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EFFICIENCY AND GENERALISED CONVEXITY IN VECTOR OPTIMISATION PROBLEMS

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Abstract

This paper gives a necessary and sufficient condition for a Kuhn-Tucker point of a non-smooth vector optimisation problem subject to inequality and equality constraints to be an efficient solution. The main tool we use is an alternative theorem which is quite different to a corresponding result by Xu.

1. Introduction

A vector optimisation problem is a problem where two or more objectives are to be minimised on some set of feasible solutions. In such a problem we often deal with conflicts amongst objectives and hence in most cases cannot find a feasible solution which is optimal in the sense that it minimises all the objectives simultaneously. So in vector optimisation we must use concepts different from this requirement of optimality. In this paper, we restrict ourselves to the concept of an efficient solution: this is a feasible solution such that there does not exist another feasible solution at which all objectives are the same or better, with at least one being strictly better. From a mathematical viewpoint it can be formulated as follows. Let us consider a set S_1 of an Euclidean space $\mathbb{R}^n$ and m functions f_i $(i = 1, 2, \dots, m)$ defined on $\mathbb{R}^n$. The set S_1 can be interpreted as the set of feasible solutions and the functions f_i can be regarded as our objectives which we want to minimise. Then a point $x_0 \in S_1$ is an efficient solution if we cannot find another point $x \in S_1$ such that $f_i(x) \leqq f_i(x_0)$ for all i and, in addition, at least one of these inequalities is strict. This efficiency property originated with Pareto [16] and plays a crucial role in economics, game theory and statistical decision theory (see [1, 3, 7, 16, 22, 24]). In many practical situations S_1 is

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given as a subset of a closed set S which consists of all points x of S satisfying a system of inequalities and equalities:

$$g_j(x) \leqq 0 \ (j = 1, 2, \ldots, k), \quad h_s(x) = 0 \ (s = 1, 2, \ldots, l).$$

In addition, the functions f_i, g_j and h_s are not differentiable in the classical sense. In this paper they are assumed to be locally Lipschitz. Such functions are often encountered in economics, engineering design and various branches of analysis. Examples of locally Lipschitz functions arising in these fields can be found in [5] which is a basic book for everyone interested in nonsmooth analysis. The problem of finding efficient solutions for the objectives f_i on the above feasible set S_1 is referred to as problem (VOP) and is written as follows:

$$\begin{aligned} \text{Minimise } & f(x) := \big(f_1(x), f_2(x), \ldots, f_m(x)\big) \text{ subject to} \\ & g_j(x) \leqq 0 \quad (j = 1, 2, \ldots, k), & (1.1) \\ & h_s(x) = 0 \quad (s = 1, 2, \ldots, l), & (1.2) \\ & \quad x \in S. & (1.3) \end{aligned}$$

It is well-known [12] that the convexity of functions involved in a minimisation problem with inequality constraints (that is, problem (VOP) where $p = 1$, $S = \mathbb{R}^n$ and h_s are absent) assures the optimality of a point satisfying the Kuhn-Tucker conditions [12] and the validity of the Wolfe duality theorems [12]. In 1981, Hanson [9] was the first to show that a generalised convexity requirement, later called invexity, is an appropriate substitute for the usual convexity condition in proving these facts. The invexity idea is also useful for establishing necessary optimality conditions [10, 11] and alternative theorems [4]. In [13] the invexity property was extended to KT-invexity to prove that a Kuhn-Tucker point (that is, a point satisfying the Kuhn-Tucker necessary optimality conditions) is a minimiser of a minimisation problem with differentiable data if and only if this program is KT-invex at this point. A generalisation of invexity to locally Lipschitz functions was introduced in [6, 17, 18]. It has been noted [21] that invexity is not suitable for problems with equality constraints since the Kuhn-Tucker multipliers associated with these constraints are not necessarily nonnegative. So a new notion of infine functions was defined and was shown in [21] to be an adequate tool for equality constraints. Observe from [21, Remark 3.5] (see also Remark 4.2 of the present paper) that introducing a new terminology for infineness is needed since the class of locally Lipschitz infine functions does not coincide with the class of cone-invex functions defined by Craven [6]. The invex-infineness property (that is, the requirement of invexity for objectives and inequality constraints, and of infineness for equality constraints) is used in [21] to establish a necessary and sufficient condition for an efficient solution to be a Geoffrion properly efficient solution [8] in problem (VOP) with locally Lipschitz data.

The aim of this paper is to extend the invex-infineness to KT-pseudoinvex-infineness and GKT-pseudoinvex-infineness such that under suitable assumptions a Kuhn-Tucker point x_0 is an efficient solution of (VOP) if and only if (VOP) is KT-pseudoinvex-infine (or GKT-pseudoinvex-infine) at x_0. Roughly speaking, we want to generalise a known result of Martin [13, Theorem 2.1] to the case of efficient solutions of nonsmooth multiobjective problems subject to mixed constraints (1.1)–(1.3). The proof of the above result and other related facts in Section 4 is based on an alternative theorem which is established in Section 3 for a system of inequalities and equalities given by the support function of nonempty convex compact sets. When each of these sets is a singleton we obtain a result (Corollary 3.2) which is quite different from the nonhomogeneous Farkas lemma of Xu [23]: Xu restricts himself to the case when equalities are absent and some additional hypothesis is required for the validity of his Farkas lemma while our Corollary 3.2 is true without these restrictions. (The formulation of Xu's result and that of our own is also different.) Section 3 also contains an application of our alternative result to a concave vector optimisation problem subject to several concave inequality constraints and abstract linear constraints. Observe that applications of concave programming problems arise more frequently in areas such as inventory, production and transportation planning, site selection, Leontiev substitution systems, assignment problems, decision theory, network flows and so forth. The reader is referred to [2] for a comprehensive survey of the theory of concave programming and an overview of its applications.

To conclude this introduction let us observe from Remark 4.4 that our paper contains results which can be applied to any practical problem with inequality constraints. This shows the applicability of some of our theoretical results to a wide class of problems often encountered in practice.

2. Preliminaries

Let $\mathbb{R}^n$ be an Euclidean space. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ we will use the following notation:

$$\begin{aligned}
&x = y \Leftrightarrow x_i = y_i, \quad \text{for all } i;\\
&x < y \Leftrightarrow x_i < y_i, \quad \text{for all } i;\\
&x \leqq y \Leftrightarrow x_i \leqq y_i, \quad \text{for all } i;\\
&x \leq y \Leftrightarrow x \leqq y \text{ and } x \neq y;\\
&x \not\leq y : \text{the negation of } x \leq y.
\end{aligned}$$

Let us observe that if $n = 1$, that is, if x and y are real numbers then the above notation shows that $x \leq y \Leftrightarrow x < y$.

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If I is a nonempty subset of $\{1, 2, \ldots, n\}$ we will denote by λ_I or $(\lambda_i)_{i\in I}$ the vector with components λ_i $(i \in I)$. Similarly, if $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i \in I)$ is a function then we will use the symbol f_I or $(f_i)_{i\in I}$ to denote the vector-valued map with components f_i $(i \in I)$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function, that is, for any $z \in \mathbb{R}^n$, there exist $\alpha > 0$, $\beta > 0$ such that for any $x, x' \in \mathbb{R}^n$ with $\|x - z\| < \alpha$, $\|x' - z\| < \alpha$, $|f(x) - f(x')| \leqq \beta\|x - x'\|$, and let $x_0 \in \mathbb{R}^n$. Then the Clarke directional derivative of f at x_0 in the direction v is defined by

$$f^0(x_0; v) = \limsup_{y\to x_0 \lambda\downarrow 0} \frac{f(y + \lambda v) - f(y)}{\lambda}$$

and the Clarke subdifferential of f at x_0 is defined by

$$\partial f(x_0) = \{\xi \in \mathbb{R}^n : f^0(x_0; v) \geqq \langle \xi, v\rangle \ \forall v \in \mathbb{R}^n\},$$

where $\langle \cdot, \cdot\rangle$ denotes the inner product in $\mathbb{R}^n$.

It is well-known [5] that for any $v \in \mathbb{R}^n$

$$f^0(x_0; v) = \max_{\xi\in\partial f(x_0)} \langle \xi, v\rangle$$

and $\partial f(x_0)$ is a nonempty compact convex subset of $\mathbb{R}^n$. When f is of class C^1 then $\partial f(x_0)$ coincides with the Fréchet derivative f'_{x_0} of f at x_0 (see [5]). If f_I is a vector-valued map with locally Lipschitz components f_i $(i \in I)$ then we denote by $f_I^0(x_0; \cdot)$ the vector-valued map with components $f_i^0(x_0; \cdot)$ $(i \in I)$. The symbol f'_{x_0} is used to denote the matrix with row vectors f'_{ix_0} $(i \in I)$. Thus $f'_{x_0}\eta$ is simply the vector with components $f'_{ix_0}\eta := \langle f'_{ix_0}, \eta\rangle$ $(i \in I)$.

Let S be a closed subset of $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$. The Clarke [5] tangent cone of S at x_0 is defined by

$$T_S(x_0) := \{v \in \mathbb{R}^n : d_S^0(x_0; v) = 0\},$$

where $d_S(x) = \inf_{z\in S} \|z - x\|$, and the Clarke [5] normal cone of S at x_0 is defined by

$$N_S(x_0) := \{w \in \mathbb{R}^n : \langle v, w\rangle \leqq 0 \ \forall v \in T_S(x_0)\}.$$

A subset $A \subset \mathbb{R}^n$ is said to be a cone if $\lambda x \in A$ for all $x \in A$ and $\lambda \geqq 0$. A cone which is a convex set is said to be a convex cone. For any nonempty subset $A \subset \mathbb{R}^n$ denote by cone A the intersection of all *convex* cones containing A. It is easy to check that cone A is a convex cone consisting of all points of the form $\sum_{i=1}^m \lambda_i x_i$ where m is a positive integer, $x_i \in A$ and $\lambda_i \geqq 0$. Also cone A = cone(co A), where co A stands for the convex hull of A. When A is a convex set, cone $A = \{\lambda x : \lambda \geqq 0,\ x \in A\}$. It has been proved [5] that $N_S(x_0) = \mathrm{cl\ cone}\,\partial d_S(x_0)$, where cl A denotes the closure of A. Also we denote by *sp* A the intersection of all subspaces of $\mathbb{R}^n$ containing A. Observe that sp A = cone A − cone A.

3. An alternative theorem

In this section we give an alternative theorem which is needed to prove the results of Section 4. Let

$$f_i(x) = \max_{v \in B_i} \langle v, x \rangle, \quad i \in I := \{1, 2, \dots, m\},$$
$$g_j(x) = \max_{v \in C_j} \langle v, x \rangle, \quad j \in J := \{1, 2, \dots, k\},$$
$$h_s(x) = \max_{v \in A_s} \langle v, x \rangle, \quad s \in L := \{1, 2, \dots, l\},$$

where B_i $(i \in I)$, C_j $(j \in J)$ and A_s $(s \in L)$ are nonempty convex compact subsets of $\mathbb{R}^n$. Let S be a closed convex cone in $\mathbb{R}^n$. Setting $\tilde{h}_s(x) = \max_{v \in -A_s} \langle v, x \rangle$, we see that

$$\left. \begin{array}{l} h_s(x) \leqq a_s \\ \tilde{h}_s(x) \leqq -a_s \end{array} \right\} \Longrightarrow h_s(x) = a_s. \tag{3.1}$$

Let $S^- := \{\xi \in \mathbb{R}^n : \langle \xi, x \rangle \leqq 0 \text{ for any } x \in S\}$. Then we can check that $x \in S$ if and only if

$$q(x) \leqq 0, \tag{3.2}$$

where $q(x) = \max_{\xi \in D} \langle \xi, x \rangle$, $D := S^- \cap B^n$ and B^n is the closed unit ball of $\mathbb{R}^n$.

The following lemma will be needed for obtaining our alternative theorem.

LEMMA 3.1 ([19, 20]). *The system of inequalities* $f_i(x) < 0$ $(i \in I)$, $g_j(x) \leqq 0$ $(j \in J)$ *has a solution if and only if* $0 \notin \operatorname{co} \bigcup_{i \in I} B_i + \operatorname{cl} \operatorname{cone} \bigcup_{j \in J} C_j$.

Now let $a = (a_1, a_2, \dots, a_l)$, $b = (b_1, b_2, \dots, b_m)$ and $c = (c_1, c_2, \dots, c_k)$ and let $A'_s = A_s \times \{-a_s\} \subset \mathbb{R}^n \times \mathbb{R}$, $B'_i = B_i \times \{-b_i\} \subset \mathbb{R}^n \times \mathbb{R}$ and $C'_j = C_j \times \{-c_j\} \subset \mathbb{R}^n \times \mathbb{R}$. We will need the following closedness assumption (H).

(H) For each $p \in I$ the set

$$\operatorname{cone} \left\{ \bigcup_{i \neq p} B'_i, \bigcup_{j \in J} C'_j \right\} + \operatorname{sp} \bigcup_{s \in L} A'_s + [S^- \times \{0\}]$$

is closed.

REMARK 3.1. Assumption (H) automatically holds if each of the sets B_i, C_j and A_s is a singleton and if S^- is a polyhedral cone.

THEOREM 3.1. *Assume that the closedness assumption* (H) *holds. Consider the following statements*:

(a) *The system*

$$f(x) \leq b, \quad g(x) \leqq c, \quad h(x) = a, \quad x \in S \tag{3.3}$$

has a solution, where $f = (f_i)_{i\in I}$, $g = (g_j)_{j\in J}$ *and* $h = (h_s)_{s\in L}$;

(b)

$$\text{(I)} \quad \begin{cases} (\lambda_i)_{i\in I} > 0, \quad (\mu_j)_{j\in J} \geqq 0, \quad (\delta_s)_{s\in L} \geqq 0, \quad (\bar{\delta}_s)_{s\in L} \geqq 0, & (3.4) \\ 0 \in \sum_{i\in I} \lambda_i B_i + \sum_{j\in J} \mu_j C_j + \sum_{s\in L} \delta_s A_s - \sum_{s\in L} \bar{\delta}_s A_s + S^-, & (3.5) \\ 0 = \sum_{i\in I} \lambda_i b_i + \sum_{j\in J} \mu_j c_j + \sum_{s\in L} \delta_s a_s - \sum_{s\in L} \bar{\delta}_s a_s, & (3.6) \end{cases}$$

or

$$\text{(II)} \quad \begin{cases} (\lambda_i)_{i\in I} \geqq 0, \quad (\mu_j)_{j\in J} \geqq 0, \quad (\delta_s)_{s\in L} \geqq 0, \quad (\bar{\delta}_s)_{s\in L} \geqq 0, & (3.7) \\ 0 \in \sum_{i\in I} \lambda_i B_i + \sum_{j\in J} \mu_j C_j + \sum_{s\in L} \delta_s A_s - \sum_{s\in L} \bar{\delta}_s A_s + S^-, & (3.8) \\ 0 > \sum_{i\in I} \lambda_i b_i + \sum_{j\in J} \mu_j c_j + \sum_{s\in L} \delta_s a_s - \sum_{s\in L} \bar{\delta}_s a_s, & (3.9) \end{cases}$$

has a solution.

Then

(1) (a) *does not hold* $\Longrightarrow$ (b) *holds.*

(2) *If we assume additionally that for any* $s \in L$, A_s *is a singleton, then we can state that either* (a) *or* (b) *holds, but never both.*

PROOF. (1) Suppose that (a) does not hold. Then system (3.3) has no solution. Using (3.1) and (3.2) we derive that for each fixed index $p \in I$ the following system in the variable $x' = (x, r) \in \mathbb{R}^n \times \mathbb{R}$ has no solution:

$$\text{(III)} \quad \begin{cases} \langle 0, x \rangle - 1r < 0, \\ f_p(x) - b_p r < 0, \\ f_i(x) - b_i r \leqq 0, \quad i \neq p, \\ g_j(x) - c_j r \leqq 0, \quad j \in J, \\ h_s(x) - a_s r \leqq 0, \quad s \in L, \\ \tilde{h}_s(x) + a_s r \leqq 0, \quad s \in L, \\ q(x) + 0r \leqq 0. \end{cases}$$

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By the closedness assumption and Lemma 3.1, we have

$$0 \in \operatorname{co}\left\{B_p \times \left\{-b_p\right\}, \{0\} \times \{-1\}\right\} + \operatorname{cone}\left\{\bigcup_{i \neq p} B_i', \bigcup_{j \in J} C_j', \bigcup_{s \in L} A_s', -\bigcup_{s \in L} A_s'\right\} + \left[S^- \times \{0\}\right].$$

Thus there exist $\lambda_p' \geqq 0$, $r_p' \geqq 0$, $\lambda_i^{(p)} \geqq 0$, $(i \neq p)$, $\mu_j^{(p)} \geqq 0$, $(j \in J)$, $\delta_s^{(p)} \geqq 0$, $\bar{\delta}_s^{(p)} \geqq 0$, $(s \in L)$ such that

$$r_p' + \lambda_p' = 1 \tag{3.10$_p$}$$

$$0 \in r_p' 0 + \lambda_p' B_p + \sum_{i \neq p} \lambda_i^{(p)} B_i + \sum_{i \in J} \mu_j^{(p)} C_j + \sum_{s \in L} \delta_s^{(p)} A_s - \sum_{s \in L} \bar{\delta}_s^{(p)} A_s + S^- \tag{3.11$_p$}$$

and

$$0 = -1 \cdot r_p' - \lambda_p' b_p - \sum_{i \neq p} \lambda_i^{(p)} b_i - \sum_{j \in J} \mu_j^{(p)} c_j - \sum_{s \in L} \delta_s^{(p)} a_s + \sum_{s \in L} \bar{\delta}_s^{(p)} a_s + 0. \tag{3.12$_p$}$$

Summing up $(3.11)_p$ over $p \in I$ and setting

$$\begin{aligned}
\lambda_1 &= \lambda_1' + \lambda_1^{(2)} + \lambda_1^{(3)} + \cdots + \lambda_1^{(m)}, \\
\lambda_2 &= \lambda_2^{(1)} + \lambda_2' + \lambda_2^{(3)} + \cdots + \lambda_2^{(m)}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\lambda_m &= \lambda_m^{(1)} + \lambda_m^{(2)} + \lambda_m^{(3)} + \cdots + \lambda_m', \\
\mu_j &= \mu_j^{(1)} + \mu_j^{(2)} + \cdots + \mu_j^{(m)}, \quad j \in J, \\
\delta_s &= \delta_s^{(1)} + \delta_s^{(2)} + \cdots + \delta_s^{(l)}, \quad s \in L, \\
\bar{\delta}_s &= \bar{\delta}_s^{(1)} + \bar{\delta}_s^{(2)} + \cdots + \bar{\delta}_s^{(l)}, \quad s \in L,
\end{aligned}$$

we obtain that

$$0 \in \sum_{i \in I} \lambda_i B_i + \sum_{j \in J} \mu_j C_j + \sum_{s \in L} \delta_s A_s - \sum_{s \in L} \bar{\delta}_s A_s + S^-. \tag{3.13}$$

Summing up $(3.12)_p$ over $p \in I$ and setting

$$r' = r_1' + r_2' + \cdots + r_m', \tag{3.14}$$

we obtain

$$0 = -1 \cdot r' - \sum_{i \in I} \lambda_i b_i - \sum_{j \in J} \mu_j c_j - \sum_{s \in L} \delta_s a_s + \sum_{s \in L} \bar{\delta}_s a_s + 0. \tag{3.15}$$

There are two cases:

(i) $\forall p \in I$, $r'_p = 0$ (hence $r' = 0$ by (3.14));

(ii) $\exists p \in I$ such that $r'_p > 0$ (hence $r' > 0$ by (3.14)).

In case (i): $\forall p \in I$, $\lambda'_p = 1$ (see $(3.10)_p$). Therefore $\lambda_p > 0$ ($\forall p \in I$) and system (I) has a solution.

In case (ii): system (II) has a solution (see (3.13), (3.15)).

(2) Suppose that A_s is a singleton for any $s \in L$. Assume to the contrary that (a) and (b) hold simultaneously and λ_i, μ_j, δ_s, $\bar{\delta}_s$ satisfy system (I). Since $(\lambda_i)_{i \in I} > 0$, we see from (3.3) that there exists $x \in S$ such that

$$\sum_{i \in I} \lambda_i f_i(x) < \sum_{i \in I} \lambda_i b_i, \qquad \sum_{s \in L} \delta_s h_s(x) = \sum_{s \in L} \delta_s a_s,$$
$$\sum_{j \in J} \mu_j g_j(x) \leqq \sum_{j \in J} \mu_j c_j, \qquad -\sum_{s \in L} \bar{\delta}_s h_s(x) = -\sum_{s \in L} \bar{\delta}_s a_s$$

and hence we have

$$\begin{aligned} \zeta(x) &:= \sum_{i \in I} \lambda_i f_i(x) + \sum_{j \in J} \mu_j g_j(x) + \sum_{s \in L} \delta_s h_s(x) - \sum_{s \in L} \bar{\delta}_s h_s(x) \\ &< \sum_{i \in I} \lambda_i b_i + \sum_{j \in J} \mu_j c_j + \sum_{s \in L} \delta_s a_s - \sum_{s \in L} \bar{\delta}_s a_s = 0 \quad \text{(by (3.6))}. \end{aligned} \tag{3.16}$$

On the other hand, since A_s is a singleton, it follows from (3.5) that there exists $\xi \in S^-$ such that

$$\langle -\xi, x \rangle \leqq \zeta(x). \tag{3.17}$$

Since $x \in S$ and $\xi \in S^-$, then $\langle -\xi, x \rangle \geqq 0$. Hence (3.17) implies that $\zeta(x) \geqq 0$, a contradiction to (3.16).

Assume now that x satisfies (3.3), and λ_i, μ_j, δ_s, $\bar{\delta}_s$ satisfy (II). Then we have

$$0 > \sum_{i \in I} \lambda_i b_i + \sum_{j \in J} \mu_j c_j + \sum_{s \in L} \delta_s a_s - \sum_{s \in L} \bar{\delta}_s a_s \quad \text{(by (3.9))} \tag{3.18}$$
$$\geqq \zeta(x) \quad \text{(by (3.3) and (3.7))} \tag{3.19}$$
$$\geqq \langle -\xi, x \rangle \quad \text{(by (3.17))} \tag{3.20}$$
$$\geqq 0. \tag{3.21}$$

This is a contradiction.

REMARK 3.2. The closedness assumption (H) is not used for proving that (a) and (b) do not hold simultaneously.

REMARK 3.3. If A_s is a singleton for any $s \in L$, then (b) in Theorem 3.1 can be replaced by (b)′, where

(b)′

$$\begin{cases} (\lambda_i)_{i\in I} > 0, \quad (\mu_j)_{j\in J} \geqq 0, \quad (\tilde{\delta}_s)_{s\in L} \in \mathbb{R}^l, \\ 0 \in \sum_{i\in I} \lambda_i B_i + \sum_{j\in J} \mu_j C_j + \sum_{s\in L} \tilde{\delta}_s A_s + S^-, \\ 0 = \sum_{i\in I} \lambda_k b_i + \sum_{j\in J} \mu_j c_j + \sum_{s\in L} \tilde{\delta}_s a_s \end{cases}$$

or

$$\begin{cases} (\lambda_i)_{i\in I} \geqq 0, \quad (\mu_j)_{j\in J} \geqq 0, \quad (\tilde{\delta}_s)_{s\in L} \in \mathbb{R}^l, \\ 0 \in \sum_{i\in I} \lambda_i B_i + \sum_{j\in J} \mu_j C_j + \sum_{s\in L} \tilde{\delta}_s A_s + S^-, \\ 0 > \sum_{i\in I} \lambda_i b_i + \sum_{j\in J} \mu_j c_j + \sum_{s\in L} \tilde{\delta}_s a_s \end{cases}$$

has a solution.

This can be obtained by setting $\tilde{\delta}_s = \delta_s - \tilde{\delta}_s$ ($\tilde{\delta}_s$ can be nonnegative or nonpositive).

COROLLARY 3.1. *In addition to the closedness assumption* (H) *of Theorem* 3.1, *we assume that the system*

$$f(x) \leqq b, \quad g(x) \leqq c, \quad h(x) = a \quad x \in S \tag{3.22}$$

has at least a solution and for any $s \in L$, A_s is a singleton.

Then either

(a) *System* (3.3) *has a solution or*
(b) *System* (I) *has a solution,*

but never both.

PROOF. It suffices to show that the consistency of (3.22) implies the inconsistency of (II). We omit the proof of this fact since it can be established by an argument similar to that used in the proof of the second part of Theorem 3.1 (see (3.18)–(3.21)).

COROLLARY 3.2. *Let S^- be a polyhedral cone. Let each of the sets B_i, C_j and A_s be a singleton: $B_i = \{u_i\}$, $C_j = \{v_j\}$ and $A_s = \{w_s\}$. Denote by $\langle u, x\rangle$ the vector with components $\langle u_i, x\rangle$ ($i \in I$) and similarly for $\langle v, x\rangle$ and $\langle w, x\rangle$.*

(A) *Then either*

(a) *the system*

$$\langle u, x\rangle \le b, \quad \langle v, x\rangle \leqq c, \quad \langle w, x\rangle = a, \quad x \in S \tag{3.23}$$

has a solution or

(b) *the system*

$$(\mathrm{I})' \quad \begin{cases} (\lambda_i)_{i\in I} > 0, \quad (\mu_j)_{j\in J} \geqq 0, \quad (\delta_s)_{s\in L} \in \mathbb{R}^l, \\ 0 \in \sum_{i\in I} \lambda_i u_i + \sum_{j\in J} \mu_j v_j + \sum_{s\in L} \delta_s w_s + S^-, \\ 0 = \sum_{i\in I} \lambda_i b_i + \sum_{j\in J} \mu_j c_j + \sum_{s\in L} \delta_s a_s \end{cases}$$

or

$$(\mathrm{II})' \quad \begin{cases} (\lambda_i)_{i\in I} \geqq 0, \quad (\mu_j)_{j\in J} \geqq 0, \quad (\delta_s)_{s\in L} \in \mathbb{R}^l, \\ 0 \in \sum_{i\in I} \lambda_i u_i + \sum_{j\in J} \mu_j v_j + \sum_{s\in L} \delta_s w_s + S^-, \\ 0 > \sum_{i\in I} \lambda_i b_i + \sum_{j\in J} \mu_j c_j + \sum_{s\in L} \delta_s a_s \end{cases}$$

has a solution,

but never both.

(B) *If we additionally assume that the system*

$$\langle u, x\rangle \leqq b, \quad \langle v, x\rangle \leqq c, \quad \langle w, x\rangle = a, \quad x \in S$$

has at least a solution, then either

(a) *System* (3.23) *has a solution or*

(b) *System* (I)$'$ *has a solution,*

but never both.

PROOF. This is a direct consequence of Theorem 3.1, Corollary 3.1 and Remark 3.1.

REMARK 3.4. Let us compare our Corollary 3.2 with [23, Theorem 2.1] under the same assumption that $S = \mathbb{R}^n$ (which implies that $S^- = \{0\}$). In this special case, part (A) of Corollary 3.2 is quite different from Theorem 2.1 of Xu [23] since in our case the equalities exist. Also, unlike [23] no assumption is imposed on our corollary. Our conclusion is quite different from that of Xu [23] and does not contradict his counterexample 1.1.

COROLLARY 3.3. *Let each of the sets A_s be a singleton: $A_s = \{w_s\}$. Consider the following statements:*

(a) *System* (3.3) *has a solution.*

(b) *For any* $u_i \in B_i$ $(i \in I)$ *and* $v_j \in C_j$ $(j \in J)$, *system* (3.23) *has a solution.*

Then (a) $\Rightarrow$ (b); *and the converse implication holds if the closedness assumption* (H) *is satisfied.*

PROOF. (a) $\Rightarrow$ (b) This implication is clear from the definition of the functions f and g.

(b) $\Rightarrow$ (a) Assume to the contrary that system (3.3) has no solution. Then by Theorem 3.1 either system (I) or system (II) has a solution. If system (I) has a solution then there exist $u_i \in B_i$ and $v_j \in C_j$ such that system (I)′ has a solution. By Theorem 3.1 and Remark 3.2, system (3.23) has no solution, a contradiction to statement (b). Similarly, the consistency of system (II) implies the consistency of system (II)′ where $u_i \in B_i$ and $v_j \in C_j$ are suitable points. By Theorem 3.1 and Remark 3.2, system (3.23) has no solution, a contradiction to statement (b).

Now we will consider an application of Corollary 3.1 to a concave vector optimisation problem. Let $f = (f_1, f_2, \dots, f_m)$ and $g = (g_1, g_2, \dots, g_k)$ be vector-valued maps with components being concave on $\mathbb{R}^n$. This means that for all $\bar{x}$ and $x_0 \in \mathbb{R}^n$

$$f_i(\bar{x}) - f_i(x_0) \leqq f_i^0(x_0; \bar{x} - x_0) \quad (i = 1, 2, \dots, m), \tag{3.24}$$

$$g_j(\bar{x}) - g_j(x_0) \leqq g_j^0(x_0; \bar{x} - x_0) \quad (j = 1, 2, \dots, k). \tag{3.25}$$

Let S and M be closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^l$, respectively. Let A be an $l \times n$-matrix, and let $c = (c_1, c_2, \dots, c_k)$ and $d = (d_1, d_2, \dots, d_l)$ be given vectors.

Consider the following concave vector optimisation problem $(VOP)_C$ with concave constraints and abstract linear constraints:

$$(VOP)_C \qquad \begin{aligned} &\text{Minimise } f(x) := (f_1(x), f_2(x), \dots, f_m(x)) \\ &\text{subject to } g(x) \leqq c, && (3.26) \\ &\qquad Ax + d \in M, && (3.27) \\ &\qquad x \in S. && (3.28) \end{aligned}$$

A point x satisfying (3.26)–(3.28) is called a feasible solution of $(VOP)_C$. We are interested in finding an efficient solution x_0 of $(VOP)_C$, that is, a feasible solution x_0 such that there is no other feasible solution x of $(VOP)_C$ with $f(x) \leq f(x_0)$. Obviously, if $x_0 \in S$ is an efficient solution for $(VOP)_C$, then the system

$$f^0(x_0; x) \leq 0, \quad g^0(x_0; x) \leqq c', \quad Ax + d' \in M, \quad x \in S,$$

is inconsistent where $c' = c - g(x_0)$ and $d' = Ax_0 + d$. Indeed, if this system has a solution x then in view of (3.24) and (3.25) we see that $\bar{x} := x + x_0$ is a feasible solution of $(VOP)_C$ such that $f(\bar{x}) \leq f(x_0)$, a contradiction to the efficiency property of x_0.

Let us set $x' = (x, y) \in \mathbb{R}^n \times \mathbb{R}^l$,

$$F_i(x') := f_i^0(x_0; x) = \max_{\xi' \in B_i'} \langle \xi', x' \rangle, \qquad H_s(x') = \langle a_s', x' \rangle,$$
$$G_j(x') := g_j^0(x_0; x) = \max_{\xi' \in C_j'} \langle \xi', x' \rangle, \qquad S' = S \times M,$$

where $B_i' = \partial f_i(x_0) \times \{0\}$, $C_j' = \partial g_j(x_0) \times \{0\}$, $a_s' = (a_s, -p_s)$, a_s is the s-th row of the matrix A and $p_s = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^l$ (1 being the s-th component of p_s).

Then the following system must be inconsistent:

$$F(x') \leq 0, \quad G(x') \leqq c', \quad H(x') = -d', \quad x' \in S'.$$

Observe that $x' = (0, d') \in \mathbb{R}^n \times \mathbb{R}^l$ is a solution of the last system with the sign $\leq$ of its first inequality being replaced by $\leqq$. So by Corollary 3.1, we find $(\lambda_i)_{i \in I} > 0$, $(\mu_j)_{j \in J} \geqq 0$ and $r_s \in \mathbb{R}$ $(s \in L)$ such that

$$0 \in \sum_{i \in I} \lambda_i B_i' + \sum_{j \in J} \mu_j C_j' + \sum_{s \in L} r_s a_s' + S'^- \tag{3.29}$$

and

$$0 = \sum_{i \in I} \lambda_i 0 + \sum_{j \in J} \mu_j c_j' - \sum_{s \in L} r_s d_s' \tag{3.30}$$

if the following closedness assumption holds: for every $i \in I$, the set

$$Q(i) = \operatorname{cone}\left\{ \bigcup_{i' \neq i} B_{i'}' \times \{0\}, \bigcup_{j \in J} C_j' \times \{-c_j\} \right\} + \operatorname{sp}\left[\bigcup_{s \in L} \{a_s'\} \times \{d_s'\} \right] + \left[S'^- \times \{0\} \right]$$

is closed.

From (3.29) and the definitions of B_i', C_j' and a_s', we can derive that

$$0 \in \sum_{i \in I} \lambda_i \partial f_i(x_0) + \sum_{j \in J} \mu_j \partial g_j(x_0) + \sum_{s \in L} r_s a_s + S^- \tag{3.31}$$

and

$$0 = \sum_{i \in I} \lambda_i 0 + \sum_{j \in J} \mu_j 0 - \sum_{s \in L} r_s p_s + M^-. \tag{3.32}$$

Setting $r = (r_1, r_2, \dots, r_l)$, we see that $\sum_{k \in L} r_k p_k = r$. Thus (3.32) means that $r \in M^-$. From (3.30), we have

$$\sum_{j \in J} \mu_j (c_j - g_j(x_0)) - \sum_{s \in L} [r_s \langle a_s, x_0 \rangle + r_s d_s] = 0 \tag{3.33}$$

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(from which we can derive the complementarity condition).

From the above discussion, we can obtain the following necessary optimality condition for $(VOP)_C$.

THEOREM 3.2. *Assume that for every $i \in I$ the set $Q(i)$ is closed. If x_0 is an efficient solution of the concave vector optimisation problem $(VOP)_C$, then there are $(\lambda_i)_{i\in I} > 0$,, $(\mu_j)_{j\in J} \geqq 0$ and $(r_s)_{s\in L} \in M^-$ satisfying* (3.31) *and* (3.33).

COROLLARY 3.4. *In addition to the above assumption that $Q(i)$ is closed for each $i \in I$, assume that $g \equiv 0$, $c = 0$ and f is of class C^1. If x_0 is an efficient solution for $(VOP)_C$, then there are $(\lambda_i)_{i\in I} > 0$, $(r_s)_{s\in L} \in M^-$ such that*

$$0 \in \sum_{i\in I} \lambda_i f'_{ix_0} + \sum_{s\in L} r_s a_s + S^- \quad \textit{and} \quad 0 = \sum_{s\in L} r_s(\langle a_s, x_0\rangle + d_s),$$

where f'_{ix_0} is the Fréchet derivative of f_i at x_0.

REMARK 3.5. In the case where $M = \mathbb{R}^l_-$ (the nonpositive orthant of $\mathbb{R}^l$) and $S = \mathbb{R}^n$, Corollary 3.4 is exactly Theorem 3.1 of [23] (the closedness assumption is automatically satisfied since the sum of polyhedral cones is closed).

4. Efficient solutions of nonsmooth problems of vector optimisation

In this section we will use the notion of a Kuhn-Tucker point for the vector optimisation problem (VOP) which coincides with the usual notion of a Kuhn-Tucker point for the case of scalar optimisation. For smooth problems with inequality constraints only, Martin [13] (see also [14, 15]) introduced a class of KT-invex problems and proved that every Kuhn-Tucker point is a global minimiser if and only if this problem is KT-invex. The main result of this section is Theorem 4.1 which shows that the above result of Martin can be extended to the case of nonsmooth vector optimisation problems with mixed constraints, that is, the case when not only inequality constraints (1.1) but also equality constraints (1.2) and a "geometric" constraint (1.3) are considered. The class of KT-pseudoinvex-infine problems (Definition 4.1) and the class of GKT-pseudoinvex-infine problems (Definition 4.1′) will be used as substitutes for the class of KT-invex problems of Martin. We will see that they are suitable for our goal. As a consequence of Theorem 4.1 we will obtain a generalisation of the above result of Martin to problems with mixed constraints (see Remark 4.3). This section will also discuss relationships between several classes of invex-infine problems (see Theorems 4.2 and 4.3).

Let $f := (f_1, f_2, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$, $g := (g_1, g_2, \dots, g_k) : \mathbb{R}^n \to \mathbb{R}^k$ and $h := (h_1, h_2, \dots, h_l) : \mathbb{R}^n \to \mathbb{R}^l$ be locally Lipschitz functions, and let S be a nonempty closed subset of $\mathbb{R}^n$. Let $I = \{1, 2, \dots, m\}$, $J = \{1, 2, \dots, k\}$ and $L = \{1, 2, \dots, l\}$.

Consider the vector optimisation problem (VOP) formulated in the introduction:

$$\text{Minimise } f(x) \text{ subject to } x \in S_1,$$

where S_1 denotes the set of all points x satisfying (1.1)–(1.3). We will be interested in efficient solutions of (VOP). Recall that $x_0 \in S_1$ is an efficient solution of (VOP) if for any $x \in S_1$, $f(x) \not\leq f(x_0)$, that is, there does not exist $x \in S_1$ such that $f_i(x) \leqq f_i(x_0)$ for all i and at least one of these inequalities is strict.

Let $J_0 = \{j \in J : g_j(x_0) = 0\}$. A point $x_0 \in S_1$ is said to be a Kuhn-Tucker point of (VOP) if there are vectors $(\lambda_i)_{i\in I} > 0$, $(\mu_j)_{j\in J_0} \geqq 0$ and $(r_s)_{s\in L} \in \mathbb{R}^l$ such that

$$0 \in \sum_{i\in I} \lambda_i \partial f_i(x_0) + \sum_{j\in J_0} \mu_j \partial g_j(x_0) + \sum_{s\in L} r_s \partial h_s(x_0) + N_S(x_0). \tag{4.1}$$

This becomes the usual notion of a Kuhn-Tucker point if $m = 1$.

DEFINITION 4.1. Problem (VOP) is KT-pseudoinvex-infine at $x_0 \in S_1$ if for any $x \in S_1$ with $f(x) \leq f(x_0)$ there is $\eta \in T_S(x_0)$ such that

$$0 \geq f^0(x_0; \eta), \tag{4.2}$$

$$0 \geqq g^0_{J_0}(x_0; \eta), \tag{4.3}$$

$$0 = h^0(x_0; \eta). \tag{4.4}$$

REMARK 4.1. Let us consider problem (VOP) when the function h is absent. In this case, it is natural to use the terminology "KT-pseudoinvex" instead of "KT-pseudoinvex-infine". We now provide an example showing that (VOP) is KT-pseudoinvex in the above sense but it is not KT-pseudoinvex in the weaker sense of [19]. Recall that the authors of [19] say that the problem (VOP) of minimising f subject to $g(x) \leqq 0$, $x \in S$ is KT-pseudoinvex at $x_0 \in S_1 := \{x \in S : g(x) \leqq 0\}$ if for any $x \in S_1$ with $f(x) < f(x_0)$ there exists $\eta \in T_S(x_0)$ such that

$$0 > f^0(x_0; \eta), \quad 0 \geqq g^0_{J_0}(x_0; \eta).$$

Consider the following vector optimisation problem:

$$(VOP) \qquad \text{Minimise } f(x) \text{ subject to } x \in S_1 := \{x \in S : g(x) \leqq 0\},$$

where $f(x) = (-x^2, x^2 - x)$, $g(x) = (x^2 - 1/4, -x)$ and $S = \mathbb{R}$. Then $S_1 = [0, 1/2]$. For $x_0 = 0$, the active constraint function is $g_2(x) = -x$ and we have $f'_{x_0} = (0, -1)$ and $g'_{2x_0} = -1$. We can check that $f(x) < f(x_0)$ for any $x \in S_1 \setminus \{0\}$. For all such x, let $\eta = 1$. Then we have $f'_{x_0}\eta = (0, -\eta) \leq 0$ and $g'_{2x_0}\eta = -\eta < 0$. Thus ($VOP$) is KT-pseudoinvex, but (VOP) is not KT-pseudoinvex in the sense of [19] since $f'_{1x_0} = 0$ and hence we cannot find η such that $f'_{x_0}\eta < 0$.

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DEFINITION 4.1$'$. Problem (VOP) is GKT-pseudoinvex-infine at $x_0 \in S_1$ if for any $x \in S_1$ with $f(x) \leq f(x_0)$ and

$$u_i \in \partial f_i(x_0)\ (i \in I), \quad v_j \in \partial g_j(x_0)\ (j \in J_0), \quad w_s \in \partial h_s(x_0)\ (s \in L), \tag{4.5}$$

there is $\eta \in T_S(x_0)$ such that

$$0 \geq \langle u, \eta \rangle, \tag{4.2$'$}$$
$$0 \geqq \langle v, \eta \rangle, \tag{4.3$'$}$$
$$0 = \langle w, \eta \rangle. \tag{4.4$'$}$$

(Recall that $\langle u, \eta \rangle$ is the vector with components $\langle u_i, \eta \rangle$ $(i \in I)$ and similarly for $\langle v, \eta \rangle$ and $\langle w, \eta \rangle$.)

Let us introduce the following closedness assumptions.

(H_1) For any $i \in I$ the set

$$\operatorname{cone}\left\{ \bigcup_{i' \neq i} \partial f_{i'}(x_0), \bigcup_{j \in J_0} \partial g_j(x_0) \right\} + \operatorname{sp} \bigcup_{s \in L} \partial h_s(x_0) + N_S(x_0)$$

is closed.

$(\mathrm{H}_1)'$ For any $i \in I$ and

$$u_{i'} \in \partial f_{i'}(x_0)\ (i' \neq i), \quad v_j \in \partial g_j(x_0)\ (j \in J_0), \quad w_s \in \partial h_s(x_0)\ (s \in L), \tag{4.6}$$

the set $\operatorname{cone}\left\{ \bigcup_{i' \neq i} u_{i'}, \bigcup_{j \in J_0} v_j \right\} + \operatorname{sp} \bigcup_{s \in L} w_s + N_S(x_0)$ is closed.

THEOREM 4.1. *Consider the following statements*:

(a) *Problem* (VOP) *is KT-pseudoinvex-infine at* $x_0 \in S_1$;
(b) *Problem* (VOP) *is GKT-pseudoinvex-infine at* $x_0 \in S_1$;
(c) *If* $x_0 \in S_1$ *is a Kuhn-Tucker point then* x_0 *is an efficient solution of problem* (VOP).

Then

(1) (a) $\Rightarrow$ (b) *if* h *is of class* C^1; *and* (b) $\Rightarrow$ (a) *if* h *is of class* C^1 *and* (H_1) *holds.*
(2) (b) $\Rightarrow$ (c); *and* (c) $\Rightarrow$ (b) *if* $(\mathrm{H}_1)'$ *holds.*
(3) (a) $\Rightarrow$ (c) *if* h *is of class* C^1; *and* (c) $\Rightarrow$ (a) *if* h *is of class* C^1 *and* (H_1) *holds.*

PROOF. (a) $\Rightarrow$ (b) (if h is of class C^1): Obviously. (Observe that $\partial h_s(x_0)$ equals the Fréchet derivative h'_{sx_0} of h_s at x_0 since h_s is of class C^1.)

(b) $\Rightarrow$ (a) (if h is of class C^1 and assumption (H_1) holds): This is a direct consequence of Definitions 4.1, 4.1$'$ and Corollary 3.3 where S is replaced by $N_S(x_0)$, J is replaced by J_0 and functions f_i, g_j and h_s are replaced by

$$\bar{f}_i(\cdot) = f_i^0(x_0; \cdot), \quad \bar{g}_j(\cdot) = g_j^0(x_0; \cdot), \quad \bar{h}_s(\cdot) = h_s^0(x_0; \cdot). \tag{4.7}$$

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(b) $\Rightarrow$ (c): Let $\lambda_i > 0$ $(i \in I)$, $\mu_j \geqq 0$, $(j \in J_0)$, $r_s \in \mathbb{R}$, $(s \in L)$, $u_i \in \partial f_i(x_0)$, $v_j \in \partial g_j(x_0)$, $w_s \in \partial h_s(x_0)$ and $y \in N_S(x_0)$ be such that

$$\zeta := \sum_{i \in I} \lambda_i u_i + \sum_{j \in J_0} \mu_j v_j + \sum_{s \in L} r_s w_s = -y, \tag{4.8}$$

that is, let x_0 be a Kuhn-Tucker point of (VOP). Suppose to the contrary that $x_0 \in S_1$ is not efficient for (VOP). Then $f(x) \leq f(x_0)$ for some $x \in S_1$. From (4.2)′–(4.4)′, we have $\langle \zeta, \eta \rangle < 0$ for suitable $\eta \in T_S(x_0)$ while (4.8) yields $\langle \zeta, \eta \rangle = \langle -y, \eta \rangle \geqq 0$. Thus we obtain a contradiction and the efficiency of x_0 is proved.

(c) $\Rightarrow$ (b) (if $(H_1)'$ holds): If $x_0 \in S_1$ is a Kuhn-Tucker point of (VOP), then by statement (c), there is no $x \in S_1$ such that $f(x) \leq f(x_0)$. This obviously means that (VOP) is GKT-pseudoinvex-infine at x_0.

Assume now that $x_0 \in S_1$ is not a Kuhn-Tucker point of (VOP). Then, for any u_i, v_j and w_s satisfying (4.5), the following system has no solution:

$$\begin{gathered}(\lambda_i)_{i \in I} > 0, \quad (\mu_j)_{j \in J_0} \geqq 0, \quad r_s \in \mathbb{R} \quad (s \in L), \\ 0 \in \sum_{i \in I} \lambda_i u_i + \sum_{j \in J_0} \mu_j v_j + \sum_{s \in L} r_s w_s + N_S(x_0).\end{gathered}$$

Observe that the system

$$\langle u, x \rangle \leqq 0, \quad \langle v, x \rangle \leqq 0, \quad \langle w, x \rangle = 0, \quad x \in T_S(x_0)$$

has a solution $x = 0$. By Corollary 3.1 the system

$$\langle u, x \rangle \leq 0, \quad \langle v, x \rangle \leqq 0, \quad \langle w, x \rangle = 0, \quad x \in T_S(x_0)$$

has a solution denoted by η. Thus, for any $x \in S_1$ with $f(x) \leq f(x_0)$, the point η satisfies all the requirements of Definition 4.1′.

(a) $\Rightarrow$ (c) (if h is of class C^1): This is obvious since (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a) (if h is of class C^1 and if (H_1) holds): If x_0 is a Kuhn-Tucker point then the conclusion is obvious since by statement (c) there is no $x \in S_1$ with $f(x) \leq f(x_0)$. If x_0 is not a Kuhn-Tucker point then the system

$$\begin{gathered}(\lambda_i)_{i \in I} > 0, \quad (\mu_j)_{j \in J_0} \geqq 0, \quad r_s \in \mathbb{R} \quad (s \in L), \\ 0 \in \sum_{i \in I} \lambda_i \partial f_i(x_0) + \sum_{j \in J_0} \mu_j \partial g_j(x_0) + \sum_{s \in L} r_s h'_{sx_0} + N_S(x_0)\end{gathered}$$

has no solution. Observe that the system

$$\bar{f}_i(x) \leqq 0 \; (i \in I), \quad \bar{g}_j(x) \leqq 0 \; (j \in J_0), \quad \bar{h}_s(x) = 0 \; (s \in L), \; x \in T_S(x_0)$$

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has a solution $x = 0$, where $\bar{f}_i$, $\bar{g}_j$ and $\bar{h}_s$ are defined by (4.7). By Corollary 3.1, with $\bar{f}_i$, $\bar{g}_j$, h_s and $T_S(x_0)$ in place of f_i, g_j, h_s and S respectively, we infer that system

$$\bar{f}(x) \leq 0, \quad \bar{g}(x) \leqq 0, \quad \bar{h}(x) = 0, \quad x \in T_S(x_0)$$

has a solution $x = \eta$. Thus for any $x \in S_1$ with $f(x) \leq f(x_0)$ the point η satisfies all the requirements of Definition 4.1.

THEOREM 4.1′. (1) *Assume that h is of class C^1 and the closedness assumption* (H_1) *holds. Then the statements* (a), (b) *and* (c) *of Theorem* 4.1 *are equivalent.*
(2) *If $N_S(x_0)$ is a polyhedral cone then the statements* (b) *and* (c) *of Theorem* 4.1 *are equivalent.*

PROOF. This is a consequence of Theorem 4.1.

Before going further let us introduce some notation which is close to Definitions 4.1 and 4.1′ (see Propositions 4.1 and 4.2).
Let $\bar{S}_1 = \{x \in S_1 : f_i(x) < f_i(x_0) \text{ for some } i \in I\}$.

DEFINITION 4.2. Problem (*VOP*) is KT-invex-infine at $x_0 \in S_1$ if for any $x \in \bar{S}_1$ there exists $\eta \in T_S(x_0)$ such that

$$f(x) - f(x_0) \geq f^0(x_0; \eta), \tag{4.9}$$
$$0 \geqq g^0_{J_0}(x_0; \eta), \tag{4.10}$$
$$0 = h^0(x_0; \eta). \tag{4.11}$$

DEFINITION 4.2′. Problem (*VOP*) is GKT-invex-infine at $x_0 \in S_1$ if for any $x \in \bar{S}_1$ and u_i, v_j, w_s satisfying (4.5) there exists $\eta \in T_S(x_0)$ such that

$$f(x) - f(x_0) \geq \langle u, \eta \rangle, \tag{4.9′}$$
$$0 \geqq \langle v, \eta \rangle, \tag{4.10′}$$
$$0 = \langle w, \eta \rangle. \tag{4.11′}$$

DEFINITION 4.3. Problem (*VOP*) is HC-invex-infine at $x_0 \in S_1$ if for any $x \in \bar{S}_1$ there exists $\eta \in T_S(x_0)$ such that

$$f(x) - f(x_0) \geq f^0(x_0; \eta), \tag{4.12}$$
$$g_{J_0}(x) - g_{J_0}(x_0) \geqq g^0_{J_0}(x_0; \eta), \tag{4.13}$$
$$0 = h^0(x_0; \eta). \tag{4.14}$$

DEFINITION 4.3′. Problem (*VOP*) is GHC-invex-infine at $x_0 \in S_1$ if for any $x \in \bar{S}_1$ and u_i, v_j, w_s satisfying (4.5) there exists $\eta \in T_S(x_0)$ such that

$$f(x) - f(x_0) \geq \langle u, \eta \rangle, \tag{4.12$'$}$$

$$g_{J_0}(x) - g_{J_0}(x_0) \geqq \langle v, \eta \rangle, \tag{4.13$'$}$$

$$0 = \langle w, \eta \rangle. \tag{4.14$'$}$$

REMARK 4.2. Let us note [21] that the appearance of equality constraints (1.2) in problem (*VOP*) leads to the introduction of a subclass of invex functions, called the class of infine functions. Recall [21] that a locally Lipschitz function $\tilde{\zeta} : \mathbb{R}^n \to \mathbb{R}$ is called infine on S at $x_0 \in S$ if

$$\forall x \in S \quad \forall \xi \in \partial\tilde{\zeta}(x_0) \quad \exists \eta \in T_S(x_0) : \quad \tilde{\zeta}(x) - \tilde{\zeta}(x_0) = \langle \xi, \eta \rangle.$$

Consider now the trivial cone $\tilde{M} = \{0\}$ of $\mathbb{R}$ and define the $\tilde{M}$-invexity of $\tilde{\zeta}$ on S at x_0 in the sense of Craven [6] (see also [17]) by requiring that

$$\forall x \in S \quad \exists \eta \in T_S(x_0) : \quad \tilde{\zeta}(x) - \tilde{\zeta}(x_0) - \tilde{\zeta}^0(x_0; \eta) \in \tilde{M}$$

(that is, $\tilde{\zeta}(x) - \tilde{\zeta}(x_0) = \tilde{\zeta}^0(x_0; \eta)$).

It is natural to ask if the class of infine functions coincides with the class of $\{0\}$-invex functions of Craven. The answer is negative: the function $\tilde{\zeta}(x) = |x|$ is $\{0\}$-invex on $S = \mathbb{R}$ at $x_0 = 0$ but it is not infine in our sense. So a separate definition of infine functions is needed. The situation is similar to that of the non-coincidence of the class of convex functions and the class of linear functions.

When dealing with both inequality and equality constraints (1.1) and (1.2) it is natural to require that each component of g is invex and each component of h is infine, with the same $\eta \in T_S(x_0)$. This is nothing more than the notion of invex-infineness of a vector-valued function introduced in [21]. Recall [21] that the vector-valued function $(g_{J_0}; h)$ is called invex-infine on S at $x_0 \in S$ if for any $x \in S$, $v_j \in \partial g_j(x_0) (j \in J_0)$ and $w_s \in \partial h_s(x_0) (s \in L)$ there exists $\eta \in T_S(x_0)$ such that $g_{J_0}(x) - g_{J_0}(x_0) \geqq \langle v, \eta \rangle$ and $h(x) - h(x_0) = \langle w, \eta \rangle$. If in Definition 4.3′ $\bar{S}_1 = S_1$ then conditions (4.13)′ and (4.14)′ mean that $(g_{J_0}; h)$ is invex-infine on S_1 at $x_0 \in S_1$ in the above sense. Indeed, for $x \in S_1$ we have $h(x) - h(x_0) = 0$ and hence a combination of this equality with (4.13)′ and (4.14)′ shows that all the requirements of the definitions of invex-infineness of $(g_{J_0}; h)$ are fulfilled.

We begin our discussion of the relationship between Definitions 4.1–4.3, and Definitions 4.1′–4.3′ with the following obvious result.

PROPOSITION 4.1.

(1) *HC-invex-infine $\Rightarrow$ KT-invex-infine $\Rightarrow$ KT-pseudo-invex-infine.*

(2) *GHC-invex-infine $\Rightarrow$ GKT-invex-infine $\Rightarrow$ GKT-pseudoinvex-infine.*

The following proposition shows that if $m = 1$ then KT-invex-infine $\Leftrightarrow$ KT-pseudoinvex-infine, and GKT-invex-infine $\Leftrightarrow$ GKT-pseudoinvex-infine.

PROPOSITION 4.2. *Assume that $m = 1$ (that is, f is a real-valued function). Then*

(1) *The following statements are equivalent:*

(a) *Problem (VOP) is KT-pseudoinvex-infine at $x_0 \in S_1$.*

(b) *Problem (VOP) is KT-invex-infine at $x_0 \in S_1$.*

(c) *For any $x \in S_1$ there exists $\eta \in T_S(x_0)$ such that*

$$f(x) - f(x_0) \geqq f^0(x_0; \eta), \tag{4.15}$$

$$0 \geqq g^0_{J_0}(x_0; \eta), \tag{4.16}$$

$$0 = h^0(x_0; \eta). \tag{4.17}$$

(2) *The following statements are equivalent:*

(a)′ *Problem (VOP) is GKT-pseudoinvex-infine at $x_0 \in S_1$.*

(b)′ *Problem (VOP) is GKT-invex-infine at $x_0 \in S_1$.*

(c)′ *For any $x \in S_1$ and u_i, v_j, w_s satisfying (4.5) there exists $\eta \in T_S(x_0)$ such that*

$$f(x) - f(x_0) \geqq \langle u, \eta \rangle, \tag{4.15)$'$}$$

$$0 \geqq \langle v, \eta \rangle, \tag{4.16)$'$}$$

$$0 = \langle w, \eta \rangle. \tag{4.17)$'$}$$

PROOF. Let us prove the first part of Proposition 4.1. The second part can be proved using a similar argument.

(a) $\Rightarrow$ (b): Since $m = 1$, $\bar{S}_1 = \{x \in S_1 : f(x) < f(x_0)\}$. Let $x \in \bar{S}_1$, that is, $f(x) < f(x_0)$. By Definition 4.1 there exists $\eta \in T_S(x_0)$ satisfying (4.3), (4.4) and the inequality $f^0(x_0; \eta) < 0$. Let $\gamma > 0$ be such that

$$f(x) - f(x_0) > \gamma f^0(x_0; \eta) = f^0(x_0; \bar{\eta}),$$

where $\bar{\eta} = \gamma\eta \in T_S(x_0)$. Thus (4.9)–(4.11) hold, with $\bar{\eta}$ in place of η.

(b) $\Rightarrow$ (a): Since $m = 1, f(x) \leq f(x_0) \Leftrightarrow f(x) < f(x_0)$ (see Section 2). Thus if $x \in S_1$ and $f(x) \leq f(x_0)$ then $x \in \bar{S}_1$. To complete the proof it remains to note that conditions (4.9)–(4.11) imply conditions (4.2)–(4.4).

(c) $\Rightarrow$ (b): Let $x \in S_1$ with $f(x) \leq f(x_0)$. Then by (c) there exists $\eta \in T_S(x_0)$ satisfying (4.15)–(4.17). Since $f(x) \leq f(x_0) \Leftrightarrow f(x) - f(x_0) < 0$ we derive from (4.15) that $f^0(x^0; \eta) < 0$. As in the proof of implication (a) $\Rightarrow$ (b) we can find $\gamma > 0$ such that (4.9)–(4.11) are satisfied, with η being replaced by $\bar{\eta} = \gamma\eta \in T_S(x_0)$.

(b) $\Rightarrow$ (c): If $x \in \bar{S}_1$ then the fact of the existence of $\eta \in T_S(x_0)$ satisfying (4.15)–(4.17) is a direct consequence of Definition 4.2. If $x \notin \bar{S}_1$, that is, $x \in S_1$ and $f(x) \geqq f(x_0)$, then $\eta = 0 \in T_S(x_0)$ satisfies (4.15)–(4.17).

COROLLARY 4.1. *Let $S = \mathbb{R}^n$. Let f be a real-valued function and f, g and h be of class C^1. Then the following statements are equivalent*:

(a) *For any $x \in S_1$ there is $\eta \in \mathbb{R}^n$ such that*

$$f(x) - f(x_0) \geqq f'_{x_0}\eta, \quad -g_{J_0}(x_0) \geqq g'_{x_0}\eta, \quad 0 = h'_{x_0}\eta.$$

(b) *If x_0 is a Kuhn-Tucker point then x_0 is a minimiser of the (scalar) optimisation problem (VOP).*

PROOF. Since f is a real-valued function an efficient point is exactly a minimiser. Therefore our corollary is a direct consequence of Theorem 4.1′ and Proposition 4.1. (Observe that for a C^1-function the Fréchet derivative coincides with the Clarke subdifferential and that $g_{J_0}(x_0) = 0$.)

REMARK 4.3. Corollary 4.1 is a generalisation of a result of Martin [13, Theorem 2.1] to programs with mixed constraints. In [13] Martin restricts himself to inequality constraints only.

Clearly, if h is of class C^1 then KT-pseudoinvex-infine $\Rightarrow$ GKT-pseudo-invex-infine, KT-invex-infine $\Rightarrow$ GKT-invex-infine and HC-invex-infine $\Rightarrow$ GHC-invex-infine. We have seen from Theorem 4.1 that under suitable assumptions KT-pseudoinvex-infine $\Leftrightarrow$ GKT-pseudoinvex-infine. It is then natural to ask under which conditions we have

$$\text{KT-invex-infine} \Leftrightarrow \text{GKT-invex-infine},$$
$$\text{HC-invex-infine} \Leftrightarrow \text{GHC-invex-infine}.$$

The remainder of this paper is devoted to giving an answer to this question. Our results (Theorems 4.2 and 4.3) are also interesting from the point of view of sufficiency conditions for the efficiency property. Indeed, from the implication (b) $\Rightarrow$ (c) of Theorem 4.1 and the first part of Proposition 4.1 it is clear that conditions equivalent to GHC-invex-infineness or GKT-invex-infineness are sufficient conditions for a Kuhn-Tucker point to be an efficient solution of (*VOP*). The same is true for the case of HC-invex-infineness and KT-invex-infineness if h is of class C^1 (see Theorem 4.1 and Proposition 4.1).

Let us introduce the following closedness conditions:

(H_2) For any $i \in I$ and $x \in \bar{S}_1$ the set

$$\operatorname{cone}\left\{\bigcup_{i' \neq i} \partial f_{i'}(x_0) \times \{f_{i'}(x_0) - f_{i'}(x)\}, \bigcup_{j \in J_0} \partial g_j(x_0) \times \{0\}\right\} + \operatorname{sp}\left(\bigcup_{s \in L} \partial h_s(x_0) \times \{0\}\right) + [N_S(x_0) \times \{0\}]$$

is closed.

$(\mathrm{H}_2)'$ For any $i \in I, x \in \bar{S}_1$ and $u_{i'}, v_j, w_s$ satisfying (4.6) the set

$$\operatorname{cone}\left\{\bigcup_{i' \neq i} u_{i'} \times \{f_{i'}(x_0) - f_{i'}(x)\}, \bigcup_{j \in J_0} v_j \times \{0\}\right\} + \operatorname{sp}\left(\bigcup_{s \in L} w_s \times \{0\}\right) + [N_S(x_0) \times \{0\}]$$

is closed.

THEOREM 4.2. *Consider the following statements*:

(a) *Problem (VOP) is KT-invex-infine at* $x_0 \in S_1$.

(b) *Problem (VOP) is GKT-invex-infine at* $x_0 \in S_1$.

(c) *If* $(\lambda_i)_{i \in I} \geqq 0$, $(\mu_j)_{j \in J_0} \geqq 0$, $(r_s)_{s \in L} \in \mathbb{R}^l$ *are such that* (4.1) *is satisfied then, for all* $x \in \bar{S}_1$, $\sum_{i \in I} \lambda_i f_i(x) \geqq \sum_{i \in I} \lambda_i f_i(x_0)$ *and, in addition, this inequality is strict in the case when* $(\lambda_i)_{i \in I} > 0$.

Then

(1) (a) $\Rightarrow$ (b) *if h is of class* C^1; *and* (b) $\Rightarrow$ (a) *if h is of class* C^1 *and* (H_2) *holds.*

(2) (b) $\Rightarrow$ (c); *and* (c) $\Rightarrow$ (b) *if* $(\mathrm{H}_2)'$ *holds.*

(3) (a) $\Rightarrow$ (c) *if h is of class* C^1; *and* (c) $\Rightarrow$ (a) *if h is of class* C^1 *and* (H_2) *holds.*

PROOF. (a) $\Rightarrow$ (b) (if h is of class C^1): Obviously.

(b) $\Rightarrow$ (a) (if h is of class C^1 and (H_2) holds): This is a direct consequence of Definitions 4.2, 4.2′ and Corollary 3.3 where $b_i = f_i(x) - f_i(x_0)$, $c_j = 0$, $a_s = 0$; S is replaced by $N_S(x_0)$; and f_i, g_j and h_s are replaced by $\bar{f}_i$, $\bar{g}_j$ and $\bar{h}_s$ (see (4.7)).

(b) $\Rightarrow$ (c): Let $(\lambda_i)_{i \in I} \geqq 0$; $(\mu_j)_{j \in J_0} \geqq 0$, $(r_s)_{s \in L} \in \mathbb{R}^l$, $u_i \in \partial f_i(x_0)$, $v_j \in \partial g_j(x_0)$, $w_s \in \partial h_s(x_0)$ and $y \in N_S(x_0)$ be such that (4.8) holds. Then using (4.8) and Definition 4.2′, for any $x \in \bar{S}_1$ we find $\eta \in T_S(x_0)$ such that

$$\sum_{i \in I} \lambda_i [f_i(x) - f_i(x_0)] \geqq \langle \zeta, \eta \rangle = \langle -y, \eta \rangle \geqq 0.$$

Thus $\sum_{i \in I} \lambda_i [f_i(x) - f_i(x_0)] \geqq 0$ and this inequality is strict if $(\lambda_i)_{i \in I} > 0$.

(c) $\Rightarrow$ (b) (if $(\mathrm{H}_2)'$ holds): Assume to the contrary that for $x \in \bar{S}_1$ and u_i, v_j, w_s satisfying (4.5) the system

$$f(x) - f(x_0) \geq \langle u, \eta \rangle, \quad 0 \geqq \langle v, \eta \rangle, \quad 0 = \langle w, \eta \rangle, \quad \eta \in T_S(x_0)$$

has no solution. Then by Theorem 3.1 either the system

$$\begin{cases} (\lambda_i)_{i \in I} > 0, \quad (\mu_j)_{j \in J_0} \geqq 0, \quad (r_s)_{s \in L} \in \mathbb{R}^l, & (4.18) \\ 0 \in \sum_{i \in I} \lambda_i u_i + \sum_{j \in J_0} \mu_j v_j + \sum_{s \in L} r_s w_s + N_S(x_0), & (4.19) \\ \sum_{i \in I} \lambda_i [f_i(x_0) - f_i(x)] = 0 & (4.20) \end{cases}$$

or the system

$$\begin{cases} (\lambda_i)_{i \in I} \geqq 0, \quad (\mu_j)_{j \in J_0} \geqq 0, \quad (r_s)_{s \in L} \in \mathbb{R}^l, & (4.21) \\ 0 \in \sum_{i \in I} \lambda_i u_i + \sum_{j \in J_0} \mu_j v_j + \sum_{s \in L} r_s w_s + N_S(x_0), & (4.22) \\ \sum_{i \in I} \lambda_i [f_i(x_0) - f_i(x)] > 0 & (4.23) \end{cases}$$

has a solution. This contradicts statement (c).

(a) $\Rightarrow$ (c) (if h is of class C^1): This implication is obvious since (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a) (if h is of class C^1 and if (H_2) holds): Assume to the contrary that for $x \in \bar{S}_1$ the system

$$f(x) - f(x_0) \geq \bar{f}(\eta), \quad 0 \geqq \bar{g}_{J_0}(\eta), \quad 0 = \bar{h}(\eta), \quad \eta \in T_S(x_0),$$

has no solution where $\bar{f}$, $\bar{g}_{J_0}$ and $\bar{h}$ are defined by (4.7). Then by Theorem 3.1 either the system (4.18)–(4.20) or the system (4.21)–(4.23) has a solution where $w_s = h'_{sx_0}$; and $u_i \in \partial f_i(x_0)$ and $v_j \in \partial g_j(x_0)$ are suitable points. (Observe that the points u_i and v_j which appear in (4.19) and (4.22) may be different points.) This contradicts statement (c).

We conclude our paper by formulating Theorem 4.3 whose proof is similar to that of Theorem 4.2 and is omitted. For this purpose, let us introduce the following closedness conditions:

(H_3) For any $i \in I$ and $x \in \bar{S}_1$ the set

$$\operatorname{cone} \left\{ \bigcup_{i' \neq i} \partial f_{i'}(x_0) \times \{f_{i'}(x_0) - f_{i'}(x)\}, \bigcup_{j \in J_0} \partial g_j(x_0) \times \{g_j(x_0) - g_j(x)\} \right\}$$
$$+ \operatorname{sp} \left(\bigcup_{s \in L} \partial h_s(x_0) \times \{0\} \right) + [N_S(x_0) \times \{0\}]$$

is closed.

$(\mathrm{H}_3)'$ For any $i \in I$, $x \in \bar{S}_1$ and $u_{i'}$, v_j, w_s satisfying (4.6) the set

$$\operatorname{cone}\left\{\bigcup_{i'\neq i} u_{i'} \times \{f_{i'}(x_0) - f_{i'}(x)\},\ \bigcup_{j\in J_0} v_j \times \{g_j(x_0) - g_j(x)\}\right\}$$
$$+ \operatorname{sp}\left(\bigcup_{s\in L} \partial h_s(x_0) \times \{0\}\right) + [N_S(x_0) \times \{0\}]$$

is closed.

THEOREM 4.3. *Consider the following statements*:

(a) *Problem* (*VOP*) *is HC-invex-infine at* $x_0 \in S_1$.

(b) *Problem* (*VOP*) *is GHC-invex-infine at* $x_0 \in S_1$.

(c) *If* $(\lambda_i)_{i\in I} \geqq 0$, $(\mu_j)_{j\in J_0} \geqq 0$, $(r_s)_{s\in L} \in \mathbb{R}^l$ *are such that* (4.1) *is satisfied then for any* $x \in \bar{S}_1$, $\zeta(x) \geqq \zeta(x_0)$ *and, in addition, this inequality is strict in the case* $(\lambda_i)_{i\in I} > 0$ *where*

$$\zeta(x) = \sum_{i\in I} \lambda_i f_i(x) + \sum_{j\in J_0} \mu_j g_j(x).$$

Then

(1) (a) $\Rightarrow$ (b) *if* h *is of class* C^1; *and* (b) $\Rightarrow$ (a) *if* h *is of class* C^1 *and* (H_3) *holds.*

(2) (b) $\Rightarrow$ (c); *and* (c) $\Rightarrow$ (b) *if* $(\mathrm{H}_3)'$ *holds.*

(3) (a) $\Rightarrow$ (c) *if* h *is of class* C^1; *and* (c) $\Rightarrow$ (a) *if* h *is of class* C^1 *and* (H_3) *holds.*

REMARK 4.4. Let us observe that the convex cone generated by a finite number of points is always closed. Hence the closedness assumptions $(\mathrm{H}_1)'$, $(\mathrm{H}_2)'$ and $(\mathrm{H}_3)'$ are automatically satisfied for the case when $S = X$ and the equality constraints are absent. This remark is useful since many practical problems involve only inequality constraints and hence for such problems we do not need to check these assumptions.

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