(X), h)$ is a complete metric space.

$\mathcal{M}(X)$ the set of Borel probability measures on $\mathcal{A}(X)$, *i.e.* $\mu(X) = 1$, where $\mathcal{A}(X)$ denotes the σ-algebra of Borel subsets of X.

d_H a metric on $\mathcal{M}(X)$, often called the *Hutchinson metric* due to its use in [2]:

$$d_H(\mu,\nu) = \sup_{f\in \mathrm{Lip}(X)} \Big[\int f\, d\mu - \int f\, d\nu\Big] \quad \mu, \nu \in \mathcal{M}(X). \tag{2.1}$$

$(\mathcal{M}(X), d_H)$ is a complete metric space. Now note the following:

(i) $f \in \mathrm{Lip}(X)$ implies that $g = f + c \in \mathrm{Lip}(X)$ for $c \in \mathbf{R}$ constant, and

(ii) $\int g\, d\mu - \int g\, d\nu = \int f\, d\mu - \int f\, d\nu$. Thus, the supremum in the definition of d_H may be restricted to the subspace $f \in \mathrm{Lip}_0(X)$, where $\mathrm{Lip}_0(X) = \{f \in \mathrm{Lip}(X), f(0) = 0\}$.

Let $(\mathbf{w}, \mathbf{p})$ denote an N-map contractive IFS on X with probabilities, that is, a set of N contraction maps, $\mathbf{w} = (w_1, w_2, \ldots, w_N)$, $w_i \in \mathrm{Con}(X)$, with associated probabilities $\mathbf{p} = (p_1, p_2, \ldots, p_N)$, $\sum_{i=1}^N p_i = 1$, and contractivity factors s_i. The contractivity factor of the IFS is defined as

$$s = \max_{1\le i\le N} s_i. \tag{2.2}$$

There are two fundamental results [2,3]:

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UNIQUE ATTRACTOR. Define a set-valued mapping $\hat{\mathbf{w}}: \mathcal{H}(X) \longrightarrow \mathcal{H}(X)$ as follows. For a subset $S \in \mathcal{H}(X)$ denote $w_i(S) = \{w_i(x), x \in S\}$, $i = 1, 2, \ldots, N$ and let the action of $\hat{\mathbf{w}}$ on S be given by

$$\hat{\mathbf{w}}(S) \equiv \bigcup_{i=1}^{N} w_i(S). \tag{2.3}$$

Then there exists a unique compact set $A \in \mathcal{H}(X)$, the *attractor* of $\mathbf{w}$ (independent of $\mathbf{p}$), such that

$$A = \hat{\mathbf{w}}(A) = \bigcup_{i=1}^{N} w_i(A) \tag{2.4}$$

and $h\big(\hat{\mathbf{w}}^n(S), A\big) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $S \in \mathcal{H}(X)$. This result follows from the fact that $\hat{\mathbf{w}}$ is a contraction mapping on $\big(\mathcal{H}(X), h\big)$, *i.e.* for $A, B \in \mathcal{H}(X)$,

$$h\big(\hat{\mathbf{w}}(A), \hat{\mathbf{w}}(B)\big) \leq sh(A, B). \tag{2.5}$$

UNIQUE INVARIANT MEASURE. Define a "Markov operator" on the probability measure space, $M: \mathcal{M}(X) \longrightarrow \mathcal{M}(X)$ as follows: For $\nu \in \mathcal{M}(X)$, let

$$M\nu = \sum_{i=1}^{N} p_i \nu \circ w_i^{-1}. \tag{2.6}$$

Then there exists a unique measure $\mu \in \mathcal{M}(X)$, the *invariant measure* of the IFS, for which $M\mu = \mu$. M is a contraction mapping on $\big(\mathcal{M}(X), d_H\big)$, *i.e.* for $\mu, \nu \in \mathcal{M}(X)$,

$$d_H(M\mu, M\nu) \leq s d_H(\mu, \nu). \tag{2.7}$$

In addition, $\operatorname{supp}(\mu) \subseteq A$.

3. **Continuity of** IFS **attractors and invariant measures.** In both cases, we make use of a simple yet important consequence of the Banach Contraction Mapping Principle which establishes the continuity of fixed points with respect to contraction maps on a complete metric space. Let (Y, d_Y) be a complete metric space and $\operatorname{Con}(Y)$ the set of contraction maps $f: Y \longrightarrow Y$. Consider the following natural metric on this function space:

$$d_{\operatorname{Con}(Y)}(f_1, f_2) = \sup_{y \in Y} d\big(f_1(y), f_2(y)\big), \quad f_1, f_2 \in \operatorname{Con}(Y). \tag{3.1}$$

Note that the metric space $\big(\operatorname{Con}(Y), d_{\operatorname{Con}(Y)}\big)$ itself is not necessarily complete.

THEOREM 3.1. *Define a mapping $F: \operatorname{Con}(Y) \longrightarrow Y$ as follows: For $f \in \operatorname{Con}(Y)$, $F: f \longrightarrow \bar{x}$, where $\bar{x} \in X$ denotes the unique fixed point of f, i.e. $f(\bar{x}) = \bar{x}$. Then F is continuous at each f in* $\operatorname{Con}(Y)$.

PROOF. Let $f \in \operatorname{Con}(Y)$, with contractivity factor s and fixed point $\bar{x}$. For a given $\epsilon > 0$, let $f_\epsilon \in \operatorname{Con}(Y)$, with fixed point $\bar{x}_\epsilon$, such that $d_{\operatorname{Con}(Y)}(f, f_\epsilon) < \epsilon(1-s)$. Then

$$\begin{aligned} d(\bar{x}, \bar{x}_\epsilon) &= d\big(f(\bar{x}), f_\epsilon(\bar{x}_\epsilon)\big) \\ &\leq d\big(f(\bar{x}), f(\bar{x}_\epsilon)\big) + d\big(f(\bar{x}_\epsilon), f_\epsilon(\bar{x}_\epsilon)\big) \\ &< s d(\bar{x}, \bar{x}_\epsilon) + \epsilon(1-s). \end{aligned}$$

A rearrangement gives the desired result,

$$d(\bar{x}, \bar{x}_\epsilon) < \epsilon.$$

We now proceed to apply this result to N-map IFS over a base space (X, d). It is convenient to consider the IFS as an element of a topological space $\mathcal{S}^N_{\text{IFS}}(X) = [\text{Con}(X)]^N \times \mathbf{H}^N$. (See the glossary at beginning of this section. Note, however, that the usual restriction that all probabilities be nonzero has been relaxed so that IFS with different numbers of maps may be compared.) Thus, $\mathcal{S}^N_{\text{IFS}}(X) \subset \mathcal{S}^{N+1}_{\text{IFS}}(X)$ for $N = 1, 2, \ldots$. In cases where more than one IFS are considered at a time, it is convenient to denote them as $(\mathbf{w}_k, \mathbf{p}_k) \in \mathcal{S}^N_{\text{IFS}}(X)$, $k = 1, 2, \ldots$, where each map and probability N-vector is given by

$$\text{(3.2)} \qquad \mathbf{w}_k = (w_{k1}, w_{k2}, \ldots, w_{kN}), \quad \mathbf{p}_k = (p_{k1}, p_{k2}, \ldots, p_{kN}).$$

Let s_{kj} denote the contractivity factors of the w_{kj}; the contractivity factor of the IFS $(\mathbf{w}_k, \mathbf{p}_k)$ is then $s_k = \max_{1\le j\le N} s_{kj}$. Now define the following natural metric on $[\text{Con}(X)]^N$:

$$\text{(3.3)} \qquad d^N_w(\mathbf{w}_1, \mathbf{w}_2) \equiv \max_{1\le i\le N} d_{\text{Con}(X)}(w_{1i}, w_{2i}).$$

(Note that this metric does not take into consideration the permutational symmetry of the IFS map vector: the IFS vectors $\mathbf{w}$ and $\pi\mathbf{w}$, where $\pi \in \mathcal{S}_N$, the symmetric group on n elements, obviously possess the same attractor A even though $d^N_w(\mathbf{w}, \pi\mathbf{w})$ is not necessarily zero. This is not a serious issue (it can be rectified with a little extra work) since we are concerned only with the continuity of A with respect to variations in the w_i and not the structure of the mappings from $\mathbf{w}$ to X or from $(\mathbf{w}, \mathbf{p})$ to $\mathcal{M}(X)$, respectively. Thus, the d^N_w metric will suffice.)

COROLLARY 3.2. *The mapping* $F_{\text{att}}\colon [\text{Con}(X)]^N \longrightarrow \mathcal{H}(X)$ *defined by* $F_{\text{att}}(\mathbf{w}) = A$, *where* $\mathbf{w}(A) = A$, *is continuous at each* $\mathbf{w}$ *in* $[\text{Con}(X)]^N$.

This is a generalization of Barnsley's result [1, Section 3.11], where the variations in the IFS maps were essentially made with respect to a single parameter in the IFS. As will be discussed in Section 4, the above result allows us to construct approximations to A by using IFS composed of approximations to the contraction maps w_{kj}.

Now let $\text{Con}^N_M\big(\mathcal{M}(X)\big)$ denote the space of contractive N-map IFS Markov operators on $\mathcal{M}(X)$. The operators $M_k \in \text{Con}^N_M\big(\mathcal{M}(X)\big)$ associated with the indexed IFS $(\mathbf{w}_k, \mathbf{p}_k)$ are given by

$$\text{(3.4)} \qquad M_k\nu = \sum_{j=1}^{N} p_{kj}\nu \circ w_{kj}^{-1}, \quad \nu \in \mathcal{M}(X).$$

Now define the following metric on this space:

$$\text{(3.5)} \qquad d^N_M(M_1, M_2) \equiv \sup_{\nu\in\mathcal{M}(X)} d_H(M_1\nu, M_2\nu).$$

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COROLLARY 3.3. *The mapping* $F_{\text{meas}}\colon \text{Con}_M^N(\mathcal{M}(X)) \longrightarrow \mathcal{M}(X)$ *defined by* $F_{\text{meas}}(M) = \mu$, *where* $M\mu = \mu$, *is continuous at each* M *in* $\text{Con}_M^N(\mathcal{M}(X))$.

3.1 *Continuity of invariant measures with respect to* IFS *maps and probabilities.* From a practical perspective, the continuity properties expressed by the two corollaries above contain no information: it would be useful to express the distance $d_M^N(M_1, M_2)$ and ultimately the Hutchinson distance between respective invariant measures, $d_H(\mu_1, \mu_2)$, in terms of the IFS components $\mathbf{w}$ and $\mathbf{p}$. This will now be done in order to show that a continuous variation of the invariant meausure μ of an IFS is possible by means of continuous variations in the contractive map vector $\mathbf{w}$ as well as the probability vector $\mathbf{p}$. Given two N-map IFS $(\mathbf{w}_k, \mathbf{p}_k)$, $k = 1, 2$, we define the following natural distance functions in $[\text{Con}(X)]^N$ and $\mathbf{H}^N$:

$$d_w^N(\mathbf{w}_1, \mathbf{w}_2) = \max_{1\le i\le N} d_{\text{Con}(X)}(w_{1i}, w_{2i}), \tag{3.3}$$

$$d_p^N(\mathbf{p}_1, \mathbf{p}_2) = \max_{1\le i\le N} |p_{1i} - p_{2i}|. \tag{3.6}$$

THEOREM 3.4. *Let* $(\mathbf{w}_1, \mathbf{p}_1) \in \mathcal{S}_{\text{IFS}}^N(X)$, *with contractivity factor* s_1, *associated Markov operator* $M_1 \in \text{Con}_M^N(\mathcal{M}(X))$ *and invariant measure* μ_1. *Then for every* $\epsilon > 0$, *there exist* δ_1, $\delta_2 > 0$ *such that for all N-map* IFS$(\mathbf{w}_2, \mathbf{p}_2) \in \mathcal{S}_{\text{IFS}}^N(X)$ *satisfying*

$$d_w^N(\mathbf{w}_1, \mathbf{w}_2) < \delta_1, \quad d_p^N(\mathbf{p}_1, \mathbf{p}_2) < \delta_2, \tag{3.7}$$

it follows that $d_H(\mu_1, \mu_2) < \epsilon$ *where* μ_2 *is the invariant measure of the* IFS$(\mathbf{w}_2, \mathbf{p}_2)$.

PROOF. Given an $\epsilon > 0$, let $(\mathbf{w}_2, \mathbf{p}_2) \in \mathcal{S}_{\text{IFS}}^N(X)$, such that

$$d_w^N(\mathbf{w}_1, \mathbf{w}_2) < \delta_1, \quad d_p^N(\mathbf{p}_1, \mathbf{p}_2) < \delta_2, \tag{3.8}$$

where $\delta_1, \delta_2 > 0$ are such that

$$\delta_1 + ND\delta_2 < \epsilon(1 - s_1). \tag{3.9}$$

$(D = \text{diam}(X))$. The existence of such a $\mathbf{w}_2$ has been dicussed elsewhere [1]. Then, from

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equation (3.5),

(3.10)

$$
\begin{aligned}
d_M^N(M_1, M_2) &= \sup_{\mu\in\mathcal{M}} \sup_{f\in \mathrm{Lip}_0(X)} \Big[\int f\, dM_1(\mu) - \int f\, dM_2(\mu)\Big] \\
&= \sup_{\mu\in\mathcal{M}} \sup_{f\in \mathrm{Lip}_0(X)} \Big[\sum_{i=1}^N \int \big[p_{1i} f\big(w_{1i}(x)\big) - p_{2i} f\big(w_{2i}(x)\big)\big]\, d\mu\Big] \\
&= \sup_{\mu\in\mathcal{M}} \sup_{f\in \mathrm{Lip}_0(X)} \Big[\sum_{i=1}^N \int \big[p_{1i} f(w_{1i}(x)) - p_{2i} f\big(w_{1i}(x)\big) + p_{2i} f\big(w_{1i}(x)\big) - p_{2i} f\big(w_{2i}(x)\big)\big]\, d\mu\Big] \\
&= \sup_{\mu\in\mathcal{M}} \sup_{f\in \mathrm{Lip}_0(X)} \Big[\sum_{i=1}^N (p_{1i} - p_{2i}) \int f\big(w_{1i}(x)\big)\, d\mu + \sum_{i=1}^N p_{2i} \int \big[f\big(w_{1i}(x)\big) - f\big(w_{2i}(x)\big)\big]\, d\mu\Big] \\
&\leq \sup_{\mu\in\mathcal{M}} \sup_{f\in \mathrm{Lip}_0(X)} \sum_{i=1}^N (p_{1i} - p_{2i}) \int f\big(w_{1i}(x)\big)\, d\mu \\
&\quad + \sup_{\mu\in\mathcal{M}} \sup_{f\in \mathrm{Lip}_0(X)} \sum_{i=1}^N p_{2i} \int \big[f\big(w_{1i}(x)\big) - f\big(w_{2i}(x)\big)\big]\, d\mu.
\end{aligned}
$$

We now proceed to simplify the two expressions on the right side of this inequality. Starting with the latter term,

$$
\begin{aligned}
\sup_{f\in \mathrm{Lip}_0(X)} \sum_{i=1}^N p_{2i} \int \big[f\big(w_{1i}(x)\big) - f\big(w_{2i}(x)\big)\big]\, d\mu \qquad & \\
\leq \sup_{f\in \mathrm{Lip}_0(X)} \sum_{i=1}^N p_{2i} \int \big|f\big(w_{1i}(x)\big) - f\big(w_{2i}(x)\big)\big|\, d\mu & \\
\leq \sum_{i=1}^N p_{2i} \int \|w_{1i}(x) - w_{2i}(x)\|\, d\mu & \qquad (3.11)\\
\leq \sum_{i=1}^N p_{2i} \sup_{x\in X} \|w_{1i}(x) - w_{2i}(x)\| \int d\mu & \\
\leq \max_{1\leq j\leq N} \sup_{x\in X} \|w_{1j}(x) - w_{2j}(x)\| \sum_{i=1}^N p_{2i} & \\
= d_w^N(\mathbf{w}_1, \mathbf{w}_2). &
\end{aligned}
$$

Returning to the first term on the right side of (3.10),

(3.12)

$$
\begin{aligned}
\sup_{\mu\in\mathcal{M}} \sup_{f\in \mathrm{Lip}_0(X)} \sum_{i=1}^N (p_{1i} - p_{2i}) \int f\big(w_{1i}(x)\big)\, d\mu \qquad & \\
\leq \max_{1\leq j\leq N} |p_{1j} - p_{2j}| \sup_{\mu\in\mathcal{M}} \sup_{f\in \mathrm{Lip}_0(X)} \sum_{i=1}^N \int f\big(w_{1i}(x)\big)\, d\mu & \\
\leq \max_{1\leq j\leq N} |p_{1j} - p_{2j}| \sup_{\mu\in\mathcal{M}} \sup_{f\in \mathrm{Lip}_0(X)} \sum_{i=1}^N \int \big|f\big(w_{1i}(x)\big)\big|\, d\mu. &
\end{aligned}
$$

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Now, $f \in \mathrm{Lip}_0(X)$ implies that $|f(x) - f(y)| \le d(x,y)$ for $x, y \in X$. Choosing $y = 0$, we have $|f(x)| \le d(x,0)$, $x \in X$. Thus $\big|f\big(w_{1i}(x)\big)\big| \le \mathrm{diam}(X) = D$. Therefore,

$$\text{(3. 13)} \qquad \sup_{\mu \in \mathcal{M}} \sup_{f \in \mathrm{Lip}_0(X)} \sum_{i=1}^{N} (p_{1i} - p_{2i}) \int f\big(w_{1i}(x)\big)\, d\mu \le N D d_p^N(\mathbf{p}_1, \mathbf{p}_2).$$

Substitution of (3.12) and (3.13) into (3.10) yields

$$\text{(3. 14)} \qquad d_M^N(M_1, M_2) \le \delta_1 + ND\delta_2 < \epsilon(1 - s_1).$$

Therefore, from equation (3.5),

$$\text{(3. 15)} \qquad d_H(M_1\nu, M_2\nu) \le d_M^N(M_1, M_2) < \epsilon(1 - s_1).$$

From the proof of Theorem 3.1, letting $Y = \mathcal{M}(X)$ and $\mathrm{Con}(Y) = \mathrm{Con}_M^N\big(\mathcal{M}(X)\big)$, we have the result

$$\text{(3. 16)} \qquad d_H(\mu_1, \mu_2) < \epsilon,$$

and the proof is complete.

4. **Some examples and applications.** Barnsley's "continuity with respect to a parameter" result [1, Section 3.11] is quite useful with regard to animation in images. In most practical cases, linear IFS transformations are employed. By varying the parameters in these maps, an IFS attractor/image can be made to vary in time in an apparently continuous fashion. Here, however, we wish to address a different problem: the approximation of attractors and measures of IFS with those of "nearby" IFS.

4.1 *Piecewise linear approximations to nonlinear* IFS. As an illustrative example, consider the following polynomial IFS on $X = [0,1]$:

$$\text{(4. 1)} \qquad w_1(x) = \frac{1}{3}x^2, \quad w_2(x) = \frac{1}{3}x^2 + \frac{2}{3}, \quad p_1 = p_2 = \frac{1}{2}.$$

The attractor A of this IFS, studied in [4], is a Cantor-like set with nonuniform invariant measure μ. Now consider a second IFS with the same probabilities, $(\mathbf{u}^{(n)}, \mathbf{p})$, whose maps $u_i^{(n)}$ are piecewise linear interpolations of the w_i, constructed as follows. For $n \ge 1$, let $0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$ define a partition of $[0,1]$. Then let

(4. 2)
$$u_i^{(n)} = \frac{w_i(x_k) - w_i(x_{k-1})}{x_k - x_{k-1}}(x - x_{k-1}) + w_i(x_{k-1}) \quad \text{for } x_{k-1} \le x \le x_k,\ k = 1, 2, \ldots, n.$$

In other words, the graph of $u_i^{(n)}(x)$ is obtained by connecting the contiguous points $P_{k-1} = \big(x_{k-1}, w_i(x_{k-1})\big)$ and $P_k = \big(x_k, w_i(x_k)\big)$, $k = 1, 2, \ldots, n$, of the graph of $w_i(x)$ with straight lines. Thus, the graph of $u_i^{(n)}(x)$ is composed of n line segments. It is convenient to construct a regular partition of $[0,1]$, *i.e.* $x_k = k/n$, $k = 0, 1, \ldots, n$. The nonlinear IFS maps w_i and their approximations $u_i^{(n)}$ for $n = 1$ and 2 are shown in Figure 1. (Note that in the case $n = 1$, *i.e.* the IFS $(\mathbf{u}^{(1)}, \mathbf{p})$, the attractor is the ternary Cantor set on $[0, 1]$, with invariant measure $\mu =$ uniform Cantor-Lebesgue measure.) For $n \geq 1$, it is easy to show that

$$d_w^2(\mathbf{w}, \mathbf{u}^{(n)}) = d_{\mathrm{Con}(X)}(w_1, u_1^{(n)}) = d_{\mathrm{Con}(X)}(w_2, u_2^{(n)}) = \frac{1}{12n^2}. \tag{4.3}$$

If we let $A^{(n)}$ and $\mu^{(n)}$ denote, respectively, the attractor and invariant measure of this IFS $(\mathbf{u}^{(n)}, \mathbf{p})$ then, by Corollary 3.2 and Theorem 3.4,

$$h(A, A^{(n)}) \longrightarrow 0, \ d_H(\mu, \mu^{(n)}) \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty. \tag{4.4}$$

The convergence of the sequence of measures $\mu^{(n)}$ to μ in Hutchinson metric implies convergence of respective power moments, *i.e.*

$$g_k^{(n)} \longrightarrow g_k \quad \text{as} \quad n \longrightarrow \infty, \tag{4.5}$$

where

$$g_k^{(n)} \equiv \int_X x^k \, d\mu^{(n)}, \ g_k \equiv \int_X x^k \, d\mu, \quad k = 1, 2, \dots. \tag{4.6}$$

This follows from the definition of the Hutchinson metric in equation (2.1): For any $f \in \mathrm{Lip}(X)$,

$$d_H(\mu, \nu) \geq \left[\int_X f(x) \, d\mu - \int_X f(x) \, d\nu \right]. \tag{4.7}$$

For a given $k > 1$, let $f(x) = x^k/k$, $\nu = \mu^{(n)}$ and take limits $n \longrightarrow \infty$ of both sides of the inequality.

The first five moments for the measures $\mu(n)$, $n = 2, 4, 8, 16, 100$ and the "target" measure μ are shown in Table 1. The convergence of measures is clearly demonstrated. In all cases the moments were computed by using the following property [5],

$$(T^n f)(x) \longrightarrow \int_A f \, d\mu, \quad x \in X. \tag{4.8}$$

Here μ is the invariant measure of an IFS$(\mathbf{w}, \mathbf{p})$. The operator $T: C(X) \longrightarrow C(X)$ is given by

$$(Tf)(x) = \sum_{i=1}^{N} p_i (f \circ w_i)(x). \tag{4.9}$$

(The Markov operator M in equation (2.6) is the adjoint of T.) The iterate in equation (4.8) is given by the nested sum

$$(T^n f)(x) = \sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} p_{i_1} \cdots p_{i_n} f(w_{i_1} \circ \cdots \circ w_{i_n})(x), \tag{4.10}$$

which involves the enumeration of an N-tree to n generations. For $n \geq 15$, the estimates $y_n = (T^n f)(x_0)$, where $f(x) = x^k$, $1 \leq k \leq 5$, were in agreement to one part in 10^7 in all cases. As well, the accuracy of these estimates was independent of the choice of starting point $x_0 \in [0,1]$. (Note: for *affine* IFS, the moments of an invariant measure may be computed recursively [3, 6]. This is not the case for *nonlinear* IFS, since the recursion relations involving the moments are not complete. The estimation of moments using (i) Hausdorff inequalities and (ii) perturbation methods has recently been investigated [4].)

4.2 *Approximations to Julia sets of complex quadratic maps.* In this final section, we present some rather novel approximations to Julia sets of the one-parameter family of quadratic complex maps $R(z) = z^2 + c$, $c \in \mathbf{C}$. (The Julia-Fatou theory of iteration of rational functions is discussed in detail by Brolin [7]. An excellent review of more recent work on complex dynamics is given by Blanchard [8]. See also the very readable book by Devaney [9].) The Julia set $J(R)$ of a rational function $R(z)$ may be defined as the closure of all repulsive k-cycles of $R(z)$. It is invariant, *i.e.* $J = R(J) = R^{-1}(J)$, perfect and non-null. For the quadratic maps studied here, some special cases of the Julia sets are:

$c = 0$: J is the unit circle $C = \{z \in \mathbf{C} \mid |z| = 1\}$,
$c = -2$: J is the real interval $[-2, 2]$,
$c < -2$: J is a Cantor-like set on the real line.

J is the *repeller* set under forward action of the map $R(z)$ and the *attractor* under the action of $R^{-1}(z)$. In this case, J is the attractor of the IFS composed of the two branches of the inverse of $R(z)$:

$$R_1^{-1}(z) = \sqrt{z-c}, \quad R_2^{-1}(z) = -\sqrt{z-c}. \tag{4.10}$$

(Here, $\sqrt{x}$ denotes the principal square root of $x \in \mathbf{C}$.) The invariant measure μ under forward iteration of $R(z)$ [7, Section III.16] is obtained when $p_1 = p_2 = 1/2$ [3]. Note that this IFS is *not* contractive so the theory developed in Section 3 does not rigorously apply. However, the existence of a unique attractor is guaranteed by the dynamics of iteration of the analytic map $R(z)$. We present below some numerical evidence to support a conjecture that similar continuity results for attractors and invariant measures will hold for this case, on compact subsets $K_c \subset \mathbf{C}$ which are large enough to contain J_c, the Julia set of the quadratic map $z^2 + c$, and for which $R_i^{-1}: K_c \longrightarrow K_c$.

In polar coordinates, the action of the map $z \longrightarrow z^2$ is given by $(r, \theta) \longrightarrow (r^2, 2\theta)$. Our approximations to $R(z)$ will consist of replacing the quadratic radial part of this map $f(r) = r^2$ by piecewise linear interpolation functions over the infinite interval $r \geq 0$. Because of the appearance of the inverse maps $R_i^{-1}(z)$ in the IFS, it is convenient to consider the approximation of the inverse radial map $f^{-1}(r) = \sqrt{r}$, $r \geq 0$. We partition each interval $[k, k+1]$, $k = 0, 1, 2, \ldots$ on the r-axis into n subintervals of equal length and construct the piecewise linear interpolation function to $f^{-1}(r)$ on this partition for $r \geq 0$. We shall denote the resulting *nonanalytic* approximation to $R_i^{-1}(z)$ (which include the angular dependence of the branches) as $u_i^{(n)}(z)$ $i = 1, 2$. (Note that n now refers to the number of linear maps *per unit interval* on the r-axis.) As before, the corresponding IFS will be denoted as $(\mathbf{u}^{(n)}, \mathbf{p})$, where $p_1 = p_2 = 1/2$. Based on the graphical and numerical evidence presented below, we make the following:

CONJECTURE 4.1. *For $c \in \mathbf{C}$, let J_c denote the Julia set for the complex map $R_c(z) = z^2 + c$, with invariant measure μ_c. Let $(\mathbf{u}^{(n)}, \mathbf{p})$, with $p_1 = p_2 = 1/2$ be the (noncontractive)* IFS *obtained by the piecewise linear interpolation of the radial function $\sqrt{r}$ as discussed in the previous paragraph. Then:*

(i) There exists a unique attractor $A^{(n)}$ and invariant measure $\mu^{(n)}$ for this IFS. *Moreover,*

(ii) $h(A^{(n)}, J_c) \longrightarrow 0$ and $d_H(\mu^{(n)}, \mu_c) \longrightarrow 0$ as $n \longrightarrow \infty$.

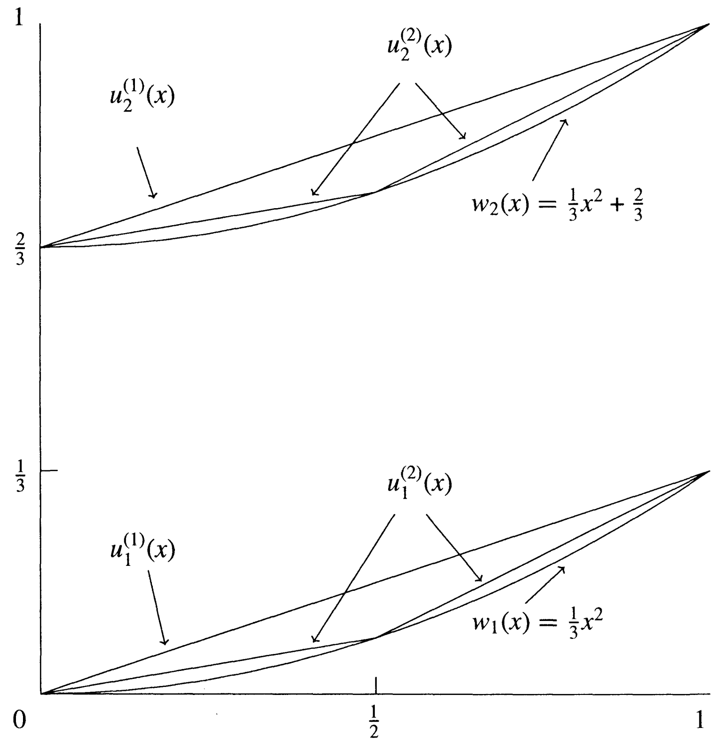

FIGURE 1. GRAPHS OF THE FOLLOWING NONLINEAR IFS MAPS ON [0, 1]:

$$w_1(x) = \frac{1}{3}x^2, \quad w_2(x) = \frac{1}{3}x^2 + \frac{2}{3}$$

AS WELL AS THEIR 1- AND 2-POINT PIECEWISE LINEAR INTERPOLATIONS, $u_i^{(1)}(x)$ AND $u_i^{(2)}(x)$, RESPECTIVELY AS DEFINED IN EQUATION 4.2.

For $c = 0$, $A^{(n)} = J = C$, the unit circle in $\mathbf{C}$, for $n \geq 2$, *i.e.* all approximations coincide with the Julia set. This property is insured by the attractive nature of the point $r = 1$ under the action of the inverse maps $u^{(n)}$ for $n \geq 2$. This is not the case for $c \neq 0$. As n increases, however, computer plots of the approximations $A^{(n)}$ appear to converge to the Julia sets J. Some converging approximations for the cases $c = -0.75$ and $-0.5 + 0.5i$ are shown in Figures 2 and 3, respectively.

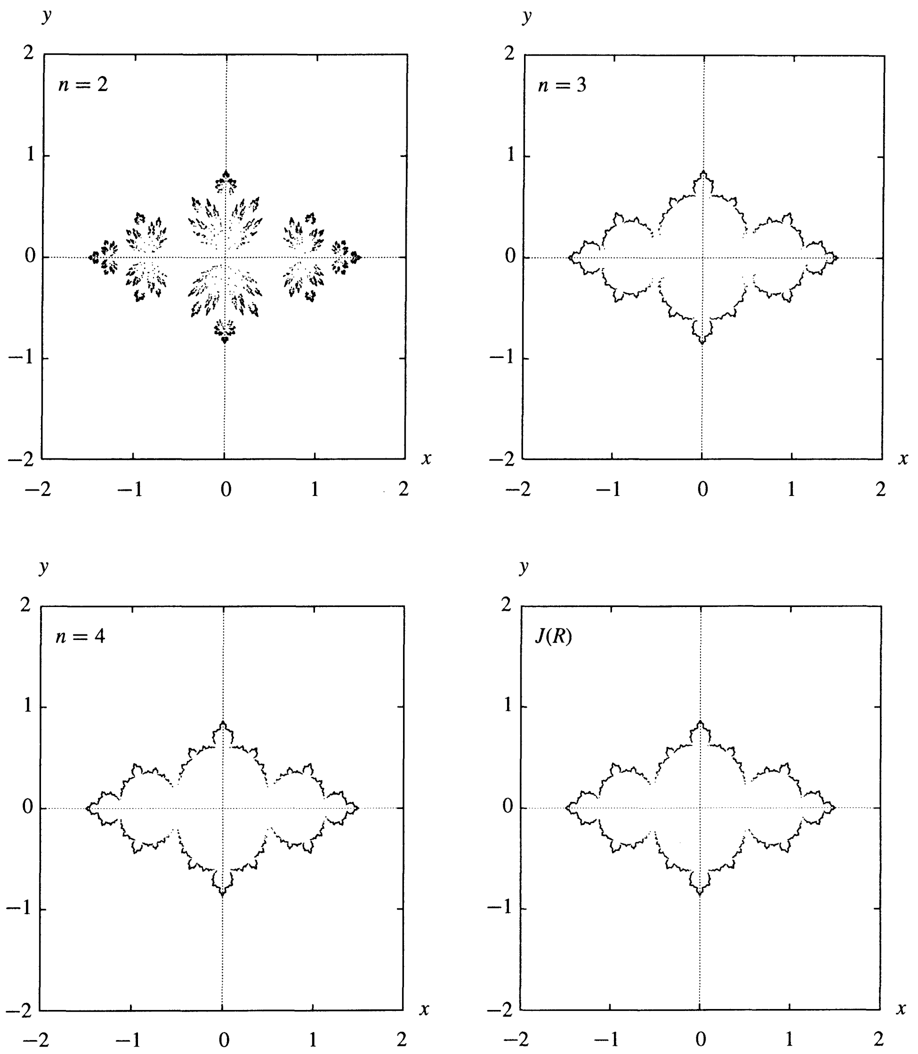

FIGURE 2. APPROXIMATIONS TO THE JULIA SET $J(r)$ OF THE COMPLEX QUADRATIC MAP $R(z) = z^2 + c$, $c = -3/4$, AS GIVEN BY THE ATTRACTORS OF THE IFS $(u_1^{(n)}, u_2^{(n)}, p_1 = p_2 = \frac{1}{2})$, WHERE THE $u_i^{(n)}$ PIECEWISE LINEAR APPROXIMATIONS (n SEGMENTS PER UNIT INTERVAL) TO THE INVERSE FUNCTIONS $R_i^{-1}(z) = (-1)^{i+1}\sqrt{z+3/4}$. THE ACTUAL JULIA SET $J(R)$ IS ALSO SHOWN FOR COMPARISON. IN ALL CASES, THE REGION OF $\mathbf{C}$ SHOWN IS $-2 \leq \mathrm{Re}(z) \leq 2$, $-2 \leq \mathrm{Im}(z) \leq 2$.

We would also expect the convergence of measures $d_H(\mu, \mu^{(n)})$ as $n \longrightarrow \infty$. (Here, $\mathcal{M}(X)$ is the space of probability measures on the σ-algebra of Borel subsets of $X \subset \mathbf{C}$ compact.) This, in turn, implies the convergence of two-dimensional complex power moments of the measures, *i.e*

$$g_{ij}^{(n)} \longrightarrow g_{ij} \quad \text{as} \quad n \longrightarrow \infty, \quad i,j \geq 0, \tag{4.11}$$

where

$$g_{ij}^{(n)} = \int_{A_c^{(n)}} z^i \bar{z}^j \, d\mu^{(n)}, \quad g_{ij} = \int_{J_c} z^i \bar{z}^j \, d\mu. \tag{4.12}$$

When $c \leq -2$, $J_c \subset \mathbf{R}$ and it is sufficient to consider the action of real valued maps $R_c(x) = x^2 + c$ on a finite interval $[-\bar{x}, \bar{x}]$, where $\bar{x} = (1 + \sqrt{1-4c})/2$ is the positive and repulsive fixed point of $R(z)$. We now present some numerical calculations to indicate the convergence of moments for the special case $c = -2$, for which $J = [-2, 2]$. The invariant measure μ is absolutely continuous with respect to Lebesgue measure with (normalized) density function [10]

$$\rho(x) = \frac{1}{\pi\sqrt{4 - x^2}}. \tag{4.13}$$

All odd moments of this measure vanish; the even moments are given by

$$g_{2k} = \frac{1}{\pi} \int_{-2}^{2} \frac{x^{2k}}{\sqrt{4 - x^2}} \, dx = \frac{(2k)!}{(k!)^2}, \quad k = 0, 1, 2, \ldots . \tag{4.14}$$

The approximation $f^{(n)}$ becomes a linear interpolation of the map $x^2 - 2$ composed of $4n$ line segments over the interval $[-2, 2]$. The first five moments for the approximating measures $\mu^{(n)}, n = 2, 4, 8, 16, 100$ as well as for the target measure μ are shown in Table 2. As in the previous case, the moments of the $\mu^{(n)}$ were computed by means of the iteration procedure of equation (4.9).

5. **Closing remarks.** Continuity of IFS attractors and invariant measures with respect to the component maps w_i and associated probabilities p_i has been established. The continuity followed naturally from the rather simple structure of IFS: the metrics employed for the map vectors $\mathbf{w}$ and the probability vectors $\mathbf{p}$ are easily and directly related to the Hutchinson metric d_H on $\mathcal{M}(X)$. This continuity is important for practical purposes like animation or the inverse problem of fractal construction [6], where it is desirable that the attractor or invariant measure be varied in a continuous fashion by varying the IFS components. As well, these elementary continuity properties provide a starting point for an approximation theory of nonlinear IFS.

The continuity properties of attractors of Iterated Fuzzy Set Systems (IFZS) [11], which have shown much promise in the treatment of the inverse problem of fractal/measure construction, have recently been derived [12]. The situation for IFZS attractors is more complicated than for the IFS case, primarily because a composition of functions is involved. One must consider continuity at each specific attractor subject to some restrictions on the variations of the so-called grey-level maps ϕ_i. More recently, however, the original IFZS method has been modified [13]. One of the modifications involves a new distance function which, in special cases, becomes the $L^1(X)$ distance with respect to a measure μ on X. The attractors of this new IFS method on $L^1(X)$ then trivially become continuous with respect to the grey-level maps ϕ_i. This new approach

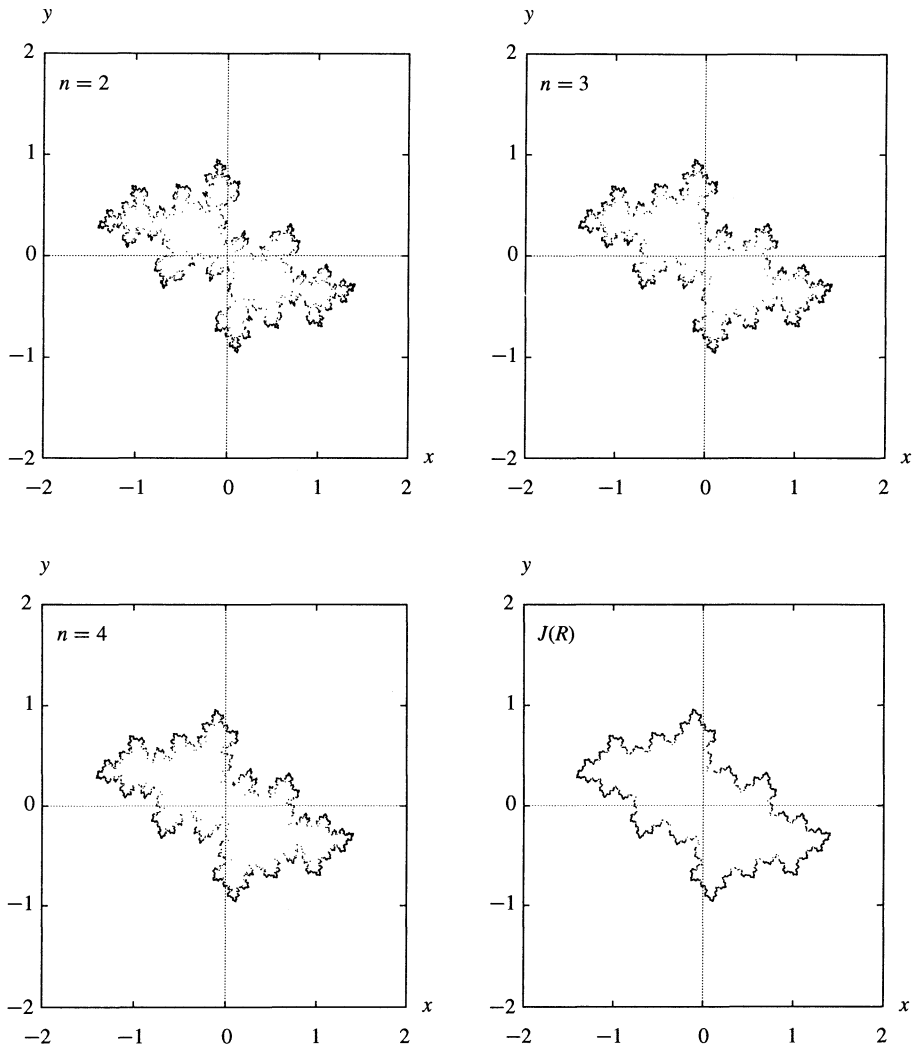

FIGURE 3. AS IN FIGURE 2, BUT APPROXIMATIONS TO THE JULIA SET $J(R)$ OF $R(z) = z^2 + c$ FOR $c = -\frac{1}{2} + \frac{1}{2}i$.

has proved to be very effective in solving the inverse problem of function approximation and image representation.

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TABLE 1. Moments of the invariant measures $\mu^{(n)}$, $n = 2, 4, 8, 16, 100$ of the IFS $(\mathbf{u}^{(n)}, \mathbf{p})$, $p_1 = p_2 = 1/2$, where the $u_i^{(n)}$, defined in equation (3.2), are piecewise linear approximations (n segments per unit interval) to the following polynomial IFS maps on $[0, 1]$:

$$w_1(x) = \frac{1}{3}x^2, \quad w_2(x) = \frac{1}{3}x^2 + \frac{2}{3}, \quad p_1 = p_2 = \frac{1}{2}.$$

Final column: Moments of the invariant measure μ of the above polynomial IFS. In all cases, the moments were computed using the iterative method of equation (3.6).

	$n=2$	$n=4$	$n=8$	$n=16$	$n=100$	g_k
g_1	0.458333	0.442725	0.439290	0.438294	0.438019	0.438011
g_2	0.333781	0.318676	0.315288	0.314296	0.314041	0.314034
g_3	0.266854	0.250234	0.246457	0.245356	0.245077	0.245069
g_4	0.221913	0.203893	0.199751	0.198542	0.198245	0.198236
g_5	0.188984	0.169980	0.165553	0.164202	0.163953	0.163943

TABLE 2. Moments of the invariant measures $\mu^{(n)}$, $n = 2, 4, 8, 16, 100$ of the IFS $(\mathbf{u}^{(n)}, \mathbf{p})$, $p_1 = p_2 = 1/2$, where the $u_i^{(n)}$, defined in equation (3.2), are piecewise linear approximations (n segments per unit interval) to the two branches of the inverse of $R(z) = z^2 - 2$ on $[-2, 2]$:

$$R_1^{-1}(x) = \sqrt{x+2}, \quad R_2^{-1}(x) = -\sqrt{x+2}.$$

The moments of the $\mu^{(n)}$ were computed using the iterative method of equation (3.6). Final column: Exact moments of the invariant measure μ of the Julia set $J(R)$ as given by equation (4.14).

	$n=2$	$n=4$	$n=8$	$n=16$	$n=100$	g_k
g_2	1.981200	1.993000	1.997547	1.999128	1.999944	2.0
g_4	5.961977	5.987957	5.996290	5.998804	5.999936	6.0
g_6	19.83123	19.94348	19.98162	19.99382	19.99964	20.0
g_8	69.29426	69.76274	69.92289	69.97410	69.99851	70.0
g_{10}	249.2348	251.0834	251.7064	251.9026	251.9945	252.0

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