

THE URYSOHN-MENGER SUM FORMULA: AN EXTENSION OF THE DYDAK-WALSH THEOREM TO DIMENSION ONE

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Abstract

Let X be a finite-dimensional separable metric space, presented as a disjoint union of subsets, $X = A \cup B$. We prove the following theorem: For every prime p , $c\text{-dim}_{\mathbb{Z}_p} X \leq c\text{-dim}_{\mathbb{Z}_p} A + c\text{-dim}_{\mathbb{Z}_p} B + 1$. This improves upon some of the earlier work by Dydak and Walsh.

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1. Introduction

Cohomological dimension theory (of separable metrizable spaces) is in many respects parallel to the classical (Lebesgue covering) dimension theory (see, for example, the survey by Kuz'minov [13]). This is particularly true for the cohomological dimension $c\text{-dim}_{\mathbb{Z}}$ over the ring of integers \mathbb{Z} . The basic reason for that is the equivalence of $c\text{-dim}_{\mathbb{Z}} X$ and $\dim X$ for finite dimensional spaces X , a fact which was established already in the 1930's by Aleksandrov, the founder of (co)homological dimension theory. However, in general, $c\text{-dim}_{\mathbb{Z}} X$ and $\dim X$ need not be the same – there exist infinite dimensional spaces X of finite cohomological dimension over \mathbb{Z} , the first such example having been found by Dranišnikov in 1987 [1, 2]. This result has had other important implications, since it provided dimension raising cell-like maps, thus solving another outstanding problem for many years in geometric (Bing) topology (see, for example, the survey by Mitchell and Repovš [14]).

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One of the classical results of the Lebesgue covering dimension theory is the Urysohn-Menger sum formula [11]: it asserts that for all subsets A and B of X such that $X = A \cup B$,

$$(1) \quad \dim X \leq \dim A + \dim B + 1.$$

It was only very recently that (1) was verified for the cohomological dimension over \mathbb{Z} : in 1992 Rubin [15] proved that

$$(2) \quad \text{c-dim}_{\mathbb{Z}} X \leq \text{c-dim}_{\mathbb{Z}} A + \text{c-dim}_{\mathbb{Z}} B + 1.$$

On the other hand, it was shown in 1992 by Dranišnikov, Repovš and Ščepin [6] that the Urysohn-Menger sum formula (1) fails for cohomological dimension over arbitrary abelian groups: they have constructed subsets A and B of \mathbb{R}^4 such that

$$(3) \quad \text{c-dim}_{\mathbb{Q}/\mathbb{Z}}(A \cup B) > \text{c-dim}_{\mathbb{Q}/\mathbb{Z}} A + \text{c-dim}_{\mathbb{Q}/\mathbb{Z}} B + 1.$$

(Subsequently, Dydak [9] presented a different approach to this construction.)

Rubin’s argument [15] for (2) is based on the resolution method, which can be traced back to the 1970’s pioneering work of Edwards (see, for example, the survey by Walsh [16]). Roughly speaking, a *resolution* of a polyhedron L for some integer n , is a replacement of all $(n + 1)$ and higher dimensional simplices of L by the Eilenberg-MacLane spaces $K(\oplus \mathbb{Z}, n)$. In the 1980’s Dranišnikov [1] adapted this method for the Bockstein rings $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}_{(p)}$, that is the localization of \mathbb{Z} at the prime p (see also the surveys [2] and [14]). Subsequently, Dydak and Walsh [10] used this method to prove the Urysohn-Menger sum formula (1) for cohomological dimension c-dim_R over all Bockstein rings $R \in \{\mathbb{Z}_{(\ell)}, \mathbb{Z}_p\}_{\ell \subset \mathcal{P}, p \in \mathcal{P}}$ (where \mathcal{P} is the set of all primes and $\mathbb{Z}_{(\ell)}$ is the localization of the integers at ℓ), however they had to impose the restriction that both $\text{c-dim}_R A \geq 2$ and $\text{c-dim}_R B \geq 2$. (Note that such a restriction was unnecessary in Rubin’s paper [15] since $\text{c-dim}_{\mathbb{Z}} X = 1$ obviously implies $\dim X = 1$.)

Let $\text{AE}(X)$ denote the class of all absolute extensors for X . Recall that the standard definition of the cohomological dimension of X over an abelian group G is

$$(4) \quad \text{c-dim}_G X \leq n \quad \text{if and only if} \quad K(G, n) \in \text{AE}(X).$$

It follows from the work by Dranišnikov [3, 4], that the Eilenberg-MacLane complexes $K(G, n)$ in the definition (4) can be replaced by the Moore spaces $M(G, n)$:

$$(5) \quad \text{c-dim}_G X \leq n \quad \text{if and only if} \quad M(G, n) \in \text{AE}(X).$$

Recall that $M(G, n)$ is a polyhedron such that

$$\tilde{H}_i(M(G, n)) \cong \begin{cases} G & \text{if } i = n, \\ 0 & \text{if } i \neq n. \end{cases}$$

The equivalence (5) is true for $n \geq 2$ and the conclusion $\text{c-dim}_G X \leq n$ also holds for $n = 1$. This result allows instead for a different type of resolution of a polyhedron L , based on Moore spaces as building blocks.

The purpose of this paper is to remove the dimensional restrictions from the Dydak-Walsh theorem [10]. We were able to achieve this by using the new type of resolutions described above, that is, using the Moore spaces $M(G, n)$. We shall only give the proof for the case of the finite groups \mathbb{Z}_p since the proof for $\mathbb{Z}_{(\ell)}$ is similar. This result was obtained in August 1992. Subsequently, Dydak [8] announced a generalization of (6) to all rings R with unity.

THEOREM 1.1. *For all subsets $A, B \subset X$ of a finite-dimensional separable metric space X and for every prime p , the following holds:*

$$(6) \quad \text{c-dim}_{\mathbb{Z}_p}(A \cup B) \leq \text{c-dim}_{\mathbb{Z}_p} A + \text{c-dim}_{\mathbb{Z}_p} B + 1.$$

2. Preliminaries

We shall require the following result from [3]:

THEOREM 2.1. *Suppose $\dim X < \infty$ and that for some $n \geq 2$ and some abelian group G , $\text{c-dim}_G X \leq n$. Then the corresponding Moore space $M(G, n)$ is an absolute extensor for X , $M(G, n) \in AE(X)$.*

We shall also need the following version of the Blakers-Massey theorem from [12, Proposition 16.30]:

PROPOSITION 2.2. *Let $(X, A) \in \text{AHEP}$ and suppose (X, A) is $(n - 1)$ -connected and A is $(s - 1)$ -connected. Then the homomorphism $\pi_r(X, A, *) \rightarrow \pi_r(X/A, *)$ is an $(n + s - 1)$ -isomorphism, for every $r > 0$.*

PROPOSITION 2.3. *Let p be any prime, $n \geq k$ and let K be any $(k - 1)$ -connected polyhedron such that $\pi_i(K) \cong \bigoplus \mathbb{Z}_p$ for all $k \leq i \leq n$. Then there exists an inclusion $K \hookrightarrow \tilde{K}$ such that:*

- (i) $\pi_i(\tilde{K}) \cong \bigoplus \mathbb{Z}_p$ for $i \leq n + 1$; and
- (ii) *the inclusion-induced homomorphism $H^*(\tilde{K}; \mathbb{Z}_p) \rightarrow H^*(K; \mathbb{Z}_p)$ is bijective for $* \leq k$ and surjective for $* > k$.*

PROOF. Let $K^1 = K \cup_{\varphi_i} B_i^{n+2}$ where $\{\varphi_i : \partial B_i^{n+2} \rightarrow K\}_{i \in \mathbb{N}}$ are the generators of $\pi_{n+1}(K)$. Let $\tilde{K} = (K^1, K)_{\mathbb{Z}_p}$, that is, \tilde{K} is obtained from K^1 by replacing all $(n + 2)$ -dimensional simplices of $K^1 \setminus K$ by the Moore spaces $M(\mathbb{Z}_p, n + 1)$. (See [2]

for more on this construction.) Join all copies of $M(\mathbb{Z}_p, n + 1)$ in \tilde{K} by arcs in $K^1 \setminus K$ to obtain the wedge $\vee M(\mathbb{Z}_p, n + 1)$.

Consider a couple $(\tilde{K}, \vee M(\mathbb{Z}_p, n + 1))$ and its exact sequence

$$\cdots \oplus \mathbb{Z}_p \cong \pi_{n+1}(\vee M(\mathbb{Z}_p, n + 1)) \xrightarrow{\alpha} \pi_{n+1}(\tilde{K}) \rightarrow \pi_{n+1}(\tilde{K}, \vee M(\mathbb{Z}_p, n + 1)) \rightarrow \cdots$$

By Proposition 2.2, we have that

$$\pi_r(\tilde{K}, \vee M) \rightarrow \pi_r(\tilde{K} / \vee M) \cong \pi_r(K^1 / \text{wedge of arcs}) \cong \pi_r(K^1)$$

is an isomorphism for $r = n + 1$, since $\vee(M(\mathbb{Z}_p, n + 1))$ is n -connected and $(\tilde{K}, \vee M)$ is 0-connected. By construction, $\pi_{n+1}(K^1) = 0$; thus α is an epimorphism, and hence the image is $\oplus \mathbb{Z}_p$.

PROPOSITION 2.4. *Let K be such that $H_k(K) = \oplus \mathbb{Z}_p$. Then there exists an inclusion $K \hookrightarrow \tilde{K}$ such that*

- (i) $\pi_i(\tilde{K}) = \oplus \mathbb{Z}_p$ for all i ; and
- (ii) $H^i(\tilde{K}; \mathbb{Z}_p) \rightarrow H^i(K; \mathbb{Z}_p)$ is an isomorphism for $i \leq k$ and is an epimorphism for $i > k$.

PROOF. Let \bar{K} be the abelianization of K , that is, \bar{K} is obtained from K by attaching 2-cells along the commutators of all the generators of the fundamental group. Then $\pi_i(\bar{K}) = \oplus \mathbb{Z}_p$. The map $H^1(\bar{K}; \mathbb{Z}_p) \rightarrow H^1(K; \mathbb{Z}_p)$ is an isomorphism because 2-disks are attached by homology trivial maps (commutators). Apply Proposition 2.3, starting from $n = 1$ and \bar{K} to get a sequence

$$K \hookrightarrow K_1 \xrightarrow{j_1} K_2 \xrightarrow{j_2} K_3 \hookrightarrow \cdots,$$

where $K_1 = \bar{K}$, $\pi_i(K_n) = \oplus \mathbb{Z}_p$ for $i \leq n$, and j_i^* is an isomorphism in dimension one, and j_i^* is an epimorphism for $i > 1$. Finally, define $\tilde{K} = \varinjlim K_i$.

PROPOSITION 2.5. *If $\dim X < \infty$ and $\text{c-dim}_{\mathbb{Z}_p} X = 1$ then $K \in \text{AE}(X)$ if $\pi_i(K) \cong \oplus \mathbb{Z}_p$ for all i .*

PROOF. See [3, 7].

3. (cd_R, n) -resolutions

DEFINITION. Let cd_R be an abbreviation for c-dim_R . Suppose that we have a polyhedron K with some triangulation τ . Then a map $\psi : \hat{K} \rightarrow K$ is called a (cd_R, n) -resolution if for every simplex $\sigma \in \tau$, $\psi^{-1}(\sigma) \in \text{AE}(\text{cd}_R \leq n, \dim < \infty)$ and $\psi^{-1}|_{K^{(n)}} : K^{(n)} \rightarrow \psi^{-1}(K^{(n)})$ is a homeomorphism.

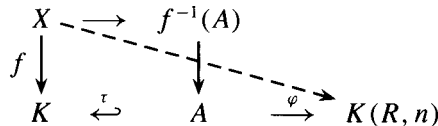
Property (*) of such a resolution means: For every simplex $\sigma \in \tau$ and for every integer $n \leq i$, the map $H^i(\psi^{-1}(\sigma); R) \rightarrow H^i(\psi^{-1}(\partial\sigma); R)$ is an epimorphism and $H^n(\psi^{-1}(\sigma); R) \rightarrow H^n(\psi^{-1}(\partial\sigma); R)$ is an isomorphism (for $i < n$ an isomorphism of trivial groups).

THEOREM 3.1. *For all p, n, K , and τ there exists a $(\text{cd}_{\mathbb{Z}_p}, n)$ -resolution $\psi : \hat{K} \rightarrow K$ with the property (*).*

PROOF. The proof is by induction on $\dim K$. For $\dim K = n + 1$, replace $(n + 1)$ -simplexes by $K(\mathbb{Z}_p, n)$'s, by identifying the boundary of the simplex with the n -skeleton of $K(\mathbb{Z}_p, n)$. For the inductive step m to $m + 1$: Suppose that $\dim K = m + 1$. Consider $\psi_i : \hat{K}^{(m)} \rightarrow K^{(m)}$. Fix an $(m + 1)$ -simplex σ . Let $L = \psi_i^{-1}(\partial\sigma)$. Then by the construction $H_n(L) \cong \bigoplus \mathbb{Z}_p$. Apply Proposition 2.4 to obtain the embedding $L \hookrightarrow \hat{L}$. Then by Proposition 2.5, $\hat{L} \in \text{AE}(\text{cd}_{\mathbb{Z}_p} \leq n, \dim < \infty)$ and so the property (*) holds.

REMARK. Such a resolution with a weaker form of the property (*) was constructed in [2], whereas in [10] a $(\text{cd}_{\mathbb{Z}_p}, n)$ -resolution with the property (*) was constructed just for $n \geq 2$. The argument in [10] is quite different from ours and it does not allow for an extension to the case $n = 1$ (hence the restrictions $\text{c-dim}_R A \geq 2$ and $\text{c-dim}_R B \geq 2$ in their proof of the special case of Theorem 1.1). The rest of our proof of Theorem 1.1 is to some extent similar to the argument in [10]; however, it is in many respects more elementary.

NOTATION. Let (K, τ) be a polyhedron with a triangulation τ , let $f : X \rightarrow K$ be a map. Then the notation $\text{c-dim}_R(f, \tau) \leq n$ means that the following extension problem has a solution for each subcomplex $A \subset K$ of K :



(cf. the survey of Dranišnikov [2]).

PROPOSITION 3.2. *Suppose that $f : X \rightarrow K$ is a map and that for every simplex $\sigma \in \tau$ the map $H^n(f^{-1}(\sigma); R) \rightarrow H^n(f^{-1}(\partial\sigma); R)$ is an epimorphism. Then $\text{c-dim}_R(f, \tau) \leq n$.*

The proof of Proposition 3.2 is trivial. The following two theorems are taken from [2] with only minor changes:

THEOREM 3.3. *Suppose that for every open covering ω of the metric space Y there exists an ω -map $g : Y \rightarrow K$, with τ and there exists a τ -lifting $g' : Y \rightarrow X$ for some $f : X \rightarrow K$ with $\text{c-dim}_R(f, \tau) \leq n$. Then $\text{c-dim}_R Y \leq n$.*

Here τ -lifting means that if $g(x) \in \sigma \in \tau$ then $f \circ g'(x) \in \sigma$ and the ω -map has the property that $g^{-1}(\tau) < \omega$, where $<$ means refinement.

THEOREM 3.4. *Suppose $\text{c-dim}_R Y \leq n$. Then for every map $g : Y \rightarrow K$ with a triangulation τ , and for every (cd_R, n) -resolution $\psi : \hat{K} \rightarrow K$, there is a τ -lifting $g' : Y \rightarrow \hat{K}$.*

We shall also need the following assertion from [10]:

LEMMA 3.5. *If X is $(n - 1)$ -connected and Y is $(m - 1)$ -connected then $H^{n+m+1}(X * Y; R) \cong H^n(X; H^m(Y; R))$.*

PROPOSITION 3.6. *Suppose that we have resolutions: $\hat{\psi} : \hat{K} \rightarrow K$ ((cd_R, n) -resolution) and $\hat{\psi}_1 : \hat{L} \rightarrow L$ ((cd_R, m) -resolution) and both have the property $(*)$. Let $\hat{\sigma} = \hat{\psi}^{-1}(\sigma)$, $\hat{\delta} = \hat{\psi}_1^{-1}(\delta)$. Then the following are isomorphisms:*

$$\begin{aligned} H^{n+m+1}(\hat{\sigma} * \hat{\delta}) &\rightarrow H^{n+m+1}(\hat{\sigma} * \partial\hat{\delta}), \\ H^{n+m+1}(\hat{\sigma} * \hat{\delta}) &\rightarrow H^{n+m+1}(\widehat{\partial\sigma} * \hat{\delta}) \quad \text{and} \\ H^{n+m+1}(\hat{\sigma} * \hat{\delta}) &\rightarrow H^{n+m+1}(\widehat{\partial\sigma} * \widehat{\partial\delta}) \end{aligned}$$

over the ring R , where $\widehat{\partial\delta} = \hat{\psi}_1^{-1}(\partial\delta)$ and $\widehat{\partial\sigma} = \hat{\psi}^{-1}(\partial\sigma)$.

PROOF. Notice that α is an isomorphism:

$$\begin{array}{ccc} H^{n+m+1}(\hat{\sigma} * \hat{\delta}) & \rightarrow & H^{n+m+1}(\hat{\sigma} * \partial\hat{\delta}) \\ \parallel & & \parallel \\ H^n(\hat{\sigma}; H^m(\hat{\delta})) & \xrightarrow{\alpha} & H^n(\hat{\sigma}; H^m(\partial\hat{\delta})). \end{array}$$

Similarly the second one is isomorphism. Now the third one:

$$\begin{array}{ccc} H^{n+m+1}(\hat{\sigma} * \hat{\delta}) & \longrightarrow & H^{n+m+1}(\widehat{\partial\sigma} * \widehat{\partial\delta}) \\ \parallel & & \parallel \\ H^n(\hat{\sigma}; H^m(\hat{\delta})) & & H^n(\widehat{\partial\sigma}; H^m(\widehat{\partial\delta})) \\ \parallel & & \parallel \\ H^n(\hat{\sigma}; \bigoplus R) & \xrightarrow{\beta} & H^n(\widehat{\partial\sigma}; \bigoplus R). \end{array}$$

Now β is an isomorphism, since it is for just one R ; so it follows for finite sums $\bigoplus R$.

4. The proof of Theorem 1.1

LEMMA 4.1. *The map $H^{n+m+1}(\hat{\sigma} * \hat{\delta}; R) \xrightarrow{\gamma} H^{n+m+1}(\partial(\hat{\sigma} * \hat{\delta}); R)$ is an epimorphism, where $\partial(\hat{\sigma} * \hat{\delta}) = (\hat{\sigma} * \partial\hat{\delta}) \cup (\partial\sigma * \hat{\delta}) \hookrightarrow \hat{\sigma} * \hat{\delta}$.*

PROOF. Consider the following Mayer-Vietoris cohomology sequences over R (where $s = n + m + 1$):

$$\begin{array}{ccccccc} \rightarrow & H^{n+m}(\partial\hat{\sigma} * \partial\hat{\delta}) & \rightarrow & H^s(\partial(\hat{\sigma} * \hat{\delta})) & \rightarrow & H^s(\hat{\sigma} * \partial\hat{\delta}) \oplus H^s(\partial\hat{\sigma} * \hat{\delta}) & \rightarrow & H^s(\partial\hat{\delta} * \partial\hat{\sigma}) & \rightarrow & \dots \\ & & & \cong \uparrow \gamma & & \uparrow \alpha & & \uparrow \beta & & \\ \rightarrow & 0 & \rightarrow & H^s(\hat{\sigma} * \hat{\delta}) & \rightarrow & H^s(\hat{\sigma} * \hat{\delta}) \oplus H^s(\hat{\sigma} * \hat{\delta}) & \rightarrow & H^s(\hat{\sigma} * \hat{\delta}) & \rightarrow & \dots \end{array}$$

It suffices to show that γ is an isomorphism. Suppose that $\dim \sigma \leq n$ and $\dim \delta \leq m$. Then all cohomology groups above vanish. If $\dim \sigma > n$ and $\dim \delta > m$, we have that $\partial\hat{\sigma} * \partial\hat{\delta}$ is $(n + m)$ -connected, hence $H^{n+m}(\partial\hat{\sigma} * \partial\hat{\delta}) = 0$ and the Five Lemma yields the assertion.

It remains to consider the case when $\dim \sigma \leq n$ and $\dim \delta > m$ or vice versa. Then $\hat{\sigma} \simeq \text{point}$: hence $\hat{\sigma} * \partial\hat{\delta} \simeq \text{point}$, and $\hat{\sigma} * \hat{\delta} \simeq \text{point}$. We want to show that $H^{n+m+1}(\partial(\hat{\sigma} * \hat{\delta})) = 0$. It suffices to prove that $H^{n+m}(\partial\hat{\sigma} * \partial\hat{\delta})$ maps onto, since we know it maps into. To this end, consider the following diagram:

$$\begin{array}{ccc} H^{n+m}(\partial\hat{\sigma} * \hat{\delta}) & \longrightarrow & H^{n+n+1}(\partial(\hat{\sigma} * \hat{\delta})) \\ \parallel & & \parallel \\ H^{n+m}(S^k * \hat{\delta}) & \longrightarrow & H^{n+k+1}(S^k * \partial\hat{\delta}) \\ \parallel & & \parallel \\ H^{n+m-k-1}(\hat{\delta}) & \longrightarrow & H^{n+m-k-1}(\partial\hat{\delta}), \end{array}$$

and recall that $\partial\hat{\sigma} = S^k$, so by the property (*) there exists an epimorphism. Hence we get a zero where we need it.

PROPOSITION 4.2. *Suppose that $\psi : \hat{K} \rightarrow K$ is a (cd_R, n) -resolution and $\varphi : \hat{L} \rightarrow L$ is a (cd_R, m) -resolution, both with the property (*). Then $\text{c-dim}_R(\psi * \varphi, \tau * \tau') \leq n + m + 1$.*

PROOF. Follows by Lemma 4.1 and Proposition 3.2.

DEFINITION. Suppose that $A, B \subset X$ are disjoint subsets of X , $X = A \cup B$ and suppose that $f : A \rightarrow K$ and $g : B \rightarrow L$ are any maps. Then define the map $f \cup^* g : A \cup B \rightarrow K * L$ as follows: Let $\bar{f} : U \rightarrow K$ be an extension of a map f' , which is close to, and hence homotopic to the map f , over an open neighbourhood $U \subset X$ of A in X and let $\bar{g} : V \rightarrow L$ be an extension of a map g' , which is close to, and hence homotopic to, the map g , over an open neighbourhood $V \subset X$ of B in

X (cf. [10, Lemma 4.2]). Let $d_A(x) = \rho(x, X \setminus U)$ and $d_B(x) = \rho(x, X \setminus V)$ be the distance functions. Now, define for every $x \in X$,

$$(f \dot{\cup} g)(x) = (f(x), g(x), d_B(x)/(d_A(x) + d_B(x))).$$

Let $\pi : K * L \rightarrow [0, 1]$ be the natural projection of the join $K * L$ onto the interval $[0, 1]$. (Collapse K and L to a point, respectively.)

LEMMA 4.3. *Suppose that $X = A \cup B$. Then for every cover ω of X there exist maps $\varphi_A : A \rightarrow K$ with a triangulation τ and $\varphi_B : B \rightarrow L$ with a triangulation τ' such that $\varphi_A \dot{\cup} \varphi_B$ is an ω -map onto $K * L$ with respect to the triangulation $\tau * \tau'$.*

PROOF. Choose a cover ω_A (respectively ω_B) which is a star-refinement of ω and consider the projection onto nerves, φ_A (respectively φ_B).

PROPOSITION 4.4. *Suppose that $X = A \cup B$ and that there are maps $\psi : \hat{K} \rightarrow K$, $\varphi : \hat{L} \rightarrow L$, $f : A \rightarrow K$ with a τ -lifting $f' : A \rightarrow \hat{K}$ and $g : B \rightarrow L$ with a τ' -lifting $g' : B \rightarrow \hat{L}$. Then the map $f \dot{\cup} g : A \cup B \rightarrow K * L$ has a $(\tau * \tau')$ -lifting $q : A \cup B \rightarrow \hat{K} * \hat{L}$.*

PROOF. Define the lifting as follows: $q(x) = (f'(x), g'(x), (\pi(f \dot{\cup} g))(x))$.

PROOF OF THEOREM 1.1. It suffices to prove Theorem 1.1 for the case when the subsets $A, B \subset X$ are disjoint; $A \cap B = \emptyset$. Indeed, if $A \cap B \neq \emptyset$ we define $B' = B \setminus A$ and it follows that

$$\begin{aligned} \text{c-dim}_R(A \cup B) &= \text{c-dim}_R(A \cup B') \leq \text{c-dim}_R A + \text{c-dim}_R B' + 1 \\ &\leq \text{c-dim}_R A + \text{c-dim}_R B + 1. \end{aligned}$$

So suppose now that $A \cap B = \emptyset$, $A \cup B = X$, $\text{c-dim}_R A \leq n$ and $\text{c-dim}_R B \leq m$. We shall prove that $\text{c-dim}_R(A \cup B) \leq n + m + 1$.

To this end, consider an arbitrary cover ω of X and apply Lemma 4.3 to get maps $\varphi_A : A \rightarrow K$ and $\varphi_B : B \rightarrow L$. Next, apply Theorem 3.1 to obtain the corresponding resolutions of K and L , that is, a (cd_R, n) -resolution $\psi : \hat{K} \rightarrow K$ with the property (*) and a (cd_R, m) -resolution $\varphi : \hat{L} \rightarrow L$ with the property (*).

By Proposition 4.2, $\text{c-dim}_R(\psi * \varphi, \tau * \tau') \leq n + m + 1$, and by Theorem 3.4 and Proposition 4.4, there exists a lifting $q : A \cup B \rightarrow \hat{K} * \hat{L}$ of $\varphi_A \dot{\cup} \varphi_B$ which is a $(\tau * \tau')$ -lifting. Since ω as an arbitrary covering, it follows by Theorem 3.3 that $\text{c-dim}_R(A \cup B) \leq n + m + 1$ as asserted.

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