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TOPICS SURROUNDING THE COMBINATORIAL ANABELIAN GEOMETRY OF HYPERBOLIC CURVES IV: DISCRETENESS AND SECTIONS

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Abstract. Let Σ be a nonempty subset of the set of prime numbers which is either equal to the entire set of prime numbers or of cardinality one. In the present paper, we continue our study of the pro-Σ fundamental groups of hyperbolic curves and their associated configuration spaces over algebraically closed fields in which the primes of Σ are invertible. The present paper focuses on the topic of *comparison* between the theory developed in earlier papers concerning *pro-*Σ fundamental groups and various *discrete* versions of this theory. We begin by developing a theory concerning certain combinatorial analogues of the *section conjecture* and *Grothendieck conjecture*. This portion of the theory is *purely combinatorial* and essentially follows from a result concerning the *existence of fixed points* of actions of finite groups on finite graphs (satisfying certain conditions). We then examine various applications of this purely combinatorial theory to *scheme theory*. Next, we verify various results in the theory of discrete fundamental groups of hyperbolic topological surfaces to the effect that various properties of *(discrete) subgroups* of such groups hold if and only if analogous properties hold for the closures of these subgroups in the *profinite completions* of the discrete fundamental groups under consideration. These results make possible a fairly *straightforward translation*, into *discrete versions*, of pro-Σ results obtained in previous papers by the authors. Finally, we discuss a construction that was considered previously by M. Boggi in the discrete case from the point of view of the present paper.

Contents

Introduction

Let $\Sigma \subseteq \mathfrak{Primes}$ be a subset of the set of prime numbers $\mathfrak{Primes}$ which is either equal to $\mathfrak{Primes}$ or of cardinality one. In the present paper, we continue our study of the *pro-*

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Σ *fundamental groups* of hyperbolic curves and their associated configuration spaces over algebraically closed fields in which the primes of Σ are invertible (cf. [8]–[11], [18], [20], [23]). The present paper focuses on the topic of understanding the relationship between the theory developed in earlier papers concerning pro-Σ fundamental groups and various *discrete* versions of this theory. This topic of comparison of pro-Σ and discrete versions of the theory turns out to be closely related, in many situations, to the theory of *sections* of various natural surjections of profinite groups. Indeed, this relationship with the theory of sections is, in some sense, not surprising, inasmuch as sections typically amount to some sort of *fixed point* within a *profinite continuum*. That is to say, such fixed points are often closely related to the identification of a *rigid discrete structure* within the profinite continuum.

In §§1 and 2, we study two different aspects of this topic of comparison of pro-Σ and discrete structures. Both §§1 and 2 follow the same pattern: we begin by proving an abstract and somewhat technical *combinatorial* result and then proceed to discuss various *applications* of this combinatorial result.

In §1, the main technical combinatorial result is summarized in Theorem A below (where Σ is allowed to be an arbitrary nonempty set of prime numbers). This result consists of versions of the *section conjecture* and *Grothendieck conjecture*—that is, the central issues of concern in *anabelian geometry*—for *outer representations of ENN-type* (cf. Definition 1.7(i)). Here, we remark that outer representations of ENN-type are generalizations of the outer representations of NN-type studied in [8]. Just as an outer representation of NN-type may be described, roughly speaking, as a purely combinatorial object modeled on the outer Galois representation arising from a hyperbolic curve over a complete discretely valued field whose residue field is separably closed, an outer representation of ENN-type may be described, again roughly speaking, as an analogous sort of purely combinatorial object that arises in the case where the residue field is not necessarily separably closed. The pro-Σ section conjecture portion of Theorem A (i.e., Theorem 1.13(i)) is then obtained by combining

- the essential *uniqueness* of *fixed points* of certain group actions on profinite graphs given in [8, Prop. 3.9(i)–(iii)] with
- an essentially classical result concerning the *existence* of *fixed points* (cf. Lemma 1.6 and Remarks 1.6.1 and 1.6.2), which amounts, in essence, to a geometric reformulation of the well-known fact that free pro-Σ groups are *torsion-free* (cf. Remarks 1.13.1 and 1.15.2(i)).

The argument applied to prove this pro-Σ section conjecture portion of Theorem A is essentially similar to the argument applied in the tempered case discussed in [17, Ths. 3.7 and 5.4], which is reviewed (in slightly greater generality) in the tempered section conjecture portion of Theorem A (cf. Theorem 1.13(ii)). These section conjecture portions of Theorem A imply, under suitable conditions, that there is a natural *bijection* between conjugacy classes of *pro-*Σ and *tempered* sections (cf. Theorem 1.13(iii)). This implication may be regarded as an important example of the phenomenon discussed above, that is, that considerations concerning sections are closely related to the topic of comparison of pro-Σ and discrete structures. Finally, by combining the pro-Σ section conjecture portion of Theorem A with the combinatorial version of the Grothendieck conjecture obtained in [10, Th. 1.9(i)], one obtains the Grothendieck conjecture portion of Theorem A (cf. Corollary 1.14).

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THEOREM A (Combinatorial versions of the section conjecture and Grothen dieck conjecture). *Let Σ be a nonempty set of prime numbers, let $\mathcal{G}$ be a semi-graph of anabelioids of pro-Σ PSC-type, let G be a profinite group, and let $\rho\colon G \to \mathrm{Aut}(\mathcal{G})$ be a continuous homomorphism that is of ENN-type for a conducting subgroup $I_G \subseteq G$ (cf. Definition 1.7(i)). Write $\Pi_{\mathcal{G}}$ for the (pro-Σ) fundamental group of $\mathcal{G}$ and $\Pi_{\mathcal{G}}^{\mathrm{tp}}$ for the tempered fundamental group of $\mathcal{G}$ (cf. [17, Exam. 2.10] and the discussion preceding [17, Prop. 3.6]). (Thus, we have a natural outer injection $\Pi_{\mathcal{G}}^{\mathrm{tp}} \hookrightarrow \Pi_{\mathcal{G}}$—cf. [11, Lem. 3.2(i)] and the proof of [11, Prop. 3.3(i) and (ii)].) Write $\Pi_G \overset{\mathrm{def}}{=} \Pi_{\mathcal{G}} \overset{\mathrm{out}}{\rtimes} G$ (cf. the discussion entitled "Topological groups" in [9, §0]); $\Pi_G^{\mathrm{tp}} \overset{\mathrm{def}}{=} \Pi_{\mathcal{G}}^{\mathrm{tp}} \overset{\mathrm{out}}{\rtimes} G$; $\widetilde{\mathcal{G}} \to \mathcal{G}$, $\widetilde{\mathcal{G}}^{\mathrm{tp}} \to \mathcal{G}$ for the universal pro-Σ and pro-tempered coverings of $\mathcal{G}$ corresponding to $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{G}}^{\mathrm{tp}}$; $\mathrm{VCN}(-)$ for the set of vertices, cusps, and nodes of the underlying (pro-)semi-graph of a (pro-)semi-graph of anabelioids (cf. Definition 1.1(i)). Thus, we have a natural commutative diagram*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_{\mathcal{G}}^{\mathrm{tp}} & \longrightarrow & \Pi_G^{\mathrm{tp}} & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Pi_{\mathcal{G}} & \longrightarrow & \Pi_G & \longrightarrow & G & \longrightarrow & 1
\end{array}$$

—where the horizontal sequences are exact, and the vertical arrows are outer injections; Π_G^{tp} acts naturally on $\widetilde{\mathcal{G}}^{\mathrm{tp}}$; Π_G acts naturally on $\widetilde{\mathcal{G}}$. Then the following hold:

(i) *Suppose that ρ is l-cyclotomically full (cf. Definition 1.7(ii)) for some $l \in \Sigma$. Let $s\colon G \to \Pi_G$ be a continuous section of the natural surjection $\Pi_G \twoheadrightarrow G$. Then, relative to the action of Π_G on $\mathrm{VCN}(\widetilde{\mathcal{G}})$ via conjugation of VCN-subgroups, the image of s stabilizes some element of $\mathrm{VCN}(\widetilde{\mathcal{G}})$.*

(ii) *Let $s\colon G \to \Pi_G^{\mathrm{tp}}$ be a continuous section of the natural surjection $\Pi_G^{\mathrm{tp}} \twoheadrightarrow G$. Then, relative to the action of Π_G^{tp} on $\mathrm{VCN}(\widetilde{\mathcal{G}}^{\mathrm{tp}})$ via conjugation of VCN-subgroups (cf. Definition 1.9), the image of s stabilizes some element of $\mathrm{VCN}(\widetilde{\mathcal{G}}^{\mathrm{tp}})$.*

(iii) *Write $\mathrm{Sect}(\Pi_G/G)$ for the set of $\Pi_{\mathcal{G}}$-conjugacy classes of continuous sections of the natural surjective homomorphism $\Pi_G \twoheadrightarrow G$ and $\mathrm{Sect}(\Pi_G^{\mathrm{tp}}/G)$ for the set of $\Pi_{\mathcal{G}}^{\mathrm{tp}}$-conjugacy classes of continuous sections of the natural surjective homomorphism $\Pi_G^{\mathrm{tp}} \twoheadrightarrow G$. Then the natural map*

$$\mathrm{Sect}(\Pi_G^{\mathrm{tp}}/G) \longrightarrow \mathrm{Sect}(\Pi_G/G)$$

is injective. If, moreover, ρ is l-cyclotomically full for some $l \in \Sigma$, then this map is bijective.

(iv) *Let $\mathcal{H}$ be a semi-graph of anabelioids of pro-Σ PSC-type, let H be a profinite group, and let $\rho_{\mathcal{H}}\colon H \to \mathrm{Aut}(\mathcal{H})$ be a continuous homomorphism that is of ENN-type for a conducting subgroup $I_H \subseteq H$. Write $\Pi_{\mathcal{H}}$ for the (pro-Σ) fundamental group of $\mathcal{H}$. Suppose further that ρ is verticially quasi-split (cf. Definition 1.7(i)). Let $\beta\colon G \xrightarrow{\sim} H$ be a continuous isomorphism such that $\beta(I_G) = I_H$; let $l \in \Sigma$ be a prime number such that $\rho_{\mathcal{G}} \overset{\mathrm{def}}{=} \rho$ and $\rho_{\mathcal{H}}$ are l-cyclotomically full; let $\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$ be a continuous isomorphism such that the diagram*

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$$\begin{array}{ccccc} G & \xrightarrow{\rho_{\mathcal{G}}} & \mathrm{Aut}(\mathcal{G}) & \hookrightarrow & \mathrm{Out}(\Pi_{\mathcal{G}}) \\ {\scriptstyle\beta}\downarrow & & & & \downarrow \\ H & \xrightarrow{\rho_{\mathcal{H}}} & \mathrm{Aut}(\mathcal{H}) & \hookrightarrow & \mathrm{Out}(\Pi_{\mathcal{H}}) \end{array}$$

—where the right-hand vertical arrow is the isomorphism obtained by conjugating by α—commutes. Then α is graphic (cf. [18, Def. 1.4(i)]).

The purely combinatorial theory of §1—that is, the theory surrounding and including Theorem A—has important applications to scheme theory—that is, to the theory of *hyperbolic curves* over quite general *complete discretely valued fields*—as follows:

(A-1) We observe that a quite general result in the style of the main results of [26] concerning *valuations* fixed by sections of the arithmetic fundamental group follows formally, in the case of hyperbolic curves over quite general complete discretely valued fields, from Theorem A (cf. Corollary 1.15(iii) and Remark 1.15.2(i) and (ii)). The quite substantial generality of this result is a reflection of the *purely combinatorial* nature of Theorem A. This approach contrasts substantially with the approach of [26] via essentially scheme-theoretic techniques such as the local–global principle for the Brauer group (cf. Remark 1.15.2(i)). The approach of the present paper also differs substantially from [26] in that the transition from fixed points of graphs to fixed valuations is treated as a formal consequence of well-known elementary properties of Berkovich spaces, that is, in essence the compactness of the unit interval $[0,1] \subseteq \mathbb{R}$ (cf. Remark 1.15.2(ii)).

(A-2) We observe that the natural bijection between conjugacy classes of *pro-*Σ and *tempered sections* discussed in the purely combinatorial setting of Theorem A implies a similar *bijection* in the case of hyperbolic curves over quite general complete discretely valued fields (cf. Corollary 1.15(vi)). This portion of the theory was partially motivated by discussions between the second author and Y. André.

In the context of (A-1), we remark that, in the Appendix to the present paper, we give an elementary exposition from the point of view of two-dimensional log regular log schemes of the phenomenon of *convergence of valuations*, without applying the language or notions, such as Stone–Čech compactifications, typically applied in expositions of the theory of Berkovich spaces.

In §2, we turn to the task of formulating *discrete analogues* of a substantial portion of the theory developed in earlier papers. This formulation centers around the notion of a *semi-graph of temperoids of HSD-type* (i.e., "hyperbolic surface decomposition type"—cf. Definition 2.3(iii)), which may be thought of as a natural discrete analogue of the notion of a semi-graph of anabelioids of pro-Σ PSC-type (cf. [18, Def. 1.1(i)]). As the name suggests, this notion may be thought of as referring to the sort of collection of discrete combinatorial data that one may associate with a decomposition of a hyperbolic surface into hyperbolic subsurfaces. Alternatively, it may be thought of as referring to the sort of collection of combinatorial data that arise from systems of topological coverings of the system of topological spaces naturally associated with a stable log curve over a log point whose underlying scheme is the spectrum of the field of complex numbers (cf.

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Example 2.4(i)). After discussing various basic properties and terms related to semi-graphs of temperoids of HSD-type (cf. Proposition 2.5 and Definitions 2.6 and 2.7), we observe that the fundamental operations of *restriction*, *partial compactification*, *resolution*, and *generization* discussed in [9, §2], admit natural compatible analogues for semi-graphs of temperoids of HSD-type (cf. Definitions 2.8 and 2.9 and Proposition 2.10).

The main technical combinatorial result of §2 is summarized in Theorem B below. This result asserts, in effect, that discrete subgroups of the discrete fundamental group of a semi-graph of temperoids of HSD-type satisfy various properties of interest if and only if the profinite completions of these discrete subgroups satisfy analogous properties (cf. Theorem 2.15 and Corollary 2.19(i)). The main technical tool that is applied in order to derive this result is the fact that any inclusion of a finitely generated group into a (finitely generated) free discrete group is, after possibly passing to a suitable finite index subgroup, necessarily **split** (cf. [17, Cor. 1.6(ii)], which is applied in the proof of Lemma 2.14(i) of the present paper). Here, we recall that in [17], this fact (i.e., [17, Cor. 1.6(ii)]) is obtained as an immediate consequence of "*Zariski's main theorem for semi-graphs*" (cf. [17, Th. 1.2]).

THEOREM B (Profinite versus discrete subgroups). *Let $\mathcal{G}$, $\mathcal{H}$ be semi-graphs of temperoids of HSD-type (cf. Definition 2.3(iii)). Write $\widehat{\mathcal{G}}$, $\widehat{\mathcal{H}}$ for the semi-graphs of anabelioids of pro-$\mathfrak{Primes}$ PSC-type determined by $\mathcal{G}$, $\mathcal{H}$ (cf. Proposition 2.5(iii) in the case where $\Sigma = \mathfrak{Primes}$), respectively; $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{H}}$ for the respective fundamental groups of $\mathcal{G}$, $\mathcal{H}$ (cf. Proposition 2.5(i)); $\Pi_{\widehat{\mathcal{G}}}$, $\Pi_{\widehat{\mathcal{H}}}$ for the respective (profinite) fundamental groups of $\widehat{\mathcal{G}}$, $\widehat{\mathcal{H}}$. Then the following hold:*

(i) *Let H, $J \subseteq \Pi_{\mathcal{G}}$ be subgroups. Since $\Pi_{\mathcal{G}}$ injects into its pro-l completion for any $l \in \mathfrak{Primes}$ (cf. Remark 2.5.1), let us regard subgroups of $\Pi_{\mathcal{G}}$ as subgroups of the profinite completion $\widehat{\Pi}_{\mathcal{G}}$ of $\Pi_{\mathcal{G}}$. Write $\overline{H}$, $\overline{J} \subseteq \widehat{\Pi}_{\mathcal{G}}$ for the closures of H, J in $\widehat{\Pi}_{\mathcal{G}}$, respectively. Suppose that the following conditions are satisfied:*

(a) *The subgroups H and J are finitely generated.*
(b) *If J is of infinite index in $\Pi_{\mathcal{G}}$, then $\overline{J}$ is of infinite index in $\widehat{\Pi}_{\mathcal{G}}$.*

(Here, we note that condition (b) is automatically satisfied whenever $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$—cf. [17, Cor. 1.6(ii)].) Then the following hold:

(1) *It holds that $J = \overline{J} \cap \Pi_{\mathcal{G}}$.*
(2) *Suppose that there exists an element $\widehat{\gamma} \in \widehat{\Pi}_{\mathcal{G}}$ such that*

$$H \subseteq \widehat{\gamma} \cdot \overline{J} \cdot \widehat{\gamma}^{-1}.$$

Then there exists an element $\delta \in \Pi_{\mathcal{G}}$ such that

$$H \subseteq \delta \cdot J \cdot \delta^{-1}.$$

(ii) *Let*

$$\alpha \colon \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$$

be an outer isomorphism. Write $\widehat{\alpha} \colon \Pi_{\widehat{\mathcal{G}}} \xrightarrow{\sim} \Pi_{\widehat{\mathcal{H}}}$ for the outer isomorphism determined by α and the natural outer isomorphisms $\widehat{\Pi}_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\widehat{\mathcal{G}}}$, $\widehat{\Pi}_{\mathcal{H}} \xrightarrow{\sim} \Pi_{\widehat{\mathcal{H}}}$ of Proposition 2.5(iii).

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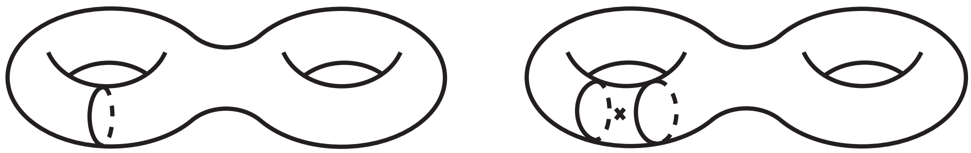

Figure 1.
A cycle and lifting cycles.

Then the outer isomorphism α is group-theoretically verticial (resp. group-theoretically cuspidal; group-theoretically nodal; graphic) (cf. Definition 2.7(i) and (ii)) if and only if the outer isomorphism $\widehat{\alpha}$ is group-theoretically verticial (cf. [18, Def. 1.4(iv)]) (resp. group-theoretically cuspidal [cf. [18, Def. 1.4(iv)]]; group-theoretically nodal [cf. [8, Def. 1.12]]; graphic [cf. [18, Def. 1.4(i)]]).

The significance of Theorem B lies in the fact that it renders possible a fairly straightforward translation of a substantial portion of the profinite results obtained in earlier papers by the authors into discrete versions, as follows:

(B-1) the *partial combinatorial cuspidalization* obtained in [9, Th. A] and [10, Th. A] (cf. Corollary 2.20 of the present paper);
(B-2) the theory of *Dehn multi-twists* summarized in [9, Th. B] (cf. Corollary 2.21 of the present paper);
(B-3) the theory of the *tripod homomorphism* and *metric-admissibility* summarized in [10, Th. C] and [11, Ths. A, C, and D] (cf. Theorem 2.24 of the present paper);
(B-4) the *archimedean* analogue (cf. Corollary 2.25 of the present paper) of the *characterization*, given in [11, Th. B], of *nonarchimedean local Galois groups* in the *global Galois image* associated with a hyperbolic curve.

Finally, in §3, we examine the theory of *canonical liftings of cycles* discussed in [5] from the point of view of the profinite theory developed so far by the authors. This approach contrasts substantially with the intuitive topological approach of [5] in the discrete case. From a naive topological point of view, the canonical liftings of cycles in question amount to *once-punctured tubular neighborhoods* of the given cycles (cf. Figure 1), that is, to the construction of a *tripod* (i.e., a copy of the projective line minus three points) canonically and functorially associated with the cycle. This tripod satisfies a *remarkable rigidity property*, that is, it admits a canonical isomorphism, subject to almost no indeterminacies, with a given fixed tripod that is independent of the choice of the cycle. Moreover, this canonical isomorphism is functorial with respect to "geometric" outer automorphisms of the profinite fundamental group of the stable log curve under consideration that lift to automorphisms of the profinite fundamental group of a configuration space (associated with the stable log curve) of sufficiently high dimension. Here, by "geometric," we mean that the outer automorphism under consideration lies in the kernel of the tripod homomorphism studied in [10, §3]. Indeed, this remarkable rigidity property is obtained as an immediate consequence of the theory of *tripod synchronization* developed in [10, §3].

The *profinite* version of the theory of canonical liftings of cycles developed in §3 is summarized in Theorem C below (cf. Theorem 3.10). By applying the translation apparatus

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developed in §2 to this profinite version of the theory, we also obtain a corresponding *discrete* version of the theory of canonical liftings of cycles (cf. Theorem 3.14).

THEOREM C (Canonical liftings of cycles). *Let (g,r) be a pair of nonnegative integers such that $2g-2+r>0$; let Σ be a set of prime numbers which is either equal to the entire set of prime numbers or of cardinality one; let k be an algebraically closed field of characteristic $\notin \Sigma$; let $S^{\log} \overset{\mathrm{def}}{=} \mathrm{Spec}(k)^{\log}$ be the log scheme obtained by equipping $S \overset{\mathrm{def}}{=} \mathrm{Spec}(k)$ with the log structure determined by the fs chart $\mathbb{N} \to k$ that maps $1 \mapsto 0$; let $X^{\log} = X_1^{\log}$ be a stable log curve of type (g,r) over $S^{\log}$. For positive integers $m \leq n$, write*

$$X_n^{\log}$$

for the nth log configuration space of the stable log curve $X^{\log}$ (cf. the discussion entitled "Curves" in [9, §0]);

$$\Pi_n$$

for the maximal pro-Σ quotient of the kernel of the natural surjection $\pi_1(X_n^{\log}) \twoheadrightarrow \pi_1(S^{\log})$;

$$p_{n/m}^{\log}\colon X_n^{\log} \longrightarrow X_m^{\log}, \quad p_{n/m}^{\Pi}\colon \Pi_n \twoheadrightarrow \Pi_m,$$

$$\Pi_{n/m} \overset{\mathrm{def}}{=} \mathrm{Ker}(p_{n/m}^{\Pi}) \subseteq \Pi_n, \quad \mathcal{G}, \quad \Pi_{\mathcal{G}}$$

for the objects defined in the discussion at the beginning of [10, §3] and [10, Def. 3.1]. Let $I \subseteq \Pi_{2/1} \subseteq \Pi_2$ be a cuspidal inertia group associated with the diagonal cusp of a fiber of $p_{2/1}^{\log}$; let $\Pi_{\mathrm{tpd}} \subseteq \Pi_3$ be a 3-central $\{1,2,3\}$-tripod of Π_3 (cf. [10, Def. 3.7(ii)]); let $I_{\mathrm{tpd}} \subseteq \Pi_{\mathrm{tpd}}$ be a cuspidal subgroup of Π_{tpd} that does not arise from a cusp of a fiber of $p_{3/2}^{\log}$; let J_{tpd}^{*}, $J_{\mathrm{tpd}}^{**} \subseteq \Pi_{\mathrm{tpd}}$ be cuspidal subgroups of Π_{tpd} such that I_{tpd}, J_{tpd}^{*}, and J_{tpd}^{**} determine three distinct Π_{tpd}-conjugacy classes of closed subgroups of Π_{tpd}. (Note that one verifies immediately from the various definitions involved that such cuspidal subgroups I_{tpd}, J_{tpd}^{*}, and J_{tpd}^{**} always exist.) For positive integers $n \geq 2$, $m \leq n$, and $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)$ (cf. [20, Def. 1.1(ii)]), write

$$\alpha_m \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_m)$$

for the automorphism of Π_m determined by α;

$$\mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I) \subseteq \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)$$

for the subgroup consisting of $\beta \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)$ such that $\beta_2(I) = I$;

$$\mathrm{Aut}^{\mathrm{FC}}(\Pi_n)^{\mathrm{G}} \subseteq \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)$$

for the subgroup consisting of $\beta \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)$ such that the image of β via the composite $\mathrm{Aut}^{\mathrm{FC}}(\Pi_n) \twoheadrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1) \to \mathrm{Out}(\Pi_{\mathcal{G}})$—where the second arrow is the natural injection of [8, Th. B] and the third arrow is the homomorphism induced by the natural outer isomorphism $\Pi_1 \xrightarrow{\sim} \Pi_{\mathcal{G}}$—is graphic (cf. [18, Def. 1.4(i)]);

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$$\mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I) \cap \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)^{\mathrm{G}};$$

$$\mathrm{Cycle}^n(\Pi_1)$$

for the set of n-cuspidalizable cycle-subgroups of Π_1 *(cf. Definition 3.5(i) and (ii));*

$$\mathrm{Tpd}_I(\Pi_{2/1})$$

for the set of closed subgroups $T \subseteq \Pi_{2/1}$ *such that* T *is a tripodal subgroup associated with some* 2*-cuspidalizable cycle-subgroup of* Π_1 *(cf. Definition 3.6(i)), and, moreover,* I *is a distinguished cuspidal subgroup (cf. Definition 3.6(ii)) of* T*. Then the following hold:*

(i) *Let* $n \geq 3$ *be a positive integer. Then* $\mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}}$ *acts naturally on* $\mathrm{Cycle}^n(\Pi_1)$, $\mathrm{Tpd}_I(\Pi_{2/1})$*; there exists a unique* $\mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}}$*-equivariant map*

$$\mathfrak{C}_I \colon \mathrm{Cycle}^n(\Pi_1) \longrightarrow \mathrm{Tpd}_I(\Pi_{2/1})$$

such that, for every $J \in \mathrm{Cycle}^n(\Pi_1)$, $\mathfrak{C}_I(J)$ *is a tripodal subgroup associated with* J *(cf. Definition 3.6(i)). Moreover, there exists an assignment*

$$\mathrm{Cycle}^n(\Pi_1) \ni J \mapsto \mathfrak{syn}_{I,J}$$

—where $\mathfrak{syn}_{I,J}$ *denotes an* I*-conjugacy class of isomorphisms* $\Pi_{\mathrm{tpd}} \xrightarrow{\sim} \mathfrak{C}_I(J)$*— such that:*

(a) $\mathfrak{syn}_{I,J}$ *maps* I_{tpd} *bijectively onto* I*,*
(b) $\mathfrak{syn}_{I,J}$ *maps the subgroups* J^*_{tpd}, J^{**}_{tpd} *bijectively onto lifting cycle-subgroups of* $\mathfrak{C}_I(J)$ *(cf. Definition 3.6(ii)), and*
(c) *for* $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}}$*, the diagram (of* I_{tpd}*-,* I*-conjugacy classes of isomorphisms)*

$$\begin{tikzcd} \Pi_{\mathrm{tpd}} \arrow[r] \arrow[d, "\mathfrak{syn}_{I,J}"'] & \Pi_{\mathrm{tpd}} \arrow[d, "\mathfrak{syn}_{I,\alpha_1(J)}"] \\ \mathfrak{C}_I(J) \arrow[r] & \mathfrak{C}_I(\alpha_1(J)) \end{tikzcd}$$

—where the upper horizontal arrow is the (uniquely determined—cf. the commensurable terminality of I_{tpd} *in* Π_{tpd} *discussed in [18, Prop. 1.2(ii)])* I_{tpd}*-conjugacy class of automorphisms of* Π_{tpd} *that lifts* $\mathfrak{T}_{\Pi_{\mathrm{tpd}}}(\alpha)$ *(cf. [10, Def. 3.19]) and preserves* I_{tpd}*; the lower horizontal arrow is the* I*-conjugacy class of isomorphisms induced by* α_2 *(cf. the "*$\mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}}$*-equivariance" mentioned above)—commutes up to possible composition with the cycle symmetry of* $\mathfrak{C}_I(\alpha_1(J))$ *associated with* I *(cf. Definition 3.8).*

Finally, the assignment

$$J \mapsto \mathfrak{syn}_{I,J}$$

is uniquely determined, up to possible composition with cycle symmetries, by these conditions (a)–(c).

(ii) *Let* $n \geq 4$ *be a positive integer,* $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}}$*, and* $J \in \mathrm{Cycle}^n(\Pi_1)$*. Then there exists an automorphism* $\beta \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}}$ *such that the FC-admissible outer automorphism of* Π_3 *determined by* β_3 *lies in the kernel of the tripod homomorphism* $\mathfrak{T}_{\Pi_{\mathrm{tpd}}}$ *of [10, Def. 3.19], and, moreover,* $\alpha_1(J) = \beta_1(J)$*. Finally, the diagram (of* I_{tpd}*-, I-conjugacy classes of isomorphisms)*

$$\begin{tikzcd} \Pi_{\mathrm{tpd}} \arrow[r, equal] \arrow[d, "\mathfrak{syn}_{I,J}"'] & \Pi_{\mathrm{tpd}} \arrow[d, "\mathfrak{syn}_{I,\alpha_1(J)} = \mathfrak{syn}_{I,\beta_1(J)}"] \\ \mathfrak{C}_I(J) \arrow[r] & \mathfrak{C}_I(\alpha_1(J)) = \mathfrak{C}_I(\beta_1(J)) \end{tikzcd}$$

—where the lower horizontal arrow is the isomorphism induced by β_2 *(cf. the "*$\mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}}$*-equivariance" mentioned in (i))—commutes up to possible composition with the cycle symmetry of* $\mathfrak{C}_I(\alpha_1(J)) = \mathfrak{C}_I(\beta_1(J))$ *associated with I.*

§0. Notations and conventions

Sets: Let S be a finite set. Then we shall write $S^\sharp$ for the cardinality of S.

Let S be a set equipped with an action by a group G. Then we shall write $S^G \subseteq S$ for the subset consisting of elements of S fixed by the action of G on S.

Numbers: Write $\mathfrak{Primes}$ for the set of all prime numbers. Let Σ be a set of prime numbers. Then we shall refer to a nonzero integer n as a Σ-*integer* if every prime divisor of n is contained in Σ. The notation $\mathbb{R}$ will be used to denote the set, additive group, or field of real numbers, each of which we regard as being equipped with its usual topology. The notation $\mathbb{C}$ will be used to denote the set, additive group, or field of complex numbers, each of which we regard as being equipped with its usual topology.

Groups: Let Σ be a set of prime numbers, and let $f\colon G \to H$ be a homomorphism (resp. outer homomorphism) of groups. Then we shall say that f is Σ-*compatible* if the homomorphism (resp. outer homomorphism) $f^\Sigma\colon G^\Sigma \to H^\Sigma$ between pro-Σ completions induced by f is injective. Note that one verifies easily that if G is a group, and $H \subseteq G$ is a subgroup of G of finite index, then the natural inclusion $H \hookrightarrow G$ is $\mathfrak{Primes}$-compatible. If G is a topological group, then we shall write

$$G^{\mathrm{ab}}$$

for the *abelianization* of G, that is, the quotient of G by the closed normal subgroup of G generated by the commutators of G. If G is a profinite group, then we shall write

$$G \twoheadrightarrow G^{\Sigma\text{-ab-free}}$$

for the maximal pro-Σ abelian torsion-free quotient of G. We shall use the terms *normally terminal* and *commensurably terminal* as they are defined in the discussion entitled "Topological groups" in [9, §0]. If I, $J \subseteq G$ are closed subgroups of a topological group G, then we shall write

$$I \prec J$$

if some open subgroup of I is contained in J.

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§1. The combinatorial section conjecture

In the present section, we study outer representations of ENN-type (cf. Definition 1.7(i) below) on the fundamental group of a semi-graph of anabelioids of PSC-type (cf. [18, Def. 1.1(i)]). Roughly speaking, such outer representations may be thought of as an abstract combinatorial version of the natural outer representation of the maximal tamely ramified quotient of the absolute Galois group of a complete local field on the logarithmic fundamental group of the geometric special fiber of a stable model of a pointed stable curve over the complete local field. By comparison to the outer representation of NN-type studied in [8], outer representations of ENN-type correspond to the situation in which the residue field of the complete local field under consideration is not necessarily separably closed. Such outer representations of ENN-type give rise to a surjection of profinite groups, which corresponds, in the case of pointed stable curves over complete local fields, to the surjection from the arithmetic fundamental group to (some quotient of) the absolute Galois group of the base field. Our first main result (cf. Theorem 1.13(i) below) asserts that, under the additional assumption that the associated cyclotomic character has open image, any section of this surjection necessarily admits a fixed point (i.e., a fixed vertex or edge). This "combinatorial section conjecture" is obtained as an immediate consequence of an essentially classical result concerning fixed points of group actions on graphs (cf. Lemma 1.6 below). By applying this existence of fixed points, we show that there is a natural bijection between conjugacy classes of profinite sections and conjugacy classes of tempered sections (cf. Theorem 1.13(iii) below) and derive a rather strong version of the combinatorial Grothendieck conjecture (cf. [8, Th. A] and [10, Th. 1.9]) for cyclotomically full outer representations of ENN-type (cf. Corollary 1.14 below). We also observe in passing that a generalization of the main result of [26] may be obtained as a consequence of the theory discussed in the present section (cf. Corollary 1.15 below).

In the present section, let Σ be a nonempty set of prime numbers and let $\mathcal{G}$ be a semi-graph of anabelioids of pro-Σ PSC-type (cf. [18, Def. 1.1(i)]). Write $\mathbb{G}$ for the underlying semi-graph of $\mathcal{G}$, $\Pi_{\mathcal{G}}$ for the (pro-Σ) fundamental group of $\mathcal{G}$, and $\Pi_{\mathcal{G}}^{\mathrm{tp}}$ for the tempered fundamental group of $\mathcal{G}$ (cf. [17, Exam. 2.10] and the discussion preceding [17, Prop. 3.6]). Thus, we have a natural outer injection

$$\Pi_{\mathcal{G}}^{\mathrm{tp}} \hookrightarrow \Pi_{\mathcal{G}}$$

(cf. [11, Lem. 3.2(i)] and the proof of [11, Prop. 3.3(i) and (ii)]). Let us write

$$\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}, \quad \widetilde{\mathcal{G}}^{\mathrm{tp}} \longrightarrow \mathcal{G}$$

for the universal pro-Σ and pro-tempered coverings of $\mathcal{G}$ corresponding to $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{G}}^{\mathrm{tp}}$, and

$$\mathrm{VCN}(\widetilde{\mathcal{G}}) \overset{\mathrm{def}}{=} \varprojlim \mathrm{VCN}(\mathcal{H}), \quad \mathrm{VCN}(\widetilde{\mathcal{G}}^{\mathrm{tp}}) \overset{\mathrm{def}}{=} \varprojlim \mathrm{VCN}(\mathcal{H}^{\mathrm{tp}}),$$

where $\mathcal{H}$ (resp. $\mathcal{H}^{\mathrm{tp}}$) ranges over the subcoverings of $\widetilde{\mathcal{G}} \to \mathcal{G}$ (resp. $\widetilde{\mathcal{G}}^{\mathrm{tp}} \to \mathcal{G}$) corresponding to open subgroups of $\Pi_{\mathcal{G}}$ (resp. $\Pi_{\mathcal{G}}^{\mathrm{tp}}$), and $\mathrm{VCN}(-)$ denotes the "$\mathrm{VCN}(-)$" of the underlying semi-graph of the semi-graph of anabelioids in parentheses (cf. Definition 1.1(i) below and [8, Def. 1.1(iii)]).

We begin by reviewing certain well-known facts concerning semi-graphs and group actions on semi-graphs.

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Definition 1.1. Let Γ be a semi-graph (cf. the discussion at the beginning of [17, §1]).

(i) We shall write $\mathrm{Vert}(\Gamma)$ (resp. $\mathrm{Cusp}(\Gamma)$; $\mathrm{Node}(\Gamma)$) for the set of vertices (resp. open edges, i.e., "cusps"; closed edges, i.e., "nodes") of Γ. We shall write $\mathrm{Edge}(\Gamma) \stackrel{\mathrm{def}}{=} \mathrm{Cusp}(\Gamma) \sqcup \mathrm{Node}(\Gamma)$; $\mathrm{VCN}(\Gamma) \stackrel{\mathrm{def}}{=} \mathrm{Vert}(\Gamma) \sqcup \mathrm{Edge}(\Gamma)$.

(ii) We shall write

$$\mathcal{V}_\Gamma \colon \mathrm{Edge}(\Gamma) \longrightarrow 2^{\mathrm{Vert}(\Gamma)}$$
$$(\text{respectively}, \mathcal{C}_\Gamma \colon \mathrm{Vert}(\Gamma) \longrightarrow 2^{\mathrm{Cusp}(\Gamma)};$$
$$\mathcal{N}_\Gamma \colon \mathrm{Vert}(\Gamma) \longrightarrow 2^{\mathrm{Node}(\Gamma)};$$
$$\mathcal{E}_\Gamma \colon \mathrm{Vert}(\Gamma) \longrightarrow 2^{\mathrm{Edge}(\Gamma)})$$

(cf. (i); the discussion entitled "Sets" in [9, §0]) for the map obtained by sending $e \in \mathrm{Edge}(\Gamma)$ (resp. $v \in \mathrm{Vert}(\Gamma)$; $v \in \mathrm{Vert}(\Gamma)$; $v \in \mathrm{Vert}(\Gamma)$) to the set of vertices (resp. open edges; closed edges; edges) of Γ to which e abuts (resp. which abut to v; which abut to v; which abut to v). For simplicity, we shall write $\mathcal{V}$ (resp $\mathcal{C}$; $\mathcal{N}$; $\mathcal{E}$) instead of $\mathcal{V}_\Gamma$ (resp $\mathcal{C}_\Gamma$; $\mathcal{N}_\Gamma$; $\mathcal{E}_\Gamma$) when there is no danger of confusion.

(iii) Let n be a nonnegative integer; v, $w \in \mathrm{Vert}(\Gamma)$ (cf. (i)). Then we shall write $\delta(v,w) \le n$ if the following conditions are satisfied:

- If $n = 0$, then $v = w$.
- If $n \ge 1$, then either $\delta(v,w) \le n-1$ or there exist n closed edges $e_1, \ldots, e_n \in \mathrm{Node}(\Gamma)$ of Γ (cf. (i)) and $n+1$ vertices $v_0, \ldots, v_n \in \mathrm{Vert}(\Gamma)$ of Γ such that $v_0 = v$, $v_n = w$, and, for $1 \le i \le n$, it holds that $\mathcal{V}(e_i) = \{v_{i-1}, v_i\}$ (cf. (ii)).

Moreover, we shall write $\delta(v,w) = n$ if $\delta(v,w) \le n$, but $\delta(v,w) \not\le n-1$. If $\delta(v,w) = n$, then we shall say that the *distance between v and w* is equal to n.

Definition 1.2. Let Γ be a semi-graph.

(i) Let G be a group that acts on Γ. Then (by a slight abuse of notation, relative to the notation defined in the discussion entitled "Sets" in §0) we shall write

$$\Gamma^G$$

for the semi-graph (i.e., the "G-invariant portion of Γ") defined as follows:

- We take $\mathrm{Vert}(\Gamma^G)$ to be $\mathrm{Vert}(\Gamma)^G$ (cf. Definition 1.1(i); the discussion entitled "Sets" in §0).
- We take $\mathrm{Edge}(\Gamma^G)$ to be $\mathrm{Edge}(\Gamma)^G$ (cf. Definition 1.1(i); the discussion entitled "Sets" in §0).
- Let $e \in \mathrm{Edge}(\Gamma^G) = \mathrm{Edge}(\Gamma)^G$. Then the *coincidence map*

$$\zeta_e \colon e \longrightarrow \mathrm{Vert}(\Gamma^G) \cup \{\mathrm{Vert}(\Gamma^G)\}$$

of Γ^G (cf. item (3) of the discussion at the beginning of [17, §1]) is defined as follows: write $\zeta^\Gamma_e \colon e \to \mathrm{Vert}(\Gamma) \cup \{\mathrm{Vert}(\Gamma)\}$ for the coincidence map associated with Γ. Then, for $b \in e$, if $b \in e^G$ and $\zeta^\Gamma_e(b) \in \mathrm{Vert}(\Gamma)^G$ (resp. if either $b \notin e^G$ or $\zeta^\Gamma_e(b) \notin \mathrm{Vert}(\Gamma)^G$), then we set $\zeta_e(b) \stackrel{\mathrm{def}}{=} \zeta^\Gamma_e(b)$ (resp. $\stackrel{\mathrm{def}}{=} \mathrm{Vert}(\Gamma^G)$). In particular, it holds that $\mathcal{V}_{\Gamma^G}(e) = \mathcal{V}_\Gamma(e) \cap \mathrm{Vert}(\Gamma)^G$ (cf. Definition 1.1(ii)) whenever it holds either that Γ is *untangled* (i.e., every node abuts to two distinct vertices—cf. the discussion entitled "Semi-graphs" in [8, §0]) or that G acts on Γ without inversion (i.e., that if $e \in \mathrm{Edge}(\Gamma)^G$, then $e = e^G$).

(ii) We shall write

$$\Gamma^{\div}$$

for the semi-graph (i.e., the result of "subdividing" Γ) defined as follows:

- We take $\mathrm{Vert}(\Gamma^{\div})$ to be $\mathrm{Vert}(\Gamma) \sqcup \mathrm{Edge}(\Gamma)$.
- We take $\mathrm{Edge}(\Gamma^{\div})$ to be the set of branches of Γ.
- Let b be a branch of an edge e of Γ. Write $e(b) \in \mathrm{Edge}(\Gamma^{\div})$, $v(e) \in \mathrm{Vert}(\Gamma^{\div})$ for the edge and vertex of $\Gamma^{\div}$ corresponding to b, e, respectively. If b abuts, relative to Γ, to a vertex $v \in \mathrm{Vert}(\Gamma)$, then we take the edge $e(b)$ to be a node that abuts to $v(e)$ and the vertex of $\Gamma^{\div}$ corresponding to $v \in \mathrm{Vert}(\Gamma)$. If b does not abut, relative to Γ, to a vertex of Γ, then we take the edge $e(b)$ to be a cusp that abuts to $v(e)$.

DEFINITION 1.3. Let Γ be a semi-graph, and let $\Gamma_0 \subseteq \Gamma$ be a sub-semi-graph (cf. [17, the discussion following the figure entitled "A Typical Semi-graph"]) of Γ.

(i) We shall write

$$\Gamma_0^{\multimap} \subseteq \Gamma$$

for the sub-semi-graph of Γ (i.e., whenever a suitable condition is satisfied [cf. Lemma 1.4(v) below], a sort of "open neighborhood" of Γ_0) whose sets of vertices and edges are defined as follows. (Here, we recall that it follows immediately from the definition of a sub-semi-graph that a sub-semi-graph is completely determined by its sets of vertices and edges.)

- We take $\mathrm{Vert}(\Gamma_0^{\multimap})$ to be $\mathrm{Vert}(\Gamma_0)$.
- We take $\mathrm{Edge}(\Gamma_0^{\multimap})$ to be the set of edges e of Γ such that $\mathcal{V}_\Gamma(e) \cap \mathrm{Vert}(\Gamma_0) \neq \emptyset$.

(ii) We shall write

$$\Gamma_0^{\notin} \subseteq \Gamma$$

for the sub-semi-graph of Γ whose sets of vertices and edges are taken to be $\mathrm{Vert}(\Gamma) \setminus \mathrm{Vert}(\Gamma_0)$, $\mathrm{Edge}(\Gamma) \setminus \mathrm{Edge}(\Gamma_0)$, respectively.

(iii) We shall write $\Gamma_0^{\notin\multimap} \stackrel{\mathrm{def}}{=} (\Gamma_0^{\notin})^{\multimap}$ (cf. (i) and (ii)).

(iv) We shall say that an edge e of Γ is a Γ_0-*bridge* if $\mathcal{V}_\Gamma(e) \cap \mathrm{Vert}(\Gamma_0)$, $\mathcal{V}_\Gamma(e) \cap \mathrm{Vert}(\Gamma_0^{\notin}) \neq \emptyset$. (Thus, one verifies easily that every Γ_0-bridge is a node.) We shall write $\mathrm{Brdg}(\Gamma_0 \subseteq \Gamma) \subseteq \mathrm{Node}(\Gamma)$ for the set of Γ_0-bridges of Γ. By abuse of notation, we shall write $\mathrm{Brdg}(\Gamma_0 \subseteq \Gamma) \subseteq \Gamma$ for the sub-semi-graph of Γ whose sets of vertices and edges are taken to be $\emptyset$ (i.e., the empty set), $\mathrm{Brdg}(\Gamma_0 \subseteq \Gamma) \subseteq \mathrm{Node}(\Gamma)$, respectively.

LEMMA 1.4 (Basic properties of sub-semi-graphs). *Let Γ be a semi-graph, $\Gamma_0 \subseteq \Gamma$ a sub-semi-graph (cf. [17, the discussion following the figure entitled "A Typical Semi-graph"]) of Γ, G a group, and $\rho\colon G \to \mathrm{Aut}(\Gamma)$ an action of G on Γ. Then the following hold:*

(i) *Suppose either that Γ is untangled or that G acts on Γ without inversion. Then the semi-graph Γ^G (cf. Definition 1.2(i)) may be regarded, in a natural way, as a sub-semi-graph of Γ.*

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(ii) *Suppose that G acts on Γ without inversion, and that every edge of Γ abuts to at least one vertex of Γ. Then every edge of Γ^G abuts to at least one vertex of Γ^G.*

(iii) *The semi-graph $\Gamma^{\div}$ (cf. Definition 1.2(ii)) is untangled.*

(iv) *There exists a natural injection $\mathrm{Aut}(\Gamma) \hookrightarrow \mathrm{Aut}(\Gamma^{\div})$. Moreover, the resulting action $\rho^{\div}$ of G on $\Gamma^{\div}$ (i.e., the composite $G \xrightarrow{\rho} \mathrm{Aut}(\Gamma) \hookrightarrow \mathrm{Aut}(\Gamma^{\div})$) is an action without inversion. Finally, it holds that $\Gamma^G = \emptyset$ if and only if $(\Gamma^{\div})^G = \emptyset$.*

(v) *Suppose that every edge of Γ_0 abuts to at least one vertex of Γ_0. Then Γ_0 may be regarded, in a natural way, as a sub-semi-graph of $\Gamma_0^{-\circ}$ (cf. Definition 1.3(i)).*

(vi) *We have an equality of subsets of $\mathrm{Edge}(\Gamma)$:*

$$\mathrm{Edge}(\Gamma_0^{-\circ}) \;\cap\; \mathrm{Edge}(\Gamma_0^{\not\in -\circ}) \;=\; \mathrm{Brdg}(\Gamma_0 \subseteq \Gamma).$$

Proof. The assertions of Lemma 1.4 follow immediately from the various definitions involved. ∎

LEMMA 1.5 (Sub-semi-graphs of invariants). *In the situation of Lemma 1.4, suppose either that Γ is untangled or that G acts on Γ without inversion. Suppose, moreover, that the sub-semi-graph $\Gamma_0 \subseteq \Gamma$ is a connected component of the sub-semi-graph $\Gamma^G \subseteq \Gamma$ (cf. Lemma 1.4(i)). Then the following hold:*

(i) *The action ρ naturally determines actions of G on Γ_0, $\Gamma_0^{-\circ}$, $\Gamma_0^{\not\in -\circ}$, respectively.*

(ii) *The intersection of $\Gamma_0^{-\circ} \subseteq \Gamma$ with any connected component of $\Gamma^G \subseteq \Gamma$ that is $\neq \Gamma_0$ is empty.*

(iii) *We have an equality of subsets of $\mathrm{Edge}(\Gamma)$:*

$$\mathrm{Edge}(\Gamma^G) \;\cap\; \mathrm{Brdg}(\Gamma_0^{-\circ} \subseteq \Gamma) \;=\; \emptyset.$$

Proof. The assertions of Lemma 1.5 follow immediately from the various definitions involved. ∎

LEMMA 1.6 (Existence of fixed points). *Let Γ be a finite connected (hence nonempty) semi-graph, let G be a finite solvable group whose order is a Σ-integer (cf. the discussion entitled "Numbers" in §0), and let*

$$\rho\colon G \longrightarrow \mathrm{Aut}(\Gamma)$$

be an action of G on Γ. Write $\Pi_\Gamma^{\mathrm{disc}}$ for the (discrete) topological fundamental group of Γ; Π_Γ^{Σ} for the pro-Σ completion of $\Pi_\Gamma^{\mathrm{disc}}$; $\widetilde{\Gamma}^{\mathrm{disc}} \to \Gamma$, $\widetilde{\Gamma}^{\Sigma} \to \Gamma$ for the discrete, pro-Σ universal coverings of Γ corresponding to $\Pi_\Gamma^{\mathrm{disc}}$, Π_Γ^{Σ}, respectively. Let $\square \in \{\mathrm{disc}, \Sigma\}$. Write $\mathrm{Aut}(\widetilde{\Gamma}^{\square} \to \Gamma) \subseteq \mathrm{Aut}(\widetilde{\Gamma}^{\square})$ for the group of automorphisms $\widetilde{\alpha}$ of $\widetilde{\Gamma}^{\square}$ such that $\widetilde{\alpha}$ lies over a(n) (necessarily unique) automorphism α of Γ;

$$\begin{array}{ccc} \mathrm{Aut}(\widetilde{\Gamma}^{\square} \to \Gamma) & \longrightarrow & \mathrm{Aut}(\Gamma) \\ \widetilde{\alpha} & \mapsto & \alpha \end{array}$$

for the resulting natural homomorphism;

$$\Pi_{\Gamma//G}^{\square} \overset{\mathrm{def}}{=} \mathrm{Aut}(\widetilde{\Gamma}^{\square} \to \Gamma) \times_{\mathrm{Aut}(\Gamma)} G$$

for the fiber product of the natural homomorphism $\mathrm{Aut}(\widetilde{\Gamma}^{\square} \to \Gamma) \to \mathrm{Aut}(\Gamma)$ and the action $\rho\colon G \to \mathrm{Aut}(\Gamma)$. Thus, one verifies easily that $\Pi_{\Gamma//G}^{\square}$ fits into an exact sequence

$$1 \longrightarrow \Pi_\Gamma^{\square} \longrightarrow \Pi_{\Gamma//G}^{\square} \longrightarrow G \longrightarrow 1.$$

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Let $s\colon G \to \Pi^{\square}_{\Gamma//G}$ *be a section of the above exact sequence. Write* $\widetilde{\rho}^{\square}_{s}\colon G \to \mathrm{Aut}(\widetilde{\Gamma}^{\square})$ *for the action obtained by forming the composite* $G \xrightarrow{s} \Pi^{\square}_{\Gamma//G} \xrightarrow{\mathrm{pr}_1} \mathrm{Aut}(\widetilde{\Gamma}^{\square} \to \Gamma) \hookrightarrow \mathrm{Aut}(\widetilde{\Gamma}^{\square})$*. We shall say that a connected finite subcovering* $\Gamma_* \to \Gamma$ *of* $\widetilde{\Gamma}^{\Sigma} \to \Gamma$ *is G-compatible if* $\Gamma_* \to \Gamma$ *is Galois, and, moreover, the corresponding normal open subgroup of* Π^{Σ}_{Γ} *is preserved by the outer action of G, via* ρ*, on* Π^{Σ}_{Γ}*. If* $\Gamma_* \to \Gamma$ *is a G-compatible connected finite subcovering of* $\widetilde{\Gamma}^{\Sigma} \to \Gamma$*, then let us write* $\rho_{s,*}\colon G \to \mathrm{Aut}(\Gamma_*)$ *for the action of G on* Γ_* *determined by* $\widetilde{\rho}^{\square}_{s}$*;* Γ^{G}_{*} *for the semi-graph defined in Definition 1.2(i), with respect to the action* $\rho_{s,*}$*. (Thus, if* Γ*, hence also* Γ_**, is untangled, then* Γ^{G}_{*} *is a sub-semi-graph of* Γ_**—cf. Lemma 1.4(i).) Then the following hold:*

(i) *Suppose that* Γ *is untangled. Then, for each G-compatible connected finite subcovering* $\Gamma_* \to \Gamma$ *of* $\widetilde{\Gamma}^{\Sigma} \to \Gamma$*, the sub-semi-graph* $\Gamma^{G}_{*} \subseteq \Gamma_*$ *coincides with the disjoint union of some (possibly empty) collection of connected components of* $\Gamma_*|_{\Gamma^G} \stackrel{\mathrm{def}}{=} \Gamma_* \times_{\Gamma} \Gamma^{G} \subseteq \Gamma_*$*.*

(ii) *Suppose that* Γ *is untangled, and that G is isomorphic to* $\mathbb{Z}/l\mathbb{Z}$ *for some prime number* $l \in \Sigma$*. Then, for every G-compatible connected finite subcovering* $\Gamma_* \to \Gamma$ *of* $\widetilde{\Gamma}^{\Sigma} \to \Gamma$*, the sub-semi-graph* $\Gamma^{G}_{*} \subseteq \Gamma_*$ *is nonempty.*

(iii) *Suppose that* $\square = \mathrm{disc}$*. Write* $(\widetilde{\Gamma}^{\mathrm{disc}})^{G}$ *for the sub-semi-graph (cf. Lemma 1.4(i)) of (the necessarily untangled semi-graph!)* $\widetilde{\Gamma}^{\mathrm{disc}}$ *defined in Definition 1.2(i), with respect to the action* $\widetilde{\rho}^{\mathrm{disc}}_{s}$*. Then* $(\widetilde{\Gamma}^{\mathrm{disc}})^{G}$ *is nonempty and connected. If, moreover, we write* $(\Gamma^{G})_0 \subseteq \Gamma^{G}$ *for the image of the composite* $(\widetilde{\Gamma}^{\mathrm{disc}})^{G} \hookrightarrow \widetilde{\Gamma}^{\mathrm{disc}} \to \Gamma$*, then the resulting morphism* $(\widetilde{\Gamma}^{\mathrm{disc}})^{G} \to (\Gamma^{G})_0$ *is a (discrete) universal covering of* $(\Gamma^{G})_0$*.*

(iv) *Suppose that* $\square = \mathrm{disc}$ *(resp.* $\square = \Sigma$*). Then the set*

$$\mathrm{VCN}(\widetilde{\Gamma}^{\mathrm{disc}})^{G} \quad (\text{resp. } \mathrm{VCN}(\widetilde{\Gamma}^{\Sigma})^{G} \stackrel{\mathrm{def}}{=} \varprojlim \mathrm{VCN}(\Gamma_*)^{G}),$$

where, in the resp'd case, the projective limit is taken over the G-compatible connected finite subcoverings $\Gamma_* \to \Gamma$ *of* $\widetilde{\Gamma}^{\Sigma} \to \Gamma$*, is nonempty.*

(v) *Suppose that* $\square = \Sigma$*, that* Γ *is untangled, and that G is isomorphic to* $\mathbb{Z}/l\mathbb{Z}$ *for some prime number* $l \in \Sigma$*. Let* $(\Gamma^{G})_0 \subseteq \Gamma^{G}$ *be a (nonempty) connected component of* Γ^{G} *such that*

$$\mathrm{VCN}((\Gamma^{G})_0) \;\cap\; \mathrm{Im}\big(\mathrm{VCN}(\widetilde{\Gamma}^{\Sigma})^{G} \to \mathrm{VCN}(\Gamma)\big) \;\neq\; \emptyset$$

(cf. (iv)). Then there exists a G-compatible connected finite subcovering $\Gamma_* \to \Gamma$ *of* $\widetilde{\Gamma}^{\Sigma} \to \Gamma$ *such that the image of* $\Gamma^{G}_{*} \subseteq \Gamma_*$ *in* Γ *coincides with* $(\Gamma^{G})_0 \subseteq \Gamma^{G}$*.*

(vi) *Suppose that* $\square = \Sigma$*, and that* Γ *is untangled. Then the sub-pro-semi-graph* $(\widetilde{\Gamma}^{\Sigma})^{G}$ *of* $\widetilde{\Gamma}^{\Sigma}$ *determined by the projective system of sub-semi-graphs* Γ^{G}_{*}*—where* $\Gamma_* \to \Gamma$ *ranges over the G-compatible connected finite subcoverings of* $\widetilde{\Gamma}^{\Sigma} \to \Gamma$*—is nonempty and connected. If, moreover, we write* $(\Gamma^{G})_0 \subseteq \Gamma^{G}$ *for the image of the composite* $(\widetilde{\Gamma}^{\Sigma})^{G} \hookrightarrow \widetilde{\Gamma}^{\Sigma} \to \Gamma$*, then the resulting morphism* $(\widetilde{\Gamma}^{\Sigma})^{G} \to (\Gamma^{G})_0$ *is a pro-*Σ *universal covering of* $(\Gamma^{G})_0$*.*

Proof. First, we verify assertion (i). Let us first observe that one verifies immediately that there is an inclusion of sub-semi-graphs $\Gamma^{G}_{*} \subseteq \Gamma_*|_{\Gamma^G}$ (cf. Lemma 1.4(i)). Next, let us observe that it follows immediately from Lemma 1.4(iii) and (iv) that, by replacing Γ by $\Gamma^{\div}$, we may assume without loss of generality that G acts without inversion on Γ (which implies that G acts trivially on Γ^{G}—cf. Definition 1.2(i)). Thus, to complete the verification of assertion (i), it suffices to verify that the following assertion holds:

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Claim 1.6.A: Let $(\Gamma_*|_{\Gamma^G})_0 \subseteq \Gamma_*|_{\Gamma^G}$ be a connected component of $\Gamma_*|_{\Gamma^G}$ such that $(\Gamma_*|_{\Gamma^G})_0 \cap \Gamma_*^G \neq \emptyset$. Then $(\Gamma_*|_{\Gamma^G})_0 \subseteq \Gamma_*^G$.

To verify Claim 1.6.A, let us observe that since $(\Gamma_*|_{\Gamma^G})_0 \cap \Gamma_*^G \neq \emptyset$, the action $\rho_{s,*}$ of G on Γ_* stabilizes $(\Gamma_*|_{\Gamma^G})_0 \subseteq \Gamma_*$. In particular, we obtain an action of G on $(\Gamma_*|_{\Gamma^G})_0$ over Γ^G. Thus, since the action of G on Γ^G is trivial, and the composite $(\Gamma_*|_{\Gamma^G})_0 \hookrightarrow \Gamma_*|_{\Gamma^G} \to \Gamma^G$ is a connected finite covering of some connected component of Γ^G, again by our assumption that $(\Gamma_*|_{\Gamma^G})_0 \cap \Gamma_*^G \neq \emptyset$, we conclude that the action of G on $(\Gamma_*|_{\Gamma^G})_0$ is trivial, that is, that there is an inclusion of sub-semi-graphs $(\Gamma_*|_{\Gamma^G})_0 \subseteq \Gamma_*^G$. This completes the proof of Claim 1.6.A, hence also of assertion (i).

Next, we verify assertion (ii). One verifies immediately that we may assume without loss of generality that $\Gamma_* = \Gamma$. Now suppose that $\Gamma^G = \emptyset$. Then since $G \cong \mathbb{Z}/l\mathbb{Z}$, it follows that the action of G on Γ is free, which thus implies that the quotient map $\Gamma \twoheadrightarrow \Gamma/G$ is a covering of Γ/G. In particular, $\Pi^{\Sigma}_{\Gamma//G}$ is isomorphic to the pro-Σ completion of the topological fundamental group of the semi-graph Γ/G. Thus, the pro-Σ group $\Pi^{\Sigma}_{\Gamma//G}$ is free, hence, in particular, torsion-free. But this contradicts the existence of the section of the surjection $\Pi^{\Sigma}_{\Gamma//G} \twoheadrightarrow G$ determined by s. This completes the proof of assertion (ii).

Next, we verify the resp'd portion of assertion (iv) (i.e., the assertion that $\mathrm{VCN}(\widetilde{\Gamma}^{\Sigma})^G \neq \emptyset$) in the case where G is isomorphic to $\mathbb{Z}/l\mathbb{Z}$ for some prime number $l \in \Sigma$. Let us first observe that it follows immediately from Lemma 1.4(iii) and (iv) that, by replacing Γ by $\Gamma^{\div}$, we may assume without loss of generality that Γ is untangled. Thus, the assertion that $\mathrm{VCN}(\widetilde{\Gamma}^{\Sigma})^G \neq \emptyset$ follows immediately from assertion (ii), together with the well-known elementary fact that a projective limit of nonempty finite sets is nonempty. This completes the proof of the assertion that $\mathrm{VCN}(\widetilde{\Gamma}^{\Sigma})^G \neq \emptyset$ in the case where G is isomorphic to $\mathbb{Z}/l\mathbb{Z}$ for some prime number $l \in \Sigma$.

Next, we verify assertion (iii). Let us first observe that since $\widetilde{\Gamma}^{\mathrm{disc}}$ is a tree, hence untangled, it follows from Lemma 1.4(i) that $(\widetilde{\Gamma}^{\mathrm{disc}})^G$ is a sub-semi-graph of $\widetilde{\Gamma}^{\mathrm{disc}}$. Next, let us observe that it follows immediately from Lemma 1.4(iv) that, by replacing Γ by $\Gamma^{\div}$, we may assume without loss of generality that G acts without inversion on Γ. Thus, the assertion that $(\widetilde{\Gamma}^{\mathrm{disc}})^G$ is nonempty and connected follows immediately from [17, Lem. 1.8(ii)]. The remainder of assertion (iii) follows from a similar argument to the argument applied in the proof of assertion (i). This completes the proof of assertion (iii). In particular, the unresp'd portion of assertion (iv) (i.e., the assertion that $\mathrm{VCN}(\widetilde{\Gamma}^{\mathrm{disc}})^G \neq \emptyset$) holds.

Next, we verify assertion (v). Let us first observe that, to verify assertion (v), it follows immediately from Lemma 1.4(iii) and (iv) that, by replacing Γ by $\Gamma^{\div}$, we may assume without loss of generality that the action ρ is an action without inversion, and that every edge of Γ abuts to at least one vertex of Γ. In particular, since (we have assumed that) $(\Gamma^G)_0 \neq \emptyset$, it follows from Lemma 1.4(ii) and (v) that $(\Gamma^G)_0^{-\circ} \neq \emptyset$ (cf. Definition 1.3(i)). Now if Γ^G is connected, then one verifies immediately that the trivial covering $\Gamma \xrightarrow{\mathrm{id}} \Gamma$ satisfies the condition imposed on "$\Gamma_* \to \Gamma$" in the statement of assertion (v). Thus, to complete the verification of assertion (v), we may assume without loss of generality that Γ^G is not connected, hence (cf. Lemma 1.4(ii)) contains at least one vertex $\notin \mathrm{Vert}((\Gamma^G)_0)$. In particular, $(\Gamma^G)_0^{\notin -\circ} \neq \emptyset$ (cf. Definition 1.3(iii)).

Write $((\Gamma^G)_0^{-\circ})^{\amalg} \to (\Gamma^G)_0^{-\circ}$ for the trivial $\mathbb{Z}/l\mathbb{Z}$-covering obtained by taking a disjoint union of copies of $(\Gamma^G)_0^{-\circ}$ indexed by the elements of $\mathbb{Z}/l\mathbb{Z}$; $((\Gamma^G)_0^{\notin -\circ})^{\amalg} \to (\Gamma^G)_0^{\notin -\circ}$ for the trivial $\mathbb{Z}/l\mathbb{Z}$-covering obtained by taking a disjoint union of copies of $(\Gamma^G)_0^{\notin -\circ}$ indexed by

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the elements of $\mathbb{Z}/l\mathbb{Z}$. Then the natural actions of G on $((\Gamma^G)_0^{-\circ})^{\amalg}$, $((\Gamma^G)_0^{\notin -\circ})^{\amalg}$ (cf. Lemma 1.5(i)) determine natural actions of $G \times \mathbb{Z}/l\mathbb{Z}$ on $((\Gamma^G)_0^{-\circ})^{\amalg}$, $((\Gamma^G)_0^{\notin -\circ})^{\amalg}$, that is, we have homomorphisms

$$\rho^{-\circ}\colon G \times \mathbb{Z}/l\mathbb{Z} \longrightarrow \mathrm{Aut}\big(((\Gamma^G)_0^{-\circ})^{\amalg}\big),$$

$$\rho^{\notin -\circ}\colon G \times \mathbb{Z}/l\mathbb{Z} \longrightarrow \mathrm{Aut}\big(((\Gamma^G)_0^{\notin -\circ})^{\amalg}\big).$$

Let $\phi\colon G \xrightarrow{\sim} \mathbb{Z}/l\mathbb{Z}$ be an isomorphism. Write

$$\begin{array}{rccccc} \rho_\phi^{\notin -\circ}\colon & G \times \mathbb{Z}/l\mathbb{Z} & \longrightarrow & G \times \mathbb{Z}/l\mathbb{Z} & \xrightarrow{\rho^{\notin -\circ}} & \mathrm{Aut}\big(((\Gamma^G)_0^{\notin -\circ})^{\amalg}\big) \\ & (a,b) & \mapsto & (a, \phi(a)+b) & & \end{array}$$

for the composite of $\rho^{\notin -\circ}$ with the homomorphism described in the second line of the display.

Next, for $e \in \mathrm{Brdg} \stackrel{\mathrm{def}}{=} \mathrm{Brdg}((\Gamma^G)_0 \subseteq \Gamma)$ (cf. Definition 1.3(iv)), write $G \cdot e \subseteq \mathrm{Edge}((\Gamma^G)_0^{-\circ})$ for the G-orbit of e. Then it is immediate that $G \cdot e \subseteq \mathrm{Brdg}$; moreover, since $G \cong \mathbb{Z}/l\mathbb{Z}$, it follows immediately from Lemma 1.5(iii) that $G \cdot e$ is a G-torsor. Next, let us write

$$((\Gamma^G)_0^{-\circ})^{\amalg}|_{G\cdot e} \stackrel{\mathrm{def}}{=} ((\Gamma^G)_0^{-\circ})^{\amalg} \times_{(\Gamma^G)_0^{-\circ}} G \cdot e,$$

$$((\Gamma^G)_0^{\notin -\circ})^{\amalg}|_{G\cdot e} \stackrel{\mathrm{def}}{=} ((\Gamma^G)_0^{\notin -\circ})^{\amalg} \times_{(\Gamma^G)_0^{\notin -\circ}} G \cdot e$$

(cf. Lemma 1.4(vi)). Then one verifies easily from the various definitions involved that the following hold:

(a) The actions $\rho^{-\circ}$, $\rho_\phi^{\notin -\circ}$ of $G \times \mathbb{Z}/l\mathbb{Z}$ on $((\Gamma^G)_0^{-\circ})^{\amalg}$, $((\Gamma^G)_0^{\notin -\circ})^{\amalg}$ determine actions on these fibers $((\Gamma^G)_0^{-\circ})^{\amalg}|_{G\cdot e}$, $((\Gamma^G)_0^{\notin -\circ})^{\amalg}|_{G\cdot e}$.

(b) These fibers $((\Gamma^G)_0^{-\circ})^{\amalg}|_{G\cdot e}$, $((\Gamma^G)_0^{\notin -\circ})^{\amalg}|_{G\cdot e}$ are $(G \times \mathbb{Z}/l\mathbb{Z})$-torsors with respect to the actions of (a).

(c) There is a natural isomorphism of semi-graphs $((\Gamma^G)_0^{-\circ})^{\amalg}|_{G\cdot e} \xrightarrow{\sim} ((\Gamma^G)_0^{\notin -\circ})^{\amalg}|_{G\cdot e}$ (cf. Lemma 1.4(vi)), which we shall use to identify these two semi-graphs.

(d) Let $e_{\mathrm{base}} \in ((\Gamma^G)_0^{-\circ})^{\amalg}|_{G\cdot e} = ((\Gamma^G)_0^{\notin -\circ})^{\amalg}|_{G\cdot e}$ (cf. (c)) be a lifting of $e \in \mathrm{Brdg}$. Then there is a uniquely determined (cf. (b)) isomorphism

$$\iota_{e_{\mathrm{base}}}\colon ((\Gamma^G)_0^{-\circ})^{\amalg}|_{G\cdot e} \xrightarrow{\sim} ((\Gamma^G)_0^{\notin -\circ})^{\amalg}|_{G\cdot e}$$

of $(G \times \mathbb{Z}/l\mathbb{Z})$-torsors (cf. (b)) that maps e_{base} to e_{base}.

Let $\mathbb{B}$ be a collection of elements "e_{base}" as in (d) such that the map $e_{\mathrm{base}} \mapsto e$ determines a bijection between $\mathbb{B}$ and the set of G-orbits of Brdg. Thus, by gluing $((\Gamma^G)_0^{-\circ})^{\amalg}$ to $((\Gamma^G)_0^{\notin -\circ})^{\amalg}$ by means of the collection of isomorphisms $\{\iota_{e_{\mathrm{base}}}\}_{e_{\mathrm{base}} \in \mathbb{B}}$ of (d) (cf. Lemma 1.4(vi)), we obtain a finite covering $\Gamma_* \to \Gamma$, together with an action of $G \times \mathbb{Z}/l\mathbb{Z}$ on Γ_* (i.e., obtained by gluing the actions $\rho^{-\circ}$, $\rho_\phi^{\notin -\circ}$), such that the morphism $\Gamma_* \to \Gamma$ is equivariant with respect to this action of $G \times \mathbb{Z}/l\mathbb{Z}$ on Γ_* and the action of $G \times \mathbb{Z}/l\mathbb{Z}$ on Γ obtained by composing the projection $G \times \mathbb{Z}/l\mathbb{Z} \to G$ with the given action of G on Γ. Next, let us observe that since ϕ is an isomorphism, and both $(\Gamma^G)_0$ and $(\Gamma^G)_0^{\notin -\circ}$ contain vertices fixed by G (cf. the discussion at the beginning of the present proof of assertion (v)), one verifies immediately—for example, by considering the orbit by the action of $G \times \{1\} (\subseteq G \times \mathbb{Z}/l\mathbb{Z})$

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of some lifting to Γ_* (which may be chosen to pass through an element of $\mathbb{B}$) of a path of minimal length between such vertices fixed by G—that Γ_* is connected. Moreover, it follows from the definition of Γ_* that the covering $\Gamma_* \to \Gamma$ is Galois, G-compatible, and equipped with a natural isomorphism $\mathrm{Gal}(\Gamma_*/\Gamma) \xrightarrow{\sim} \mathbb{Z}/l\mathbb{Z}$; in particular, $\widetilde{\Gamma}^{\Sigma} \to \Gamma$ factors as a composite $\widetilde{\Gamma}^{\Sigma} \to \Gamma_* \to \Gamma$.

Next, let us observe that, for each $g \in G$, the automorphism α_g of Γ_* obtained by considering the difference between $\rho_{s,*}(g)$ and the action of g (i.e., $(g,0) \in G \times \mathbb{Z}/l\mathbb{Z}$) on Γ_* defined above is an automorphism over Γ. Moreover, it follows immediately from our assumption that

$$\mathrm{VCN}((\Gamma^G)_0) \ \cap \ \mathrm{Im}\big(\mathrm{VCN}(\widetilde{\Gamma}^{\Sigma})^G \to \mathrm{VCN}(\Gamma)\big) \ \neq \ \emptyset$$

that α_g fixes an element of $\mathrm{VCN}(\Gamma_*)$ that maps to $\mathrm{VCN}((\Gamma^G)_0) \subseteq \mathrm{VCN}(\Gamma)$. But this implies that α_g is trivial, that is, that the action $\rho_{s,*}$ of G coincides with the action of $G(= G \times \{0\} \subseteq G \times \mathbb{Z}/l\mathbb{Z})$ on Γ_* defined above.

On the other hand, since ϕ is an isomorphism, it follows that $(\Gamma_*)^G \subseteq \Gamma_*$ is contained in the sub-semi-graph of Γ_* determined by $((\Gamma^G)_0^{-\circ})^{\coprod}$. In particular, it follows immediately from Lemma 1.5(ii) that the image of $\Gamma_*^G \subseteq \Gamma_*$ in Γ is contained in $(\Gamma^G)_0 \subseteq \Gamma^G$. Thus, it follows immediately from assertion (i) that the image of $\Gamma_*^G \subseteq \Gamma_*$ in Γ coincides with $(\Gamma^G)_0 \subseteq \Gamma^G$. This completes the proof of assertion (v).

Next, we verify assertion (vi). First, we claim that the following assertion holds:

> Claim 1.6.B: If G is isomorphic to $\mathbb{Z}/l\mathbb{Z}$ for some prime number $l \in \Sigma$, then assertion (vi) holds.

Indeed, it follows from the resp'd portion of assertion (iv) (i.e., the assertion that $\mathrm{VCN}(\widetilde{\Gamma}^{\Sigma})^G \neq \emptyset$) in the case where G is isomorphic to $\mathbb{Z}/l\mathbb{Z}$ for some prime number $l \in \Sigma$ (i.e., the case that has already been verified!) that $(\widetilde{\Gamma}^{\Sigma})^G \neq \emptyset$. On the other hand, it follows immediately from assertion (v) (i.e., by allowing "Γ" to vary among the G-compatible connected finite subcoverings of $\widetilde{\Gamma}^{\Sigma} \to \Gamma$) that $(\widetilde{\Gamma}^{\Sigma})^G$ is connected. Thus, the final portion of assertion (vi) (in the case where G is isomorphic to $\mathbb{Z}/l\mathbb{Z}$ for some prime number $l \in \Sigma$) follows immediately from assertion (i) (and the evident pro-Σ version of [17, Prop. 2.5(i)]). This completes the proof of Claim 1.6.B.

Next, we verify assertion (vi) for arbitrary finite solvable G by induction on $G^\sharp$. Let us first observe that it follows immediately from Lemma 1.4(iii) and (iv) that, by replacing Γ by $\Gamma^{\div}$, we may assume without loss of generality that the action ρ is an action without inversion. Next, observe that since G is finite and solvable, there exists a normal subgroup $N \subseteq G$ of G such that G/N is a nontrivial finite group of prime order. Then it follows from the induction hypothesis that if we write $(\Gamma^N)_0 \subseteq \Gamma^N$ for the (nonempty, connected!) image of the composite $(\widetilde{\Gamma}^{\Sigma})^N \hookrightarrow \widetilde{\Gamma}^{\Sigma} \to \Gamma$, then the resulting morphism $(\widetilde{\Gamma}^{\Sigma})^N \to (\Gamma^N)_0$ is a pro-Σ universal covering of $(\Gamma^N)_0$, and, moreover, (since the action ρ is an action without inversion) N acts trivially on $(\widetilde{\Gamma}^{\Sigma})^N$. Next, let us observe that since N is normal in G, (one verifies immediately that) the action $\widetilde{\rho}_s^{\Sigma}$ of G on $\widetilde{\Gamma}^{\Sigma}$ preserves $(\widetilde{\Gamma}^{\Sigma})^N \subseteq \widetilde{\Gamma}^{\Sigma}$. Thus, by replacing $(\widetilde{\Gamma}^{\Sigma} \to \Gamma, G)$ by $((\widetilde{\Gamma}^{\Sigma})^N \to (\Gamma^N)_0, G/N)$ and applying Claim 1.6.B, we conclude that assertion (vi) holds for the given G. This completes the proof of assertion (vi).

Finally, we verify the resp'd portion of assertion (iv) (i.e., the assertion that $\mathrm{VCN}(\widetilde{\Gamma}^{\Sigma})^G \neq \emptyset$). Let us first observe that, to verify the assertion that $\mathrm{VCN}(\widetilde{\Gamma}^{\Sigma})^G \neq \emptyset$, it follows immediately from Lemma 1.4(iii) and (iv) that, by replacing Γ by $\Gamma^{\div}$, we may assume

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without loss of generality that Γ is untangled. Thus, the assertion that $\mathrm{VCN}(\widetilde{\Gamma}^{\Sigma})^G \neq \emptyset$ follows immediately from assertion (vi). This completes the proof of Lemma 1.6. $\square$

Remark 1.6.1. The conclusion of Lemma 1.6(vi) follows for an arbitrary (i.e., not necessarily solvable!) finite group G from [30, Ths. 2.8 and 2.10]. That is to say, the proof given above of Lemma 1.6(vi) may be regarded as an alternative proof of these results of [30] in the case where G is solvable. In this context, it is also perhaps of interest to observe that, by considering Lemma 1.6(vi) in the case where $\Sigma = \mathfrak{Primes}$ and "Γ" is taken to be some finite connected sub-semi-graph of $\widetilde{\Gamma}^{\mathrm{disc}}$ that is stabilized by the action of G (where we note that one verifies easily that $\widetilde{\Gamma}^{\mathrm{disc}}$ is a union of such sub-semi-graphs), one obtains an alternative proof of the classical result concerning actions of finite groups on trees quoted in the proofs of Lemma 1.6(iii) and [17, Lem. 1.8(ii)]—hence also alternative proofs of Lemma 1.6(iii) and [17, Lem. 1.8(ii)]—in the case where the finite group under consideration is solvable.

Remark 1.6.2.

(i) In the situation of Lemma 1.6, if G is isomorphic to $\mathbb{Z}/l^n\mathbb{Z}$ for some prime number $l \in \Sigma$ and a positive integer n, then the conclusion of the resp'd portion of Lemma 1.6(iv) may be verified by the following easier argument: since (as is well known) a projective limit of nonempty finite sets is nonempty, to verify the assertion that $\mathrm{VCN}(\widetilde{\Gamma}^{\Sigma})^G \neq \emptyset$, it suffices to verify that $\mathrm{VCN}(\Gamma_*)^G \neq \emptyset$ for every G-compatible connected finite subcovering $\Gamma_* \to \Gamma$ of $\widetilde{\Gamma}^{\Sigma} \to \Gamma$. Moreover, one verifies immediately that we may assume without loss of generality that $\Gamma_* = \Gamma$. Next, let us observe that it follows immediately from Lemma 1.4(iv) that, by replacing Γ by $\Gamma^{\div}$, we may assume without loss of generality that G acts on Γ without inversion. Write $H \subseteq G$ for the unique subgroup such that $Q \stackrel{\mathrm{def}}{=} G/H$ is of order l; $\Gamma_Q \stackrel{\mathrm{def}}{=} \Gamma/H$ for the "quotient semi-graph," that is, the semi-graph whose vertices, edges, and branches are, respectively, the H-orbits of the vertices, edges, and branches of Γ (cf. the fact that G acts on Γ without inversion). Then one verifies immediately that the natural morphism of semi-graphs $\Gamma \twoheadrightarrow \Gamma_Q$ determines an outer homomorphism

$$\Pi^{\Sigma}_{\Gamma//G} \longrightarrow \Pi^{\Sigma}_{\Gamma_Q//Q}$$

(cf. the notation of the statement of Lemma 1.6). Now since $\Pi^{\Sigma}_{\Gamma_Q}$ is a free pro-Σ group, hence torsion-free, it follows that the restriction $s(H) \to \Pi^{\Sigma}_{\Gamma_Q//Q}$ (which clearly factors through $\Pi^{\Sigma}_{\Gamma_Q} \subseteq \Pi^{\Sigma}_{\Gamma_Q//Q}$) of the outer homomorphism $\Pi^{\Sigma}_{\Gamma//G} \to \Pi^{\Sigma}_{\Gamma_Q//Q}$ to $s(H) \subseteq \Pi^{\Sigma}_{\Gamma//G}$ is trivial, hence that s determines a section $s_Q\colon Q \to \Pi^{\Sigma}_{\Gamma_Q//Q}$ of the natural surjection $\Pi^{\Sigma}_{\Gamma_Q//Q} \twoheadrightarrow Q$. In particular, by applying Lemma 1.6(ii), we thus conclude that $\mathrm{VCN}(\Gamma_Q)^Q \neq \emptyset$. Let $z_Q \in \mathrm{VCN}(\Gamma_Q)^Q$, let $z \in \mathrm{VCN}(\Gamma)$ be a lifting of z_Q, and let $g \in G$ be a generator of G. Then since Q fixes z_Q, it follows that $z^g = z^h$, for some $h \in H$, hence that z is fixed by $g \cdot h^{-1} \in G$. On the other hand, since $g \cdot h^{-1}$ generates G, we thus conclude that z is fixed by G, that is, that $\mathrm{VCN}(\Gamma_*)^G \neq \emptyset$, as desired.

(ii) The proof of Lemma 1.6(ii), as well as the argument of (i) above, is essentially the same as the argument applied in [19] to prove [19, Lem. 2.1(iii)].

Remark 1.6.3. In the respective situations of Lemma 1.6(iii) and (vi), the sub-semi-graph $(\widetilde{\Gamma}^{\mathrm{disc}})^G$ and the sub-pro-semi-graph $(\widetilde{\Gamma}^{\Sigma})^G$ are necessarily connected (cf. Lemma 1.6(iii) and (vi)). On the other hand, Γ^G is not, in general, connected. This phenomenon may be seen in the following example: suppose that $2 \in \Sigma$, and that $\widetilde{\Gamma}^{\mathrm{disc}}$ is the graph given by the integral points of the real line $\mathbb{R}$, that is, the vertices are given by the elements of $\mathbb{Z} \subseteq \mathbb{R}$, and the edges are given by the closed line segments joining adjacent elements of $\mathbb{Z}$. For $N = 2M$ a positive even integer, write Γ_N for the quotient of $\widetilde{\Gamma}^{\mathrm{disc}}$ by the evident action of $N \in \mathbb{Z}$ on $\widetilde{\Gamma}^{\mathrm{disc}}$ via translations. Thus, we have a diagram of natural covering maps

$$\widetilde{\Gamma}^{\mathrm{disc}} \longrightarrow \Gamma_N \longrightarrow \Gamma \overset{\mathrm{def}}{=} \Gamma_2,$$

and the group $G = \mathbb{Z}/2\mathbb{Z}$ acts equivariantly on this diagram via multiplication by ± 1. Here, we observe that since N is even, one verifies immediately that G acts on Γ_N without inversion. Then one computes easily that

$$(\widetilde{\Gamma}^{\mathrm{disc}})^G \;=\; \{0\}, \quad \Gamma_N^G \;=\; M\mathbb{Z}/N\mathbb{Z}.$$

In particular, the pro-semi-graph $(\widetilde{\Gamma}^{\Sigma})^G$ corresponds to the inverse limit

$$\varprojlim \; M\mathbb{Z}/N\mathbb{Z},$$

hence consists of a single pro-vertex and no pro-edge and, in particular, is nonempty and connected. On the other hand, each Γ_N^G consists of precisely two vertices and no edges, hence is not connected.

Definition 1.7. Let G be a profinite group, and let $\rho\colon G \to \mathrm{Aut}(\mathcal{G})$ be a continuous homomorphism.

(i) We shall say that ρ is *of ENN-type* (where the "ENN" stands for "extended NN") (resp. *of EPIPSC-type* (where the "EPIPSC" stands for "extended PIPSC"]) if there exists a normal closed subgroup $I_G \subseteq G$ of G such that, for every open subgroup $J \subseteq I_G$ of I_G, the composite $J \hookrightarrow G \xrightarrow{\rho} \mathrm{Aut}(\mathcal{G})$ factors as a composite $J \twoheadrightarrow J^{\Sigma\text{-ab-free}} \to \mathrm{Aut}(\mathcal{G})$ (cf. the discussion entitled "Groups" in §0), where the second arrow is of NN-type (cf. [8, Def. 2.4(iii)]) (resp. of PIPSC-type (cf. [11, Def. 1.3]). In this situation, we shall refer to I_G as a *conducting subgroup*. Suppose that ρ is of ENN-type for some conducting subgroup $I_G \subseteq G$. Then we shall say that ρ is *verticially quasi-split* if there exists an open subgroup $H \subseteq G$ that acts as the identity (i.e., relative to the action induced by ρ) on the underlying semi-graph $\mathbb{G}$ of $\mathcal{G}$ and, moreover, for every $v \in \mathrm{Vert}(\mathcal{G})$, satisfies the following condition: the extension of profinite groups (cf. the discussion entitled "Topological groups" in [9], §0)

$$1 \longrightarrow \Pi_v \longrightarrow \Pi_v \overset{\mathrm{out}}{\rtimes} H \longrightarrow H \longrightarrow 1,$$

where $\Pi_v \subseteq \Pi_{\mathcal{G}}$ is a verticial subgroup associated with $v \in \mathrm{Vert}(\mathcal{G})$, associated with the outer action of H on Π_v determined by ρ (cf. [18, Prop. 1.2(ii)]; [9, Lem. 2.12]) admits a splitting $s_v : H \to \Pi_v \overset{\mathrm{out}}{\rtimes} H$ such that the image of the restriction of s_v to $I_G \cap H$ commutes with Π_v.

(ii) Let $l \in \Sigma$. Then we shall say that ρ is *l-cyclotomically full* if the image of the composite $G \xrightarrow{\rho} \mathrm{Aut}(\mathcal{G}) \xrightarrow{\chi_{\mathcal{G}}} (\widehat{\mathbb{Z}}^{\Sigma})^{\times} \twoheadrightarrow \mathbb{Z}_l^{\times}$ (cf. [9, Def. 3.8(ii)]) is open.

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REMARK 1.7.1. It follows immediately from [11, Rem. 1.6.2] that the following implication holds:

$$\text{EPIPSC-type} \implies \text{ENN-type}.$$

LEMMA 1.8 (Outer representations induced on pro-l completions). *Let G be a profinite group, and let $\rho\colon G \to \mathrm{Aut}(\mathcal{G})$ be a continuous homomorphism of ENN-type (resp. of EPIPSC-type) for a conducting subgroup $I_G \subseteq G$ (cf. Definition 1.7(i)). For $l \in \Sigma$, write $\mathcal{G}^{\{l\}}$ for the semi-graph of anabelioids of pro-$\{l\}$ PSC-type obtained by forming the pro-l completion of $\mathcal{G}$ (cf. [17, Def. 2.9(ii)]). Then the composite $G \xrightarrow{\rho} \mathrm{Aut}(\mathcal{G}) \to \mathrm{Aut}(\mathcal{G}^{\{l\}})$ is of ENN-type (resp. of EPIPSC-type) for some conducting subgroup $\subseteq G$, which may be taken to be a normal open subgroup of I_G.*

Proof. This follows immediately from the various definitions involved (cf. also [9, Th. 4.8(iv)]; [9, Cor. 5.9(ii) and (iii)]). ∎

DEFINITION 1.9. Let $z \in \mathrm{VCN}(\mathcal{G})$. If $z \in \mathrm{Vert}(\mathcal{G})$ (resp. $z \in \mathrm{Edge}(\mathcal{G})$), then we shall refer to a verticial (resp. an edge-like) subgroup of $\Pi_{\mathcal{G}}^{\mathrm{tp}}$ associated with z (cf. [17, Th. 3.7(i) and (iii)]) as a *VCN-subgroup of* $\Pi_{\mathcal{G}}^{\mathrm{tp}}$ *associated with* z. For $\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}}^{\mathrm{tp}})$, we shall also speak of *VCN-subgroups of* $\Pi_{\mathcal{G}}^{\mathrm{tp}}$ *associated with* $\widetilde{z}$.

DEFINITION 1.10.

(i) Let Γ be a semi-graph and $v \in \mathrm{Vert}(\Gamma)$. Then we shall write $\mathcal{V}^{\delta \le 1}(v) \subseteq \mathrm{Vert}(\Gamma)$ for the subset consisting of $w \in \mathrm{Vert}(\Gamma)$ such that either $w = v$ or $\mathcal{N}(v) \cap \mathcal{N}(w) \neq \emptyset$. Also, we shall write $\mathrm{Star}(v) \stackrel{\mathrm{def}}{=} \mathcal{V}^{\delta \le 1}(v) \sqcup \mathcal{E}(v) \subseteq \mathrm{VCN}(\Gamma)$.

(ii) Let $v \in \mathrm{Vert}(\mathcal{G})$. Then we shall write $\mathcal{V}^{\delta \le 1}(v) \subseteq \mathrm{Vert}(\mathcal{G})$, $\mathrm{Star}(v) \subseteq \mathrm{VCN}(\mathcal{G})$ for the respective subsets of (i) applied to the underlying semi-graph of $\mathcal{G}$.

(iii) Let $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. Then we shall write $\mathcal{V}^{\delta \le 1}(\widetilde{v}) \subseteq \mathrm{Vert}(\widetilde{\mathcal{G}})$, $\mathrm{Star}(\widetilde{v}) \subseteq \mathrm{VCN}(\widetilde{\mathcal{G}})$ for the respective projective limits of the subsets of (ii), that is, where the constructions of these subsets are applied to the images of $\widetilde{v}$ in the connected finite etale subcoverings of $\widetilde{\mathcal{G}} \to \mathcal{G}$.

LEMMA 1.11 (VCN-subgroups and sections). *Let G be a profinite group, let $\rho\colon G \to \mathrm{Aut}(\mathcal{G})$ be a continuous homomorphism, let $\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$, let $\widetilde{z}^{\mathrm{tp}} \in \mathrm{VCN}(\widetilde{\mathcal{G}}^{\mathrm{tp}})$, let $\Pi_{\widetilde{z}} \subseteq \Pi_{\mathcal{G}}$ be a VCN-subgroup of $\Pi_{\mathcal{G}}$ associated with $\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$, and let $\Pi_{\widetilde{z}^{\mathrm{tp}}} \subseteq \Pi_{\mathcal{G}}^{\mathrm{tp}}$ be a VCN-subgroup of $\Pi_{\mathcal{G}}^{\mathrm{tp}}$ associated with $\widetilde{z}^{\mathrm{tp}}$ (cf. Definition 1.9). Write $\Pi_G \stackrel{\mathrm{def}}{=} \Pi_{\mathcal{G}} \overset{\mathrm{out}}{\rtimes} G$, $\Pi_G^{\mathrm{tp}} \stackrel{\mathrm{def}}{=} \Pi_{\mathcal{G}}^{\mathrm{tp}} \overset{\mathrm{out}}{\rtimes} G$ (cf. the discussion entitled "Topological groups" in [9, §0]). Thus, we have a natural commutative diagram*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_{\mathcal{G}}^{\mathrm{tp}} & \longrightarrow & \Pi_G^{\mathrm{tp}} & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Pi_{\mathcal{G}} & \longrightarrow & \Pi_G & \longrightarrow & G & \longrightarrow & 1
\end{array}$$

—where the horizontal sequences are exact, the vertical arrows are outer injections, Π_G^{tp} acts naturally on $\widetilde{\mathcal{G}}^{\mathrm{tp}}$, and Π_G acts naturally on $\widetilde{\mathcal{G}}$. Then the following hold:

(i) *It holds that*

$$\Pi_{\widetilde{z}} = N_{\Pi_G}(\Pi_{\widetilde{z}}) \cap \Pi_{\mathcal{G}} = C_{\Pi_G}(\Pi_{\widetilde{z}}) \cap \Pi_{\mathcal{G}},$$

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$$D_{\widetilde{z}} \stackrel{\text{def}}{=} N_{\Pi_G}(\Pi_{\widetilde{z}}) = C_{\Pi_G}(\Pi_{\widetilde{z}}) = N_{\Pi_G}(D_{\widetilde{z}}) = C_{\Pi_G}(D_{\widetilde{z}}),$$

$$\Pi_{\widetilde{z}^{\mathrm{tp}}} = N_{\Pi_G^{\mathrm{tp}}}(\Pi_{\widetilde{z}^{\mathrm{tp}}}) \cap \Pi_{\mathcal{G}}^{\mathrm{tp}} = C_{\Pi_G^{\mathrm{tp}}}(\Pi_{\widetilde{z}^{\mathrm{tp}}}) \cap \Pi_{\mathcal{G}}^{\mathrm{tp}},$$

$$D_{\widetilde{z}^{\mathrm{tp}}} \stackrel{\text{def}}{=} N_{\Pi_G^{\mathrm{tp}}}(\Pi_{\widetilde{z}^{\mathrm{tp}}}) = C_{\Pi_G^{\mathrm{tp}}}(\Pi_{\widetilde{z}^{\mathrm{tp}}}) = N_{\Pi_G^{\mathrm{tp}}}(D_{\widetilde{z}^{\mathrm{tp}}}) = C_{\Pi_G^{\mathrm{tp}}}(D_{\widetilde{z}^{\mathrm{tp}}}).$$

(ii) *Suppose that ρ is of ENN-type for a conducting subgroup $I_G \subseteq G$ (cf. Definition 1.7(i)). Let S be a nonempty subset of $\mathrm{VCN}(\widetilde{\mathcal{G}})$, and let $s\colon G \to \Pi_G$ be a section of the surjection $\Pi_G \twoheadrightarrow G$ such that, for each $\widetilde{y} \in S$, it holds that $s(I_G) \prec D_{\widetilde{y}}$ (cf. the discussion entitled "Groups" in §0). Then there exists an element $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ such that $S \subseteq \mathrm{Star}(\widetilde{v})$ (cf. Definition 1.10(iii)).*

(iii) *Suppose that ρ is of ENN-type for a conducting subgroup $I_G \subseteq G$. Let $s\colon G \to \Pi_G$ be a section of the surjection $\Pi_G \twoheadrightarrow G$ such that $s(I_G) \prec D_{\widetilde{z}}$ (cf. the discussion entitled "Groups" in §0). Write $G_s \stackrel{\text{def}}{=} C_{\Pi_G}(s(I_G))$. Then there exists an element $\widetilde{z}' \in \mathrm{VCN}(\widetilde{\mathcal{G}})$ such that $s(G) \subseteq G_s \subseteq D_{\widetilde{z}'}$.*

(iv) *Suppose that ρ is of ENN-type for a conducting subgroup $I_G \subseteq G$. Let $s\colon G \to \Pi_G^{\mathrm{tp}}$ be a section of the surjection $\Pi_G^{\mathrm{tp}} \twoheadrightarrow G$ such that $s(I_G) \prec D_{\widetilde{z}^{\mathrm{tp}}}$ (cf. the discussion entitled "Groups" in §0). Write $G_s \stackrel{\text{def}}{=} C_{\Pi_G^{\mathrm{tp}}}(s(I_G))$. Then there exists an element $(\widetilde{z}')^{\mathrm{tp}} \in \mathrm{VCN}(\widetilde{\mathcal{G}}^{\mathrm{tp}})$ such that $s(G) \subseteq G_s \subseteq D_{(\widetilde{z}')^{\mathrm{tp}}}$. In particular, G_s is contained in a profinite subgroup of Π_G^{tp} (cf. (i)).*

Proof. First, we verify assertion (i). The two equalities of the first (resp. third) line of the display and the first "=" of the second (resp. fourth) line of the display follow immediately from [18, Prop. 1.2(i) and (ii)] (resp. [18, Prop. 1.2(i) and (ii)], together with the injection reviewed at the beginning of the present §1). Thus, the second and third "=" of the second (resp. fourth) line of the display follow immediately from the chain of inclusions

$$D_{\widetilde{z}} \subseteq N_{\Pi_G}(D_{\widetilde{z}}) \subseteq C_{\Pi_G}(D_{\widetilde{z}}) \subseteq C_{\Pi_G}(D_{\widetilde{z}} \cap \Pi_{\mathcal{G}}) = C_{\Pi_G}(\Pi_{\widetilde{z}}) = D_{\widetilde{z}}$$

(resp.

$$D_{\widetilde{z}^{\mathrm{tp}}} \subseteq N_{\Pi_G^{\mathrm{tp}}}(D_{\widetilde{z}^{\mathrm{tp}}}) \subseteq C_{\Pi_G^{\mathrm{tp}}}(D_{\widetilde{z}^{\mathrm{tp}}}) \subseteq C_{\Pi_G^{\mathrm{tp}}}(D_{\widetilde{z}^{\mathrm{tp}}} \cap \Pi_{\mathcal{G}}^{\mathrm{tp}}) = C_{\Pi_G^{\mathrm{tp}}}(\Pi_{\widetilde{z}^{\mathrm{tp}}}) = D_{\widetilde{z}^{\mathrm{tp}}}),$$

where the third "$\subseteq$" follows immediately from [10, Lem. 3.9(i)] (resp. the [easily verified] tempered version of [10, Lem. 3.9(i)]). This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us first observe that it follows from the definition of the term "ENN-type" that the restriction of ρ to $I_G \subseteq G$ factors through the quotient $I_G \twoheadrightarrow I_G^{\Sigma\text{-ab-free}}$ (cf. the discussion entitled "Groups" in §0). Write $\Pi_{I_G} \stackrel{\text{def}}{=} \Pi_{\mathcal{G}} \overset{\text{out}}{\rtimes} I_G$ and $\Pi_{I_G^{\Sigma\text{-ab-free}}} \stackrel{\text{def}}{=} \Pi_{\mathcal{G}} \overset{\text{out}}{\rtimes} I_G^{\Sigma\text{-ab-free}}$. Thus, we have a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_{\mathcal{G}} & \longrightarrow & \Pi_G & \longrightarrow & G & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Pi_{\mathcal{G}} & \longrightarrow & \Pi_{I_G} & \longrightarrow & I_G & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Pi_{\mathcal{G}} & \longrightarrow & \Pi_{I_G^{\Sigma\text{-ab-free}}} & \longrightarrow & I_G^{\Sigma\text{-ab-free}} & \longrightarrow & 1
\end{array}$$

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—where the horizontal sequences are exact, the upper vertical arrows are injective, the lower vertical arrows are surjective, and the two right-hand squares are Cartesian. Next, let us observe that we may assume without loss of generality that S is equal to the set of all $\widetilde{y} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$ such that $s(I_G) \prec D_{\widetilde{y}}$. Now since $s(I_G) \prec D_{\widetilde{y}} = C_{\Pi_G}(\Pi_{\widetilde{y}})$ (cf. assertion (i)) for every $\widetilde{y} \in S$, it holds that, for each $\widetilde{y} \in S$, some open subgroup of the image $J \subseteq \Pi_{I_G^{\Sigma\text{-ab-free}}}$ of $I_G \xrightarrow{s} \Pi_{I_G} \twoheadrightarrow \Pi_{I_G^{\Sigma\text{-ab-free}}}$ is contained in $C_{\Pi_{I_G^{\Sigma\text{-ab-free}}}}(\Pi_{\widetilde{y}})$. In particular, it follows from [8, Props. 3.8(i) and 3.9(i)–(iii)] that:

- every pair of edges $\in S$ abut to a common vertex $\in S$;
- the distance between any two vertices $\in S$ is ≤ 2 (cf. Definition 1.1(iii)), and the edges "$e_1, \ldots, e_n$" and vertices "$v_0, \ldots, v_n$" of Definition 1.1(iii) may be taken to be $\in S$;
- if $\widetilde{e} \in S$ is an edge, then $\mathcal{V}(\widetilde{e}) \subseteq S$.

It is now a matter of elementary combinatorial graph theory (cf. also [8, Lem. 1.8]) to conclude that $S \subseteq \mathrm{Star}(\widetilde{v})$ for some $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, as desired. This completes the proof of assertion (ii).

Next, we verify assertion (iii). Since $s(I_G) \prec D_{\widetilde{z}}$, the action of some open subgroup of I_G on $\widetilde{\mathcal{G}}$ determined by $s|_{I_G}$ fixes $\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$. Thus, it follows from the definition of G_s that, if, for $\gamma \in G_s$, we write $\widetilde{z}^{\gamma} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$ for the image of $\widetilde{z}$ by the action of $\gamma \in G_s$, then the action of some open subgroup of I_G on $\widetilde{\mathcal{G}}$ fixes $\widetilde{z}^{\gamma} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$, that is, $s(I_G) \prec D_{\widetilde{z}^{\gamma}}$ for every $\gamma \in G_s$.

Now suppose that $\widetilde{z} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$. Then it follows from assertion (ii) that there exists a vertex $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ such that $\{\widetilde{z}^{\gamma} \mid \gamma \in G_s\} \subseteq \mathcal{E}(\widetilde{v})$. Now if $\{\widetilde{z}^{\gamma} \mid \gamma \in G_s\}^{\sharp} = 1$, then it is immediate that $G_s \subseteq D_{\widetilde{z}}$. On the other hand, if $\{\widetilde{z}^{\gamma} \mid \gamma \in G_s\}^{\sharp} \geq 2$, then one verifies immediately from the various definitions involved (cf. also [8, Lem. 1.8]) that the action of G_s fixes $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, which thus implies that $G_s \subseteq D_{\widetilde{v}}$. This completes the proof of assertion (iii) in the case where $\widetilde{z} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$.

Next, suppose that $\widetilde{z} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. Then it follows from assertion (ii) that the set S_δ of vertices $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ such that

- $S_{\widetilde{z}} \stackrel{\mathrm{def}}{=} \{\widetilde{z}^{\gamma} \mid \gamma \in G_s\} \subseteq \mathcal{V}^{\delta \leq 1}(\widetilde{v})$;
- any edge $\in \mathrm{Edge}(\widetilde{\mathcal{G}})$ that abuts to two distinct elements of $S_{\widetilde{z}}$ (hence is fixed by the action, determined by $s|_{I_G}$, of some open subgroup of I_G—cf. [8, Prop. 3.9(ii)]) necessarily abuts to $\widetilde{v}$

is nonempty. If the action of G_s fixes some $\widetilde{y} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$, then $G_s \subseteq D_{\widetilde{y}}$. Thus, we may assume without loss of generality that the action of G_s does not fix any element of $\mathrm{VCN}(\widetilde{\mathcal{G}})$. In particular, it follows that the (nonempty!) sets $S_{\widetilde{z}}$ and S_δ—both of which are clearly preserved by the action of G_s —are of cardinality ≥ 2. In a similar vein, $S_\delta \setminus (S_\delta \cap S_{\widetilde{z}})$ is either empty or of cardinality ≥ 2. Moreover, the latter case contradicts [8, Lem. 1.8. Thus, we conclude that $S_\delta \subseteq S_{\widetilde{z}}$, which, by the definition of $S_{\widetilde{z}}$ and S_δ, implies that $S_\delta = S_{\widetilde{z}}$, that is, that, for any two distinct $\widetilde{z}_1$, $\widetilde{z}_2 \in S_{\widetilde{z}}$, there exists a (unique, by [8, Lem. 1.8]) $\widetilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$ such that $\mathcal{V}(\widetilde{e}) = \{\widetilde{z}_1, \widetilde{z}_2\}$. But, in light of the definition of S_δ, this implies that $S_{\widetilde{z}}^{\sharp} = 2$, and hence that $\mathrm{Edge}(\widetilde{\mathcal{G}})$ contains an element fixed by the action of G_s —a contradiction! This completes the proof of assertion (iii) in the case where $\widetilde{z} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, hence also of assertion (iii). Assertion (iv) follows immediately from a similar argument to the argument applied in the proof of assertion (iii). This completes the proof of Lemma 1.11. $\square$

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LEMMA 1.12 (Triviality via passage to abelianizations). *Let G and J be profinite groups, and let $\phi\colon J \to G$ be a continuous homomorphism. Then the following hold:*

(i) *Let $\gamma \in G$ be such that, for every open subgroup $H \subseteq G$ of G that contains γ, the image of γ in H^{ab} is trivial. Then γ is trivial.*

(ii) *Suppose that, for every open subgroup $H \subseteq G$ of G, the composite $\phi^{-1}(H) \xrightarrow{\phi} H \twoheadrightarrow H^{\mathrm{ab}}$ is trivial. Then ϕ is trivial.*

Proof. First, we verify assertion (i). Assume that γ is nontrivial. Then it is immediate that there exists a normal open subgroup $N \subseteq G$ of G such that $\gamma \notin N$. Write $H \subseteq G$ for the closed subgroup of G topologically generated by N and γ. Then the image of γ in the abelian quotient $H \twoheadrightarrow H/N$ is nontrivial. This completes the proof of assertion (i). Assertion (ii) follows immediately from assertion (i). This completes the proof of Lemma 1.12. ∎

THEOREM 1.13 (The combinatorial section conjecture for outer representations of ENN-type). *Let Σ be a nonempty set of prime numbers, let $\mathcal{G}$ be a semi-graph of anabelioids of pro-Σ PSC-type, let G be a profinite group, and let $\rho\colon G \to \mathrm{Aut}(\mathcal{G})$ be a continuous homomorphism that is of ENN-type for a conducting subgroup $I_G \subseteq G$ (cf. Definition 1.7(i)). Write $\Pi_{\mathcal{G}}$ for the (pro-Σ) fundamental group of $\mathcal{G}$ and $\Pi_{\mathcal{G}}^{\mathrm{tp}}$ for the tempered fundamental group of $\mathcal{G}$ (cf. [17, Exam. 2.10]; the discussion preceding [17, Prop. 3.6]). (Thus, we have a natural outer injection $\Pi_{\mathcal{G}}^{\mathrm{tp}} \hookrightarrow \Pi_{\mathcal{G}}$—cf. [11, Lem. 3.2(i)]; the proof of [11, Prop. 3.3(i) and (ii)].) Write $\Pi_G \stackrel{\mathrm{def}}{=} \Pi_{\mathcal{G}} \stackrel{\mathrm{out}}{\rtimes} G$ (cf. the discussion entitled "Topological groups" in [9, §0]); $\Pi_G^{\mathrm{tp}} \stackrel{\mathrm{def}}{=} \Pi_{\mathcal{G}}^{\mathrm{tp}} \stackrel{\mathrm{out}}{\rtimes} G$; $\widetilde{\mathcal{G}} \to \mathcal{G}$, $\widetilde{\mathcal{G}}^{\mathrm{tp}} \to \mathcal{G}$ for the universal pro-Σ and pro-tempered coverings of $\mathcal{G}$ corresponding to $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{G}}^{\mathrm{tp}}$; $\mathrm{VCN}(-)$ for the set of vertices, cusps, and nodes of the underlying (pro-)semi-graph of a (pro-)semi-graph of anabelioids (cf. Definition 1.1(i)). Thus, we have a natural commutative diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{\mathcal{G}}^{\mathrm{tp}} & \longrightarrow & \Pi_G^{\mathrm{tp}} & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Pi_{\mathcal{G}} & \longrightarrow & \Pi_G & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

—where the horizontal sequences are exact, the vertical arrows are outer injections, Π_G^{tp} acts naturally on $\widetilde{\mathcal{G}}^{\mathrm{tp}}$, and Π_G acts naturally on $\widetilde{\mathcal{G}}$. Then the following hold:

(i) *Suppose that ρ is l-cyclotomically full (cf. Definition 1.7(ii)) for some $l \in \Sigma$. Let $s\colon G \to \Pi_G$ be a continuous section of the natural surjection $\Pi_G \twoheadrightarrow G$. Then, relative to the action of Π_G on $\mathrm{VCN}(\widetilde{\mathcal{G}})$ via conjugation of VCN-subgroups, the image of s stabilizes some element of $\mathrm{VCN}(\widetilde{\mathcal{G}})$.*

(ii) *Let $s\colon G \to \Pi_G^{\mathrm{tp}}$ be a continuous section of the natural surjection $\Pi_G^{\mathrm{tp}} \twoheadrightarrow G$. Then, relative to the action of Π_G^{tp} on $\mathrm{VCN}(\widetilde{\mathcal{G}}^{\mathrm{tp}})$ via conjugation of VCN-subgroups (cf. Definition 1.9), the image of s stabilizes some element of $\mathrm{VCN}(\widetilde{\mathcal{G}}^{\mathrm{tp}})$.*

(iii) *Write $\mathrm{Sect}(\Pi_G/G)$ for the set of $\Pi_{\mathcal{G}}$-conjugacy classes of continuous sections of the natural surjective homomorphism $\Pi_G \twoheadrightarrow G$ and $\mathrm{Sect}(\Pi_G^{\mathrm{tp}}/G)$ for the set of*

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$\Pi_{\mathcal{G}}^{\mathrm{tp}}$-conjugacy classes of continuous sections of the natural surjective homomorphism $\Pi_{G}^{\mathrm{tp}} \twoheadrightarrow G$. Then the natural map

$$\mathrm{Sect}(\Pi_G^{\mathrm{tp}}/G) \longrightarrow \mathrm{Sect}(\Pi_G/G)$$

is injective. If, moreover, ρ is l-cyclotomically full for some $l \in \Sigma$, then this map is bijective.

Proof. First, we verify assertion (i). Let us first observe that by replacing I_G by a suitable open subgroup of I_G and $\mathcal{G}$ by the pro-l completion of the finite étale covering of $\mathcal{G}$ determined by a varying normal open subgroup $H \subseteq \Pi_G$ such that $s(G) \subseteq H$ (cf. Lemma 1.8; [11, Lem. 1.5]), it follows immediately from the well-known fact that a projective limit of nonempty finite sets is nonempty that we may assume without loss of generality that $\Sigma = \{l\}$.

Next, let us observe that we may assume without loss of generality that $\mathcal{G}$ has at least one node. In particular, it follows immediately from Lemma 1.11(iii) that, to verify assertion (i), by replacing Π_G by a suitable open subgroup of Π_G, we may assume without loss of generality—that is, by arguing as in the discussion entitled "Curves" in [21, §0]—that the pro-l completion $\Pi_{\mathbb{G}}$ of the topological fundamental group of the underlying semi-graph $\mathbb{G}$ of $\mathcal{G}$ is a free pro-l group of rank ≥ 2, hence, in particular, center-free.

Then we claim that the following assertion holds:

> Claim 1.13.A: For every connected finite étale Galois subcovering $\mathcal{H} \to \mathcal{G}$ of $\widetilde{\mathcal{G}} \to \mathcal{G}$ that determines a normal open subgroup of Π_G, the action of I_G on $\mathcal{H}$, via s, fixes an element of $\mathrm{VCN}(\mathcal{H})$.

To verify Claim 1.13.A, let us observe that, by replacing $\mathcal{H}$ by $\mathcal{G}$ (cf. [11, Lem. 1.5]), we may assume without loss of generality that $\mathcal{H} = \mathcal{G}$. Next, let us observe that since the underlying semi-graph $\mathbb{G}$ of $\mathcal{G}$ is finite, the continuous action of G on $\mathbb{G}$ factors through a finite quotient $G \twoheadrightarrow Q$, that is, by a normal open subgroup of G. Write $\Pi_{\mathbb{G}//Q} \stackrel{\mathrm{def}}{=} \Pi_{\mathbb{G}} \stackrel{\mathrm{out}}{\rtimes} Q$ (i.e., notation which is well-defined since $\Pi_{\mathbb{G}}$ is center-free—cf. the discussion entitled "Topological groups" in [9, §0]; the notational conventions of Lemma 1.6, in the case where "Σ" is taken to be $\{l\}$). Thus, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{\mathcal{G}} & \longrightarrow & \Pi_G & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Pi_{\mathbb{G}} & \longrightarrow & \Pi_{\mathbb{G}//Q} & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

—where the horizontal sequences are exact, and the vertical arrows are surjective. Write $I_G \twoheadrightarrow I_Q$ for the (finite) quotient of I_G determined by the quotient $G \twoheadrightarrow Q$, $N_G \stackrel{\mathrm{def}}{=} \mathrm{Ker}(G \twoheadrightarrow Q)$, and $N_I \stackrel{\mathrm{def}}{=} \mathrm{Ker}(I_G \twoheadrightarrow I_Q)$. Now let us observe that

(a) since Q is finite, it is immediate that N_G, N_I are open in G, I_G, respectively, and, moreover,

(b) it follows from the definition of the term "ENN-type" that, by replacing $G \twoheadrightarrow Q$ by a suitable quotient of Q if necessary, we may assume without loss of generality that the quotient $I_G \twoheadrightarrow I_Q$ factors through the quotient $I_G \twoheadrightarrow I_G^{\{l\}\text{-ab-free}}$ (cf. the discussion entitled "Groups" in §0), hence is cyclic of order a power of l.

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Next, let us observe that the composite $N_G \hookrightarrow G \xrightarrow{s} \Pi_G \twoheadrightarrow \Pi_{\mathbb{G}//Q}$ determines a commutative diagram

$$\begin{array}{ccc} N_I & \hookrightarrow & N_G \\ \downarrow & & \downarrow \\ \Pi_{\mathbb{G}} & = & \Pi_{\mathbb{G}} \end{array}$$

—where the upper horizontal arrow is the natural inclusion. Now we claim that the following assertion holds:

> Claim 1.13.B: The left-hand vertical arrow $N_I \to \Pi_{\mathbb{G}}$ of the above diagram is the trivial homomorphism.

Indeed, let $H \subseteq \Pi_{\mathbb{G}}$ be an open subgroup and write $N_{I,H} \subseteq N_I$ and $N_{G,H} \subseteq N_G$ for the open subgroups obtained by forming the inverse image of $H \subseteq \Pi_{\mathbb{G}}$ via the vertical arrows of the above commutative diagram. Thus, $N_{G,H}$ normalizes $N_{I,H}$; the action of $N_{G,H}$ on H by conjugation induces the trivial action of $N_{G,H}$ on H^{ab}. Next, let us observe that since H^{ab} is a free $\mathbb{Z}_l$-module, the left-hand vertical arrow under consideration determines a homomorphism $N_{I,H}^{\{l\}\text{-ab-free}} \to H^{\mathrm{ab}}$ of free $\mathbb{Z}_l$-modules of finite rank (cf. Definition 1.7(i)), which is $N_{G,H}$-equivariant (with respect to the actions of $N_{G,H}$ by conjugation). On the other hand, since the action of $N_{G,H}$ on H^{ab} is trivial, the $N_{G,H}$-equivariant homomorphism $N_{I,H}^{\{l\}\text{-ab-free}} \to H^{\mathrm{ab}}$ factors through a quotient of $N_{I,H}^{\{l\}\text{-ab-free}}$ on which $N_{G,H}$ acts trivially. Thus, since ρ is l-cyclotomically full, and $N_{G,H}$ acts on $N_{I,H}^{\{l\}\text{-ab-free}}$ via the cyclotomic character (cf. Definition 1.7(i); [9, Lem. 5.2(ii)]), we conclude that the $N_{G,H}$-equivariant homomorphism $N_{I,H}^{\{l\}\text{-ab-free}} \to H^{\mathrm{ab}}$ is trivial. In particular, Claim 1.13.B follows from Lemma 1.12(ii). This completes the proof of Claim 1.13.B.

Next, let us observe that it follows immediately from Claim 1.13.B that the section s determines a section of the natural surjection

$$\Pi_{\mathbb{G}//I_Q} \stackrel{\text{def}}{=} \Pi_{\mathbb{G}//Q} \times_Q I_Q \stackrel{\mathrm{pr}_2}{\twoheadrightarrow} I_Q.$$

Thus, it follows immediately from the resp'd portion of Lemma 1.6(iv) together with the observation (b) discussed above (cf. also Remark 1.13.1 below), that Claim 1.13.A holds. This completes the proof of Claim 1.13.A.

Now, by allowing the subcovering $\mathcal{H}$ in Claim 1.13.A to vary, we conclude immediately (from the well-known fact that a projective limit of nonempty finite sets is nonempty) that $s(I_G)$ stabilizes some element of $\mathrm{VCN}(\widetilde{\mathcal{G}})$. Thus, it follows from Lemma 1.11(iii) that the image $s(G)$ stabilizes some element of $\mathrm{VCN}(\widetilde{\mathcal{G}})$. This completes the proof of assertion (i).

Assertion (ii) follows, by applying [8, Prop. 3.9(i)], from a similar argument to the argument applied to prove [17, Ths. 3.7 and 5.4]. That is to say, instead of considering "subjoints" (i.e., paths of length 2) as in the proof of [17, Th. 3.7], the content of [8, Prop. 3.9(i)] requires us to consider paths of length 3. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). Let $s, t\colon G \to \Pi_G^{\mathrm{tp}}$ be sections of the surjection $\Pi_G^{\mathrm{tp}} \twoheadrightarrow G$ such that there exists an element $\gamma \in \Pi_{\mathcal{G}}$ such that the composite $\widehat{s}\colon G \xrightarrow{s} \Pi_G^{\mathrm{tp}} \hookrightarrow \Pi_G$ is the conjugate by $\gamma \in \Pi_{\mathcal{G}}$ of the composite $\widehat{t}\colon G \xrightarrow{t} \Pi_G^{\mathrm{tp}} \hookrightarrow \Pi_G$. Thus, it follows from assertion (ii) (applied to both s and t) that there exist elements $\widetilde{y}, \widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}}^{\mathrm{tp}})$ such that if we write $\widetilde{z}^{\gamma} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$ for the image of $\widetilde{z}$ by the action of γ, then $\widehat{s}$ stabilizes both $\widetilde{y}$ and $\widetilde{z}^{\gamma}$. In particular, we conclude from Lemma 1.11(ii) that the distance between $\widetilde{y}$ and $\widetilde{z}^{\gamma}$ is finite,

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hence that, for each subcovering $\mathcal{H} \to \mathcal{G}$ of $\widetilde{\mathcal{G}}^{\mathrm{tp}} \to \mathcal{G}$ that arises from an open subgroup of $\Pi_{\mathcal{G}}^{\mathrm{tp}}$, the distance between the images of $\widetilde{z}$ and $\widetilde{z}^{\gamma}$ in $\mathcal{H}$ is finite, which implies that $\gamma \in \Pi_{\mathcal{G}}^{\mathrm{tp}}$. This completes the proof of the injectivity portion of assertion (iii). Since (one verifies immediately that) every element of $\mathrm{VCN}(\widetilde{\mathcal{G}})$ lies in the $\Pi_{\mathcal{G}}$-orbit of an element of $\mathrm{VCN}(\widetilde{\mathcal{G}}^{\mathrm{tp}})$, the final portion of assertion (iii) follows immediately from assertion (i). This completes the proof of Theorem 1.13. ∎

REMARK 1.13.1. We observe in passing, with regard to the application of Lemma 1.6(iv) in the proof of Theorem 1.13(i) that, in fact, Lemma 1.6(iv) is only applied in the case where the group "G" of Lemma 1.6 is cyclic and of order a power of l. That is to say, we only apply Lemma 1.6(iv) in the case that, as discussed in Remark 1.6.2(i) admits a relatively simple proof.

COROLLARY 1.14 (A combinatorial version of the Grothendieck conjecture for outer representations of ENN-type). *Let Σ be a nonempty set of prime numbers; $\mathcal{G}$, $\mathcal{H}$ semi-graphs of anabelioids of pro-Σ PSC-type; $G_{\mathcal{G}}$, $G_{\mathcal{H}}$ profinite groups; $\beta\colon G_{\mathcal{G}} \xrightarrow{\sim} G_{\mathcal{H}}$ a continuous isomorphism; $\rho_{\mathcal{G}}\colon G_{\mathcal{G}} \to \mathrm{Aut}(\mathcal{G})$, $\rho_{\mathcal{H}}\colon G_{\mathcal{H}} \to \mathrm{Aut}(\mathcal{H})$ continuous homomorphisms that are of ENN-type for conducting subgroups $I_{G_{\mathcal{G}}} \subseteq G_{\mathcal{G}}$, $I_{G_{\mathcal{H}}} \subseteq G_{\mathcal{H}}$ (cf. Definition 1.7(i)) such that $\beta(I_{G_{\mathcal{G}}}) = I_{G_{\mathcal{H}}}$; $l \in \Sigma$ a prime number such that $\rho_{\mathcal{G}}$ and $\rho_{\mathcal{H}}$ are l-cyclotomically full (cf. Definition 1.7(ii)). Suppose further that $\rho_{\mathcal{G}}$ is verticially quasi-split (cf. Definition 1.7(i)). Write $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{H}}$ for the (pro-Σ) fundamental groups of $\mathcal{G}$, $\mathcal{H}$, respectively. Let $\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$ be a continuous isomorphism such that the diagram*

$$\begin{array}{ccccc}
G_{\mathcal{G}} & \xrightarrow{\rho_{\mathcal{G}}} & \mathrm{Aut}(\mathcal{G}) & \hookrightarrow & \mathrm{Out}(\Pi_{\mathcal{G}}) \\
{\scriptstyle\beta}\big\downarrow & & & & \big\downarrow \\
G_{\mathcal{H}} & \xrightarrow{\rho_{\mathcal{H}}} & \mathrm{Aut}(\mathcal{H}) & \hookrightarrow & \mathrm{Out}(\Pi_{\mathcal{H}})
\end{array}$$

—where the right-hand vertical arrow is the isomorphism obtained by conjugating by α —commutes. Then α is graphic (cf. [18, Def. 1.4(i)]).

Proof. First, let us observe that by [18, Cor. 2.7(i)], it follows from our assumption that $\rho_{\mathcal{G}}$ and $\rho_{\mathcal{H}}$ are l-cyclotomically full that $\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$ is group-theoretically cuspidal. Thus, by applying [18, Prop. 1.5(ii)] and [8, Lem. 1.14], we conclude that it suffices to verify that α is group-theoretically verticial under the additional assumption that $\mathcal{G}$ and $\mathcal{H}$ are noncuspidal. Write $\Pi_{G_{\mathcal{G}}}$, $\Pi_{G_{\mathcal{H}}}$ for the profinite groups "Π_G" (cf. Theorem 1.13) associated with $\rho_{\mathcal{G}}$, $\rho_{\mathcal{H}}$. Then it follows immediately from our assumption that $\rho_{\mathcal{G}}$ is verticially quasi-split that we may assume, after possibly replacing $G_{\mathcal{G}}$ and $G_{\mathcal{H}}$ by corresponding open subgroups, that there exists a section $s_{\mathcal{G}}\colon G_{\mathcal{G}} \to \Pi_{G_{\mathcal{G}}}$ such that the image of the restriction of $s_{\mathcal{G}}$ to $I_{G_{\mathcal{G}}}$ commutes with some verticial subgroup of $\Pi_{\mathcal{G}}$. In particular, $s_{\mathcal{G}}$ satisfies the conditions imposed on the section "$s\colon G \to \Pi_G$" in Lemma 1.11(ii), for some nonempty subset "S." Moreover, it follows from Theorem 1.13(i) that the isomorphism $\Pi_{G_{\mathcal{G}}} \xrightarrow{\sim} \Pi_{G_{\mathcal{H}}}$ determined by α and β maps $s_{\mathcal{G}}$ to a section $s_{\mathcal{H}}\colon G_{\mathcal{H}} \to \Pi_{G_{\mathcal{H}}}$ that is contained in the normalizer in $\Pi_{G_{\mathcal{H}}}$ of a VCN-subgroup of $\Pi_{\mathcal{H}}$. In particular, after possibly replacing $G_{\mathcal{G}}$ and $G_{\mathcal{H}}$ by corresponding open subgroups, we may assume (cf. [18, Prop 1.2(ii)] and [8, Rem. 2.7.1]) that the image of the restriction of $s_{\mathcal{H}}$ to $I_{G_{\mathcal{H}}}$ commutes with some nontrivial verticial element of $\Pi_{\mathcal{H}}$ (cf. [10, Def. 1.1]). Thus, by restricting these sections $s_{\mathcal{G}}$, $s_{\mathcal{H}}$ to the respective

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conducting subgroups and forming appropriate centralizers [cf. [8, Lem. 3.6(i)], applied to the restriction of $s_{\mathcal{G}}$ to $I_{G_{\mathcal{G}}}$], we conclude from the assumption that β is compatible with the respective conducting subgroups that $\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$ maps some nontrivial verticial element of $\Pi_{\mathcal{G}}$ to a nontrivial verticial element of $\Pi_{\mathcal{H}}$. In particular, it follows from the implication (3) $\Rightarrow$ (1) of [10, Th. 1.9(i)] that α is group-theoretically verticial, as desired. ∎

Remark 1.14.1. It is not difficult to verify that the assumption in the statement of Corollary 1.14 that $\beta(I_{G_{\mathcal{G}}}) = I_{G_{\mathcal{H}}}$ cannot be omitted. Indeed, if one omits this assumption, then a counterexample to the graphicity asserted in Corollary 1.14 may be obtained as follows: let $\mathcal{J}$ be a semi-graph of anabelioids of pro-Σ PSC-type and $e_{\mathcal{G}}$, $e_{\mathcal{H}}$ distinct nodes of $\mathcal{J}$. Write $\mathcal{G}$ (resp. $\mathcal{H}$) for the semi-graph of anabelioids of pro-Σ PSC-type $\mathcal{J}_{\rightsquigarrow \mathrm{Node}(\mathcal{J})\setminus\{e_{\mathcal{G}}\}}$ (resp. $\mathcal{J}_{\rightsquigarrow \mathrm{Node}(\mathcal{J})\setminus\{e_{\mathcal{H}}\}}$) obtained by deforming the nodes of $\mathcal{J}$ that are $\neq e_{\mathcal{G}}$ (resp. $\neq e_{\mathcal{H}}$) (cf. [9, Def. 2.8]); $I_{G_{\mathcal{G}}}$ (resp. $I_{G_{\mathcal{H}}}$) for the (necessarily normal—cf. [9, Th. 4.8(i) and (v)]) closed subgroup of $\mathrm{Aut}^{|\{e_{\mathcal{G}},e_{\mathcal{H}}\}|}(\mathcal{J})$ (cf. [9, Def. 2.6(i)]) generated by the profinite Dehn twists that arise from the direct summand of the direct sum decomposition in the display of [9, Th. 4.8(iv)], labeled by $e_{\mathcal{G}}$ (resp. $e_{\mathcal{H}}$). Next, let $G_{\mathcal{G}} = G_{\mathcal{H}}$ be a closed subgroup of $\mathrm{Aut}^{|\{e_{\mathcal{G}},e_{\mathcal{H}}\}|}(\mathcal{J})$ such that:

- $G_{\mathcal{G}} = G_{\mathcal{H}}$ contains both $I_{G_{\mathcal{G}}}$ and $I_{G_{\mathcal{H}}}$,
- the natural inclusion $G_{\mathcal{G}} = G_{\mathcal{H}} \hookrightarrow \mathrm{Aut}(\mathcal{J})$ is l-cyclotomically full for some $l \in \Sigma$, and, moreover,
- if we write $\rho_{\mathcal{G}}$ (resp. $\rho_{\mathcal{H}}$) for the continuous injection $G_{\mathcal{G}} \hookrightarrow \mathrm{Aut}(\mathcal{G})$ (resp. $G_{\mathcal{H}} \hookrightarrow \mathrm{Aut}(\mathcal{H})$) obtained by forming the composite of the natural inclusion $G_{\mathcal{G}} = G_{\mathcal{H}} \hookrightarrow \mathrm{Aut}^{|\{e_{\mathcal{G}},e_{\mathcal{H}}\}|}(\mathcal{J})$ and the injection $\mathrm{Aut}^{|\{e_{\mathcal{G}},e_{\mathcal{H}}\}|}(\mathcal{J}) \hookrightarrow \mathrm{Aut}(\mathcal{G})$ (resp. $\mathrm{Aut}^{|\{e_{\mathcal{G}},e_{\mathcal{H}}\}|}(\mathcal{J}) \hookrightarrow \mathrm{Aut}(\mathcal{H})$) (cf. [9, Prop. 2.9(ii)]), then $\rho_{\mathcal{G}}$ is verticially quasi-split.

(Note that one verifies easily the existence of such a closed subgroup of $\mathrm{Aut}^{|\{e_{\mathcal{G}},e_{\mathcal{H}}\}|}(\mathcal{J})$ by considering, for instance, a homomorphism $G_{\mathcal{G}} = G_{\mathcal{H}} \hookrightarrow \mathrm{Aut}(\mathcal{J})$ of EPIPSC-type that arises from a suitable stable log curve—cf. also Remark 1.7.1, [9, Lem. 5.4(ii)], and [9, Prop. 5.6(ii)].) Then if one takes the "α" of Corollary 1.14 to be the outer isomorphism determined by the specialization outer isomorphisms $\Phi_{\mathcal{J}_{\rightsquigarrow \mathrm{Node}(\mathcal{J})\setminus\{e_{\mathcal{G}}\}}}$, $\Phi_{\mathcal{J}_{\rightsquigarrow \mathrm{Node}(\mathcal{J})\setminus\{e_{\mathcal{H}}\}}}$ (cf. [9, Def. 2.10]) and the "β" of Corollary 1.14 to be the identity isomorphism, then one verifies immediately from [9, Cor. 3.9(i)] and [9, Cor. 5.9(iii)] that one obtains a counterexample as desired.

Let R be a complete discrete valuation ring whose residue characteristic we denote by p (so p may be zero); $\overline{K}$ a separable closure of the field of fractions K of R;

$$\mathcal{X}^{\log}$$

a stable log curve (cf. the discussion entitled "Curves" in [9, §0]) over the log regular log scheme $\mathrm{Spec}(R)^{\log}$ obtained by equipping $\mathrm{Spec}(R)$ with the log structure determined by the maximal ideal $\mathfrak{m}_R \subseteq R$ of R. Suppose, for simplicity, that $\mathcal{X}^{\log}$ is *split*, that is, that the natural action of $\mathrm{Gal}(\overline{K}/K)$ on the dual semi-graph $\Gamma_{X^{\log}}$ associated with the geometric special fiber of $\mathcal{X}^{\log}$ is trivial. Write $X^{\log} \stackrel{\mathrm{def}}{=} \mathcal{X}^{\log} \times_R K$; $\mathrm{Vert}(X^{\log})$ (resp. $\mathrm{Cusp}(X^{\log})$; $\mathrm{Node}(X^{\log})$) for the set of vertices (resp. open edges; closed edges) of $\Gamma_{X^{\log}}$, that is, the set of connected components of the complement of the cusps and nodes (resp. the set of cusps;

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the set of nodes) of the special fiber of $\mathcal{X}^{\log}$;

$$\mathrm{VCN}(X^{\log}) \stackrel{\mathrm{def}}{=} \mathrm{Vert}(X^{\log}) \sqcup \mathrm{Cusp}(X^{\log}) \sqcup \mathrm{Node}(X^{\log}).$$

Before proceeding, we recall that

> to each element $z \in \mathrm{VCN}(X^{\log})$, one may associate, in a way that is functorial with respect to arbitrary automorphisms of the log scheme $X^{\log}$, a discrete valuation that dominates R on the residue field of some point of X, which is closed if and only if z is a cusp.

Indeed, this is immediate if z is a vertex, since a vertex corresponds to a prime of height 1 of $\mathcal{X}$. This is also immediate if z is a cusp, since the residue field of the closed point of X that corresponds to z is finite over (the complete discrete valuation field) K, which implies that the discrete valuation of K extends uniquely to a discrete valuation on the residue field of a cusp. Now suppose that z is a node that is locally defined by an equation of the form $s_1 s_2 - a$, for some $a \in \mathfrak{m}_R$ (cf., e.g., the discussion of [9, Def. 5.3(ii)]). By descent, we may assume without loss of generality that a admits a square root b in R. Then one associates with z the discrete valuation determined by the exceptional divisor of the blowup of $\mathcal{X}$ at the locus (s_1, s_2, b). (One verifies immediately that this construction is compatible with arbitrary automorphisms of $X^{\log}$.)

Corollary 1.15 (Fixed points associated with Galois sections). *Let Σ be a set of prime numbers; $\Sigma^\dagger \subseteq \Sigma$ a subset; $l \in \Sigma^\dagger$; R a complete discrete valuation ring of residue characteristic $p \notin \Sigma^\dagger$ (so p may be zero); $\overline{K}$ a separable closure of the field of fractions K of R;*

$$\mathcal{X}^{\log}$$

a stable log curve (cf. the discussion entitled "Curves" in [9, §0]) over the log regular log scheme $\mathrm{Spec}(R)^{\log}$ obtained by equipping $\mathrm{Spec}(R)$ with the log structure determined by the maximal ideal of R. Write $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K)$ for the absolute Galois group of K; $I_K \subseteq G_K$ for the inertia subgroup of G_K; $X^{\log} \stackrel{\mathrm{def}}{=} \mathcal{X}^{\log} \times_R K$; $X^{\log}_{\overline{K}} \stackrel{\mathrm{def}}{=} \mathcal{X}^{\log} \times_R \overline{K}$;

$$\Delta_{X^{\log}}$$

for the pro-Σ log fundamental group of $X^{\log}_{\overline{K}}$ (i.e., the maximal pro-Σ quotient of the log fundamental group of $X^{\log}_{\overline{K}}$);

$$\Pi_{X^{\log}}$$

for the geometrically pro-Σ log fundamental group of $X^{\log}$ (i.e., the quotient of the log fundamental group of $X^{\log}$ by the kernel of the natural surjection from the log fundamental group of $X^{\log}_{\overline{K}}$ onto $\Delta_{X^{\log}}$). Thus, we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Delta_{X^{\log}} \longrightarrow \Pi_{X^{\log}} \longrightarrow G_K \longrightarrow 1.$$

Write $\widetilde{X}^{\log} \to X^{\log}$ for the profinite log étale covering of $X^{\log}$ corresponding to $\Pi_{X^{\log}}$. If $Y^{\log} \to X^{\log}$ is a finite connected subcovering of $\widetilde{X}^{\log} \to X^{\log}$ that admits a stable model $\mathcal{Y}^{\log}$ over the normalization R_Y of R in Y, then let us write $\Gamma_{Y^{\log}}$ for the dual semi-graph determined by the geometric special fiber of $\mathcal{Y}^{\log}$ over R_Y; $\mathrm{Vert}(Y^{\log})$ (resp. $\mathrm{Cusp}(Y^{\log})$;

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$\mathrm{Node}(Y^{\log})$*) for the set of vertices (resp. open edges; closed edges) of* $\Gamma_{Y^{\log}}$*, that is, the set of connected components of the complement of the cusps and nodes (resp. the set of cusps; the set of nodes) of the geometric special fiber of* $\mathcal{Y}^{\log}$ *over* R_Y*;*

$$\mathrm{Edge}(Y^{\log}) \stackrel{\mathrm{def}}{=} \mathrm{Cusp}(Y^{\log}) \sqcup \mathrm{Node}(Y^{\log});$$

$$\mathrm{VCN}(Y^{\log}) \stackrel{\mathrm{def}}{=} \mathrm{Vert}(Y^{\log}) \sqcup \mathrm{Edge}(Y^{\log});$$

$$\mathrm{VCN}(\widetilde{X}^{\log}) \stackrel{\mathrm{def}}{=} \varprojlim \mathrm{VCN}(Y^{\log}),$$

where the projective limit is over all finite connected subcoverings $Y^{\log} \to X^{\log}$ *of* $\widetilde{X}^{\log} \to X^{\log}$ *as above, and, moreover, for each finite connected subcovering* $Y_1^{\log} \to X^{\log}$ *of* $\widetilde{X}^{\log} \to X^{\log}$ *that admits a stable model* $\mathcal{Y}_1^{\log}$ *over the normalization of* R *in* Y_1*, the transition map for a finite connected subcovering* $Y_2^{\log} \to Y_1^{\log}$ *of* $\widetilde{X}^{\log} \to Y_1^{\log}$ *that admits a stable model* $\mathcal{Y}_2^{\log}$ *over the normalization of* R *in* Y_2 *is defined, for* $z \in \mathrm{VCN}(Y_2^{\log})$*, as follows:*

- *If the connected component/cusp/node corresponding to* z *maps, via the extension* $\mathcal{Y}_2^{\log} \to \mathcal{Y}_1^{\log}$ *of* $Y_2^{\log} \to Y_1^{\log}$ *(cf., e.g., [14, Th. C]), to a cusp or node of the geometric special fiber of* $\mathcal{Y}_1$*, then the image of* $z \in \mathrm{VCN}(Y_2^{\log})$ *in* $\mathrm{VCN}(Y_1^{\log})$ *is defined to be the element of* $\mathrm{Edge}(Y_1^{\log})$ *corresponding to the cusp or node.*
- *If the generic point of the connected component/cusp/node corresponding to* z *maps, via the extension* $\mathcal{Y}_2^{\log} \to \mathcal{Y}_1^{\log}$ *of* $Y_2^{\log} \to Y_1^{\log}$*, to a point of the geometric special fiber of* $\mathcal{Y}_1$ *that is neither a cusp nor node, then the image of* $z \in \mathrm{VCN}(Y_2^{\log})$ *in* $\mathrm{VCN}(Y_1^{\log})$ *is defined to be the element of* $\mathrm{Vert}(Y_1^{\log})$ *corresponding to the connected component on which the point lies.*

If $\widetilde{z} \in \mathrm{VCN}(\widetilde{X}^{\log})$*, and* $Y^{\log} \to X^{\log}$ *is a finite connected subcovering of* $\widetilde{X}^{\log} \to X^{\log}$ *that admits a stable model* $\mathcal{Y}^{\log}$ *over the normalization of* R *in* Y*, then let us write* $\widetilde{z}(Y^{\log}) \in \mathrm{VCN}(Y^{\log})$ *for the element of* $\mathrm{VCN}(Y^{\log})$ *determined by* $\widetilde{z}$*. Let* $H \subseteq G_K$ *be a closed subgroup such that the image of*

$$I_H \stackrel{\mathrm{def}}{=} H \cap I_K \subseteq I_K$$

via the natural surjection $I_K \twoheadrightarrow I_K^{\Sigma^\dagger}$ *to the pro-*$\Sigma^\dagger$ *completion* $I_K^{\Sigma^\dagger}$ *of* I_K *is an open subgroup of* $I_K^{\Sigma^\dagger}$ *and*

$$s\colon H \longrightarrow \Pi_{X^{\log}}$$

a section of the restriction to $H \subseteq G_K$ *of the above exact sequence* $1 \to \Delta_{X^{\log}} \to \Pi_{X^{\log}} \to G_K \to 1$*. Then the following hold:*

(i) *If we write* $\Delta^\dagger_{X^{\log}}$ *for the maximal pro-*$\Sigma^\dagger$ *quotient of* $\Delta_{X^{\log}}$ *and regard, via the specialization outer isomorphism with respect to* $\mathcal{X}^{\log}$*, the pro-*$\Sigma^\dagger$ *group* $\Delta^\dagger_{X^{\log}}$ *as the (pro-*$\Sigma^\dagger$*) fundamental group of the semi-graph of anabelioids of pro-*$\Sigma^\dagger$ *PSC-type determined by the geometric special fiber of the stable model* $\mathcal{X}^{\log}$ *(cf. [18, Exam. 2.5]), then the natural outer Galois action*

$$H \longrightarrow \mathrm{Out}(\Delta^\dagger_{X^{\log}})$$

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determined by the above exact sequence is of EPIPSC-type for the conducting subgroup $I_H \subseteq H$ *(cf. Definition 1.7(i)). If, moreover, H is l-cyclotomically full, that is, the image of* $H \subseteq G_K$ *via the l-adic cyclotomic character on* G_K *is open, then the above outer Galois action is l-cyclotomically full (cf. Definition 1.7(ii)).*

(ii) *Suppose that the residue field of R is countable. Let* $\widetilde{z} \in \mathrm{VCN}(\widetilde{X}^{\log})$ *and* $\mathcal{S} = \{Y^{\log} \to X^{\log}\}$ *a cofinal system consisting of finite Galois subcoverings* $Y^{\log} \to X^{\log}$ *of* $\widetilde{X}^{\log} \to X^{\log}$ *such that* $Y^{\log}$ *admits a split stable model over the normalization* R_Y *of R in Y. Then there exist a valuation* $v_{\widetilde{z}}$ *on the residue field of some point of the underlying scheme* $\widetilde{X}$ *of* $\widetilde{X}^{\log}$ *(i.e., a bounded multiplicative seminorm—cf., e.g., [2, §§1.1 and 1.2]) and a countably indexed cofinal subsystem* $\mathcal{S}'$ *of* $\mathcal{S}$ *such that if* $Z^{\log} \to X^{\log}$ *is a member of* $\mathcal{S}'$*, then, as* $Y^{\log} \to X^{\log}$ *ranges over the members of* $\mathcal{S}'$ *that lie over* $Z^{\log}$*, the discrete valuations on residue fields of points of the underlying scheme Z of* $Z^{\log}$ *determined by the elements* $\widetilde{z}(Y^{\log}) \in \mathrm{VCN}(Y^{\log})$ *(cf. the discussion preceding the present Corollary 1.15) converge in the "Berkovich space topology"—that is, as bounded multiplicative seminorms—to the valuation on the residue field of some point of Z determined by* $v_{\widetilde{z}}$*.*

(iii) *Write* $\mathrm{Stab}(s) \subseteq \mathrm{VCN}(\widetilde{X}^{\log})$ *for the subset consisting of elements* $\widetilde{z} \in \mathrm{VCN}(\widetilde{X}^{\log})$ *such that the image of s stabilizes* $\widetilde{z}$*. Suppose that H is l-cyclotomically full (cf. (i)). Then it holds that*

$$\mathrm{Stab}(s) \neq \emptyset.$$

In particular, if $\widetilde{z} \in \mathrm{Stab}(s)$*, and the residue field of R is countable, then the image of s lies in the decomposition group of any valuation* $v_{\widetilde{z}}$ *as in (ii).*

(iv) *Let* $Y^{\log} \to X^{\log}$ *be a finite connected subcovering of* $\widetilde{X}^{\log} \to X^{\log}$ *that admits a stable model over the normalization* R_Y *of R in Y;* $\widetilde{z}_1, \widetilde{z}_2 \in \mathrm{Stab}(s)$ *(cf. (iii)). Then one of the following four (mutually exclusive) conditions is satisfied:*

(a) $\widetilde{z}_1(Y^{\log}), \widetilde{z}_2(Y^{\log}) \in \mathrm{Vert}(Y^{\log})$*, and* $\delta(\widetilde{z}_1(Y^{\log}), \widetilde{z}_2(Y^{\log})) \leq 2$ *(cf. Definition 1.1(iii)).*

(b) $\widetilde{z}_1(Y^{\log}), \widetilde{z}_2(Y^{\log}) \in \mathrm{Edge}(Y^{\log})$*, and, moreover,* $\mathcal{V}(\widetilde{z}_1(Y^{\log})) \cap \mathcal{V}(\widetilde{z}_2(Y^{\log})) \neq \emptyset$*.*

(c) $\widetilde{z}_1(Y^{\log}) \in \mathrm{Vert}(Y^{\log})$*,* $\widetilde{z}_2(Y^{\log}) \in \mathrm{Edge}(Y^{\log})$*, and, moreover,* $\mathcal{V}^{\delta \leq 1}(\widetilde{z}_1(Y^{\log})) \cap \mathcal{V}(\widetilde{z}_2(Y^{\log})) \neq \emptyset$ *(cf. Definition 1.10(i)).*

(d) $\widetilde{z}_1(Y^{\log}) \in \mathrm{Edge}(Y^{\log})$*,* $\widetilde{z}_2(Y^{\log}) \in \mathrm{Vert}(Y^{\log})$*, and, moreover,* $\mathcal{V}(\widetilde{z}_1(Y^{\log})) \cap \mathcal{V}^{\delta \leq 1}(\widetilde{z}_2(Y^{\log})) \neq \emptyset$*.*

(v) *In the situation of (iv), suppose, moreover, that the following assertion—that is, concerning "resolution of nonsingularities" (cf. Remark 1.15.1 below)—holds:*

> ($\dagger^{\mathrm{RNS}}$)*: Let* $Y^{\log} \to X^{\log}$ *be a finite connected subcovering of* $\widetilde{X}^{\log} \to X^{\log}$ *that admits a stable model* $\mathcal{Y}^{\log}$ *over* R_Y *and* $y \in \mathcal{Y}$ *a node of* $\mathcal{Y}$*. Then there exists a finite connected subcovering* $Z^{\log} \to Y^{\log}$ *of* $\widetilde{X}^{\log} \to Y^{\log}$ *that admits a stable model* $\mathcal{Z}^{\log}$ *over* R_Z *such that the fiber over y of the morphism* $\mathcal{Z} \to \mathcal{Y}$ *determined by* $Z^{\log} \to Y^{\log}$ *is not finite.*

Then every finite connected subcovering $Y^{\log} \to X^{\log}$ *of* $\widetilde{X}^{\log} \to X^{\log}$ *that admits a stable model over* R_Y *satisfies one of the following four (mutually exclusive) conditions:*

(a′) $\widetilde{z}_1(Y^{\log}), \widetilde{z}_2(Y^{\log}) \in \mathrm{Vert}(Y^{\log})$*, and* $\widetilde{z}_1(Y^{\log}) = \widetilde{z}_2(Y^{\log})$*.*

(b′) $\widetilde{z}_1(Y^{\log}), \widetilde{z}_2(Y^{\log}) \in \mathrm{Edge}(Y^{\log})$*, and, moreover,* $\mathcal{V}(\widetilde{z}_1(Y^{\log})) \cap \mathcal{V}(\widetilde{z}_2(Y^{\log})) \neq \emptyset$*.*

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(c′) $\widetilde{z}_1(Y^{\log}) \in \mathrm{Vert}(Y^{\log})$, $\widetilde{z}_2(Y^{\log}) \in \mathrm{Edge}(Y^{\log})$, *and, moreover,* $\widetilde{z}_1(Y^{\log}) \in \mathcal{V}(\widetilde{z}_2(Y^{\log}))$.

(d′) $\widetilde{z}_1(Y^{\log}) \in \mathrm{Edge}(Y^{\log})$, $\widetilde{z}_2(Y^{\log}) \in \mathrm{Vert}(Y^{\log})$, *and, moreover,* $\widetilde{z}_2(Y^{\log}) \in \mathcal{V}(\widetilde{z}_1(Y^{\log}))$.

(vi) *Write* $\Delta^{\mathrm{tp}}_{X^{\log}}$ *for the* Σ*-tempered fundamental group of* $X^{\log}_{\overline{K}}$ *(cf. [11, Def. 3.1(ii)]);* $\Pi^{\mathrm{tp}}_{X^{\log}}$ *for the geometrically* Σ*-tempered fundamental group of* $X^{\log}$ *(i.e., the quotient of the tempered fundamental group of* $X^{\log}$ *by the kernel of the natural surjection from the tempered fundamental group of* $X^{\log}_{\overline{K}}$ *onto* $\Delta^{\mathrm{tp}}_{X^{\log}}$*). Thus, we have a natural exact sequence of topological groups*

$$1 \longrightarrow \Delta^{\mathrm{tp}}_{X^{\log}} \longrightarrow \Pi^{\mathrm{tp}}_{X^{\log}} \longrightarrow G_K \longrightarrow 1.$$

Write $\mathrm{Sect}(\Pi_{X^{\log}}/H)$ *for the set of* $\Delta_{X^{\log}}$*-conjugacy classes of continuous sections of the restriction to* $H \subseteq G_K$ *of the natural surjection* $\Pi_{X^{\log}} \twoheadrightarrow G_K$ *and* $\mathrm{Sect}(\Pi^{\mathrm{tp}}_{X^{\log}}/H)$ *for the set of* $\Delta^{\mathrm{tp}}_{X^{\log}}$*-conjugacy classes of continuous sections of the restriction to* $H \subseteq G_K$ *of the natural surjection* $\Pi^{\mathrm{tp}}_{X^{\log}} \twoheadrightarrow G_K$*. Then the natural map*

$$\mathrm{Sect}(\Pi^{\mathrm{tp}}_{X^{\log}}/H) \longrightarrow \mathrm{Sect}(\Pi_{X^{\log}}/H)$$

is injective. If, moreover, H *is* l*-cyclotomically full (cf. (i)), then this map is bijective.*

Proof. Assertion (i) follows immediately from the definition of the term "IPSC-type" (cf. [8, Def. 2.4(i)]), together with the well-known structure of the maximal pro-$\Sigma^\dagger$ quotient of I_K. Next, we verify assertion (ii). Let us first observe that it follows immediately from our countability assumption on the residue field of R that the following three assertions hold:

- If $Y^{\log} \to X^{\log}$ is a member of $\mathcal{S}$, and $\widetilde{z}(Y^{\log}) \notin \mathrm{Cusp}(Y^{\log})$, then the function field of Y admits a subset which is countable and dense, that is, with respect to the topology determined by the discrete valuation determined by the element $\widetilde{z}(Y^{\log}) \in \mathrm{VCN}(Y^{\log})$.
- If $Y^{\log} \to X^{\log}$ is a member of $\mathcal{S}$, and $\widetilde{z}(Y^{\log}) \in \mathrm{Cusp}(Y^{\log})$, then the normalization R_Y of R in Y admits a subset which is countable and dense, that is, with respect to the topology determined by the discrete valuation determined by the element $\widetilde{z}(Y^{\log}) \in \mathrm{VCN}(Y^{\log})$.
- There exists a countably indexed cofinal subsystem of $\mathcal{S}$ (cf., e.g., [21, Lem. 2.1]).

Thus, assertion (ii) follows immediately, by applying a standard argument involving Cantor diagonalization, from the well-known (local) compactness of Berkovich spaces (cf., e.g., [2, Th. 1.2.1]). Here, we recall in passing that this compactness is, in essence, a consequence of the compactness of a product of copies of the closed interval $[0,1] \subseteq \mathbb{R}$. This completes the proof of assertion (ii). We refer to Theorem A.7 in the Appendix for another approach to proving assertion (ii).

Assertion (iii) follows immediately from the observation that, by applying Theorem 1.13(i) (cf. also Remark 1.7.1; assertion (i) of the present Corollary 1.15; [18, Prop. 1.2(i)]), together with the well-known fact that a projective limit of nonempty finite sets is nonempty, to the various finite connected subcoverings of $\widetilde{X}^{\log} \to X^{\log}$, one may conclude that the action of G_K, via s, on $\widetilde{X}^{\log}$ fixes some element $\widetilde{z}_s \in \mathrm{VCN}(\widetilde{X}^{\log})$ of $\mathrm{VCN}(\widetilde{X}^{\log})$. (Here, we note that when one applies Theorem 1.13(i) to the various finite connected subcoverings of $\widetilde{X}^{\log} \to X^{\log}$, the conducting subgroup "I_G" of Theorem 1.13(i) must be allowed to vary among suitable open subgroups of the original conducting subgroup I_G.) Assertion (iv)

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follows immediately (cf. also Remark 1.7.1; assertion (i) of the present Corollary 1.15) from Lemma 1.11(ii).

Next, we verify assertion (v). Let us first observe that it follows immediately from assertion (iv) that if $Y^{\log} \to X^{\log}$ is a finite connected subcovering of $\widetilde{X}^{\log} \to X^{\log}$ that admits a stable model over R_Y, then $\widetilde{z}_1(Y^{\log})$ and $\widetilde{z}_2(Y^{\log})$ lie in a connected sub-semi-graph Γ^* of $\Gamma_{Y^{\log}}$ such that

$$\mathrm{VCN}(\Gamma^*)^\sharp = \mathrm{Vert}(\Gamma^*)^\sharp + \mathrm{Edge}(\Gamma^*)^\sharp \leq 3+2=5.$$

Now one verifies immediately that this uniform bound "5" implies that there exists a cofinal system $\mathcal{S} = \{Y^{\log} \to X^{\log}\}$ consisting of finite Galois subcoverings $Y^{\log} \to X^{\log}$ of $\widetilde{X}^{\log} \to X^{\log}$ such that $Y^{\log}$ admits a stable model over R_Y, and, moreover, $\Gamma_{Y^{\log}}$ admits a connected sub-semi-graph $\Gamma^*_{Y^{\log}}$ such that:

- $\widetilde{z}_1(Y^{\log})$ and $\widetilde{z}_2(Y^{\log})$ lie in $\Gamma^*_{Y^{\log}}$;
- $\mathrm{VCN}(\Gamma^*_{Y^{\log}})^\sharp \leq 5$;
- the semi-graphs $\Gamma^*_{Y^{\log}}$ map isomorphically to one another as one varies $Y^{\log} \to X^{\log}$.

Write $\mathcal{V}^*(Y^{\log}) \stackrel{\mathrm{def}}{=} \mathrm{Vert}(\Gamma^*_{Y^{\log}})$. Then it follows immediately from assertion (iv) that, to complete the verification of assertion (v), it suffices to verify that the following assertion holds:

Claim 1.15.A: $\mathcal{V}^*(Y^{\log})^\sharp \leq 1$.

Indeed, suppose that $\mathcal{V}^*(Y^{\log})^\sharp \geq 2$. Then it follows immediately that there exists a compatible system of nodes $e(Y^{\log})$ of $\Gamma^*_{Y^{\log}}$ (i.e., compatible as one varies $Y^{\log} \to X^{\log}$ in $\mathcal{S}$), each of which abuts to distinct vertices $v_\alpha(Y^{\log})$, $v_\beta(Y^{\log})$ of $\Gamma^*_{Y^{\log}}$. (Thus, one may assume that the vertices $v_\alpha(-)$ [resp. $v_\beta(-)$] form a compatible system of vertices.) But this implies that for every $Z^{\log} \to X^{\log}$ in $\mathcal{S}$ that lies over $Y^{\log} \to X^{\log}$ in $\mathcal{S}$, if we write $\mathcal{Y}^{\log}$, $\mathcal{Z}^{\log}$ for the respective stable models of $Y^{\log}$, $Z^{\log}$ (so the morphism $Z^{\log} \to Y^{\log}$ extends to a morphism $\mathcal{Z}^{\log} \to \mathcal{Y}^{\log}$—cf., e.g., [14, Th. C]), then the inverse image in $\mathcal{Z}^{\log}$ of the node $e(Y^{\log})$ admits at least one isolated point (i.e., $e(Z^{\log})$), and hence (since the covering $Z^{\log} \to Y^{\log}$ is Galois) the entire inverse image in $\mathcal{Z}^{\log}$ of $e(Y^{\log})$ is of dimension zero. On the other hand, this contradicts the assertion ($\dagger^{\mathrm{RNS}}$) in the statement of assertion (v). This completes the proof of assertion (v).

Finally, we verify assertion (vi). The injectivity portion of assertion (v) follows immediately from the injectivity portion of Theorem 1.13(iii) (cf. also Remark 1.7.1; assertion (i) of the present Corollary 1.15), applied to the various finite connected subcoverings of $\widetilde{X}^{\log} \to X^{\log}$, where we take the "$\Sigma$" of Theorem 1.13 to be $\Sigma^\dagger$ (cf. also the fact that, in the notation of Theorem 1.13, "$\Pi^{\mathrm{tp}}_{\mathcal{G}}$" is dense in "$\Pi_{\mathcal{G}}$" in the profinite topology). Here, we note that

- when one applies Theorem 1.13(iii) to the various finite connected subcoverings of $\widetilde{X}^{\log} \to X^{\log}$, the conducting subgroup "I_G" of Theorem 1.13(iii) must be allowed to vary among suitable open subgroups of the original conducting subgroup I_G, and that
- it follows immediately from the final portion of Lemma 1.11(iv) that the resulting conjugacy indeterminacies that occur at various subcoverings are uniquely determined up to profinite centralizers of the sections that appear, hence converge in $\Delta^{\mathrm{tp}}_{X^{\log}}$ (i.e., if one passes to an appropriate subsequence of the system of subcoverings under consideration).

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If H is l-cyclotomically full, then the surjectivity of the map

$$\mathrm{Sect}(\Pi^{\mathrm{tp}}_{X^{\log}}/H) \to \mathrm{Sect}(\Pi_{X^{\log}}/H)$$

follows formally (cf. the proof of the final portion of Theorem 1.13(iii)) from the nonemptiness verified in assertion (iii). This completes the proof of assertion (vi). $\square$

REMARK 1.15.1. It follows from [29, Th. 0.2(v)] that if K is of characteristic zero, the residue field of R is algebraic over $\mathbb{F}_p$, and $\Sigma = \mathfrak{Primes}$, then the assertion $(\dagger^{\mathrm{RNS}})$ in the statement of Corollary 1.15(v) holds.

REMARK 1.15.2.

(i) Corollary 1.15(iii) and (v) (cf. also [17, Lem. 5.5]) may be regarded as a generalization of the Main Result of [26]. These results are obtained in the present paper (cf. the proof of Theorem 1.13(i)) by, in essence, combining, via a similar argument to the argument applied in the tempered case treated in [17, Ths. 3.7 and 5.4] (cf. also the proof of Theorem 1.13(ii) of the present paper), the uniqueness result given in [8, Props. 3.8(i) and 3.9(i)–(iii)] (cf. the proof of Lemma 1.11(ii)), with the existence of fixed points of actions of finite groups on graphs that follows as a consequence of the classical fact that (discrete or pro-Σ) free groups are torsion-free (cf. Remarks 1.6.2 and 1.13.1; the proof of Lemma 1.6(ii)). One slight difference between the profinite and tempered cases is that, whereas, in the tempered case, it follows from the discreteness of the fundamental groups of graphs that appear that the actions of profinite groups on universal coverings of such graphs necessarily factor through finite quotients, the corresponding fact in the profinite case is obtained as a consequence of the fact that, under a suitable assumption on the cyclotomic characters that appear, any homomorphism from a "positive slope" module to a torsion-free "slope zero" module necessarily vanishes (cf. the proof of Claim 1.13.B in Theorem 1.13(i)). That is to say, in a word, these results are obtained in the present paper as a consequence of

> abstract considerations concerning *abstract profinite groups* acting on *abstract semi-graphs* that may, for instance, arise as dual semi-graphs of geometric special fibers of stable models of curves that appear in scheme theory, but, a priori, have *nothing to do with scheme theory.*

This a priori irrelevance of scheme theory to such abstract considerations is reflected both in the variety of the results obtained in the present §1 as consequences of Theorem 1.13, as well as in the generality of Corollary 1.15. This approach contrasts quite substantially with the approach of [26], that is, where the main results are derived as a consequence of highly scheme-theoretic considerations concerning stable curves over complete discrete valuation rings, in which the theory of the Brauer group of the function field of such a curve plays a central role (cf. [26, §4]).

(ii) The essential equivalence between the issue of considering valuations fixed by Galois actions and the issue of considering vertices or edges of associated dual semi-graphs fixed by Galois actions may be seen in the well-known functorial homotopy equivalence between the Berkovich space associated with a stable curve over a complete discrete valuation ring and the associated dual graph (cf. [3, Ths. 8.1 and 8.2]). Moreover, the issue of convergence of (sub)sequences of valuations fixed by Galois actions is an easy consequence of the well-known (local) compactness of Berkovich spaces (cf. the proof of

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Corollary 1.15(ii); [2, Th. 1.2.1]), that is, in essence, a consequence of the well-known compactness of a product of copies of the closed interval $[0,1] \subseteq \mathbb{R}$. That is to say, there is no need to consider the quite complicated (and, at the time of writing, not well understood!) structure of inductive limits of local rings, as discussed in [26, §1.6].

REMARK 1.15.3. Recall that in Corollary 1.15(ii) and the final portion of Corollary 1.15(iii), we assume that the residue field of R is countable. In fact, however, it is not difficult to see that, in the situation of Corollary 1.15, there exists a complete discrete valuation ring $R^\dagger$ that is dominated by R, and whose residue field is countable such that

- the smooth log curve $X^{\log}$,
- the closed subgroup $H \subseteq G_K$, and
- the section $s\colon H \to \Pi_{X^{\log}}$

descend to the field of fractions of $R^\dagger$. Indeed, let us first observe that since the moduli stack of pointed stable curves of a given type over $\mathbb{Z}$ is of finite type over $\mathbb{Z}$, there exists a complete discrete valuation ring $R^\ddagger$ that is dominated by R, and whose residue field is countable such that the smooth log curve $X^{\log}$ descends to the field of fractions of $R^\ddagger$. Next, let us observe that since (cf., e.g., [15, Prop. 2.3(ii)]) the geometric fundamental group "$\Delta_{X^{\log}}$" associated with the smooth log curve $X^{\log}$ (i.e., over the field of fractions of R) is naturally isomorphic to the geometric fundamental group "$\Delta_{X^{\log}}$" associated with the descended smooth log curve (i.e., over the field of fractions of $R^\ddagger$), it follows that both of these geometric fundamental groups are topologically finitely generated (cf., e.g., [23, Prop. 2.2(ii)]), and hence that there exists a countably indexed open basis

$$\cdots \subseteq U_{n+1} \subseteq U_n \subseteq \cdots \subseteq U_2 \subseteq U_1 \subseteq U_0 = \Delta_{X^{\log}}$$

of characteristic open subgroups of $\Delta_{X^{\log}}$. In particular, there exists a complete discrete valuation ring $R^\dagger$ that is dominated by R, and whose residue field is countable such that, for each positive integer n, the finite collection of finite étale coverings (which are defined by means of finitely many polynomials, with finitely many coefficients) corresponding to

- the finite quotient $\Pi_{X^{\log}} \twoheadrightarrow Q_n$ determined by the image of the composite of the conjugation action $\Pi_{X^{\log}} \to \mathrm{Aut}(\Delta_{X^{\log}})$ and the natural homomorphism $\mathrm{Aut}(\Delta_{X^{\log}}) \to \mathrm{Aut}(\Delta_{X^{\log}}/U_n)$ and
- the subgroup $H_n \subseteq Q_n$ obtained by forming the image of the composite of the section $s\colon H \to \Pi_{X^{\log}}$ and the natural surjective homomorphism $\Pi_{X^{\log}} \twoheadrightarrow Q_n$

descends to the field of fractions of $R^\dagger$.

§2. Discrete combinatorial anabelian geometry

In the present section, we introduce the notion of a semi-graph of temperoids of HSD-type (i.e., "hyperbolic surface decomposition type"—cf. Definition 2.3(iii)) and discuss discrete versions of the profinite results obtained in [8], [9], [10], [11]. A semi-graph of temperoids of HSD-type arises naturally from a decomposition (satisfying certain properties) of a hyperbolic topological surface and may be regarded as a discrete analogue of the notion of a semi-graph of anabelioids of PSC-type. The main technical result of the present section is Theorem 2.15, one immediate consequence of which is the following (cf. Corollary 2.19):

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An isomorphism of groups between the discrete fundamental groups of a pair of semi-graphs of temperoids of HSD-type arises from an isomorphism between the semi-graphs of temperoids of HSD-type if and only if the induced isomorphism between profinite completions of fundamental groups arises from an isomorphism between the associated semi-graphs of anabelioids of pro-$\mathfrak{Primes}$ PSC-type.

In the present §2, let Σ be a nonempty set of prime numbers.

DEFINITION 2.1.

(i) We shall refer to as a *semi-graph of temperoids* $\mathcal{G}$ a collection of data as follows:

- a semi-graph $\mathbb{G}$ (cf. the discussion at the beginning of [17, §1]),
- for each vertex v of $\mathbb{G}$, a connected temperoid $\mathcal{G}_v$ (cf. [17, Def. 3.1(ii)]),
- for each edge e of $\mathbb{G}$, a connected temperoid $\mathcal{G}_e$, together with, for each branch $b \in e$ abutting to a vertex v, a morphism of temperoids $b_* : \mathcal{G}_e \to \mathcal{G}_v$ (cf. [17, Def. 3.1(iii)]).

We shall refer to a semi-graph of temperoids whose underlying semi-graph is connected as a connected semi-graph of temperoids. Given two semi-graphs of temperoids, there is an evident notion of (1-)morphism (cf. [17, Def. 2.1]; [17, Rem. 2.4.2]) between semi-graphs of temperoids.

(ii) Let $\mathcal{T}$ be a connected temperoid. We shall say that a connected object H of $\mathcal{T}$ is Σ-*finite* if there exists a morphism $J \to H$ in $\mathcal{T}$ such that J is Galois (hence connected—cf. [17, Def. 3.1(iv)]), and, moreover, $\mathrm{Aut}_{\mathcal{T}}(J)$ is a finite group whose order is a Σ-integer (cf. the discussion entitled "Numbers" in §0). We shall say that an object H of $\mathcal{T}$ is Σ-*finite* if H is isomorphic to a disjoint union of finitely many connected Σ-finite objects. We shall say that an object H of $\mathcal{T}$ is a *finite object* if H is $\mathfrak{Primes}$-finite. We shall write

$$\mathcal{T}^{\Sigma}$$

for the connected anabelioid (cf. [16, Def. 1.1.1]) obtained by forming the full subcategory of $\mathcal{T}$ whose objects are the Σ-finite objects of $\mathcal{T}$. Thus, we have a natural morphism of temperoids (cf. Remark 2.1.1 below)

$$\mathcal{T} \longrightarrow \mathcal{T}^{\Sigma}.$$

We shall write

$$\widehat{\mathcal{T}} \stackrel{\mathrm{def}}{=} \mathcal{T}^{\mathfrak{Primes}}$$

(cf. the discussion entitled "Numbers" in §0). Finally, we observe that if $\mathcal{T} = \mathcal{B}^{\mathrm{tp}}(\Pi)$, where Π is a tempered group (cf. [17, Def. 3.1(i)]), and "$\mathcal{B}^{\mathrm{tp}}(-)$" denotes the category "$\mathcal{B}^{\mathrm{temp}}(-)$" of the discussion at the beginning of [17, §3], then $\mathcal{T}^{\Sigma}$ may be naturally identified with $\mathcal{B}(\Pi^{\Sigma})$, that is, the connected anabelioid (cf. [16, Def. 1.1.1]; the discussion at the beginning of [16, §1]) determined by the pro-Σ completion Π^{Σ} of Π.

(iii) Let $\mathcal{G}$ be a semi-graph of temperoids (cf. (i)). Then, by replacing the connected temperoids "$\mathcal{G}_{(-)}$" corresponding to the vertices and edges "$(-)$" by the connected anabelioids "$\mathcal{G}^{\Sigma}_{(-)}$" (cf. (ii)), we obtain a semi-graph of anabelioids, which we denote by

$$\mathcal{G}^{\Sigma}$$

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(cf. [17, Def. 2.1]). Thus, it follows immediately from the various definitions involved that the various morphisms "$\mathcal{G}_{(-)} \to \mathcal{G}^{\Sigma}_{(-)}$" of (ii) determine a natural morphism of semi-graphs of temperoids (cf. Remark 2.1.1 below)

$$\mathcal{G} \longrightarrow \mathcal{G}^{\Sigma}.$$

We shall write $\widehat{\mathcal{G}} \stackrel{\mathrm{def}}{=} \mathcal{G}^{\mathfrak{Primes}}$. One verifies easily that if $\mathcal{G}$ is a connected semi-graph of temperoids (cf. (i)), then $\mathcal{G}^{\Sigma}$ is a connected semi-graph of anabelioids.

(iv) Let $\mathcal{G}$ be a connected semi-graph of temperoids (cf. (i)). Suppose that (the underlying semi-graph of) $\mathcal{G}$ has at least one vertex. Then we shall write

$$\mathcal{B}(\mathcal{G}) \stackrel{\mathrm{def}}{=} \mathcal{B}(\widehat{\mathcal{G}})$$

(cf. (iii); the discussion following [17, Def. 2.1]) for the connected anabelioid determined by the connected semi-graph of anabelioids $\widehat{\mathcal{G}}$.

(v) Let $\mathcal{G}$ be a semi-graph of temperoids. Then we shall write $\mathrm{Vert}(\mathcal{G})$, $\mathrm{Cusp}(\mathcal{G})$, $\mathrm{Node}(\mathcal{G})$, $\mathrm{Edge}(\mathcal{G})$, $\mathrm{VCN}(\mathcal{G})$, $\mathcal{V}$, $\mathcal{C}$, $\mathcal{N}$, $\mathcal{E}$, and δ for the Vert, Cusp, Node, Edge, VCN, $\mathcal{V}$, $\mathcal{C}$, $\mathcal{N}$, $\mathcal{E}$, and δ of Definition 1.1(i)–(iii), applied to the underlying semi-graph of $\mathcal{G}$.

(vi) Let $\mathcal{G}$ be a connected semi-graph of temperoids (cf. (i)). Suppose that (the underlying semi-graph of) $\mathcal{G}$ has at least one vertex. Then we shall write

$$\mathcal{B}^{\mathrm{tp}}(\mathcal{G})$$

for the category whose objects are given by collections of data

$$\{S_v, \phi_e\}$$

where v (resp. e) ranges over the elements of $\mathrm{Vert}(\mathcal{G})$ (resp. $\mathrm{Edge}(\mathcal{G})$) (cf. (v)); for each $v \in \mathrm{Vert}(\mathcal{G})$, S_v is an object of the temperoid $\mathcal{G}_v$ corresponding to v; for each $e \in \mathrm{Edge}(\mathcal{G})$, with branches b_1, b_2 abutting to vertices v_1, v_2, respectively, $\phi_e \colon ((b_1)_*)^* S_{v_1} \xrightarrow{\sim} ((b_2)_*)^* S_{v_2}$ is an isomorphism in the temperoid $\mathcal{G}_e$ corresponding to e—and whose morphisms are given by morphisms (in the evident sense) between such collections of data. In particular, the category (i.e., connected anabelioid) $\mathcal{B}(\mathcal{G})$ of (iv) may be regarded as a full subcategory

$$\mathcal{B}(\mathcal{G}) \subseteq \mathcal{B}^{\mathrm{tp}}(\mathcal{G})$$

of $\mathcal{B}^{\mathrm{tp}}(\mathcal{G})$. One verifies immediately that any object G' of $\mathcal{B}^{\mathrm{tp}}(\mathcal{G})$ determines, in a natural way, a semi-graph of temperoids $\mathcal{G}'$, together with a morphism of semi-graphs of temperoids $\mathcal{G}' \to \mathcal{G}$. We shall refer to this morphism $\mathcal{G}' \to \mathcal{G}$ as the covering of $\mathcal{G}$ associated with G'. We shall say that a morphism of semi-graphs of temperoids is a *covering* (resp. *finite étale covering*) *of* $\mathcal{G}$ if it factors as the post-composite of an isomorphism of semi-graphs of temperoids with the covering of $\mathcal{G}$ associated with some object of $\mathcal{B}^{\mathrm{tp}}(\mathcal{G})$ (resp. of $\mathcal{B}(\mathcal{G})(\subseteq \mathcal{B}^{\mathrm{tp}}(\mathcal{G}))$). We shall say that a covering of $\mathcal{G}$ is connected if the underlying semi-graph of the domain of the covering is connected.

REMARK 2.1.1. Since every profinite group is tempered (cf. [17, Def. 3.1(i)]; [17, Rem. 3.1.1]), it follows immediately that a connected anabelioid (cf. [16, Def. 1.1.1]) determines, in a natural way (i.e., by considering formal countable coproducts, as in the discussion entitled "Categories" in [17, §0]), a connected temperoid (cf. [17, Def. 3.1(ii)]). In particular, a semi-graph of anabelioids (cf. [17, Def. 2.1]) determines, in a natural way, a semi-graph of

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temperoids (cf. Definition 2.1(i)). By abuse of notation, we shall often use the same notation for the connected temperoid (resp. semi-graph of temperoids) naturally associated with a connected anabelioid (resp. semi-graph of anabelioids).

Definition 2.2.

(i) Let T be a topological space. Then we shall say that a closed subspace of T (resp. a closed subspace of T; an open subspace of T) is a *circle* (resp. a *closed disk*; an *open disk*) on T if it is homeomorphic to the set $\{(s,t) \in \mathbb{R}^2 \,|\, s^2+t^2=1\}$ (resp. $\{(s,t) \in \mathbb{R}^2 \,|\, s^2+t^2 \leq 1\}$; $\{(s,t) \in \mathbb{R}^2 \,|\, s^2+t^2 < 1\}$) equipped with the topology induced by the topology of $\mathbb{R}^2$. If $D \subseteq T$ is a closed disk on T, then we shall write $\partial D \subseteq D$ for the circle on T determined by the boundary of D regarded as a two-dimensional topological manifold with boundary (i.e., the closed subspace of D corresponding to the closed subspace $\{(s,t) \in \mathbb{R}^2 \,|\, s^2+t^2=1\} \subseteq \{(s,t) \in \mathbb{R}^2 \,|\, s^2+t^2 \leq 1\}$) and $D^\circ \stackrel{\mathrm{def}}{=} D \setminus \partial D \subseteq D$ for the open disk on T obtained by forming the complement of ∂D in D.

(ii) Let (g,r) be a pair of nonnegative integers. Then we shall say that a pair $X = (\overline{X}, \{D_i\}_{i=1}^r)$ consisting of a connected orientable compact topological surface $\overline{X}$ of genus g and a collection of r disjoint closed disks $D_i \subseteq \overline{X}$ of $\overline{X}$ (cf. (i)) is *of HS-type* (where the "HS" stands for "hyperbolic surface") if $2g-2+r>0$.

(iii) Let $X = (\overline{X}, \{D_i\}_{i=1}^r)$ be a pair of HS-type (cf. (ii)). Then we shall write

$$U_X \stackrel{\mathrm{def}}{=} \overline{X} \setminus \Big(\bigcup_{i=1}^r D_i^\circ\Big)$$

(cf. (i)) and refer to U_X as the *interior* of X. We shall refer to a circle on U_X determined by some $\partial D_i \subseteq U_X$ (cf. (i)) as a *cusp* of U_X, or alternatively, X. Write $\partial U_X \subseteq U_X$ for the union of the cusps of U_X; I_X for the group of homeomorphisms $\phi\colon \overline{X} \xrightarrow{\sim} \overline{X}$ such that ϕ restricts to the identity on U_X. Suppose that $Y = (\overline{Y}, \{E_i\}_{j=1}^s)$ is also a pair of HS-type. Then we define an *isomorphism* $X \xrightarrow{\sim} Y$ *of pairs of HS-type* to be an I_X-orbit of homeomorphisms $\overline{X} \xrightarrow{\sim} \overline{Y}$ such that each homeomorphism ψ that belongs to the I_X-orbit induces a homeomorphism $U_X \xrightarrow{\sim} U_Y$.

(iv) Let $X = (\overline{X}, \{D_i\}_{i=1}^r)$ be a pair of HS-type (cf. (ii)) and $\{Y_j\}_{j\in J}$ a finite collection of pairs of HS-type. For each $j \in J$, let $\iota_j\colon U_{Y_j} \hookrightarrow U_X$ (cf. (iii)) be a local immersion (i.e., a map that restricts to a homeomorphism between some open neighborhood of each point of the domain and the image of the open neighborhood, equipped with the induced topology, in the codomain) of topological spaces. Then we shall say that a pair $(\{Y_j\}_{j\in J}, \{\iota_j\}_{j\in J})$ is an *HS-decomposition* of X if the following conditions are satisfied:

(1) $U_X = \bigcup_{j\in J} \iota_j(U_{Y_j})$.

(2) For any $j \in J$, the complement of the diagonal in $U_{Y_j} \times_{U_X} U_{Y_j}$ is a disjoint union of circles, each of which maps homeomorphically, via the two projections to U_{Y_j}, to two distinct cusps of U_{Y_j} (cf. (iii)). (Thus, by "Brouwer invariance of domain," it follows that ι_j restricts to an open immersion on the complement of the cusps of U_{Y_j}.)

(3) For any j, $j' \in J$ such that $j \neq j'$, every connected component of $U_{Y_j} \times_{U_X} U_{Y_{j'}}$ projects homeomorphically onto cusps of U_{Y_j} and $U_{Y_{j'}}$.

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(4) For any (i.e., possibly equal) j, $j' \in J$, we shall refer to a circle of $U_{Y_j} \times_{U_X} U_{Y_{j'}}$ that forms a connected component of $U_{Y_j} \times_{U_X} U_{Y_{j'}}$ as a *pre-node* (of the HS-decomposition $(\{Y_j\}_{j\in J}, \{\iota_j\}_{j\in J})$) and to the cusps of U_{Y_j}, $U_{Y_{j'}}$ that arise as the images of such a pre-node via the projections to U_{Y_j}, $U_{Y_{j'}}$ as the *branch cusps* of the pre-node. Then we suppose further that every pre-node maps injectively into U_X, and that the image in U_X of the pre-node has empty intersection with ∂U_X, as well as with the image via $\iota_{j''}$, for $j'' \in J$, of any cusp of $U_{Y_{j''}}$ which is not a branch cusp of the pre-node. We shall refer to the image in U_X of a pre-node as a *node* (of the HS-decomposition $(\{Y_j\}_{j\in J}, \{\iota_j\}_{j\in J})$). Thus, (one verifies easily that) every node arises from a unique pre-node. We shall refer to the branch cusps of the pre-node that gives rise to a node as the *branch cusps* of the node. (Thus, by "Brouwer invariance of domain," it follows that, for any pre-node of $U_{Y_j} \times_{U_X} U_{Y_{j'}}$, the maps ι_j, $\iota_{j'}$ determine a homeomorphism of the topological space obtained by gluing, along the associated node, suitable open neighborhoods of the branch cusps of U_{Y_j}, $U_{Y_{j'}}$ onto the topological space constituted by a suitable open neighborhood of the associated node in U_X.)

(5) For any $j \in J$, every cusp of U_{Y_j} maps homeomorphically onto either a cusp of U_X or a node of $(\{Y_j\}_{j\in J}, \{\iota_j\}_{j\in J})$ (cf. (4)). Moreover, every cusp of U_X arises in this way from a cusp of U_{Y_j} for some (necessarily uniquely determined) $j \in J$. (Thus, by "Brouwer invariance of domain"—together with a suitable gluing argument as in (4)—it follows that every cusp of U_X admits an open neighborhood that arises, for some $j \in J$, as the homeomorphic image, via ι_j, of a suitable open neighborhood of a cusp of U_{Y_j}.)

If $(\{Y_j\}, \{\iota_j\})$ is an HS-decomposition of X, then we shall refer to the triple $(X, \{Y_j\}, \{\iota_j\})$ as a *collection of HSD-data* (where the "HSD" stands for "hyperbolic surface decomposition"). If $\mathbb{X} = (X, \{Y_j\}, \{\iota_j\})$ is a collection of HSD-data, then we shall refer to the topological space U_X (resp. [the closed subspace of U_X corresponding to] an element of the [finite] set $\{Y_j\}$; a cusp of U_X; a node of $(\{Y_j\}, \{\iota_j\})$ [cf. (4)]) as the *underlying surface* (resp. a *vertex*; a *cusp*; a *node*) of $\mathbb{X}$. Also, we shall refer to a cusp or node of $\mathbb{X}$ as an *edge* of $\mathbb{X}$.

Definition 2.3. Let $\mathbb{X} = (X, \{Y_j\}, \{\iota_j\})$ be a collection of HSD-data (cf. Definition 2.2(iv)).

(i) We shall refer to the semi-graph

$$\mathbb{G}_{\mathbb{X}}$$

defined as follows as the *dual semi-graph* of $\mathbb{X}$: we take the set of vertices (resp. open edges; closed edges) of $\mathbb{G}_{\mathbb{X}}$ is the (finite) set of vertices (resp. cusps; nodes) of $\mathbb{X}$ (cf. Definition 2.2(iv)). For a vertex v and an edge e of $\mathbb{X}$, we take the set of branches of e that abut to v to be the set of natural inclusions (i.e., that arise from $\mathbb{X}$—cf. Definition 2.2(iv)) from the edge of $\mathbb{X}$ corresponding to e into the topological space U_{Y_j} associated with the Y_j corresponding to the vertex v.

(ii) We shall refer to the connected semi-graph

$$\mathcal{G}_{\mathbb{X}}$$

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of temperoids (cf. Definition 2.1(i)) defined as follows as the *semi-graph of temperoids associated with* $\mathbb{X}$: we take the underlying semi-graph of $\mathcal{G}_{\mathbb{X}}$ to be $\mathbb{G}_{\mathbb{X}}$ (cf. (i)). For each vertex v of $\mathbb{G}_{\mathbb{X}}$, we take the connected temperoid of $\mathcal{G}_{\mathbb{X}}$ corresponding to v to be the connected temperoid determined by the category of topological coverings with countably many connected components of the topological space U_{Y_j} (cf. Definition 2.2(iii)) associated with the Y_j corresponding to the vertex v. For each edge e of $\mathbb{G}_{\mathbb{X}}$, we take the connected temperoid of $\mathcal{G}_{\mathbb{X}}$ corresponding to e to be the connected temperoid determined by the category of topological coverings with countably many connected components of the circle (cf. Definition 2.2(i)) on U_X corresponding to the edge e. For each branch b of $\mathbb{G}_{\mathbb{X}}$, we take the morphism of temperoids corresponding to b to be the morphism obtained by pulling back topological coverings of the topological spaces under consideration.

(iii) We shall say that a semi-graph of temperoids is *of HSD-type* if it is isomorphic to the semi-graph of temperoids associated with some collection of HSD-data (cf. (ii)).

Example 2.4 (Semi-graphs of temperoids of HSD-type associated with stable log curves). Let (g,r) be a pair of nonnegative integers such that $2g-2+r>0$. Write $S \stackrel{\text{def}}{=} \operatorname{Spec}(\mathbb{C})$. In the following, we shall apply the notation and terminology of the discussion entitled "Curves" in [9, §0].

(i) Let $S \to (\overline{\mathcal{M}}_{g,r})_{\mathbb{C}}$ be a $\mathbb{C}$-valued point of $(\overline{\mathcal{M}}_{g,r})_{\mathbb{C}}$. Write $S^{\log}$ for the fs log scheme obtained by equipping S with the log structure induced by the log structure of $(\overline{\mathcal{M}}^{\log}_{g,r})_{\mathbb{C}}$; $X^{\log} \to S^{\log}$ for the stable log curve over $S^{\log}$ corresponding to the resulting strict (1-)morphism $S^{\log} \to (\overline{\mathcal{M}}^{\log}_{g,r})_{\mathbb{C}}$; d for the rank of the group-characteristic of $S^{\log}$ (cf. [23, Def. 5.1(i)]), that is, the number of nodes of $X^{\log}$; $X^{\log}_{\mathrm{an}} \to S^{\log}_{\mathrm{an}}$ for the morphism of fs log analytic spaces determined by the morphism $X^{\log} \to S^{\log}$; $X_{\mathrm{an}} \to S_{\mathrm{an}}$ for the underlying morphism of analytic spaces of $X^{\log}_{\mathrm{an}} \to S^{\log}_{\mathrm{an}}$; $X^{\log}_{\mathrm{an}}(\mathbb{C})$, $S^{\log}_{\mathrm{an}}(\mathbb{C})$ for the respective topological spaces "$X^{\log}$" defined in [12, (1.2)] in the case where we take the "X" of [12, (1.2)] to be $X^{\log}_{\mathrm{an}}$, $S^{\log}_{\mathrm{an}}$, that is, for $T \in \{X,S\}$,

$$T^{\log}_{\mathrm{an}}(\mathbb{C}) \stackrel{\text{def}}{=} \{(t,h)\,|\,t \in T_{\mathrm{an}},\, h \in \operatorname{Hom}_{\mathrm{gp}}(M^{\mathrm{gp}}_{T_{\mathrm{an}},t},\mathbb{S}^1) \text{ such that}$$

$$h(f) = f(t)/|f(t)| \text{ for every } f \in \mathcal{O}^{\times}_{T_{\mathrm{an}},t} \subseteq M^{\mathrm{gp}}_{T_{\mathrm{an}},t}\}$$

—where we write $\mathbb{S}^1 \stackrel{\text{def}}{=} \{u \in \mathbb{C}\,|\,|u| = 1\}$ and $M_{T_{\mathrm{an}}}$ for the sheaf of monoids on T_{an} that defines the log structure of $T^{\log}_{\mathrm{an}}$. Then, by considering the functoriality discussed in [12, (1.2.5)] and the respective maps $X^{\log}_{\mathrm{an}}(\mathbb{C}) \to X_{\mathrm{an}}$, $S^{\log}_{\mathrm{an}}(\mathbb{C}) \to S_{\mathrm{an}}$ induced by the first projections, we obtain a commutative diagram of topological spaces and continuous maps

$$\begin{tikzcd} X^{\log}_{\mathrm{an}}(\mathbb{C}) \arrow[r] \arrow[d] & X_{\mathrm{an}} \arrow[d] \\ S^{\log}_{\mathrm{an}}(\mathbb{C}) \arrow[r] & S_{\mathrm{an}}. \end{tikzcd}$$

Now one verifies immediately from the various definitions involved that $S^{\log}_{\mathrm{an}}(\mathbb{C})$ is homeomorphic to a product $(\mathbb{S}^1)^{\times d}$ of d copies of $\mathbb{S}^1$; moreover, it follows from [24, Th. 5.1] that the left-hand vertical arrow of the above diagram is a topological fiber bundle.

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Let $s \in S^{\log}_{\mathrm{an}}(\mathbb{C})$. Thus, since (one verifies easily that) $(\mathbb{S}^1)^{\times d}$ is an *Eilenberg–Maclane space* (i.e., its universal covering space is contractible), the left-hand vertical arrow of the above diagram determines an exact sequence

$$1 \longrightarrow \pi_1(X^{\log}_{\mathrm{an}}(\mathbb{C})|_s) \longrightarrow \pi_1(X^{\log}_{\mathrm{an}}(\mathbb{C})) \longrightarrow \pi_1(S^{\log}_{\mathrm{an}}(\mathbb{C}))\ (\cong \mathbb{Z}^{\oplus d}) \longrightarrow 1,$$

where we write $X^{\log}_{\mathrm{an}}(\mathbb{C})|_s$ for the fiber of the left-hand vertical arrow $X^{\log}_{\mathrm{an}}(\mathbb{C}) \to S^{\log}_{\mathrm{an}}(\mathbb{C})$ of the above diagram at s, which thus determines an outer action

$$\pi_1(S^{\log}_{\mathrm{an}}(\mathbb{C}))\ (\cong \mathbb{Z}^{\oplus d}) \longrightarrow \mathrm{Out}(\pi_1(X^{\log}_{\mathrm{an}}(\mathbb{C})|_s)).$$

Write $N \subseteq X_{\mathrm{an}}$ for the finite subset consisting of the nodes of $X^{\log}_{\mathrm{an}}$, $C \subseteq X_{\mathrm{an}}$ for the finite subset consisting of the cusps of $X^{\log}_{\mathrm{an}}$, $U \stackrel{\mathrm{def}}{=} X_{\mathrm{an}} \setminus (N \cup C) \subseteq X_{\mathrm{an}}$, and $\pi_0(U)$ for the finite set of connected components of U. For each node $x \in N$ (resp. cusp $y \in C$; connected component $F \in \pi_0(U)$ of U), write C_x (resp. C_y; Y_F) $\subseteq X^{\log}_{\mathrm{an}}(\mathbb{C})|_s$ for the closure of the inverse image of $\{x\}$ (resp. $\{y\}$; F) $\subseteq X_{\mathrm{an}}$ via the composite $X^{\log}_{\mathrm{an}}(\mathbb{C})|_s \stackrel{\mathrm{pr}_1}{\to} X^{\log}_{\mathrm{an}}(\mathbb{C}) \to X_{\mathrm{an}}$—where the second arrow is the upper horizontal arrow of the above diagram. Then one verifies immediately from the various definitions involved that there exists a uniquely determined, up to unique isomorphism (in the evident sense), collection of data as follows:

- a pair of HS-type $Z = (\overline{Z}, \{D_i\}_{i=1}^r)$ of type (g,r) (cf. Definition 2.2(ii));
- a homeomorphism $\phi\colon X^{\log}_{\mathrm{an}}(\mathbb{C})|_s \stackrel{\sim}{\to} U_Z$ of $X^{\log}_{\mathrm{an}}(\mathbb{C})|_s$ with the interior U_Z of Z (cf. Definition 2.2(iii)) such that ϕ restricts to a homeomorphism of $\bigsqcup_{y \in C} C_y \subseteq X^{\log}_{\mathrm{an}}(\mathbb{C})|_s$ with $\partial U_Z = \bigsqcup_{i=1}^r \partial D_i \subseteq U_Z$ (cf. Definition 2.2(iii)).

Moreover, there exists a uniquely determined, up to unique isomorphism (in the evident sense), HS-decomposition of Z (cf. Definition 2.2(iv)) such that the set of vertices (resp. nodes; cusps) (cf. Definition 2.2(iv)) of the resulting collection of HSD-data (cf. Definition 2.2(iv)) is $\{\phi(Y_F)\}_{F \in \pi_0(U)}$ (resp. $\{\phi(C_x)\}_{x \in N}$; $\{\phi(C_y)\}_{y \in C}$). We shall write

$$\mathcal{G}_{X^{\log}}$$

for the semi-graph of temperoids of HSD-type associated with this collection of HSD-data (cf. Definition 2.3(ii)) and refer to $\mathcal{G}_{X^{\log}}$ as the *semi-graph of temperoids of HSD-type associated with* $X^{\log}$. Then one verifies immediately from the functoriality discussed in [12, (1.2.5)] applied to the vertices, nodes, and cusps of the data under consideration, that the locally trivial fibration $X^{\log}_{\mathrm{an}}(\mathbb{C}) \to S^{\log}_{\mathrm{an}}(\mathbb{C})$ determines an action

$$\pi_1(S^{\log}_{\mathrm{an}}(\mathbb{C}))\ (\cong \mathbb{Z}^{\oplus d}) \longrightarrow \mathrm{Aut}(\mathcal{G}_{X^{\log}}),$$

which is compatible, in the evident sense, with the outer action

$$\pi_1(S^{\log}_{\mathrm{an}}(\mathbb{C})) \longrightarrow \mathrm{Out}(\pi_1(X^{\log}_{\mathrm{an}}(\mathbb{C})|_s))$$

discussed above.

(ii) Let $S^{\log}$ be the fs log scheme obtained by equipping S with the log structure given by the fs chart $\mathbb{N} \ni 1 \mapsto 0 \in \mathbb{C}$ and $X^{\log} \to S^{\log}$ a stable log curve of type (g,r) over $S^{\log}$ (cf. [18, Exam. 2.5] in the case where $k = \mathbb{C}$). Then one verifies easily that the classifying (1-)morphism $S^{\log} \to (\overline{\mathcal{M}}^{\log}_{g,r})_{\mathbb{C}}$ of $X^{\log} \to S^{\log}$ factors as a composite $S^{\log} \to T^{\log} \to (\overline{\mathcal{M}}^{\log}_{g,r})_{\mathbb{C}}$—where the first arrow is a morphism that induces an isomorphism between the underlying schemes, and the second arrow is strict—and, moreover, if we write $Y^{\log} \to T^{\log}$ for the stable log curve determined by the strict (1-)morphism $T^{\log} \to (\overline{\mathcal{M}}_{g,r}^{\log})_{\mathbb{C}}$, then we have a natural isomorphism over $S^{\log}$

$$X^{\log} \xrightarrow{\sim} Y^{\log} \times_{T^{\log}} S^{\log}.$$

We shall write

$$\mathcal{G}_{X^{\log}} \stackrel{\text{def}}{=} \mathcal{G}_{Y^{\log}}$$

(cf. (i)) and refer to $\mathcal{G}_{X^{\log}}$ as the *semi-graph of temperoids of HSD-type associated with* $X^{\log}$. Then, by pulling back the action of the second to last display of (i) via the homomorphism $\pi_1(S_{\text{an}}^{\log}(\mathbb{C})) \to \pi_1(T_{\text{an}}^{\log}(\mathbb{C}))$ induced by the morphism $S^{\log} \to T^{\log}$, we obtain an action

$$\pi_1(S_{\text{an}}^{\log}(\mathbb{C}))\ (\cong \mathbb{Z}) \longrightarrow \mathrm{Aut}(\mathcal{G}_{X^{\log}}),$$

together with a compatible outer action

$$\pi_1(S_{\text{an}}^{\log}(\mathbb{C})) \longrightarrow \mathrm{Out}(\pi_1(X_{\text{an}}^{\log}(\mathbb{C})|_s)).$$

REMARK 2.4.1. One verifies easily that the discussion of Example 2.4(ii) generalizes immediately to the case of arbitrary fs log schemes $S^{\log}$ with underlying scheme $S = \mathrm{Spec}(\mathbb{C})$.

PROPOSITION 2.5 (Fundamental groups of semi-graphs of temperoids of HSD-type). *Let $\mathcal{G}$ be a semi-graph of temperoids of HSD-type associated (cf. Definition 2.3(ii) and (iii)) to a collection of HSD-data $\mathbb{X}$ (cf. Definition 2.2(iv)). Write $U_{\mathbb{X}}$ for the underlying surface of $\mathbb{X}$ (cf. Definition 2.2(iv)) and*

$$\mathcal{B}^{\text{tp}}(U_{\mathbb{X}})$$

for the connected temperoid (cf. [17, Def. 3.1(ii)]) determined by the category of topological coverings with countably many connected components of the topological space $U_{\mathbb{X}}$. Then the following hold:

(i) *We have a natural equivalence of categories*

$$\mathcal{B}^{\text{tp}}(U_{\mathbb{X}}) \xrightarrow{\sim} \mathcal{B}^{\text{tp}}(\mathcal{G})$$

(cf. Definition 2.1(vi)). In particular, $\mathcal{B}^{\text{tp}}(\mathcal{G})$ is a connected temperoid. Write

$$\Pi_{\mathcal{G}}$$

for the tempered fundamental group (which is well-defined, up to inner automorphism) of the connected temperoid $\mathcal{B}^{\text{tp}}(\mathcal{G})$ (cf. [17, Rem. 3.2.1]; the discussion of "Galois-countable temperoids" in [22, Rem. 2.5.3(i)]). (Thus, the tempered group $\Pi_{\mathcal{G}}$ admits a natural outer isomorphism with the topological fundamental group, equipped with the discrete topology, of the topological space $U_{\mathbb{X}}$.) We shall refer to this tempered group $\Pi_{\mathcal{G}}$ as the fundamental group of $\mathcal{G}$.

(ii) *Every connected finite étale covering $\mathcal{H} \to \mathcal{G}$ (cf. Definition 2.1(vi)) admits a natural structure of semi-graph of temperoids of HSD-type.*

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(iii) *The connected semi-graph of anabelioids $\mathcal{G}^{\Sigma}$ (cf. Definition 2.1(iii)) is of pro-Σ PSC-type (cf. [18, Def. 1.1(i)]). Write $\Pi_{\mathcal{G}^{\Sigma}}$ for the (pro-Σ) fundamental group of $\mathcal{G}^{\Sigma}$. Then the natural morphism $\mathcal{G} \to \mathcal{G}^{\Sigma}$ of semi-graphs of temperoids of Definition 2.1(iii) induces a natural outer injection*

$$\Pi_{\mathcal{G}} \hookrightarrow \Pi_{\mathcal{G}^{\Sigma}}$$

(cf. (i)). Moreover, this natural outer injection determines an outer isomorphism

$$\Pi_{\mathcal{G}}^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}^{\Sigma}},$$

where we write $\Pi_{\mathcal{G}}^{\Sigma}$ for the pro-Σ completion of $\Pi_{\mathcal{G}}$.

(iv) *Let $z \in \mathrm{VCN}(\mathcal{G})$ (cf. Definition 2.1(v)). Write $\Pi_{\mathcal{G}_z}$ for the tempered fundamental group (cf. [17, Rem. 3.2.1]) of the connected temperoid $\mathcal{G}_z$ of $\mathcal{G}$ corresponding to z. Then the natural outer homomorphism*

$$\Pi_{\mathcal{G}_z} \longrightarrow \Pi_{\mathcal{G}}$$

is a Σ-compatible injection (cf. the discussion entitled "Groups" in §0).

(v) *In the notation of (iii) and (iv), the closure of the image of the composite*

$$\Pi_{\mathcal{G}_z} \hookrightarrow \Pi_{\mathcal{G}} \hookrightarrow \Pi_{\mathcal{G}^{\Sigma}}$$

of the outer injections of (iii) and (iv) is a VCN-subgroup of $\Pi_{\mathcal{G}^{\Sigma}}$ (cf. (iii); [9, Def. 2.1(i)]) associated with $z \in \mathrm{VCN}(\mathcal{G}) = \mathrm{VCN}(\mathcal{G}^{\Sigma})$.

Proof. A natural equivalence of categories as in assertion (i) may be obtained by observing that, after sorting through the various definitions involved, an object of $\mathcal{B}^{\mathrm{tp}}(U_{\mathbb{X}})$ (i.e., a topological covering of $U_{\mathbb{X}}$) amounts to the same data as an object of $\mathcal{B}^{\mathrm{tp}}(\mathcal{G})$. Assertion (ii) follows immediately from the various definitions involved.

Next, we verify assertion (iii). The assertion that $\mathcal{G}^{\Sigma}$ is of pro-Σ PSC-type, as well as the assertion that the morphism $\mathcal{G} \to \mathcal{G}^{\Sigma}$ determines an outer isomorphism $\Pi_{\mathcal{G}}^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}^{\Sigma}}$, follows immediately from the various definitions involved. Thus, the assertion that the morphism $\mathcal{G} \to \mathcal{G}^{\Sigma}$ determines an outer injection $\Pi_{\mathcal{G}} \hookrightarrow \Pi_{\mathcal{G}^{\Sigma}}$ follows from the well-known fact that the discrete group $\Pi_{\mathcal{G}}$ injects into its pro-l completion for any $l \in \mathfrak{Primes}$ (cf., e.g., [27, Prop. 3.3.15]; [17, Th. 1.7]).

Next, we verify the injectivity portion of assertion (iv). Let us first observe that it follows immediately from the various definitions involved that the composite

$$\Pi_{\mathcal{G}_z} \to \Pi_{\mathcal{G}} \hookrightarrow \Pi_{\widehat{\mathcal{G}}}$$

(cf. Definition 2.1(iii)) of the outer homomorphism under consideration and the outer injection of assertion (iii) (in the case where $\Sigma = \mathfrak{Primes}$) factors as the composite

$$\Pi_{\mathcal{G}_z} \to \Pi_{\widehat{\mathcal{G}}_z} \hookrightarrow \Pi_{\widehat{\mathcal{G}}}$$

of the outer homomorphism $\Pi_{\mathcal{G}_z} \to \Pi_{\widehat{\mathcal{G}}_z}$ induced by the morphism $\mathcal{G}_z \to \widehat{\mathcal{G}}_z$ of Definition 2.1(ii) and the natural outer inclusion $\Pi_{\widehat{\mathcal{G}}_z} \hookrightarrow \Pi_{\widehat{\mathcal{G}}}$ (cf. [17, Prop. 2.5(i)]). Thus, to complete the verification of the injectivity portion of assertion (iv), it suffices to verify that the outer homomorphism $\Pi_{\mathcal{G}_z} \to \Pi_{\widehat{\mathcal{G}}_z}$ is injective. On the other hand, this follows from the well-known fact that $\Pi_{\mathcal{G}_z}$ injects into its pro-l completion for any $l \in \mathfrak{Primes}$ (cf., e.g., [27, Prop. 3.3.15]; [17, Th. 1.7]). This completes the proof of the injectivity portion of assertion (iv). Assertion

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(v) follows immediately from the various definitions involved. Finally, it follows immediately from assertions (iii) and (v), together with the evident pro-Σ analogue of [17, Prop. 2.5(i)], that the natural outer injection of assertion (iv) is Σ-compatible. This completes the proof of assertion (iv), hence also of Proposition 2.5. $\square$

REMARK 2.5.1. In the notation of Proposition 2.5, as is discussed in Proposition 2.5(i), the fundamental group $\Pi_{\mathcal{G}}$ of the semi-graph of temperoids of HSD-type $\mathcal{G}$ is naturally isomorphic, up to inner automorphism, to the topological fundamental group, equipped with the discrete topology, of the compact orientable hyperbolic topological surface with compact boundary $U_{\mathbb{X}}$. In particular, $\Pi_{\mathcal{G}}$ is finitely generated, torsion-free, and center-free and injects into its pro-l completion for any $l \in \mathfrak{Primes}$ (cf. Proposition 2.5(iii)). Moreover, it holds that $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$ (cf. Definition 2.1(v)) if and only if $\Pi_{\mathcal{G}}$ is free.

REMARK 2.5.2. In the situation of Example 2.4(ii), write $\mathcal{G}_{X^{\log}}$ for the semi-graph of temperoids of HSD-type associated with $X^{\log}$; $\mathcal{G}^{\Sigma}_{X^{\log}}$ for the semi-graph of anabelioids of pro-Σ PSC-type of Proposition 2.5(iii) in the case where we take the "$\mathcal{G}$" of Proposition 2.5(iii), to be $\mathcal{G}_{X^{\log}}$; $\mathcal{G}^{\text{PSC-}\Sigma}_{X^{\log}}$ for the semi-graph of anabelioids of pro-Σ PSC-type associated with $X^{\log}$ (cf. [18, Exam. 2.5]). Then it follows from Proposition 2.5(iii) that we have a natural outer isomorphism $\Pi^{\Sigma}_{\mathcal{G}_{X^{\log}}} \xrightarrow{\sim} \Pi_{\mathcal{G}^{\Sigma}_{X^{\log}}}$. On the other hand, by associating finite étale coverings of $X^{\log}_{\mathrm{an}}(\mathbb{C})$ to log étale coverings of Kummer type of $X^{\log}$ (cf. [12, Lem. 2.2]) and then restricting such finite étale coverings to $X^{\log}_{\mathrm{an}}(\mathbb{C})|_s$ (cf. Example 2.4(i)), we obtain an outer homomorphism $\Pi^{\Sigma}_{\mathcal{G}_{X^{\log}}} \to \Pi_{\mathcal{G}^{\text{PSC-}\Sigma}_{X^{\log}}}$. Then one verifies immediately from the various definitions involved that the composite of the two outer homomorphisms

$$\Pi_{\mathcal{G}^{\Sigma}_{X^{\log}}} \xleftarrow{\sim} \Pi^{\Sigma}_{\mathcal{G}_{X^{\log}}} \longrightarrow \Pi_{\mathcal{G}^{\text{PSC-}\Sigma}_{X^{\log}}}$$

is a graphic outer isomorphism (cf. [18, Def. 1.4(i)]), that is, arises from a uniquely determined isomorphism of semi-graphs of anabelioids

$$\mathcal{G}^{\Sigma}_{X^{\log}} \xrightarrow{\sim} \mathcal{G}^{\text{PSC-}\Sigma}_{X^{\log}}.$$

Finally, one verifies easily that the above discussion generalizes immediately to the case of arbitrary fs log schemes $S^{\log}$ with underlying scheme $S = \mathrm{Spec}(\mathbb{C})$ (cf. Remark 2.4.1).

DEFINITION 2.6. Let $\mathcal{G}$ be a semi-graph of temperoids of HSD-type. Write $\Pi_{\mathcal{G}}$ for the fundamental group of $\mathcal{G}$.

(i) Let $z \in \mathrm{VCN}(\mathcal{G})$ (cf. Definition 2.1(v)). Then we shall refer to a closed subgroup of $\Pi_{\mathcal{G}}$ that belongs to the $\Pi_{\mathcal{G}}$-conjugacy class of closed subgroups determined by the image of the outer injection of the display of Proposition 2.5(iv) as a *VCN-subgroup* of $\Pi_{\mathcal{G}}$ associated with $z \in \mathrm{VCN}(\mathcal{G})$. If, moreover, $z \in \mathrm{Vert}(\mathcal{G})$ (resp. $\in \mathrm{Cusp}(\mathcal{G})$; $\in \mathrm{Node}(\mathcal{G})$; $\in \mathrm{Edge}(\mathcal{G})$) (cf. Definition 2.1(v)), then we shall refer to a VCN-subgroup of $\Pi_{\mathcal{G}}$ associated with z as a *verticial* (resp. a *cuspidal*; a *nodal*; an *edge-like*) *subgroup* of $\Pi_{\mathcal{G}}$ associated with z.

(ii) Write $\widetilde{\mathcal{G}} \to \mathcal{G}$ for the universal covering of $\mathcal{G}$ corresponding to $\Pi_{\mathcal{G}}$. Let $\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$ (cf. Definition 2.1(v)). Then we shall refer to the VCN-subgroup $\Pi_{\widetilde{z}} \subseteq \Pi_{\mathcal{G}}$ (cf. (i)) determined by $\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$ as the *VCN-subgroup* of $\Pi_{\mathcal{G}}$ associated with $\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$. If, moreover, $\widetilde{z} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ (resp. $\in \mathrm{Cusp}(\widetilde{\mathcal{G}})$; $\in \mathrm{Node}(\widetilde{\mathcal{G}})$; $\in \mathrm{Edge}(\widetilde{\mathcal{G}})$) (cf. Definition 2.1(v)), then we shall refer to the VCN-subgroup of $\Pi_{\mathcal{G}}$ associated with $\widetilde{z}$ as the *verticial* (resp. *cuspidal*; *nodal*; *edge-like*) *subgroup* of $\Pi_{\mathcal{G}}$ associated with $\widetilde{z}$.

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(iii) Let (g,r) be a pair of nonnegative integers such that $2g-2+r>0$ and $v \in \mathrm{Vert}(\mathcal{G})$. Then we shall say that v is *of type* (g,r) if the "(g,r)" appearing in Definition 2.2(ii) for the pair of HS-type corresponding to v coincides with (g,r). Thus, one verifies easily that v is of type (g,r) if and only if the number of the branches of edges of $\mathcal{G}$ that abut to v is equal to r, and, moreover,

$$\mathrm{rank}_{\mathbb{Z}}(\Pi_v^{\mathrm{ab}}) = 2g + \max\{0, r-1\},$$

where we use the notation Π_v to denote a verticial subgroup associated with v.

Remark 2.6.1. In the notation of Definition 2.6, it follows from Proposition 2.5(iv) that every verticial subgroup of $\Pi_{\mathcal{G}}$ is naturally isomorphic, up to inner automorphism, to the topological fundamental group, equipped with the discrete topology, of a compact orientable hyperbolic topological surface with compact boundary. In particular, every verticial subgroup of $\Pi_{\mathcal{G}}$ is finitely generated, torsion-free, and center-free and injects into its pro-l completion for any $l \in \mathfrak{Primes}$ (cf. Proposition 2.5(iii)). Moreover, it follows from Proposition 2.5(iv) that every edge-like subgroup of $\Pi_{\mathcal{G}}$ is naturally isomorphic, up to inner automorphism, to the topological fundamental group, equipped with the discrete topology, of a unit circle (hence isomorphic to $\mathbb{Z}$).

Definition 2.7. Let $\mathcal{G}$ and $\mathcal{H}$ be semi-graphs of temperoids of HSD-type. Write $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{H}}$ for the fundamental groups of $\mathcal{G}$, $\mathcal{H}$, respectively.

(i) We shall say that an isomorphism of groups $\Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$ is *group-theoretically verticial* (resp. *group-theoretically cuspidal*; *group-theoretically nodal*) if the isomorphism induces a bijection between the set of the verticial (resp. cuspidal; nodal) subgroups (cf. Definition 2.6(i)) of $\Pi_{\mathcal{G}}$ and the set of the verticial (resp. cuspidal; nodal) subgroups of $\Pi_{\mathcal{H}}$. We shall say that an outer isomorphism $\Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$ is *group-theoretically verticial* (resp. *group-theoretically cuspidal*; *group-theoretically nodal*) if it arises from an isomorphism $\Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$ that is group-theoretically verticial (resp. group-theoretically cuspidal; group-theoretically nodal).

(ii) We shall say that an outer isomorphism $\Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$ is *graphic* if it arises from an isomorphism $\mathcal{G} \xrightarrow{\sim} \mathcal{H}$. We shall say that an isomorphism $\Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$ is *graphic* if the outer isomorphism $\Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$ determined by it is graphic.

Definition 2.8. Let $\mathcal{G}$ be a semi-graph of temperoids of HSD-type. Write $\mathbb{G}$ for the underlying semi-graph of $\mathcal{G}$. Also, for each $z \in \mathrm{VCN}(\mathcal{G})$, write $\mathcal{G}_z$ for the connected temperoid of $\mathcal{G}$ corresponding to z.

(i) Let $\mathbb{H}$ be a sub-semi-graph of PSC-type (cf. [9, Def. 2.2(i)]) of $\mathbb{G}$. Then one may define a semi-graph of temperoids of HSD-type

$$\mathcal{G}|_{\mathbb{H}}$$

as follows (cf. Figure 2 of [9]): we take the underlying semi-graph of $\mathcal{G}|_{\mathbb{H}}$ to be $\mathbb{H}$; for each vertex v (resp. edge e) of $\mathbb{H}$, we take the temperoid corresponding to v (resp. e) to be $\mathcal{G}_v$ (resp. $\mathcal{G}_e$); for each branch b of an edge e of $\mathbb{H}$ that abuts to a vertex v of $\mathbb{H}$, we take the morphism associated with b to be the morphism $\mathcal{G}_e \to \mathcal{G}_v$ associated with the branch of $\mathcal{G}$ corresponding to b. We shall refer to $\mathcal{G}|_{\mathbb{H}}$ as the *semi-graph of temperoids*

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of HSD-type obtained by restricting $\mathcal{G}$ to $\mathbb{H}$. Thus, one has a natural morphism

$$\mathcal{G}|_{\mathbb{H}} \longrightarrow \mathcal{G}$$

of semi-graphs of temperoids.

(ii) Let $S \subseteq \mathrm{Cusp}(\mathcal{G})$ be a subset of $\mathrm{Cusp}(\mathcal{G})$ (cf. Definition 2.1(v)) which is omittable (cf. [9, Def. 2.4(i)]) as a subset of the set of cusps $\mathrm{Cusp}(\widehat{\mathcal{G}})$ of the semi-graph of anabelioids of pro-$\mathfrak{Primes}$ PSC-type $\widehat{\mathcal{G}}$ (cf. Proposition 2.5(iii) in the case where $\Sigma = \mathfrak{Primes}$) relative to the natural identification $\mathrm{Cusp}(\mathcal{G}) = \mathrm{Cusp}(\widehat{\mathcal{G}})$. Then, by eliminating the cusps contained in S, and, for each vertex v of $\mathcal{G}$, replacing the temperoid $\mathcal{G}_v$ by the temperoid of coverings of $\mathcal{G}_v$ that restrict to a trivial covering over the cusps contained in S that abut to v, we obtain a semi-graph of temperoids of HSD-type

$$\mathcal{G}_{\bullet S}$$

(cf. Figure 3 of [9]). We shall refer to $\mathcal{G}_{\bullet S}$ as the *partial compactification of $\mathcal{G}$ with respect to S.*

(iii) Let $S \subseteq \mathrm{Node}(\mathcal{G})$ be a subset of $\mathrm{Node}(\mathcal{G})$ (cf. Definition 2.1(v)) such that the semi-graph obtained by removing the closed edges corresponding to the elements of S from the underlying semi-graph of $\mathcal{G}$ is connected, that is, in the terminology of [9, Def. 2.5(i)] that is not of separating type as a subset of the set of nodes $\mathrm{Node}(\widehat{\mathcal{G}})$ of the semi-graph of anabelioids of pro-$\mathfrak{Primes}$ PSC-type $\widehat{\mathcal{G}}$ (cf. Proposition 2.5(iii) in the case where $\Sigma = \mathfrak{Primes}$) relative to the natural identification $\mathrm{Node}(\mathcal{G}) = \mathrm{Node}(\widehat{\mathcal{G}})$. Then one may define a semi-graph of temperoids of HSD-type

$$\mathcal{G}_{\succ S}$$

as follows (cf. Figure 4 of [9]): we take the underlying semi-graph of $\mathcal{G}_{\succ S}$ to be the semi-graph obtained by replacing each node e of $\mathbb{G}$ contained in S such that $\mathcal{V}(e) = \{v_1, v_2\} \subseteq \mathrm{Vert}(\mathcal{G})$ (cf. Definition 2.1(v))—where v_1, v_2 are not necessarily distinct—by two cusps that abut to v_1, $v_2 \in \mathrm{Vert}(\mathcal{G})$, respectively, which we think as corresponding to the two branches of e. We take the temperoid corresponding to a vertex v (resp. node e) of $\mathcal{G}_{\succ S}$ to be $\mathcal{G}_v$ (resp. $\mathcal{G}_e$). (Note that the set of vertices (resp. nodes) of $\mathcal{G}_{\succ S}$ may be naturally identified with $\mathrm{Vert}(\mathcal{G})$ (resp. $\mathrm{Node}(\mathcal{G}) \setminus S$).) We take the temperoid corresponding to a cusp of $\mathcal{G}_{\succ S}$ arising from a cusp e of $\mathcal{G}$ to be $\mathcal{G}_e$. We take the temperoid corresponding to a cusp of $\mathcal{G}_{\succ S}$ arising from a node e of $\mathcal{G}$ to be $\mathcal{G}_e$. For each branch b of $\mathcal{G}_{\succ S}$ that abuts to a vertex v of a node e (resp. of a cusp e that does not arise from a node of $\mathcal{G}$), we take the morphism associated with b to be the morphism $\mathcal{G}_e \to \mathcal{G}_v$ associated with the branch of $\mathcal{G}$ corresponding to b. For each branch b of $\mathcal{G}_{\succ S}$ that abuts to a vertex v of a cusp of $\mathcal{G}_{\succ S}$ that arises from a node e of $\mathcal{G}$, we take the morphism associated with b to be the morphism $\mathcal{G}_e \to \mathcal{G}_v$ associated with the branch of $\mathcal{G}$ corresponding to b. We shall refer to $\mathcal{G}_{\succ S}$ as the *semi-graph of temperoids of HSD-type obtained from $\mathcal{G}$ by resolving S.* Thus, one has a natural morphism

$$\mathcal{G}_{\succ S} \longrightarrow \mathcal{G}$$

of semi-graphs of temperoids.

REMARK 2.8.1. One verifies immediately that the operations of restriction, partial compactification, and resolution discussed in Definition 2.8(i)–(iii) are compatible (in the

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evident sense) with the corresponding pro-Σ operations—that is, as discussed in [9, Defs. 2.2(ii), 2.4(ii), and 2.5(ii)]—relative to the operation of passing to the associated semi-graph of anabelioids of pro-Σ PSC-type (cf. Proposition 2.5(iii)).

Remark 2.8.2. We take this opportunity to correct an unfortunate misprint in [9, Def. 2.5(ii)]: the phrase "by two cusps that abut to v_1, $v_2 \in \mathrm{Vert}(\mathcal{G})$, respectively" of [9, Def. 2.5(ii)] should read "by two cusps that abut to v_1, $v_2 \in \mathrm{Vert}(\mathcal{G})$, respectively, which we think as corresponding to the two branches of e."

Definition 2.9. In the notation of Definition 2.8, let $S \subseteq \mathrm{Node}(\mathcal{G})$ be a subset of $\mathrm{Node}(\mathcal{G})$ (cf. Definition 2.1(v)). Then we define the semi-graph of temperoids of HSD-type

$$\mathcal{G}_{\rightsquigarrow S}$$

as follows (cf. Figure 5 of [9]):

(i) We take $\mathrm{Cusp}(\mathcal{G}_{\rightsquigarrow S}) \stackrel{\mathrm{def}}{=} \mathrm{Cusp}(\mathcal{G})$ (cf. Definition 2.1(v)).

(ii) We take $\mathrm{Node}(\mathcal{G}_{\rightsquigarrow S}) \stackrel{\mathrm{def}}{=} \mathrm{Node}(\mathcal{G}) \setminus S$ (cf. Definition 2.1(v)).

(iii) We take $\mathrm{Vert}(\mathcal{G}_{\rightsquigarrow S})$ (cf. Definition 2.1(v)) to be the set of connected components of the semi-graph obtained from $\mathbb{G}$ by omitting the edges $e \in \mathrm{Edge}(\mathcal{G}) \setminus S$ (cf. Definition 2.1(v)). Alternatively, one may take $\mathrm{Vert}(\mathcal{G}_{\rightsquigarrow S})$ to be the set of equivalence classes of elements of $\mathrm{Vert}(\mathcal{G})$ with respect to the equivalence relation "$\sim$" defined as follows: for v, $w \in \mathrm{Vert}(\mathcal{G})$, $v \sim w$ if either $v = w$ or there exist n elements $e_1, \ldots, e_n \in S$ of S and $n+1$ vertices $v_0, v_1, \ldots, v_n \in \mathrm{Vert}(\mathcal{G})$ of $\mathcal{G}$ such that $v_0 \stackrel{\mathrm{def}}{=} v$, $v_n \stackrel{\mathrm{def}}{=} w$, and, for $1 \leq i \leq n$, it holds that $\mathcal{V}(e_i) = \{v_{i-1}, v_i\}$ (cf. Definition 2.1(v)).

(iv) For each branch b of an edge $e \in \mathrm{Edge}(\mathcal{G}_{\rightsquigarrow S})$ ($= \mathrm{Edge}(\mathcal{G}) \setminus S$—cf. (i) and (ii)) and each vertex $v \in \mathrm{Vert}(\mathcal{G}_{\rightsquigarrow S})$ of $\mathcal{G}_{\rightsquigarrow S}$, b abuts, relative to $\mathcal{G}_{\rightsquigarrow S}$, to v if b abuts, relative to $\mathcal{G}$, to an element of the equivalence class v (cf. (iii)).

(v) For each edge $e \in \mathrm{Edge}(\mathcal{G}_{\rightsquigarrow S}) (= \mathrm{Edge}(\mathcal{G}) \setminus S$—cf. (i) and (ii)) of $\mathcal{G}_{\rightsquigarrow S}$, we take the temperoid of $\mathcal{G}_{\rightsquigarrow S}$ corresponding to $e \in \mathrm{Edge}(\mathcal{G}_{\rightsquigarrow S})$ to be the temperoid $\mathcal{G}_e$.

(vi) Let $v \in \mathrm{Vert}(\mathcal{G}_{\rightsquigarrow S})$ be a vertex of $\mathcal{G}_{\rightsquigarrow S}$. Then one verifies easily that there exists a unique sub-semi-graph of PSC-type (cf. [9, Def. 2.2(i)]) $\mathbb{H}_v$ of the underlying semi-graph of $\mathcal{G}$ whose set of vertices consists of the elements of the equivalence class v (cf. (iii)). Write

$$T_v \stackrel{\mathrm{def}}{=} \mathrm{Node}(\mathcal{G}|_{\mathbb{H}_v}) \setminus (S \cap \mathrm{Node}(\mathcal{G}|_{\mathbb{H}_v}))$$

(cf. Definition 2.8(i)). Then we take the temperoid of $\mathcal{G}_{\rightsquigarrow S}$ corresponding to $v \in \mathrm{Vert}(\mathcal{G}_{\rightsquigarrow S})$ to be the temperoid $\mathcal{B}^{\mathrm{tp}}((\mathcal{G}|_{\mathbb{H}_v})_{\succ T_v})$ (cf. Definition 2.1(vi); Proposition 2.5(i); Definition 2.8(iii)).

(vii) Let b be a branch of an edge $e \in \mathrm{Edge}(\mathcal{G}_{\rightsquigarrow S}) (= \mathrm{Edge}(\mathcal{G}) \setminus S$—cf. (i) and (ii)) that abuts to a vertex $v \in \mathrm{Vert}(\mathcal{G}_{\rightsquigarrow S})$. Then since b abuts to v, one verifies easily that there exists a unique vertex w of $\mathcal{G}$ which belongs to the equivalence class v (cf. (iii)) such that b abuts to w relative to $\mathcal{G}$. We take the morphism of temperoids associated with b, relative to $\mathcal{G}_{\rightsquigarrow S}$, to be the morphism naturally determined by post-composing the morphism of temperoids $\mathcal{G}_e \to \mathcal{G}_w$ corresponding to the branch b relative to $\mathcal{G}$ with the natural morphism of temperoids $\mathcal{G}_w \to \mathcal{B}^{\mathrm{tp}}((\mathcal{G}|_{\mathbb{H}_v})_{\succ T_v})$ (cf. (vi)).

We shall refer to this semi-graph of temperoids of HSD-type $\mathcal{G}_{\rightsquigarrow S}$ as the *generization of $\mathcal{G}$ with respect to S.*

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REMARK 2.9.1. One verifies immediately that the operation of generization discussed in Definition 2.9 is compatible (in the evident sense) with the corresponding pro-Σ operation—that is, as discussed in [9, Def. 2.8]—relative to the operation of passing to the associated semi-graph of anabelioids of pro-Σ PSC-type (cf. Proposition 2.5(iii)).

REMARK 2.9.2. We take this opportunity to correct an unfortunate misprint in [9, Def. 2.8(vii)]: the phrase "equivalent class" should read "equivalence class."

PROPOSITION 2.10 (Specialization outer isomorphisms). *Let $\mathcal{G}$ be a semi-graph of temperoids of HSD-type, and let $S \subseteq \mathrm{Node}(\mathcal{G})$ be a subset of $\mathrm{Node}(\mathcal{G})$. Write $\Pi_{\mathcal{G}}$ for the fundamental group of $\mathcal{G}$ and $\Pi_{\mathcal{G}_{\rightsquigarrow S}}$ for the fundamental group of the generization $\mathcal{G}_{\rightsquigarrow S}$ of $\mathcal{G}$ with respect to S (cf. Definition 2.9). Then there exists a natural outer isomorphism*

$$\Phi_{\mathcal{G}_{\rightsquigarrow S}} : \Pi_{\mathcal{G}_{\rightsquigarrow S}} \xrightarrow{\sim} \Pi_{\mathcal{G}}$$

which is functorial, in the evident sense, with respect to isomorphisms of the pair $(\mathcal{G}, S)$ and satisfies the following three conditions:

(a) *$\Phi_{\mathcal{G}_{\rightsquigarrow S}}$ induces a bijection between the set of cuspidal subgroups (cf. Definition 2.6(i)) of $\Pi_{\mathcal{G}_{\rightsquigarrow S}}$ and the set of cuspidal subgroups of $\Pi_{\mathcal{G}}$.*
(b) *$\Phi_{\mathcal{G}_{\rightsquigarrow S}}$ induces a bijection between the set of nodal subgroups (cf. Definition 2.6(i)) of $\Pi_{\mathcal{G}_{\rightsquigarrow S}}$ and the set of nodal subgroups of $\Pi_{\mathcal{G}}$ associated with the elements of $\mathrm{Node}(\mathcal{G}) \setminus S$.*
(c) *Let $v \in \mathrm{Vert}(\mathcal{G}_{\rightsquigarrow S})$ be a vertex of $\mathcal{G}_{\rightsquigarrow S}$; $\mathbb{H}_v$, T_v as in Definition 2.9(vi). Then $\Phi_{\mathcal{G}_{\rightsquigarrow S}}$ induces a bijection between the $\Pi_{\mathcal{G}_{\rightsquigarrow S}}$-conjugacy class of any verticial subgroup (cf. Definition 2.6(i)) $\Pi_v \subseteq \Pi_{\mathcal{G}_{\rightsquigarrow S}}$ of $\Pi_{\mathcal{G}_{\rightsquigarrow S}}$ associated with $v \in \mathrm{Vert}(\mathcal{G}_{\rightsquigarrow S})$ and the $\Pi_{\mathcal{G}}$-conjugacy class of subgroups obtained by forming the image of the outer homomorphism*

$$\Pi_{(\mathcal{G}|_{\mathbb{H}_v})_{\succ T_v}} \longrightarrow \Pi_{\mathcal{G}}$$

induced by the natural morphism $(\mathcal{G}|_{\mathbb{H}_v})_{\succ T_v} \to \mathcal{G}$ (cf. Definition 2.8(i) and (iii)) of semi-graphs of temperoids.

We shall refer to this natural outer isomorphism $\Phi_{\mathcal{G}_{\rightsquigarrow S}}$ as the specialization outer isomorphism with respect to S.

Proof. An outer isomorphism that satisfies the three conditions in the statement of Proposition 2.10 may be obtained by observing that, after sorting through the various definitions involved, an object of $\mathcal{B}^{\mathrm{tp}}(\mathcal{G}_{\rightsquigarrow S})$ amounts to the same data as an object of $\mathcal{B}^{\mathrm{tp}}(\mathcal{G})$. This completes the proof of Proposition 2.10. ∎

REMARK 2.10.1. One verifies immediately that the specialization outer isomorphism discussed in Proposition 2.10 is compatible (in the evident sense) with the corresponding pro-Σ outer isomorphism—that is, as discussed in [9, Prop. 2.9]—relative to the operation of passing to the associated semi-graph of anabelioids of pro-Σ PSC-type (cf. Proposition 2.5(iii)).

LEMMA 2.11 (Infinite cyclic coverings). *Let $\mathcal{G}$ be a semi-graph of temperoids of HSD-type. Suppose that $(\mathrm{Vert}(\mathcal{G})^{\sharp}, \mathrm{Node}(\mathcal{G})^{\sharp}) = (1,1)$, that is, the semi-graph of anabelioids of pro-$\mathfrak{Primes}$ PSC-type $\widehat{\mathcal{G}}$ (cf. Proposition 2.5(iii) in the case where $\Sigma = \mathfrak{Primes}$) is cyclically primitive (cf. [9, Def. 4.1]). Write $\Pi_{\mathcal{G}}$ for the fundamental group of $\mathcal{G}$; $\mathbb{G}$ for the underlying semi-graph of $\mathcal{G}$; $\Pi_{\mathbb{G}} (\cong \mathbb{Z})$ for the discrete topological fundamental group of $\mathbb{G}$; $\mathcal{G}_{\infty} \to \mathcal{G}$ for*

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the connected covering of $\mathcal{G}$ *(cf. Definition 2.1(vi)) corresponding to the natural surjection* $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathbb{G}}$*;* $\Pi_{\mathcal{G}_\infty} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathbb{G}})$*. Then the following hold:*

(i) *Fix an isomorphism* $\Pi_{\mathbb{G}} \xrightarrow{\sim} \mathbb{Z}$*. Then there exists a triple of bijections*

$$V : \mathbb{Z} \xrightarrow{\sim} \mathrm{Vert}(\mathcal{G}_\infty),\ N : \mathbb{Z} \xrightarrow{\sim} \mathrm{Node}(\mathcal{G}_\infty),$$

$$C : \mathbb{Z} \times \mathrm{Cusp}(\mathcal{G}) \xrightarrow{\sim} \mathrm{Cusp}(\mathcal{G}_\infty)$$

(cf. Definition 2.1(v)) that satisfies the following properties:

- *The bijections are equivariant with respect to the action of* $\Pi_{\mathbb{G}} \xrightarrow{\sim} \mathbb{Z}$ *on* $\mathbb{Z}$ *by translations and the natural action of* $\Pi_{\mathbb{G}}$ *on "*$\mathrm{Vert}(-)$*," "*$\mathrm{Node}(-)$*," "*$\mathrm{Cusp}(-)$*."*
- *The post-composite of* C *with the natural map* $\mathrm{Cusp}(\mathcal{G}_\infty) \to \mathrm{Cusp}(\mathcal{G})$ *coincides with the projection* $\mathbb{Z} \times \mathrm{Cusp}(\mathcal{G}) \to \mathrm{Cusp}(\mathcal{G})$ *to the second factor.*
- *For each* $a \in \mathbb{Z}$*, it holds that* $\mathcal{E}(V(a)) = \{N(a), N(a+1)\} \sqcup \{\, C(a,z) \mid z \in \mathrm{Cusp}(\mathcal{G})\,\}$ *(cf. Definition 2.1(v)).*

Moreover, such a triple of bijections is unique, up to post-composition with the automorphisms of "$\mathrm{Vert}(-)$*," "*$\mathrm{Node}(-)$*," "*$\mathrm{Cusp}(-)$*" determined by the action of a (single!) element of* $\Pi_{\mathbb{G}}$*.*

(ii) *Let* $a \leq b$ *be integers. Write* $\mathbb{G}_{[a,b]}$ *for the (uniquely determined) sub-semi-graph of PSC-type (cf. [9, Def. 2.2(i)]) of the underlying semi-graph of* $\mathcal{G}_\infty$ *whose set of vertices is equal to* $\{V(a), V(a+1), \ldots, V(b)\}$ *(cf. (i)). Also, write* $\mathcal{G}_{[a,b]}$ *for the semi-graph of temperoids obtained by restricting* $\mathcal{G}_\infty$ *to* $\mathbb{G}_{[a,b]}$ *(in the evident sense—cf. also the procedure discussed in Definition 2.8(i)). Then* $\mathcal{G}_{[a,b]}$ *is a semi-graph of temperoids of HSD-type.*

(iii) *Let* $a \leq b$ *be integers. For an integer* c *such that* $a \leq c \leq b$ *(resp.* $a+1 \leq c \leq b$*), let* $\Pi_{V(c)} \subseteq \Pi_{\mathcal{G}_{[a,b]}}$ *(resp.* $\Pi_{N(c)} \subseteq \Pi_{\mathcal{G}_{[a,b]}}$*) be a verticial (resp. nodal) subgroup of* $\Pi_{\mathcal{G}_{[a,b]}}$ *associated with* $V(c) \in \mathrm{Vert}(\mathcal{G}_{[a,b]})$ *(resp.* $N(c) \in \mathrm{Node}(\mathcal{G}_{[a,b]})$*) (cf. (i) and (ii)) such that, for* $a+1 \leq c \leq b$*, it holds that* $\Pi_{N(c)} \subseteq \Pi_{V(c-1)} \cap \Pi_{V(c)}$*. Then the inclusions* $\Pi_{V(c)}, \Pi_{N(c)} \hookrightarrow \Pi_{\mathcal{G}_{[a,b]}}$ *determine an isomorphism*

$$\varinjlim \left(\Pi_{V(a)} \hookleftarrow \Pi_{N(a+1)} \hookrightarrow \Pi_{V(a+1)} \hookleftarrow \cdots \hookrightarrow \Pi_{V(b-1)} \hookleftarrow \Pi_{N(b)} \hookrightarrow \Pi_{V(b)}\right)$$

$$\xrightarrow{\sim} \Pi_{\mathcal{G}_{[a,b]}},$$

where $\varinjlim$ *denotes the inductive limit in the category of groups.*

(iv) *Let* $a \leq b$ *be integers. Then the composite* $\mathcal{G}_{[a,b]} \to \mathcal{G}_\infty \to \mathcal{G}$ *determines an outer injection* $\Pi_{\mathcal{G}_{[a,b]}} \hookrightarrow \Pi_{\mathcal{G}}$*. Moreover, the image of this outer injection is contained in the normal subgroup* $\Pi_{\mathcal{G}_\infty} \subseteq \Pi_{\mathcal{G}}$*.*

(v) *There exists a collection*

$$\{D_{[-a,a]}\}_{1 \leq a \in \mathbb{Z}}$$

of subgroups $D_{[-a,a]} \subseteq \Pi_{\mathcal{G}_\infty}$ *indexed by the positive integers which satisfy the following properties:*

- $D_{[-a,a]} \subseteq \Pi_{\mathcal{G}_\infty}$ *belongs to the* $\Pi_{\mathcal{G}}$*-conjugacy class (of subgroups of* $\Pi_{\mathcal{G}}$*) obtained by forming the image of the outer injection* $\Pi_{\mathcal{G}_{[-a,a]}} \hookrightarrow \Pi_{\mathcal{G}}$ *of (iv).*
- $D_{[-a,a]} \subseteq D_{[-a-1,a+1]}$*.*

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- *The inclusions $D_{[-a,a]} \hookrightarrow \Pi_{\mathcal{G}}$ (where a ranges over the positive integers) determine an isomorphism*

$$\varinjlim \left(D_{[-1,1]} \hookrightarrow D_{[-2,2]} \hookrightarrow D_{[-3,3]} \hookrightarrow \cdots\right) \xrightarrow{\sim} \Pi_{\mathcal{G}_\infty},$$

where $\varinjlim$ denotes the inductive limit in the category of groups.

(vi) *In the situation of (v), since $\Pi_{\mathcal{G}}$ injects into its pro-l completion for any $l \in \mathfrak{Primes}$ (cf. Remark 2.5.1), let us regard subgroups of $\Pi_{\mathcal{G}}$ as subgroups of the pro-Σ completion $\Pi_{\mathcal{G}}^{\Sigma}$ of $\Pi_{\mathcal{G}}$. Let a be a positive integer. Write $\overline{D}_{[-a,a]} \subseteq \Pi_{\mathcal{G}}^{\Sigma}$ for the closure of $D_{[-a,a]}$ in $\Pi_{\mathcal{G}}^{\Sigma}$. Let $\widehat{\gamma} \in \Pi_{\mathcal{G}}^{\Sigma}$. Suppose that $\overline{D}_{[a,-a]} \cap \widehat{\gamma} \cdot \overline{D}_{[a,-a]} \cdot \widehat{\gamma}^{-1} \neq \{1\}$. Then the image of $\widehat{\gamma} \in \Pi_{\mathcal{G}}^{\Sigma}$ in the pro-Σ completion $\Pi_{\mathbb{G}}^{\Sigma}$ of $\Pi_{\mathbb{G}}$ is contained in $\Pi_{\mathbb{G}} \subseteq \Pi_{\mathbb{G}}^{\Sigma}$.*

(vii) *In the situation of (vi), suppose, moreover, that $\widehat{\gamma}$ is contained in the closure $\overline{\Pi}_{\mathcal{G}_\infty} \subseteq \Pi_{\mathcal{G}}^{\Sigma}$ of $\Pi_{\mathcal{G}_\infty}$ in $\Pi_{\mathcal{G}}^{\Sigma}$. Then $\widehat{\gamma} \in \overline{D}_{[a,-a]}$.*

Proof. Assertions (i) and (ii) follow immediately from the various definitions involved. Assertion (iii) follows immediately from a similar argument to the argument applied in the proof of [20, Prop. 1.5(iii)]. Next, we verify assertion (iv). The injectivity portion of assertion (iv) follows immediately—by considering a suitable finite étale subcovering of $\mathcal{G}_\infty \to \mathcal{G}$ and applying a suitable specialization outer isomorphism (cf. Proposition 2.10)—from Proposition 2.5(iv). The remainder of assertion (iv) follows immediately from the various definitions involved. This completes the proof of assertion (iv). Assertion (v) follows immediately from assertion (iii).

Next, we verify assertion (vi). Write $\mathcal{G}^{\Sigma}$ for the semi-graph of anabelioids of pro-Σ PSC-type determined by $\mathcal{G}$ (cf. Proposition 2.5(iii)), $\widetilde{\mathcal{G}}^{\Sigma} \to \mathcal{G}^{\Sigma}$ for the universal covering of the semi-graph of anabelioids of pro-Σ PSC-type $\mathcal{G}^{\Sigma}$ corresponding to (the torsion-free group) $\Pi_{\mathcal{G}}^{\Sigma}$ (cf. Proposition 2.5(iii); [23, Rem. 1.2.2]), and $\widetilde{\mathbb{G}}^{\Sigma}$ for the underlying pro-semi-graph of $\widetilde{\mathcal{G}}^{\Sigma}$. Then it follows immediately—that is, by considering a suitable finite étale subcovering of $\mathcal{G}_\infty \to \mathcal{G}$ and applying a suitable specialization outer isomorphism (cf. Proposition 2.10)—from [8, Lem. 1.9(ii)] that our assumption that $\overline{D}_{[a,-a]} \cap \widehat{\gamma} \cdot \overline{D}_{[a,-a]} \cdot \widehat{\gamma}^{-1} \neq \{1\}$ implies that the respective sub-pro-semi-graphs of $\widetilde{\mathbb{G}}^{\Sigma}$ determined by $\overline{D}_{[a,-a]}$, $\widehat{\gamma} \cdot \overline{D}_{[a,-a]} \cdot \widehat{\gamma}^{-1} \subseteq \Pi_{\mathcal{G}}^{\Sigma}$ (cf. Proposition 2.5(v)) either contain a common pro-vertex or may be joined to one another by a single pro-edge. But this implies that $\widehat{\gamma}$ maps $\mathbb{G}_{[-a,a]}$ to some $\Pi_{\mathbb{G}}$-translate of $\mathbb{G}_{[-a,a]}$, hence, in particular, that the image of $\widehat{\gamma} \in \Pi_{\mathcal{G}}^{\Sigma}$ in $\Pi_{\mathbb{G}}^{\Sigma}$ is contained in $\Pi_{\mathbb{G}} \subseteq \Pi_{\mathbb{G}}^{\Sigma}$, as desired. This completes the proof of assertion (vi). Assertion (vii) follows immediately—that is, by considering a suitable finite étale subcovering of $\mathcal{G}_\infty \to \mathcal{G}$ and applying a suitable specialization outer isomorphism (cf. Proposition 2.10)—from the commensurable terminality (cf. [18, Prop. 1.2(ii)]) of $\overline{D}_{[a,-a]}$ in a suitable open subgroup of $\Pi_{\mathcal{G}}^{\Sigma}$ containing $\overline{\Pi}_{\mathcal{G}_\infty}$ (cf. also [8, Lem. 1.9(ii)]). This completes the proof of Lemma 2.11. ∎

The content of the following lemma is entirely elementary and well-known.

LEMMA 2.12 (Action of the symplectic group). *Let g be a positive integer. For each positive integer n and $v = (v_1, \ldots, v_n) \in \mathbb{Z}^{\oplus n}$, write $\mathrm{vol}(v) \in \mathbb{Z}$ for the (uniquely determined) nonnegative integer that generates the ideal $\mathbb{Z} \cdot v_1 + \cdots + \mathbb{Z} \cdot v_n \subseteq \mathbb{Z}$; $M_n(\mathbb{Z})$ for the set of n by n matrices with coefficients in $\mathbb{Z}$; $\mathrm{GL}_n(\mathbb{Z}) \subseteq M_n(\mathbb{Z})$ for the group of matrices $A \in M_n(\mathbb{Z})$*

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such that $\det(A) \in \{1,-1\}$*;* $\mathrm{Sp}_{2g}(\mathbb{Z}) \subseteq \mathrm{GL}_{2g}(\mathbb{Z})$ *for the subgroup of* $2g$ *by* $2g$ *symplectic matrices, that is,* $B \in \mathrm{GL}_{2g}(\mathbb{Z})$ *such that*

$$B \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot {}^t B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(Note that one verifies immediately that, for every $A \in \mathrm{GL}_n(\mathbb{Z})$*, it holds that* $\mathrm{vol}(v) = \mathrm{vol}(vA)$*.) Then the following hold:*

(i) *Let* $v = (v_1, \ldots, v_g) \in \mathbb{Z}^{\oplus g}$*. Then there exists an invertible matrix* $A \in \mathrm{GL}_g(\mathbb{Z})$ *such that* $vA = (\mathrm{vol}(v), \overbrace{0, \ldots, 0}^{g-1})$*.*
(ii) *Let* $v = (v_1, \ldots, v_{2g}) \in \mathbb{Z}^{\oplus 2g}$*. Then there exists a symplectic matrix* $B \in \mathrm{Sp}_{2g}(\mathbb{Z})$ *such that* $vB = (\mathrm{vol}(v), \overbrace{0, \ldots, 0}^{2g-1})$*.*
(iii) *Let* $N \subseteq \mathbb{Z}^{\oplus 2g}$ *be a submodule of* $\mathbb{Z}^{\oplus 2g}$ *and* $v \in \mathbb{Z}^{\oplus 2g}$*. Suppose that* $N \neq \{0\}$*. Then there exist a nonzero integer* $n \in \mathbb{Z} \setminus \{0\}$ *and a symplectic matrix* $B \in \mathrm{Sp}_{2g}(\mathbb{Z})$ *such that* $n \cdot vB \in N$*.*
(iv) *Let* $N \subseteq \mathbb{Z}^{\oplus 2g}$ *be a submodule of* $\mathbb{Z}^{\oplus 2g}$ *and* $\pi \colon \mathbb{Z}^{\oplus 2g} \twoheadrightarrow \mathbb{Z}$ *a surjection. Suppose that* N *is of infinite index in* $\mathbb{Z}^{\oplus 2g}$*. Then there exists a symplectic matrix* $B \in \mathrm{Sp}_{2g}(\mathbb{Z})$ *such that* $N \cdot B \subseteq \mathrm{Ker}(\pi)$*.*

Proof. First, we verify assertion (i). Let us first observe that if $v = 0$ (i.e., $\mathrm{vol}(v) = 0$), then assertion (i) is immediate. Thus, to verify assertion (i), we may assume without loss of generality that $v \neq 0$. In particular, to verify assertion (i), by replacing v by $\mathrm{vol}(v)^{-1} \cdot v \in \mathbb{Z}^{\oplus g}$, we may assume without loss of generality that $\mathrm{vol}(v) = 1$. On the other hand, since $\mathrm{vol}(v) = 1$, one verifies immediately that $\mathbb{Z}^{\oplus g}/(\mathbb{Z} \cdot v)$ is a free $\mathbb{Z}$-module of rank $g-1$, hence that there exists an injection $\mathbb{Z}^{\oplus g-1} \hookrightarrow \mathbb{Z}^{\oplus g}$ that induces an isomorphism $(\mathbb{Z} \cdot v) \oplus \mathbb{Z}^{\oplus g-1} \xrightarrow{\sim} \mathbb{Z}^{\oplus g}$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Since (one verifies easily that) $\mathrm{Sp}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) = \{B \in \mathrm{GL}_2(\mathbb{Z}) \mid \det(B) = 1\}$, assertion (ii) in the case where $g = 1$ follows immediately from assertion (i) (in the case where we take "g" in assertion (i) to be 2), together with the (easily verified) fact that

$$\left\{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det \begin{pmatrix} a & -b \\ c & -d \end{pmatrix}\right\} = \{1, -1\} \text{ for every } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}).$$

For $i \in \{1, \ldots, g\}$, write M_i for the submodule of $\mathbb{Z}^{\oplus 2g}$ generated by

$$(0, \ldots, 0, 1, 0, \ldots, 0),\ (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{\oplus 2g},$$

where the "1's" lie, respectively, in the ith and $(g+i)$th components. Then, by applying assertion (ii) in the case where $g = 1$ (already verified above) to the M_i's, we conclude that, to complete the verification of assertion (ii), we may assume without loss of generality that $v_i = 0$ for every $g+1 \leq i \leq 2g$. Write $v_{\leq g} \stackrel{\mathrm{def}}{=} (v_1, \ldots, v_g) \in \mathbb{Z}^{\oplus g}$. Then let us observe that it follows from assertion (i) that there exists an invertible matrix $A \in \mathrm{GL}_g(\mathbb{Z})$ such that $v_{\leq g} A = (\mathrm{vol}(v_{\leq g}), 0, \ldots, 0) = (\mathrm{vol}(v), 0, \ldots, 0)$. Thus, assertion (ii) follows immediately from

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the (easily verified) fact that

$$\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z}).$$

This completes the proof of assertion (ii).

Assertion (iii) follows immediately from assertion (ii). Assertion (iv) follows immediately—by applying the self-duality of $\mathbb{Z}^{\oplus 2g}$ with respect to the symplectic form determined by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$—from assertion (iii). This completes the proof of Lemma 2.12. ∎

LEMMA 2.13 (Automorphisms of surface groups). *Let g be a positive integer, Π the topological fundamental group of a connected orientable compact topological surface of genus g, $\pi\colon \Pi \twoheadrightarrow \mathbb{Z}$ a surjection, and $J \subseteq \Pi$ a subgroup of Π such that the image of J in Π^{ab} is of infinite index in Π^{ab}. (For example, this will be the case if J is generated by $2g-1$ elements.) Then there exists an automorphism σ of Π such that $\sigma(J) \subseteq \mathrm{Ker}(\pi)$.*

Proof. Write $H \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\Pi,\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\Pi^{\mathrm{ab}},\mathbb{Z})$. Let us fix isomorphisms $H \xrightarrow{\sim} \mathbb{Z}^{\oplus 2g}$ and $H^2(\Pi,\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$. Then it follows from the well-known theory of Poincaré duality that the cup product in group cohomology

$$H \times H = H^1(\Pi,\mathbb{Z}) \times H^1(\Pi,\mathbb{Z}) \longrightarrow H^2(\Pi,\mathbb{Z}) \cong \mathbb{Z}$$

determines a perfect pairing on H; moreover, if we write $\mathrm{Aut}_{\mathrm{PD}}(H) \subseteq \mathrm{Aut}(H)$ ($\xrightarrow{\sim} \mathrm{GL}_{2g}(\mathbb{Z})$—cf. the notation of Lemma 2.12) for the subgroup of automorphisms of H that are compatible with this perfect pairing, then—by replacing the isomorphism $H \xrightarrow{\sim} \mathbb{Z}^{\oplus 2g}$ by a suitable isomorphism if necessary—the isomorphism $\mathrm{Aut}(H) \xrightarrow{\sim} \mathrm{GL}_{2g}(\mathbb{Z})$ determines an isomorphism $\mathrm{Aut}_{\mathrm{PD}}(H) \xrightarrow{\sim} \mathrm{Sp}_{2g}(\mathbb{Z})$ (cf. the notation of Lemma 2.12). On the other hand, recall (cf., e.g., the discussion preceding [7, Th. 5.13]) that the natural homomorphism $\mathrm{Aut}(\Pi) \to \mathrm{Aut}(H)$ determines a surjection $\mathrm{Aut}(\Pi) \twoheadrightarrow \mathrm{Aut}_{\mathrm{PD}}(H) (\subseteq \mathrm{Aut}(H))$. Thus, Lemma 2.13 follows immediately from Lemma 2.12(iv). This completes the proof of Lemma 2.13. ∎

LEMMA 2.14 (Finitely generated subgroups of surface groups). *Let $\mathcal{G}$ be a semi-graph of temperoids of HSD-type and $J \subseteq \Pi_{\mathcal{G}}$ a finitely generated subgroup of the fundamental group $\Pi_{\mathcal{G}}$ of $\mathcal{G}$. Then the following hold:*

(i) *Suppose that $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$. Then there exist a subgroup $F \subseteq \Pi_{\mathcal{G}}$ of finite index and a surjection $F \twoheadrightarrow J$ such that $J \subseteq F$, and, moreover, the restriction of the surjection $F \twoheadrightarrow J$ to $J \subseteq F$ is the identity automorphism of J.*

(ii) *Suppose that $(\mathrm{Vert}(\mathcal{G})^\sharp, \mathrm{Cusp}(\mathcal{G})^\sharp, \mathrm{Node}(\mathcal{G})^\sharp) = (1,0,1)$. Thus, since we are in the situation of Lemma 2.11, we shall apply the notational conventions established in Lemma 2.11. Suppose that the image of J in $\Pi_{\mathcal{G}}^{\mathrm{ab}}$ is of infinite index in $\Pi_{\mathcal{G}}^{\mathrm{ab}}$. (For example, this will be the case if J is generated by $\mathrm{rank}_{\mathbb{Z}}(\Pi_{\mathcal{G}}^{\mathrm{ab}})-1$ elements.) Then there exists an automorphism $\sigma \in \mathrm{Aut}(\Pi_{\mathcal{G}})$ of $\Pi_{\mathcal{G}}$ such that $\sigma(J) \subseteq \Pi_{\mathcal{G}_\infty}$.*

(iii) *In the situation of (ii), suppose, moreover, that $J \subseteq \Pi_{\mathcal{G}_\infty}$. Then there exists a positive integer $a \in \mathbb{Z}$ such that $J \subseteq D_{[-a,a]}$ (cf. Lemma 2.11(v)).*

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Proof. Assertion (i) follows from [17, Cor. 1.6(ii)] together with the fact that $\Pi_{\mathcal{G}}$ is a finitely generated free group (cf. Remark 2.5.1). Assertion (ii) follows from Lemma 2.13. Assertion (iii) follows from Lemma 2.11(v) together with our assumption that J is finitely generated. This completes the proof of Lemma 2.14. ∎

THEOREM 2.15 (Profinite conjugates of finitely generated $\mathfrak{Primes}$-compatible subgroups). *Let Π be the topological fundamental group of a compact orientable hyperbolic topological surface with compact boundary (cf. Remark 2.5.1) and $H,\ J \subseteq \Pi$ subgroups. Since Π injects into its pro-l completion for any $l \in \mathfrak{Primes}$ (cf., e.g., [27, Prop. 3.3.15]; [25, Th. 1.7]), let us regard subgroups of Π as subgroups of the profinite completion $\widehat{\Pi}$ of Π. Write $\overline{H},\ \overline{J} \subseteq \widehat{\Pi}$ for the closures of $H,\ J$ in $\widehat{\Pi}$, respectively. Suppose that the following conditions are satisfied:*

(a) *The subgroups H and J are finitely generated.*
(b) *If J is of infinite index in Π, then $\overline{J}$ is of infinite index in $\widehat{\Pi}$.*

(Here, we note that condition (b) is automatically satisfied whenever Π is free—cf. [17, Cor. 1.6(ii)].) Then the following hold:

(i) *It holds that $J = \overline{J} \cap \Pi$.*
(ii) *Suppose that there exists an element $\widehat{\gamma} \in \widehat{\Pi}$ such that*

$$H \subseteq \widehat{\gamma} \cdot \overline{J} \cdot \widehat{\gamma}^{-1}.$$

Then there exists an element $\delta \in \Pi$ such that

$$H \subseteq \delta \cdot J \cdot \delta^{-1}.$$

Proof. Let us first observe that, to verify Theorem 2.15, we may assume without loss of generality that Π is the fundamental group $\Pi_{\mathcal{G}}$ of a semi-graph of temperoids of HSD-type $\mathcal{G}$ (cf. Definition 2.3).

Next, we claim that the following assertion holds:

Claim 2.15.A: Theorem 2.15 holds in the case where J is of finite index in $\Pi_{\mathcal{G}}$.

Indeed, write $N \subseteq \Pi_{\mathcal{G}}$ for the normal subgroup of $\Pi_{\mathcal{G}}$ obtained by forming the intersection of all $\Pi_{\mathcal{G}}$-conjugates of J. Then since J is of finite index in $\Pi_{\mathcal{G}}$, it is immediate that N is of finite index in $\Pi_{\mathcal{G}}$. Thus, by considering the images in $\Pi_{\mathcal{G}}/N$ of the various groups involved, one verifies immediately that Theorem 2.15 holds in the case where J is of finite index in $\Pi_{\mathcal{G}}$. This completes the proof of Claim 2.15.A. Thus, in the remainder of the proof of Theorem 2.15, we may assume without loss of generality that J is of infinite index in $\Pi_{\mathcal{G}}$, which implies that $\overline{J}$ is of infinite index in $\widehat{\Pi}_{\mathcal{G}}$ (cf. condition (b)).

Next, we claim that the following assertion holds:

Claim 2.15.B: Let $F \subseteq \Pi_{\mathcal{G}}$ be a subgroup of finite index such that $J \subseteq F$. Suppose that the assertion obtained by replacing $\Pi_{\mathcal{G}}$ in assertion (i) by F holds. Then assertion (i) holds, and, in the situation of assertion (ii), there exists a $\Pi_{\mathcal{G}}$-conjugate of H that is contained in F. If, moreover, the assertion obtained by replacing $\Pi_{\mathcal{G}}$ in assertion (ii) by F holds, then assertion (ii) holds.

Indeed, let us first observe that since the natural inclusion $F \hookrightarrow \Pi_{\mathcal{G}}$ is $\mathfrak{Primes}$-compatible (cf. the discussion entitled "Groups" in §0), the profinite completion $\widehat{F}$ of F may be identified

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with the closure $\overline{F}$ of F in $\widehat{\Pi}_{\mathcal{G}}$. In particular, the closure of J in $\widehat{F}$ is naturally isomorphic to the closure $\overline{J}$ of J in $\widehat{\Pi}_{\mathcal{G}}$. Thus, it follows from Claim 2.15.A applied to F that the assertion obtained by replacing $\Pi_{\mathcal{G}}$ in assertion (i) by F implies assertion (i). Next, let us observe that in the situation of assertion (ii), since (one verifies immediately that) $\Pi_{\mathcal{G}} \cdot \overline{F} = \widehat{\Pi}_{\mathcal{G}}$, by replacing H by a suitable $\Pi_{\mathcal{G}}$-conjugate of H, we may assume without loss of generality that $\widehat{\gamma} \in \overline{F}$. In particular, since $H \subseteq \widehat{\gamma} \cdot \overline{J} \cdot \widehat{\gamma}^{-1} \subseteq \widehat{\gamma} \cdot \overline{F} \cdot \widehat{\gamma}^{-1} = \overline{F}$, it follows that $H \subseteq \overline{F} \cap \Pi_{\mathcal{G}} = F$ (cf. Claim 2.15.A). Thus, one verifies easily that the assertion obtained by replacing $\Pi_{\mathcal{G}}$ in assertion (ii) by F implies assertion (ii). This completes the proof of Claim 2.15.B.

Next, we verify Theorem 2.15 in the case where $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$. Suppose that $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$. Then it follows from Lemma 2.14(i) that there exist a subgroup $F \subseteq \Pi_{\mathcal{G}}$ of finite index and a surjection $\pi\colon F \twoheadrightarrow J$ such that $J \subseteq F$, and, moreover, the restriction of π to $J \subseteq F$ is the identity automorphism of J. Now it follows immediately from Claim 2.15.B that, by replacing $\Pi_{\mathcal{G}}$ by F, we may assume without loss of generality that $\Pi_{\mathcal{G}} = F$. Next, let us observe that since (it is immediate that) $J \subseteq \overline{J} \cap \Pi_{\mathcal{G}}$, to complete the verification of assertion (i) in the case where $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$, it suffices to verify that $\overline{J} \cap \Pi_{\mathcal{G}} \subseteq J$. Moreover, since $J \subseteq \overline{J} \cap \Pi_{\mathcal{G}} (\subseteq \overline{J})$, it follows immediately from the equality $\widehat{\pi}|_{\overline{J}} = \mathrm{id}_{\overline{J}}$ (where we write $\widehat{\pi}\colon \widehat{\Pi}_{\mathcal{G}} \twoheadrightarrow \overline{J}$ for the surjection induced by π) that, to verify the inclusion $\overline{J} \cap \Pi_{\mathcal{G}} \subseteq J$, it suffices to verify that $\widehat{\pi}(\overline{J} \cap \Pi_{\mathcal{G}}) \subseteq \widehat{\pi}(J)$. On the other hand, one verifies easily that

$$\widehat{\pi}(\overline{J} \cap \Pi_{\mathcal{G}}) \subseteq \widehat{\pi}(\Pi_{\mathcal{G}}) = J = \widehat{\pi}(J),$$

as desired. This completes the proof of assertion (i) in the case where $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$.

Next, to verify assertion (ii) in the case where $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$, let us observe that, by replacing $\widehat{\gamma}$ by $\widehat{\gamma} \cdot \widehat{\pi}(\widehat{\gamma}^{-1})$, we may assume without loss of generality that $\widehat{\gamma} \in \mathrm{Ker}(\widehat{\pi})$. Now we claim that the following assertion holds:

Claim 2.15.C: It holds that $H \subseteq \widehat{\gamma} \cdot J \cdot \widehat{\gamma}^{-1}$.

Indeed, since (one verifies easily that) $\widehat{\gamma}^{-1} \cdot H \cdot \widehat{\gamma},\ J \subseteq \overline{J}$, it follows immediately from the equality $\widehat{\pi}|_{\overline{J}} = \mathrm{id}_{\overline{J}}$ that, to verify Claim 2.15.C, it suffices to verify that $\widehat{\pi}(\widehat{\gamma}^{-1} \cdot H \cdot \widehat{\gamma}) \subseteq \widehat{\pi}(J)$. On the other hand, since $\widehat{\gamma} \in \mathrm{Ker}(\widehat{\pi})$, it holds that

$$\widehat{\pi}(\widehat{\gamma}^{-1} \cdot H \cdot \widehat{\gamma}) = \widehat{\pi}(H) \subseteq \widehat{\pi}(\Pi_{\mathcal{G}}) = J = \widehat{\pi}(J),$$

as desired. This completes the proof of Claim 2.15.C. In particular, it follows immediately from [22, Th. 2.6] (i.e., in essence, the argument given in the proof of [1, Lem. 3.2.1]), that there exists an element $\delta \in \Pi_{\mathcal{G}}$ such that $\delta^{-1} \cdot H \cdot \delta = \widehat{\gamma}^{-1} \cdot H \cdot \widehat{\gamma} \subseteq J$. This completes the proof of assertion (ii) in the case where $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$, hence also of Theorem 2.15 in the case where $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$.

Next, we verify Theorem 2.15 in the case where $\mathrm{Cusp}(\mathcal{G}) = \emptyset$. Suppose that $\mathrm{Cusp}(\mathcal{G}) = \emptyset$. First, we observe that since $\overline{J}$ is of infinite index in $\widehat{\Pi}_{\mathcal{G}}$, it follows immediately that $[\Pi_{\mathcal{G}} : J \cdot N] \to +\infty$ as N ranges over the normal subgroups of $\Pi_{\mathcal{G}}$ of finite index, hence (cf. Claim 2.15.B; the fact that J is finitely generated) that, by replacing $\Pi_{\mathcal{G}}$ by a suitable subgroup of finite index in $\Pi_{\mathcal{G}}$ that contains J, we may assume without loss of generality that the image of J in $\Pi_{\mathcal{G}}^{\mathrm{ab}}$ is of infinite index in $\Pi_{\mathcal{G}}^{\mathrm{ab}}$ (cf. Remark 2.5.1). Moreover, by considering suitable specialization outer isomorphisms (cf. Proposition 2.10), we may assume without loss of generality that the equality $(\mathrm{Vert}(\mathcal{G})^{\sharp}, \mathrm{Cusp}(\mathcal{G})^{\sharp}, \mathrm{Node}(\mathcal{G})^{\sharp}) = (1,0,1)$ holds. Thus, since we are in the situation of Lemma 2.11, we shall apply the notational conventions

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established in Lemma 2.11. Moreover, it follows from Lemma 2.14(ii) that, by considering a suitable automorphism of $\Pi_{\mathcal{G}}$, we may assume without loss of generality that $J \subseteq \Pi_{\mathcal{G}_\infty}$. Thus, it follows from Lemma 2.14(iii) that there exists a positive integer $a \in \mathbb{Z}$ such that $J \subseteq D_{[-a,a]} \subseteq \Pi_{\mathcal{G}_\infty}$.

Next, let us observe that since $\Pi_{\mathcal{G}}/\Pi_{\mathcal{G}_\infty} \xrightarrow{\sim} \Pi_{\mathbb{G}}(\cong \mathbb{Z})$ injects into its profinite completion, it follows that $\overline{J} \cap \Pi_{\mathcal{G}} \subseteq \Pi_{\mathcal{G}_\infty}$. In particular, by applying Lemma 2.14(iii), we conclude that, for any given fixed element $\alpha \in \overline{J} \cap \Pi_{\mathcal{G}}$, we may assume, by possibly enlarging a, that $\alpha \in D_{[-a,a]}$. Next, let us observe—that is, by considering a suitable finite étale subcovering of $\mathcal{G}_\infty \to \mathcal{G}$ and applying a suitable specialization outer isomorphism (cf. Proposition 2.10)—that the natural inclusion $D_{[-a,a]} \hookrightarrow \Pi_{\mathcal{G}}$ is $\mathfrak{Primes}$-compatible (cf. Proposition 2.5(iv)). In particular, by replacing $\mathcal{G}$ by $\mathcal{G}_{[-a,a]}$ (cf. Lemma 2.11(ii)), we conclude that assertion (i) in the case where $\mathrm{Cusp}(\mathcal{G}) = \emptyset$ follows from assertion (i) in the case where $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$ (already verified above). This completes the proof of assertion (i) in the case where $\mathrm{Cusp}(\mathcal{G}) = \emptyset$.

Finally, to verify assertion (ii) in the case where $\mathrm{Cusp}(\mathcal{G}) = \emptyset$, let us observe that if $H = \{1\}$, then assertion (ii) is immediate. Thus, we may assume without loss of generality that $H \neq \{1\}$. Next, let us observe that since $J \subseteq D_{[-a,a]} \subseteq \Pi_{\mathcal{G}_\infty}$, and $\Pi_{\mathcal{G}}/\Pi_{\mathcal{G}_\infty} \xrightarrow{\sim} \Pi_{\mathbb{G}}$ $(\cong \mathbb{Z})$ injects into its profinite completion, one verifies immediately that $H \subseteq \Pi_{\mathcal{G}_\infty}$. Thus, since $H \subseteq \Pi_{\mathcal{G}_\infty}$ is finitely generated, it follows from Lemma 2.14(iii) that, by possibly enlarging a, we may assume without loss of generality that $H \subseteq D_{[-a,a]}$. Since, moreover, $\{1\} \neq H \subseteq \overline{D}_{[-a,a]} \cap \widehat{\gamma} \cdot \overline{J} \cdot \widehat{\gamma}^{-1} \subseteq \overline{D}_{[-a,a]} \cap \widehat{\gamma} \cdot \overline{D}_{[-a,a]} \cdot \widehat{\gamma}^{-1}$, it follows from Lemma 2.11(vi) that the image of $\widehat{\gamma} \in \widehat{\Pi}_{\mathcal{G}}$ in the profinite completion $\widehat{\Pi}_{\mathbb{G}}$ of $\Pi_{\mathbb{G}}$ is contained in $\Pi_{\mathbb{G}} \subseteq \widehat{\Pi}_{\mathbb{G}}$, which thus implies that there exists an element $\gamma' \in \Pi_{\mathcal{G}}$ such that $\gamma'\widehat{\gamma} \in \overline{\Pi}_{\mathcal{G}_\infty}$. In particular, by replacing H by $\gamma' \cdot H \cdot (\gamma')^{-1}$ and possibly enlarging a, we may assume without loss of generality that $\widehat{\gamma} \in \overline{\Pi}_{\mathcal{G}_\infty}$. Thus, again by applying the fact that $\{1\} \neq \overline{D}_{[-a,a]} \cap \widehat{\gamma} \cdot \overline{D}_{[-a,a]} \cdot \widehat{\gamma}^{-1}$, we conclude from Lemma 2.11(vii) that $\widehat{\gamma} \in \overline{D}_{[-a,a]}$. In particular, since, as discussed above (cf. the discussion immediately preceding the proof of assertion (i) in the case where $\mathrm{Cusp}(\mathcal{G}) = \emptyset$), the natural inclusion $D_{[-a,a]} \hookrightarrow \Pi_{\mathcal{G}}$ is $\mathfrak{Primes}$-compatible, by replacing $\mathcal{G}$ by $\mathcal{G}_{[-a,a]}$, we conclude that assertion (ii) in the case where $\mathrm{Cusp}(\mathcal{G}) = \emptyset$ follows from assertion (ii) in the case where $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$ (already verified above). This completes the proof of assertion (ii) in the case where $\mathrm{Cusp}(\mathcal{G}) = \emptyset$, hence also of Theorem 2.15. ∎

REMARK 2.15.1. In passing, we observe that the analogue of Theorem 2.15 for arbitrary $\Sigma \neq \mathfrak{Primes}$ is false. Indeed, if, in the statement of Theorem 2.15, one replaces "Π" by the group $\mathbb{Z}$, then it is easy to construct counterexamples to assertions (i) and (ii). One may then obtain counterexamples in the case of the original "Π" by considering the case where the original "Π" is the fundamental group $\Pi_{\mathcal{G}}$ of a semi-graph of temperoids of HSD-type $\mathcal{G}$ such that $\mathrm{Edge}(\mathcal{G}) \neq \emptyset$ and considering suitable edge-like subgroups (i.e., isomorphic to $\mathbb{Z}$!) of $\Pi_{\mathcal{G}}$.

LEMMA 2.16 (VCN-subgroups of infinite index). *Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-Σ PSC-type (resp. of temperoids of HSD-type). Write $J \stackrel{\mathrm{def}}{=} \Pi_{\mathcal{G}}^{\Sigma}$ (resp. $J \stackrel{\mathrm{def}}{=} \Pi_{\mathcal{G}}$) for the (pro-$\Sigma$ [resp. discrete]) fundamental group of $\mathcal{G}$. Let $H \subseteq J$ be a VCN-subgroup of J. Consider the following two (mutually exclusive) conditions:*

(1) *$H = J$.*
(2) *H is of infinite index in J.*

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Then we have equivalences

$$(1) \Longleftrightarrow (1'); \ (2) \Longleftrightarrow (2')$$

with the following two conditions:

(1′) *H is verticial, and* $\mathrm{Node}(\mathcal{G}) = \emptyset$.
(2′) *Either H is edge-like, or* $\mathrm{Node}(\mathcal{G}) \neq \emptyset$.

Proof. The implication $(1') \Rightarrow (1)$ follows immediately from the various definitions involved. Thus, one verifies immediately (by considering suitable contrapositive versions of the various implications involved) that, to complete the verification of Lemma 2.16, it suffices to verify the implication $(2') \Rightarrow (2)$. To this end, let us observe that if H is edge-like, then since H is abelian, and every closed subgroup of J of finite index is center-free (cf., e.g., Remark 2.5.1; [18, Rem. 1.1.3]), we conclude that H is of infinite index in J. Thus, we may assume without loss of generality that H is verticial and $\mathrm{Node}(\mathcal{G}) \neq \emptyset$. Now since $\mathrm{Node}(\mathcal{G}) \neq \emptyset$, it follows from a similar argument to the argument in the discussion entitled "Curves" in [21, §0] that, by replacing $\mathcal{G}$ by a suitable connected finite étale covering of $\mathcal{G}$, we may assume without loss of generality that the underlying semi-graph of $\mathcal{G}$ is loop-ample (cf. the discussion entitled "Semi-graphs" in [21, §0]). In particular, since (one verifies easily that) the abelianization of the (pro-Σ completion of the) topological fundamental group of a noncontractible semi-graph is infinite, the image of H in the abelianization of J is of infinite index, which thus implies that H is of infinite index in J, as desired. This completes the proof of Lemma 2.16. ∎

Corollary 2.17 (Profinite conjugates of VCN-subgroups). *Let $\mathcal{G}$ and $\mathcal{H}$ be semi-graphs of temperoids of HSD-type. Write $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{H}}$ for the respective fundamental groups of $\mathcal{G}$, $\mathcal{H}$. Thus, we obtain a semi-graph of anabelioids of pro-$\mathfrak{Primes}$ PSC-type $\widehat{\mathcal{H}}$ (cf. Proposition 2.5(iii) in the case where $\Sigma = \mathfrak{Primes}$). Let $z_{\mathcal{G}} \in \mathrm{VCN}(\mathcal{G})$, $z_{\mathcal{H}} \in \mathrm{VCN}(\mathcal{H})$, $\Pi_{z_{\mathcal{G}}} \subseteq \Pi_{\mathcal{G}}$ a VCN-subgroup of $\Pi_{\mathcal{G}}$ associated with $z_{\mathcal{G}} \in \mathrm{VCN}(\mathcal{G})$, $\Pi_{z_{\mathcal{H}}} \subseteq \Pi_{\mathcal{H}}$ a VCN-subgroup of $\Pi_{\mathcal{H}}$ associated with $z_{\mathcal{H}} \in \mathrm{VCN}(\mathcal{H})$,*

$$\widetilde{\alpha}\colon \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$$

an isomorphism of groups, and $\widehat{\gamma} \in \Pi_{\widehat{\mathcal{H}}}$ an element of the (profinite) fundamental group $\Pi_{\widehat{\mathcal{H}}}$ of $\widehat{\mathcal{H}}$. Let us fix an injection $\Pi_{\mathcal{H}} \hookrightarrow \Pi_{\widehat{\mathcal{H}}}$ such that the induced outer injection is the outer injection of Proposition 2.5(iii) and regard subgroups of $\Pi_{\mathcal{H}}$ as subgroups of $\Pi_{\widehat{\mathcal{H}}}$ by means of this fixed injection. Write $\overline{\Pi}_{z_{\mathcal{H}}} \subseteq \Pi_{\widehat{\mathcal{H}}}$ for the closure of $\Pi_{z_{\mathcal{H}}}$ in $\Pi_{\widehat{\mathcal{H}}}$. (Thus, $\overline{\Pi}_{z_{\mathcal{H}}} \subseteq \Pi_{\widehat{\mathcal{H}}}$ is a VCN-subgroup of $\Pi_{\widehat{\mathcal{H}}}$ associated with $z_{\mathcal{H}} \in \mathrm{VCN}(\widehat{\mathcal{H}}) = \mathrm{VCN}(\mathcal{H})$—cf. Proposition 2.5(v).) Then the following hold:

(i) *It holds that $\Pi_{z_{\mathcal{H}}} = \overline{\Pi}_{z_{\mathcal{H}}} \cap \Pi_{\mathcal{H}}$.*
(ii) *Suppose that*

$$\widetilde{\alpha}(\Pi_{z_{\mathcal{G}}}) \subseteq \widehat{\gamma} \cdot \overline{\Pi}_{z_{\mathcal{H}}} \cdot \widehat{\gamma}^{-1}.$$

Then there exists an element $\delta \in \Pi_{\mathcal{H}}$ such that

$$\widetilde{\alpha}(\Pi_{z_{\mathcal{G}}}) \subseteq \delta \cdot \Pi_{z_{\mathcal{H}}} \cdot \delta^{-1}.$$

Proof. First, let us observe that it follows immediately from Definition 2.3(ii), together with the well-known structure of topological fundamental groups of topological surfaces, that $\Pi_{z_{\mathcal{G}}}$ and $\Pi_{z_{\mathcal{H}}}$ are finitely generated. Thus, it follows immediately from Theorem 2.15

that, to complete the verification of Corollary 2.17, it suffices to verify that the following assertion holds:

If $\Pi_{z_{\mathcal{H}}} \neq \Pi_{\mathcal{H}}$, then $\overline{\Pi}_{z_{\mathcal{H}}}$ is of infinite index in $\Pi_{\widehat{\mathcal{H}}}$.

To this end, let us observe that since $\Pi_{z_{\mathcal{H}}} \neq \Pi_{\mathcal{H}}$, it follows from Lemma 2.16 (in the case where "$\mathcal{G}$" is a semi-graph of temperoids of HSD-type) that either $z_{\mathcal{H}}$ is an edge, or $\mathrm{Node}(\mathcal{H}) \neq \emptyset$. On the other hand, in either of these two cases, it follows immediately from Lemma 2.16 (in the case where "$\mathcal{G}$" is a semi-graph of anabelioids of PSC-type), together with Proposition 2.5(v), that $\overline{\Pi}_{z_{\mathcal{H}}}$ is of infinite index in $\Pi_{\widehat{\mathcal{H}}}$. This completes the proof of Corollary 2.17. ∎

Corollary 2.18 (Properties of VCN-subgroups). *Let $\mathcal{G}$ be a semi-graph of temperoids of HSD-type. Write $\Pi_{\mathcal{G}}$ for the fundamental group of $\mathcal{G}$. Also, write $\widetilde{\mathcal{G}} \to \mathcal{G}$ for the universal covering of $\mathcal{G}$ corresponding to $\Pi_{\mathcal{G}}$. Then the following hold:*

(i) *For $i = 1, 2$, let $\widetilde{v}_i \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ (cf. Definition 2.1(v)). Write $\Pi_{\widetilde{v}_i} \subseteq \Pi_{\mathcal{G}}$ for the verticial subgroup of $\Pi_{\mathcal{G}}$ associated with $\widetilde{v}_i$ (cf. Definition 2.6(ii)). Consider the following three (mutually exclusive) conditions (cf. Definition 2.1(v)):*

(1) $\delta(\widetilde{v}_1, \widetilde{v}_2) = 0$.
(2) $\delta(\widetilde{v}_1, \widetilde{v}_2) = 1$.
(3) $\delta(\widetilde{v}_1, \widetilde{v}_2) \geq 2$.

Then we have equivalences

$$(1) \iff (1'); \ (2) \iff (2'); \ (3) \iff (3')$$

with the following three conditions:

(1′) $\Pi_{\widetilde{v}_1} = \Pi_{\widetilde{v}_2}$.
(2′) $\Pi_{\widetilde{v}_1} \cap \Pi_{\widetilde{v}_2} \neq \{1\}$, *but* $\Pi_{\widetilde{v}_1} \neq \Pi_{\widetilde{v}_2}$.
(3′) $\Pi_{\widetilde{v}_1} \cap \Pi_{\widetilde{v}_2} = \{1\}$.

(ii) *In the situation of (i), suppose that condition (2), hence also condition (2′), holds. Then it holds that $(\mathcal{E}(\widetilde{v}_1) \cap \mathcal{E}(\widetilde{v}_2))^\sharp = 1$ (cf. Definition 2.1(v)), and, moreover, if we write $\widetilde{e} \in \mathcal{E}(\widetilde{v}_1) \cap \mathcal{E}(\widetilde{v}_2)$ for the unique element of $\mathcal{E}(\widetilde{v}_1) \cap \mathcal{E}(\widetilde{v}_2)$, then $\Pi_{\widetilde{v}_1} \cap \Pi_{\widetilde{v}_2} = \Pi_{\widetilde{e}}$; $\Pi_{\widetilde{e}} \neq \Pi_{\widetilde{v}_1}$; $\Pi_{\widetilde{e}} \neq \Pi_{\widetilde{v}_2}$.*

(iii) *For $i = 1, 2$, let $\widetilde{e}_i \in \mathrm{Edge}(\widetilde{\mathcal{G}})$ (cf. Definition 2.1(v)). Write $\Pi_{\widetilde{e}_i} \subseteq \Pi_{\mathcal{G}}$ for the edge-like subgroup of $\Pi_{\mathcal{G}}$ associated with $\widetilde{e}_i$ (cf. Definition 2.6(ii)). Then $\Pi_{\widetilde{e}_1} \cap \Pi_{\widetilde{e}_2} \neq \{1\}$ if and only if $\widetilde{e}_1 = \widetilde{e}_2$. In particular, $\Pi_{\widetilde{e}_1} \cap \Pi_{\widetilde{e}_2} \neq \{1\}$ if and only if $\Pi_{\widetilde{e}_1} = \Pi_{\widetilde{e}_2}$ (cf. Remark 2.6.1).*

(iv) *Let $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, $\widetilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$. Write $\Pi_{\widetilde{v}}$, $\Pi_{\widetilde{e}} \subseteq \Pi_{\mathcal{G}}$ for the VCN-subgroups of $\Pi_{\mathcal{G}}$ associated with $\widetilde{v}$, $\widetilde{e}$, respectively. Then $\Pi_{\widetilde{e}} \cap \Pi_{\widetilde{v}} \neq \{1\}$ if and only if $\widetilde{e} \in \mathcal{E}(\widetilde{v})$. In particular, $\Pi_{\widetilde{e}} \cap \Pi_{\widetilde{v}} \neq \{1\}$ if and only if $\Pi_{\widetilde{e}} \subseteq \Pi_{\widetilde{v}}$ (cf. Remark 2.6.1).*

(v) *Every VCN-subgroup of $\Pi_{\mathcal{G}}$ is commensurably terminal in $\Pi_{\mathcal{G}}$.*

Proof. Write $\widetilde{\mathcal{G}}^\wedge \to \widehat{\mathcal{G}}$ for the universal profinite étale covering of the semi-graph of anabelioids of pro-$\mathfrak{Primes}$ PSC-type $\widehat{\mathcal{G}}$ (cf. Proposition 2.5(iii) in the case where $\Sigma = \mathfrak{Primes}$) determined by $\widetilde{\mathcal{G}} \to \mathcal{G}$ and $\Pi_{\widehat{\mathcal{G}}}$ for the (profinite) fundamental group of $\widehat{\mathcal{G}}$ determined by the universal covering $\widetilde{\mathcal{G}}^\wedge \to \widehat{\mathcal{G}}$. Thus, one verifies easily that one obtains a natural morphism of (pro-)semi-graphs of temperoids (cf. Remark 2.1.1) $\widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}^\wedge$ that induces injections $\Pi_{\mathcal{G}} \hookrightarrow \Pi_{\widehat{\mathcal{G}}}$ (cf. Proposition 2.5(iii)) and $\mathrm{VCN}(\widetilde{\mathcal{G}}) \hookrightarrow \mathrm{VCN}(\widetilde{\mathcal{G}}^\wedge)$ (cf. [8, Def. 1.1(iii)]) such that

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- the injection $\mathrm{VCN}(\widetilde{\mathcal{G}}) \hookrightarrow \mathrm{VCN}(\widetilde{\mathcal{G}}^{\wedge})$ is compatible with the respective "δ's" (cf. Definition 2.1(v); [8, Def. 1.1(viii)]), and, moreover,
- for each $\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$, the closure $\overline{\Pi}_{\widetilde{z}} \subseteq \Pi_{\widehat{\mathcal{G}}}$ of the image of the VCN-subgroup $\Pi_{\widetilde{z}} \subseteq \Pi_{\mathcal{G}}$ of $\Pi_{\mathcal{G}}$ associated with $\widetilde{z}$ via the injection $\Pi_{\mathcal{G}} \hookrightarrow \Pi_{\widehat{\mathcal{G}}}$ coincides with the VCN-subgroup of $\Pi_{\widehat{\mathcal{G}}}$ (cf. [9, Def. 2.1(i)]) associated with the image of $\widetilde{z}$ via the injection $\mathrm{VCN}(\widetilde{\mathcal{G}}) \hookrightarrow \mathrm{VCN}(\widetilde{\mathcal{G}}^{\wedge})$ (cf. also Proposition 2.5(v)).

First, we verify assertion (i). The equivalence (1) $\Leftrightarrow$ (1′) follows immediately from the equivalence (1) $\Leftrightarrow$ (1′) of [8, Lem. 1.9(ii)], together with the discussion at the beginning of the present proof. Next, let us observe that, by considering the edge-like subgroup associated with an element of $\mathcal{E}(\widetilde{v}_1) \cap \mathcal{E}(\widetilde{v}_2)$, we conclude that condition (2) implies the condition that $\Pi_{\widetilde{v}_1} \cap \Pi_{\widetilde{v}_2} \neq \{1\}$. Thus, the implication (2) $\Rightarrow$ (2′) follows immediately from the equivalence (1) $\Leftrightarrow$ (1′). The implication (2′) $\Rightarrow$ (2) follows immediately from Corollary 2.17(i) and the implication (2′) $\Rightarrow$ (2) of [8, Lem. 1.9(ii)], together with the discussion at the beginning of the present proof. The equivalence (3) $\Leftrightarrow$ (3′) follows immediately from the equivalences (1) $\Leftrightarrow$ (1′) and (2) $\Leftrightarrow$ (2′). This completes the proof of assertion (i).

Assertion (iii) (resp. (iv)) follows immediately from [8, Lem. 1.5] (resp. [8, Lem. 1.7]), together with the discussion at the beginning of the present proof. Assertion (v) follows formally from assertions (i) and (iii) (cf. also the proof of [18, Prop. 1.2(ii)]).

Finally, we verify assertion (ii). Suppose that condition (2) (in the statement of assertion (i)), hence also condition (2′) (in the statement of assertion (i)), holds. Then the assertion that $(\mathcal{E}(\widetilde{v}_1) \cap \mathcal{E}(\widetilde{v}_2))^{\sharp} = 1$ follows immediately from the fact that the underlying semi-graph of $\widetilde{\mathcal{G}}$ is a tree. The remainder of assertion (ii) follows immediately—in light of assertion (iii)—from Corollary 2.17(i) and [8, Lem. 1.9(i)] (cf. also Remark 2.6.1), together with the discussion at the beginning of the present proof. This completes the proof of assertion (ii), hence also of Corollary 2.18. ∎

Corollary 2.19 (Graphicity of outer isomorphisms). *Let $\mathcal{G}$, $\mathcal{H}$ be semi-graphs of temperoids of HSD-type. Write $\widehat{\mathcal{G}}$, $\widehat{\mathcal{H}}$ for the semi-graphs of anabelioids of pro-$\mathfrak{Primes}$ PSC-type determined by $\mathcal{G}$, $\mathcal{H}$ (cf. Proposition 2.5(iii) in the case where $\Sigma = \mathfrak{Primes}$), respectively; $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{H}}$ for the respective fundamental groups of $\mathcal{G}$, $\mathcal{H}$; $\Pi_{\widehat{\mathcal{G}}}$, $\Pi_{\widehat{\mathcal{H}}}$ for the respective (profinite) fundamental groups of $\widehat{\mathcal{G}}$, $\widehat{\mathcal{H}}$. Let*

$$\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}$$

be an outer isomorphism. Write $\widehat{\alpha}\colon \Pi_{\widehat{\mathcal{G}}} \xrightarrow{\sim} \Pi_{\widehat{\mathcal{H}}}$ for the outer isomorphism determined by the outer isomorphism α and the natural outer isomorphisms $\widehat{\Pi}_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\widehat{\mathcal{G}}}$, $\widehat{\Pi}_{\mathcal{H}} \xrightarrow{\sim} \Pi_{\widehat{\mathcal{H}}}$ of Proposition 2.5(iii). Then the following hold:

(i) *The outer isomorphism α is group-theoretically verticial (resp. group-theoretically cuspidal; group-theoretically nodal; graphic) (cf. Definition 2.7(i) and (ii)) if and only if $\widehat{\alpha}$ is group-theoretically verticial (cf. [18, Def. 1.4(iv)]) (resp. group-theoretically cuspidal [cf. [18, Def. 1.4(iv)]]; group-theoretically nodal [cf. [8, Def. 1.12]]; graphic [cf. [18, Def. 1.4(i)]]).*

(ii) *The outer isomorphism α is graphic if and only if α is group-theoretically verticial, group-theoretically cuspidal, and group-theoretically nodal.*

Proof. Assertion (ii) follows immediately, in light of Corollary 2.18, from a similar argument to the argument applied in the proof of [18, Prop. 1.5(ii)]. Thus, it remains

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to verify assertion (i). The necessity portion of assertion (i) follows immediately from Proposition 2.5(v). Next, let us observe that inclusions of verticial subgroups of the fundamental group of a semi-graph of temperoids of HSD-type are necessarily equalities (cf. Corollary 2.18(i) and (ii)); a similar statement holds concerning inclusions of edge-like subgroups (cf. Corollary 2.18(iii)). Thus, the sufficiency portion of assertion (i) follows immediately—in light of assertion (ii) and [18, Prop. 1.5(ii)]—from Corollary 2.17(ii). This completes the proof of Corollary 2.19. ∎

COROLLARY 2.20 (Discrete combinatorial cuspidalization). *Let $\Sigma \subseteq \mathfrak{Primes}$ be a subset which is either equal to $\mathfrak{Primes}$ or of cardinality one, (g,r) a pair of nonnegative numbers such that $2g-2+r>0$, n a positive integer, and $\mathcal{X}$ a topological surface of type (g,r) (i.e., the complement of r distinct points in an orientable compact topological surface of genus g). For each positive integer i, write $\mathcal{X}_i$ for the ith configuration space of $\mathcal{X}$ (i.e., the topological space obtained by forming the complement of the various diagonals in the direct product of i copies of $\mathcal{X}$); Π_i for the topological fundamental group of $\mathcal{X}_i$; Π_i^{Σ} for the pro-Σ completion of Π_i; $\widehat{\Pi}_i$ for the profinite completion of Π_i;*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_i) \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_i) \subseteq \mathrm{Out}(\Pi_i)$$

for the subgroups of the group $\mathrm{Out}(\Pi_i)$ of outomorphisms of Π_i defined in the statement of [20, Cor. 5.1] (cf. also the discussion entitled "Topological groups" in [9, §0]);

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_i^{\Sigma}) \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_i^{\Sigma}) \subseteq \mathrm{Out}(\Pi_i^{\Sigma})$$

for the subgroups of the group $\mathrm{Out}(\Pi_i^{\Sigma})$ of outomorphisms of Π_i^{Σ} consisting of FC-admissible, F-admissible (cf. [20, Def. 1.1(ii)]; the discussion entitled "Topological groups" in [9, §0]) outomorphisms, respectively. Then the following hold:

(i) *The group Π_n is normally terminal in Π_n^{Σ} (cf. Proposition 2.5(iii)). In particular, the natural homomorphism*

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n) \longrightarrow \mathrm{Out}^{\mathrm{F}}(\Pi_n^{\Sigma})$$

is injective. In the following, we shall regard subgroups of $\mathrm{Out}^{\mathrm{F}}(\Pi_n)$ as subgroups of $\mathrm{Out}^{\mathrm{F}}(\Pi_n^{\Sigma})$.

(ii) *It holds that $\mathrm{Out}^{\mathrm{F}}(\Pi_n) \cap \mathrm{Out}^{\mathrm{FC}}(\widehat{\Pi}_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$.*

(iii) *Consider the commutative diagram*

$$\begin{array}{ccc}
\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) & \longrightarrow & \mathrm{Out}^{\mathrm{F}}(\widehat{\Pi}_{n+1}) \\
\downarrow & & \downarrow \\
\mathrm{Out}^{\mathrm{F}}(\Pi_n) & \longrightarrow & \mathrm{Out}^{\mathrm{F}}(\widehat{\Pi}_n)
\end{array}$$

—where the horizontal arrows are the injections of (i), and the vertical arrows are the homomorphisms induced by the projection $\mathcal{X}_{n+1} \to \mathcal{X}_n$ obtained by forgetting the $(n+1)$st factor. Suppose that the right-hand vertical arrow of the diagram is injective (cf. Remark 2.20.1 below). Then the commutative diagram of the above display is Cartesian. In particular, the left-hand vertical arrow of the diagram is injective.

(iv) *The image of the left-hand vertical arrow of the commutative diagram of (iii) (where we do not impose the assumption that the right-hand vertical arrow be injective) is contained in $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n)$.*

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(v) *Consider the commutative diagram*

$$\begin{tikzcd} \mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \arrow[r] \arrow[d] & \mathrm{Out}^{\mathrm{FC}}(\widehat{\Pi}_{n+1}) \arrow[d] \\ \mathrm{Out}^{\mathrm{FC}}(\Pi_{n}) \arrow[r] & \mathrm{Out}^{\mathrm{FC}}(\widehat{\Pi}_{n}) \end{tikzcd}$$

—where the horizontal arrows are the injections induced by the injections of (i), and the vertical arrows are the homomorphisms induced by the projection $\mathcal{X}_{n+1} \to \mathcal{X}_n$ obtained by forgetting the $(n+1)$st factor. This diagram is Cartesian, its right-hand vertical arrow is injective, and its left-hand vertical arrow is bijective.

(vi) *Write*

$$n_{\mathrm{FC}} \stackrel{\mathrm{def}}{=} \begin{cases} 2, & \text{if } (g,r) = (0,3), \\ 3, & \text{if } (g,r) \neq (0,3) \text{ and } r \neq 0, \\ 4, & \text{if } r = 0. \end{cases}$$

Suppose that $n \geq n_{\mathrm{FC}}$. Then it holds that

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) = \mathrm{Out}^{\mathrm{F}}(\Pi_n);$$

the left-hand vertical arrow

$$\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \longrightarrow \mathrm{Out}^{\mathrm{F}}(\Pi_n)$$

of the commutative diagram of (iii) is bijective.

Proof. Let us first observe that, to verify assertion (i), it suffices to verify that Π_n is normally terminal in Π_n^{Σ}. Moreover, once one proves the desired normal terminality in the case where $n = 1$, the desired normal terminality in the case where $n \geq 2$ follows immediately by induction (cf. the proof of [20, Cor. 5.1(i)]). Thus, we conclude that, to verify assertion (i), it suffices to verify the normal terminality of Π_1 in Π_1^{Σ}.

Next, we claim that the following assertion holds:

> Claim 2.20.A: Let F be a free nonabelian group. Then F is normally terminal in the pro-Σ completion of F.

Indeed, since F is conjugacy l-separable (cf. [25, Th. 3.2]) for every $l \in \Sigma$, Claim 2.20.A follows from a similar argument to the argument applied in the proof of [1, Lem. 3.2.1]. This completes the proof of Claim 2.20.A.

Next, let us observe that one verifies easily that there exist a semi-graph of temperoids of HSD-type $\mathcal{G}$ and an isomorphism of Π_1 with the fundamental group $\Pi_{\mathcal{G}}$ of $\mathcal{G}$. In the following, we shall identify $\Pi_{\mathcal{G}}$ with Π_1 by means of such an isomorphism. If $\mathcal{G}$ has a cusp, then it follows from Remark 2.5.1 that Π_1 is a free nonabelian group. Thus, the desired normal terminality follows from Claim 2.20.A. In the remainder of the proof of assertion (i), suppose that $\mathcal{G}$ has no cusp. In particular, we may assume without loss of generality, by applying a suitable specialization outer isomorphism (cf. Proposition 2.10), that $\mathcal{G}$ has a node. Let $\widehat{\gamma} \in N_{\Pi_1^{\Sigma}}(\Pi_1)$ be an element of the normalizer of Π_1 in Π_1^{Σ} and $\Pi_v \subseteq \Pi_{\mathcal{G}}$ a verticial subgroup of $\Pi_{\mathcal{G}}$. Then, by applying Corollary 2.17, (ii) (i.e., in the case where we take the "$(\mathcal{G}, \mathcal{H}, \Pi_{z_{\mathcal{H}}}, \Pi_{z_{\mathcal{G}}}, \widehat{\gamma})$" of Corollary 2.17 to be $(\mathcal{G}, \mathcal{G}, \Pi_v, \Pi_v, \widehat{\gamma})$ and the "$\widetilde{\alpha}$" of Corollary 2.17 to be the automorphism of $\Pi_{\mathcal{G}}$ obtained by conjugation by $\widehat{\gamma}$), we conclude immediately (cf.

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also Corollary 2.18(i) and (ii)) that we may assume without loss of generality, by multiplying $\widehat{\gamma}$ by a suitable element of $\Pi_{\mathcal{G}}$, that the element $\widehat{\gamma} \in \Pi_1^{\Sigma}$ normalizes Π_v, hence also the closure $\overline{\Pi}_v$ of Π_v in Π_1^{Σ}. In particular, it follows from Proposition 2.5(v) and [18, Prop. 1.2(ii)] that $\widehat{\gamma} \in \overline{\Pi}_v$. On the other hand, since $\mathcal{G}$ has a node, it follows from Proposition 2.5(iv) and Remark 2.6.1 that Π_v is a free nonabelian group, and $\overline{\Pi}_v$ may be identified with the pro-Σ completion of Π_v. Thus, it follows from Claim 2.20.A that $\widehat{\gamma} \in \Pi_v \subseteq \Pi_{\mathcal{G}}$, as desired. This completes the proof of assertion (i).

Assertion (ii) follows immediately from Corollary 2.19(i). Next, we verify assertion (iii). Let us first observe that since (we have assumed that) the right-hand vertical arrow of the diagram of assertion (iii) is injective, it follows immediately from assertion (i) that all arrows of the diagram of assertion (iii) are injective. Let $\alpha \in \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ be such that the image of α in $\mathrm{Out}^{\mathrm{F}}(\widehat{\Pi}_n)$ lies in the image of the right-hand vertical arrow of the diagram of assertion (iii). Then it follows from [9, Th. A(ii)] that the image of α in $\mathrm{Out}^{\mathrm{F}}(\widehat{\Pi}_n)$ is FC-admissible. Thus, it follows from assertion (ii) that $\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$. In particular, it follows from [8, Cor. 6.6] that there exists a uniquely determined element of $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1})$ whose image in $\mathrm{Out}^{\mathrm{F}}(\Pi_n)$ coincides with $\alpha \in \mathrm{Out}^{\mathrm{F}}(\Pi_n)$. Thus, since all arrows of the diagram of assertion (iii) are injective (as verified above), we conclude that the diagram of assertion (iii) is Cartesian. This completes the proof of assertion (iii). Assertion (iv) follows immediately from [9, Th. A(ii)], together with assertion (ii). Assertion (v) follows immediately from a similar argument to the argument applied in the proof of assertion (iii), together with the injectivity portion of [8, Th. B]. Assertion (vi) follows immediately from [10, Th. A(ii)] together with assertions (i), (ii), and (v). This completes the proof of Corollary 2.20. ∎

Remark 2.20.1. It follows from [10, Th. A(i)] that if either $n \neq 1$ or $r \neq 0$, then the right-hand vertical arrow of the diagram of Corollary 2.20(iii) is injective.

Remark 2.20.2. In the notation of Corollary 2.20, the bijectivity of the left-hand vertical arrow $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ of the diagram of Corollary 2.20(v) is proven in [8, Cor. 6.6] by applying, in essence, a well-known result concerning topological surfaces due to Dehn–Nielsen–Baer (cf. the proof of [20, Cor. 5.1(ii)]). On the other hand, the equivalences of Corollary 2.19(i) (cf. also the injection of Corollary 2.20(i)), together with a similar argument to the argument applied in the proof of the bijectivity portion of [8, Th. B]—that is, in essence, the argument applied in the proof of [20, Cor. 3.3]—allow one to give a purely algebraic alternative proof of this bijectivity result in the case where $n \geq \max\{3, n_{\mathrm{FC}}\}$ (cf. Corollary 2.20(vi)).

Corollary 2.21 (Discrete/profinite Dehn multi-twists). *In the situation of Example 2.4(i), write $\widehat{\mathcal{G}}_{X^{\log}}$ for the semi-graph of anabelioids of pro-$\mathfrak{Primes}$ PSC-type of Proposition 2.5(iii) in the case where we take "$(\mathcal{G},\Sigma)$" to be $(\mathcal{G}_{X^{\log}}, \mathfrak{Primes})$; $\Pi_{\mathcal{G}_{X^{\log}}}$, $\Pi_{\widehat{\mathcal{G}}_{X^{\log}}}$ for the respective fundamental groups of $\mathcal{G}_{X^{\log}}$, $\widehat{\mathcal{G}}_{X^{\log}}$; $\widehat{\Pi}_{\mathcal{G}_{X^{\log}}}$ for the profinite completion of $\Pi_{\mathcal{G}_{X^{\log}}}$ (so we have a natural outer isomorphism $\widehat{\Pi}_{\mathcal{G}_{X^{\log}}} \xrightarrow{\sim} \Pi_{\widehat{\mathcal{G}}_{X^{\log}}}$—cf. Proposition 2.5(iii));*

$$\mathrm{Dehn}(\mathcal{G}_{X^{\log}}) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}_{X^{\log}}})$$

for the subgroup consisting of the Dehn multi-twists of $\mathcal{G}_{X^{\log}}$, that is, of $\alpha \in \mathrm{Out}(\Pi_{\mathcal{G}_{X^{\log}}})$ such that the following conditions are satisfied:

(a) *α is graphic (cf. Definition 2.7(ii)) and induces the identity automorphism on the underlying semi-graph of $\mathcal{G}_{X^{\log}}$.*

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(b) *Let* $\Pi_v \subseteq \Pi_{\mathcal{G}_{X^{\log}}}$ *be a verticial subgroup of* $\Pi_{\mathcal{G}_{X^{\log}}}$. *Then the outomorphism of* Π_v *induced by restricting* α *(cf. (a); Corollary 2.18(v); the evident discrete analogue of [10, Lem. 3.10]) is trivial.*

Then the following hold:

(i) *The composite of natural outer homomorphisms*

$$\Pi_{\mathcal{G}_{X^{\log}}} \longrightarrow \widehat{\Pi}_{\mathcal{G}_{X^{\log}}} \xrightarrow{\sim} \Pi_{\widehat{\mathcal{G}}_{X^{\log}}}$$

determines an injection

$$\mathrm{Out}(\Pi_{\mathcal{G}_{X^{\log}}}) \hookrightarrow \mathrm{Out}(\Pi_{\widehat{\mathcal{G}}_{X^{\log}}}).$$

(ii) *If one regards subgroups of* $\mathrm{Out}(\Pi_{\mathcal{G}_{X^{\log}}})$ *as subgroups of* $\mathrm{Out}(\Pi_{\widehat{\mathcal{G}}_{X^{\log}}})$ *by means of the injection of (i), then the equality*

$$\mathrm{Dehn}(\mathcal{G}_{X^{\log}}) = \mathrm{Dehn}(\widehat{\mathcal{G}}_{X^{\log}}) \cap \mathrm{Out}(\Pi_{\mathcal{G}_{X^{\log}}})$$

(cf. [9, Def. 4.4]) holds.

(iii) *The homomorphism of the final display of Example 2.4(i) determines, relative to the natural outer isomorphism* $\pi_1(X^{\log}_{\mathrm{an}}(\mathbb{C})|_s) \xrightarrow{\sim} \Pi_{\mathcal{G}_{X^{\log}}}$, *an isomorphism*

$$\pi_1(S^{\log}_{\mathrm{an}}(\mathbb{C})) \xrightarrow{\sim} \mathrm{Dehn}(\mathcal{G}_{X^{\log}})$$

of free $\mathbb{Z}$*-modules of rank* $\mathrm{Node}(\mathcal{G}_{X^{\log}})^\sharp$. *Moreover, the image of this isomorphism is dense, relative to the profinite topology, in* $\mathrm{Dehn}(\widehat{\mathcal{G}}_{X^{\log}})$.

Proof. Assertion (i) follows from Corollary 2.20(i). Next, we verify assertion (ii). The inclusion $\mathrm{Dehn}(\mathcal{G}_{X^{\log}}) \subseteq \mathrm{Dehn}(\widehat{\mathcal{G}}_{X^{\log}}) \cap \mathrm{Out}(\Pi_{\mathcal{G}_{X^{\log}}})$ follows immediately from the various definitions involved. To verify the reverse inclusion, let $\alpha \in \mathrm{Dehn}(\widehat{\mathcal{G}}_{X^{\log}}) \cap \mathrm{Out}(\Pi_{\mathcal{G}_{X^{\log}}})$. Then it follows immediately from Corollary 2.19(i) together with the definition of $\mathrm{Dehn}(\widehat{\mathcal{G}}_{X^{\log}})$, that the outomorphism α of $\Pi_{\mathcal{G}_{X^{\log}}}$ satisfies the condition (a) in the statement of Corollary 2.21. Moreover, it follows immediately from Proposition 2.5(v) and Corollary 2.20(i), together with the definition of $\mathrm{Dehn}(\widehat{\mathcal{G}}_{X^{\log}})$, that the outomorphism α of $\Pi_{\mathcal{G}_{X^{\log}}}$ satisfies the condition (b) in the statement of Corollary 2.21. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). First, let us observe that it follows immediately from the various definitions involved that the homomorphism of the final display of Example 2.4(i) factors through $\mathrm{Dehn}(\mathcal{G}_{X^{\log}})$ and has dense image (i.e., relative to the profinite topology) in $\mathrm{Dehn}(\widehat{\mathcal{G}}_{X^{\log}})$ (cf. [9, Prop. 5.6(ii)]). Next, let us recall from [9, Th. 4.8(ii) and (iv)] that if, for $e \in \mathrm{Node}(\mathcal{G}_{X^{\log}}) = \mathrm{Node}(\widehat{\mathcal{G}}_{X^{\log}})$, we write $S_e \stackrel{\mathrm{def}}{=} \mathrm{Node}(\mathcal{G}_{X^{\log}}) \setminus \{e\}$ and $(\mathcal{G}_{X^{\log}})^{\wedge}_{\rightsquigarrow S_e}$ for the semi-graph of anabelioids of pro-$\mathfrak{Primes}$ PSC-type of Proposition 2.5(iii), in the case where we take "$(\mathcal{G}, \Sigma)$" to be $((\mathcal{G}_{X^{\log}})_{\rightsquigarrow S_e}, \mathfrak{Primes})$ (cf. Definition 2.9) and regard $\mathrm{Dehn}((\mathcal{G}_{X^{\log}})^{\wedge}_{\rightsquigarrow S_e})$ as a closed subgroup of $\mathrm{Dehn}(\widehat{\mathcal{G}}_{X^{\log}})$ via the specialization outer isomorphism of [9, Def. 2.10] (cf. also Remark 2.9.1 and Proposition 2.10 of the present paper), then we have an equality

$$\mathrm{Dehn}(\widehat{\mathcal{G}}_{X^{\log}}) = \bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})} \mathrm{Dehn}((\mathcal{G}_{X^{\log}})^{\wedge}_{\rightsquigarrow S_e}),$$

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where each direct summand is (noncanonically) isomorphic to $\widehat{\mathbb{Z}}$. Here, we note that these specialization outer isomorphisms are compatible (cf. [9, Prop. 5.6(ii)–(iv)]) with the corresponding homomorphisms of the final display of Example 2.4(i). Thus, in light of the density assertion that has already been verified, one verifies immediately that, to complete the verification of assertion (iii), it suffices to verify that the image of $\mathrm{Dehn}(\mathcal{G}_{X^{\log}})$ via the projection to any direct summand of the direct sum decomposition of the above display is contained in some submodule of the direct summand that is isomorphic to $\mathbb{Z}$. To this end, let us recall from [9, Th. 4.8(iv)] that such an image via a projection to a direct summand may be computed by considering the homomorphism of the first display of [9, Lem. 4.6(ii)], that is, which determines an isomorphism between the direct summand under consideration and any profinite nodal subgroup $\widehat{\Pi}_e$ associated with the node e corresponding to the direct summand. On the other hand, it follows immediately—in light of the definition of this isomorphism—from Proposition 2.5(v) and Corollary 2.17(i) that the image of $\mathrm{Dehn}(\mathcal{G}_{X^{\log}})$ under consideration is contained in a suitable discrete nodal subgroup $\Pi_e(\cong \mathbb{Z})$ associated with e (cf. Remark 2.6.1). This completes the proof of assertion (iii). ∎

Definition 2.22. Suppose that $\Sigma = \mathfrak{Primes}$. Let (g,r) be a pair of nonnegative integers such that $2g-2+r>0$; n a positive integer; $k \overset{\mathrm{def}}{=} \mathbb{C}$; $S^{\log} \overset{\mathrm{def}}{=} \mathrm{Spec}(k)^{\log}$ the log scheme obtained by equipping $S \overset{\mathrm{def}}{=} \mathrm{Spec}(k)$ with the log structure determined by the fs chart $\mathbb{N} \to k$ that maps $1 \mapsto 0$; $X^{\log} = X_1^{\log}$ a stable log curve of type (g,r) over $S^{\log}$. For each (possibly empty) subset $E \subseteq \{1,\ldots,n\}$, write

$$X_E^{\log}$$

for the $E^\sharp$-th log configuration space of the stable log curve $X^{\log}$ (cf. the discussion entitled "Curves" in [9, §0]), where we think of the factors as being labeled by the elements of $E \subseteq \{1,\ldots,n\}$ (cf. the discussion at the beginning of [10, §3] in the case where $(\Sigma,k) = (\mathfrak{Primes},\mathbb{C})$). For each nonnegative integer n and each (possibly empty) subset $E \subseteq \{1,\ldots,n\}$, write $(X_E^{\log})_{\mathrm{an}} \to S_{\mathrm{an}}^{\log}$ for the morphism of fs log analytic spaces determined by the morphism $X_E^{\log} \to S^{\log}$; $(X_E^{\log})_{\mathrm{an}}(\mathbb{C})$, $S_{\mathrm{an}}^{\log}(\mathbb{C})$ for the respective topological spaces "$X^{\log}$" defined in [12, (1.2)] in the case where we take the "X" of [12, (1.2)] to be $(X_E^{\log})_{\mathrm{an}}$, $S_{\mathrm{an}}^{\log}$ (cf. the notation established in Example 2.4(i)). Let $s \in S_{\mathrm{an}}^{\log}(\mathbb{C})$. Write

$$\mathfrak{X}_E \overset{\mathrm{def}}{=} (X_E^{\log})_{\mathrm{an}}(\mathbb{C})|_s$$

for the fiber of the natural morphism $(X_E^{\log})_{\mathrm{an}}(\mathbb{C}) \to S_{\mathrm{an}}^{\log}(\mathbb{C})$ at s;

$$\Pi_E^{\mathrm{disc}} \overset{\mathrm{def}}{=} \pi_1(\mathfrak{X}_E)$$

for the discrete topological fundamental group of $\mathfrak{X}_E$;

$$\mathfrak{X}_n \overset{\mathrm{def}}{=} \mathfrak{X}_{\{1,\ldots,n\}};\ \mathfrak{X} \overset{\mathrm{def}}{=} \mathfrak{X}_1;\ \Pi_n^{\mathrm{disc}} \overset{\mathrm{def}}{=} \Pi_{\{1,\ldots,n\}}^{\mathrm{disc}}.$$

Thus, for sets $E' \subseteq E \subseteq \{1,\ldots,n\}$, we have a projection

$$p_{E/E'}^{\mathrm{an}} \colon \mathfrak{X}_E \to \mathfrak{X}_{E'}$$

obtained by forgetting the factors that belong to $E \setminus E'$. For sets $E' \subseteq E \subseteq \{1,\ldots,n\}$ and nonnegative integers $m \leq n$, write

$$p_{E/E'}^{\Pi^{\mathrm{disc}}} \colon \Pi_E^{\mathrm{disc}} \twoheadrightarrow \Pi_{E'}^{\mathrm{disc}}$$

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for some fixed surjection (that belongs to the collection of surjections that constitutes the outer surjection) induced by $p^{\mathrm{an}}_{E/E'}$;

$$\Pi^{\mathrm{disc}}_{E/E'} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(p^{\Pi^{\mathrm{disc}}}_{E/E'}) \subseteq \Pi^{\mathrm{disc}}_{E}$$

$$p^{\mathrm{an}}_{n/m} \stackrel{\mathrm{def}}{=} p^{\mathrm{an}}_{\{1,\dots,n\}/\{1,\dots,m\}} \colon \mathfrak{X}_n \longrightarrow \mathfrak{X}_m;$$

$$p^{\Pi^{\mathrm{disc}}}_{n/m} \stackrel{\mathrm{def}}{=} p^{\Pi^{\mathrm{disc}}}_{\{1,\dots,n\}/\{1,\dots,m\}} \colon \Pi^{\mathrm{disc}}_{n} \twoheadrightarrow \Pi^{\mathrm{disc}}_{m};$$

$$\Pi^{\mathrm{disc}}_{n/m} \stackrel{\mathrm{def}}{=} \Pi^{\mathrm{disc}}_{\{1,\dots,n\}/\{1,\dots,m\}} \subseteq \Pi^{\mathrm{disc}}_{n}.$$

Finally, we shall write "$\widehat{\Pi}^{\mathrm{disc}}_{(-)}$" for the profinite completion of "$\Pi^{\mathrm{disc}}_{(-)}$." Thus, we have a natural outer isomorphism

$$\widehat{\Pi}^{\mathrm{disc}}_{E} \xrightarrow{\sim} \Pi_E,$$

where Π_E is as in the discussion at the beginning of [10, §3]. In the following, we shall also write $X^{\log}_n \stackrel{\mathrm{def}}{=} X^{\log}_{\{1,\dots,n\}}$; $\Pi_n \stackrel{\mathrm{def}}{=} \Pi_{\{1,\dots,n\}}$.

Definition 2.23. In the notation of Definition 2.22, let $i \in E \subseteq \{1,\dots,n\}$; $x \in X_n(\mathbb{C})$ a $\mathbb{C}$-valued geometric point of the underlying scheme X_n of $X^{\log}_n$.

(i) We shall write

$$\mathcal{G}^{\mathrm{disc}}$$

for the semi-graph of temperoids of HSD-type associated with $X^{\log}$ (cf. Example 2.4(ii));

$$\mathcal{G}^{\mathrm{disc}}_{i\in E,x}$$

for the semi-graph of temperoids of HSD-type associated with the geometric fiber (cf. Example 2.4(ii); Remark 2.4.1) of the projection $p^{\log}_{E/(E\setminus\{i\})} \colon X^{\log}_E \to X^{\log}_{E\setminus\{i\}}$ over $x^{\log}_{E\setminus\{i\}} \to X^{\log}_{E\setminus\{i\}}$ (cf. [10, Def. 3.1(i)]);

$$\Pi_{\mathcal{G}^{\mathrm{disc}}},\ \Pi_{\mathcal{G}^{\mathrm{disc}}_{i\in E,x}}$$

for the respective fundamental groups of $\mathcal{G}^{\mathrm{disc}}$, $\mathcal{G}^{\mathrm{disc}}_{i\in E,x}$ (cf. Proposition 2.5(i));

$$\widehat{\Pi}_{\mathcal{G}^{\mathrm{disc}}_{i\in E,x}}$$

for the profinite completion of $\Pi_{\mathcal{G}^{\mathrm{disc}}_{i\in E,x}}$. Thus, it follows from the discussion of Remark 2.5.2 that we have a natural graphic (cf. [18, Def. 1.4(i)]) outer isomorphism

$$\widehat{\Pi}_{\mathcal{G}^{\mathrm{disc}}_{i\in E,x}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{i\in E,x}},$$

where $\mathcal{G}_{i\in E,x}$ is the semi-graph of anabelioids of pro-$\mathfrak{Primes}$ PSC-type of [10, Def. 3.1(iii)], and hence a natural isomorphism of semi-graphs of anabelioids

$$\widehat{\mathcal{G}}^{\mathrm{disc}}_{i\in E,x} \xrightarrow{\sim} \mathcal{G}_{i\in E,x},$$

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where we write $\widehat{\mathcal{G}}^{\mathrm{disc}}_{i\in E,x}$ for the semi-graph of anabelioids of pro-$\mathfrak{Primes}$ PSC-type of Proposition 2.5(iii) in the case where we take "$(\mathcal{G},\Sigma)$" to be $(\mathcal{G}^{\mathrm{disc}}_{i\in E,x},\mathfrak{Primes})$. Moreover, it follows immediately from the discussion of Example 2.4 that we have a natural Π^{disc}_E-orbit (i.e., relative to composition with automorphisms induced by conjugation by elements of Π^{disc}_E) of isomorphisms

$$(\Pi^{\mathrm{disc}}_E \supseteq)\, \Pi^{\mathrm{disc}}_{E/(E\setminus\{i\})} \xrightarrow{\sim} \Pi_{\mathcal{G}^{\mathrm{disc}}_{i\in E,x}}.$$

One verifies immediately from the various definitions involved that the diagram

$$\begin{array}{ccc} \widehat{\Pi}^{\mathrm{disc}}_{E/(E\setminus\{i\})} & \xrightarrow{\sim} & \widehat{\Pi}_{\mathcal{G}^{\mathrm{disc}}_{i\in E,x}} \\ \wr\downarrow & & \downarrow\wr \\ \Pi_{E/(E\setminus\{i\})} & \xrightarrow{\sim} & \Pi_{\mathcal{G}_{i\in E,x}} \end{array}$$

—where the upper horizontal arrow is an element of the $\widehat{\Pi}^{\mathrm{disc}}_E$-orbit of isomorphisms induced by the Π^{disc}_E-orbit of isomorphisms of the above discussion; the lower horizontal arrow is an element of the Π_E-orbit of isomorphisms of [10, Def. 3.1(iii)]; the left-hand vertical arrow is the isomorphism obtained by forming the restriction of an isomorphism $\widehat{\Pi}^{\mathrm{disc}}_E \xrightarrow{\sim} \Pi_E$ that belongs to the outer isomorphism of the final display of Definition 2.22; the right-hand vertical arrow is an isomorphism that belongs to the outer isomorphism of the above discussion—commutes up to composition with automorphisms induced by conjugation by elements of Π_E.

(ii) We shall say that a vertex $v \in \mathrm{Vert}(\mathcal{G}^{\mathrm{disc}}_{i\in E,x})$ is a(n) $(E$-$)$*tripod* of $\mathfrak{X}_n$ if v is of type $(0,3)$ (cf. Definition 2.6(iii)). Thus, one verifies easily that $v \in \mathrm{Vert}(\mathcal{G}^{\mathrm{disc}}_{i\in E,x})$ is a(n) $(E$-$)$tripod if and only if the corresponding vertex of $\mathcal{G}_{i\in E,x}$ via the graphic outer isomorphism $\widehat{\Pi}_{\mathcal{G}^{\mathrm{disc}}_{i\in E,x}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{i\in E,x}}$ of (i) is a(n) $(E$-$)$tripod of $X^{\log}_n$ (cf. [10, Def. 3.1(v)]). We shall refer to a verticial subgroup of $\Pi_{\mathcal{G}^{\mathrm{disc}}_{i\in E,x}}$ associated with a(n) $(E$-$)$tripod of $\mathfrak{X}_n$ as a(n) $(E$-$)$*tripod* of Π^{disc}_n.

(iii) Let $\mathbb{P}$ be a property of $(E$-$)$tripods of Π_n (cf. [10, Def. 3.3(i)]) or $X^{\log}_n$ (e.g., the property of being strict—cf. [10, Def. 3.3(iii)]; the property of arising from an edge—cf. [10, Def. 3.7(i)]; the property of being central—cf. [10, Def. 3.7(ii)]). Then we shall say that a(n) $(E$-$)$tripod of Π^{disc}_n or $\mathfrak{X}_n$ (cf. (ii)) satisfies $\mathbb{P}$ if the corresponding $(E$-$)$tripod of Π_n or $X^{\log}_n$ satisfies $\mathbb{P}$.

(iv) Let $T \subseteq \Pi^{\mathrm{disc}}_E$ be an E-tripod of Π^{disc}_n (cf. (ii)). Then one may define the subgroups

$$\mathrm{Out}^{\mathrm{C}}(T),\ \mathrm{Out}^{\mathrm{C}}(T)^{\mathrm{cusp}},\ \mathrm{Out}^{\mathrm{C}}(T)^{\Delta},\ \mathrm{Out}^{\mathrm{C}}(T)^{\Delta+} \subseteq \mathrm{Out}(T)$$

of $\mathrm{Out}(T)$ in an entirely analogous fashion to the definition of the closed subgroups "$\mathrm{Out}^{\mathrm{C}}(T)$," "$\mathrm{Out}^{\mathrm{C}}(T)^{\mathrm{cusp}}$," "$\mathrm{Out}^{\mathrm{C}}(T)^{\Delta}$," "$\mathrm{Out}^{\mathrm{C}}(T)^{\Delta+}$" of "$\mathrm{Out}(T)$" given in [10, Def. 3.4(i)]. We leave the routine details to the reader.

THEOREM 2.24 (Outomorphisms preserving tripods). *In the notation of Definition 2.22, let $E \subseteq \{1,\dots,n\}$ be a subset and $T \subseteq \Pi^{\mathrm{disc}}_E$ an E-tripod of Π^{disc}_n (cf. Definition 2.23(ii)). Let us write*

$$\mathrm{Out}^{\mathrm{F}}(\Pi^{\mathrm{disc}}_n)[T] \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi^{\mathrm{disc}}_n)$$

for the subgroup of $\mathrm{Out}^{\mathrm{F}}(\Pi^{\mathrm{disc}}_n)$ (cf. the notational conventions introduced in the statement of Corollary 2.20) consisting of $\alpha \in \mathrm{Out}^{\mathrm{F}}(\Pi^{\mathrm{disc}}_n)$ such that the outomorphism of Π^{disc}_E

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determined by α *preserves the* Π_E^{disc}*-conjugacy class of* $T \subseteq \Pi_E^{\mathrm{disc}}$*;*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{disc}})[T] \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{F}}(\Pi_n^{\mathrm{disc}})[T] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{disc}}) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{disc}})$$

(cf. the notational conventions introduced in the statement of Corollary 2.20); $\Pi \stackrel{\mathrm{def}}{=} \Pi_1$*;* $\Pi^{\mathrm{disc}} \stackrel{\mathrm{def}}{=} \Pi_1^{\mathrm{disc}}$*;* $\mathrm{Out}^{\mathrm{C}}(\Pi^{\mathrm{disc}}) \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{FC}}(\Pi^{\mathrm{disc}})$*;* $\mathrm{Out}^{\mathrm{C}}(\Pi) \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{FC}}(\Pi)$*. Then the following hold:*

(i) *Write* $\widehat{T}$ *for the profinite completion of* T*. Then the natural homomorphism*

$$\mathrm{Out}(T) \longrightarrow \mathrm{Out}(\widehat{T})$$

is injective. If, moreover, one regards subgroups of $\mathrm{Out}(T)$ *as subgroups of* $\mathrm{Out}(\widehat{T})$ *via this injection, then it holds that*

$$\mathrm{Out}^{\mathrm{C}}(T) = \mathrm{Out}^{\mathrm{C}}(\widehat{T}) \cap \mathrm{Out}(T),$$

$$\mathrm{Out}^{\mathrm{C}}(T)^{\mathrm{cusp}} = \mathrm{Out}^{\mathrm{C}}(\widehat{T})^{\mathrm{cusp}} \cap \mathrm{Out}(T),$$

$$\mathrm{Out}^{\mathrm{C}}(T)^{\Delta} = \mathrm{Out}^{\mathrm{C}}(\widehat{T})^{\Delta} \cap \mathrm{Out}(T),$$

$$\mathrm{Out}^{\mathrm{C}}(T)^{\Delta+} = \mathrm{Out}^{\mathrm{C}}(\widehat{T})^{\Delta+} \cap \mathrm{Out}(T)$$

(cf. Definition 2.23(iv); [10, Def. 3.4(i)]).

(ii) *It holds that*

$$\mathrm{Out}^{\mathrm{C}}(T)^{\mathrm{cusp}} = \mathrm{Out}^{\mathrm{C}}(T)^{\Delta} = \mathrm{Out}^{\mathrm{C}}(T)^{\Delta+} \cong \mathbb{Z}/2\mathbb{Z},$$

$$\mathrm{Out}^{\mathrm{C}}(T) \cong \mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_3,$$

where we write $\mathfrak{S}_3$ *for the symmetric group on* 3 *letters.*

(iii) *The commensurator and centralizer of* $T \in \Pi_E^{\mathrm{disc}}$ *satisfy the equality*

$$C_{\Pi_E^{\mathrm{disc}}}(T) = T \times Z_{\Pi_E^{\mathrm{disc}}}(T).$$

Thus, by applying the evident discrete analogue of [10, Lem. 3.10] to outomorphisms of Π_E^{disc} *determined by elements of* $\mathrm{Out}^{\mathrm{F}}(\Pi_n^{\mathrm{disc}})[T]$*, one obtains a natural homomorphism*

$$\mathfrak{T}_T \colon \mathrm{Out}^{\mathrm{F}}(\Pi_n^{\mathrm{disc}})[T] \longrightarrow \mathrm{Out}(T).$$

(iv) *Suppose that* $n \geq 3$*, and that* T *is central (cf. Definition 2.23(iii)). Then it holds that*

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n^{\mathrm{disc}}) = \mathrm{Out}^{\mathrm{F}}(\Pi_n^{\mathrm{disc}})[T].$$

Moreover, the homomorphism

$$\mathfrak{T}_T \colon \mathrm{Out}^{\mathrm{F}}(\Pi_n^{\mathrm{disc}}) = \mathrm{Out}^{\mathrm{F}}(\Pi_n^{\mathrm{disc}})[T] \longrightarrow \mathrm{Out}(T)$$

of (iii) determines a surjection

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{disc}}) \twoheadrightarrow \mathrm{Out}^{\mathrm{C}}(T)^{\Delta+} \ (\cong \mathbb{Z}/2\mathbb{Z}).$$

We shall refer to this homomorphism as the tripod homomorphism associated with Π_n^{disc}*.*

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(v) *The profinite completion* $\widehat{T}$ *determines an E-tripod of* Π_n*, which, by abuse of notation, we denote by* $\widehat{T}$*. Now suppose that* T *is E-strict (cf. Definition 2.23(iii)). Then it holds that*

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n^{\mathrm{disc}})[T] \;=\; \mathrm{Out}^{\mathrm{F}}(\Pi_n)[\widehat{T}] \cap \mathrm{Out}^{\mathrm{F}}(\Pi_n^{\mathrm{disc}})$$

(cf. [10, Th. 3.16]).

(vi) *Suppose that the semi-graph of anabelioids of pro-$\mathfrak{Primes}$ PSC-type* $\mathcal{G}$ *associated with* $X^{\log}$ *(cf. [10, Def. 3.1(ii)]) is totally degenerate (cf. [9, Def. 2.3(iv)]). Recall that* $\mathcal{G}$ *may be naturally identified with the semi-graph of anabelioids of pro-$\mathfrak{Primes}$ PSC-type determined by* $\mathcal{G}^{\mathrm{disc}}$ *(cf. Proposition 2.5(iii); the discussion of Definition 2.23(i)). Then one has an equality*

$$\mathrm{Aut}(\mathcal{G}^{\mathrm{disc}})^{-} \;=\; \mathrm{Aut}(\mathcal{G}) \cap \mathrm{Out}^{\mathrm{C}}(\Pi^{\mathrm{disc}})^{-} \;\; (\subseteq \mathrm{Out}^{\mathrm{C}}(\Pi))$$

—where the superscript "$^-$'s" denote the closure in the profinite topology—of subgroups of $\mathrm{Out}^{\mathrm{C}}(\Pi)$ *(cf. Corollary 2.20(i)).*

Proof. First, we verify assertion (i). The injectivity portion of assertion (i) follows from Corollary 2.20(i). The first equality follows from Corollary 2.20(ii). Thus, the second and third equalities follow immediately from the various definitions involved; the fourth equality follows from Corollary 2.20(v). This completes the proof of assertion (i).

Next, we verify assertion (ii). The inclusions $\mathrm{Out}^{\mathrm{C}}(T)^{\Delta+} \subseteq \mathrm{Out}^{\mathrm{C}}(T)^{\Delta} \subseteq \mathrm{Out}^{\mathrm{C}}(T)^{\mathrm{cusp}}$ follow from assertion (i), together with [10, Lem. 3.5]. The inclusion $\mathrm{Out}^{\mathrm{C}}(T)^{\mathrm{cusp}} \subseteq \mathrm{Out}^{\mathrm{C}}(T)^{\Delta+}$ and the assertion that $\mathrm{Out}^{\mathrm{C}}(T)^{\mathrm{cusp}} \cong \mathbb{Z}/2\mathbb{Z}$ follow immediately from [20, Cor. 5.3(i)], together with a classical result of Nielsen (cf. [20, Rem. 5.3.1]). This completes the proof of the first line of the display of assertion (ii). Now since $\mathrm{Out}^{\mathrm{C}}(T)^{\Delta} = \mathrm{Out}^{\mathrm{C}}(T)^{\mathrm{cusp}}$, by considering the action of $\mathrm{Out}^{\mathrm{C}}(T)$ on the set of the T-conjugacy classes of cuspidal inertia subgroups of T, we obtain an exact sequence

$$1 \longrightarrow \mathrm{Out}^{\mathrm{C}}(T)^{\Delta} \longrightarrow \mathrm{Out}^{\mathrm{C}}(T) \longrightarrow \mathfrak{S}_3 \longrightarrow 1.$$

By considering outomorphisms of T arising from automorphisms of analytic spaces, one obtains a section of this sequence; moreover, it follows from the definition of $\mathrm{Out}^{\mathrm{C}}(T)^{\Delta}$ that this section determines an isomorphism $\mathrm{Out}^{\mathrm{C}}(T)^{\Delta} \times \mathfrak{S}_3 \xrightarrow{\sim} \mathrm{Out}^{\mathrm{C}}(T)$. This completes the proof of assertion (ii).

Next, we verify assertion (iii). Recall that every finite index subgroup of T is normally terminal in its profinite completion (cf. Corollary 2.20(i)) and center-free (cf. Remark 2.6.1). Thus, assertion (iii) follows immediately from [10, Th. 3.16(i)]. This completes the proof of assertion (iii).

Next, we verify assertion (iv). First, let us observe that it follows immediately from the definition of the notion of a central tripod (cf. Definition 2.23(iii); [10, Def. 3.7(ii)]) that we may assume without loss of generality that $n = 3$. To verify the equality of the first display of assertion (iv), we mimick the argument in the profinite case given in the proof of [20, Cor. 1.10(i)]: let $\alpha \in \mathrm{Out}^{\mathrm{F}}(\Pi_n^{\mathrm{disc}})$, $\widetilde{\alpha} \in \mathrm{Aut}(\Pi_n^{\mathrm{disc}})$ a lifting of α. Write $\widetilde{\alpha}_2 \in \mathrm{Aut}(\Pi_2^{\mathrm{disc}})$ for the automorphism induced by $\widetilde{\alpha}$. Now observe that since $\alpha \in \mathrm{Out}^{\mathrm{F}}(\Pi_n^{\mathrm{disc}})$, it follows immediately from Corollary 2.20(iv) that $\widetilde{\alpha}_2$ determines an element of $\mathrm{Out}^{\mathrm{FC}}(\Pi_2^{\mathrm{disc}})$, hence that $\widetilde{\alpha}_2$ preserves the Π_2^{disc}-conjugacy class of inertia groups associated with the diagonal cusp of any of the fibers of $p_{2/1}^{\mathrm{an}}$ (cf. Definition 2.22; the discussion of [20, Rem. 1.1.5]). Thus,

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by replacing $\widetilde{\alpha}$ by the composite of $\widetilde{\alpha}$ with a suitable inner automorphism, we may assume without loss of generality that $\widetilde{\alpha}_2$ preserves the inertia group associated with some diagonal cusp of a fiber of $p_{2/1}^{\mathrm{an}}$. Now the fact that $\alpha \in \mathrm{Out}^{\mathrm{F}}(\Pi_n^{\mathrm{disc}})[T]$ follows immediately from Corollary 2.17(ii); [10, Th. 1.9(ii)] (cf. the application of [20, Prop. 1.3(iv)] in the proof of [20, Cor. 1.10(i)]). The assertion that the restriction to $\mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{disc}})$ of the homomorphism $\mathrm{Out}^{\mathrm{F}}(\Pi_n^{\mathrm{disc}}) \to \mathrm{Out}(T)$ of assertion (iii) factors through $\mathrm{Out}^{\mathrm{C}}(T)^{\Delta+} \subseteq \mathrm{Out}(T)$ follows immediately from from assertions (i) and (ii), together with [10, Th. 3.16(v)]. The assertion that the resulting homomorphism is surjective follows immediately from the fact that the (unique) nontrivial element of $\mathrm{Out}^{\mathrm{C}}(T)^{\Delta+}$ is the outomorphism induced by complex conjugation (cf. [20, Rem. 5.3.1]), together with the (easily verified) fact that the pointed stable curve over $\mathbb{C}$ corresponding to the given stable log curve $X^{\log}$ may be assumed, without loss of generality—that is, by applying a suitable specialization isomorphism (cf. the discussion preceding [20, Def. 2.1] as well as [9, Rem. 5.6.1]) and observing that such specialization isomorphisms are compatible with the various discrete fundamental groups involved (cf. Remarks 2.9.1 and 2.10.1)—to be defined over $\mathbb{R}$. This completes the proof of assertion (iv).

Next, we verify assertion (v). It follows immediately from the classification of E-strict tripods given in [10, Lem. 3.8(ii)] that we may assume without loss of generality that $E^{\sharp} = n \leq 3$. When $n = 3$, assertion (v) follows formally from assertion (iv). When $n = 1$, assertion (v) follows immediately from Corollary 2.17(ii). Thus, it remains to consider the case where $n = 2$, that is, where the tripod T arises from an edge. In this case, assertion (v) follows from a similar argument to the argument applied in the proof of assertion (iv). That is to say, let $\alpha \in \mathrm{Out}^{\mathrm{F}}(\Pi_2^{\mathrm{disc}})$, $\widetilde{\alpha} \in \mathrm{Aut}(\Pi_2^{\mathrm{disc}})$ a lifting of α. Write $\widetilde{\alpha}_1 \in \mathrm{Aut}(\Pi_1^{\mathrm{disc}})$ for the automorphism induced by $\widetilde{\alpha}$; $\widetilde{\beta}_1 \in \mathrm{Aut}(\Pi_1)$, $\widetilde{\beta} \in \mathrm{Aut}(\Pi_2)$ for the automorphisms determined by $\widetilde{\alpha}$. Then we must verify that $\alpha \in \mathrm{Out}^{\mathrm{F}}(\Pi_2^{\mathrm{disc}})[T]$ under the assumption that $\widetilde{\beta}$ determines an element $\beta \in \mathrm{Out}^{\mathrm{F}}(\Pi_2)[\widehat{T}]$. Now observe that it follows immediately from the computation of the centralizer given in [10, Lem. 3.11(vii)] that $\widetilde{\beta}_1$ preserves the Π_1-conjugacy class of edge-like subgroups of Π_1 determined by the edge that gives rise to the tripod T. Thus, we conclude from Corollary 2.17(ii) that, by replacing $\widetilde{\alpha}$ by the composite of $\widetilde{\alpha}$ with a suitable inner automorphism, we may assume that $\widetilde{\alpha}_1$ preserves a specific edge-like subgroup of Π_1^{disc} corresponding to the edge that gives rise to the tripod T. Note that this assumption implies, in light of the commensurably terminality of edge-like subgroups (cf. [18, Prop. 1.2(ii)]), that $\widetilde{\beta}$ preserves the $\Pi_{2/1}$-conjugacy class of the tripod $\widehat{T}$. In particular, we conclude, as in the proof of assertion (iv), that is, by applying Corollary 2.17(ii), that $\alpha \in \mathrm{Out}^{\mathrm{F}}(\Pi_2^{\mathrm{disc}})[T]$, as desired. This completes the proof of assertion (v).

Finally, we verify assertion (vi). First, let us observe that it follows immediately from Corollary 2.20(v) that both sides of the equality in question are $\subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_3^{\mathrm{disc}})^{-} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_3)$ ($\subseteq \mathrm{Out}^{\mathrm{C}}(\Pi)$). Also, we observe that, by considering the case where $X^{\log}$ is defined over $\mathbb{R}$ (cf. the proof of assertion (iv)), it follows immediately that both sides of the equality in question surject, via the tripod homomorphism of assertion (iv), onto the finite group of order two that appears as the image of this tripod homomorphism (cf. also the fact that the topological group $\mathrm{Out}(\widehat{T})$ is profinite, hence, in particular, Hausdorff). In particular, to complete the proof of assertion (v), it suffices to verify that the evident inclusion

$$\mathrm{Aut}(\mathcal{G}^{\mathrm{disc}})^{-} \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_3)^{\mathrm{geo}} \subseteq \mathrm{Aut}(\mathcal{G}) \cap \mathrm{Out}^{\mathrm{C}}(\Pi^{\mathrm{disc}})^{-} \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_3)^{\mathrm{geo}}$$

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—where we write $\mathrm{Out}^{\mathrm{FC}}(\Pi_3)^{\mathrm{geo}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_3)$ for the kernel of the tripod homomorphism on $\mathrm{Out}^{\mathrm{FC}}(\Pi_3)$ (cf. [10, Def. 3.19])—of subgroups of $\mathrm{Out}^{\mathrm{C}}(\Pi)$ is, in fact, an equality. On the other hand, since $\mathrm{Dehn}(\mathcal{G})$ is a normal open subgroup of both $\mathrm{Aut}(\mathcal{G}^{\mathrm{disc}})^{-} \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_3)^{\mathrm{geo}}$ and $\mathrm{Aut}(\mathcal{G}) \cap \mathrm{Out}^{\mathrm{C}}(\Pi^{\mathrm{disc}})^{-} \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_3)^{\mathrm{geo}}$ (cf. Corollary 2.21(iii); [9, Th. 4.8(i)]; the commutative diagram of [10, Cor. 3.27(ii)]), and $\mathrm{Aut}(\mathcal{G}^{\mathrm{disc}})^{-} \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_3)^{\mathrm{geo}}$ clearly surjects onto the finite group of automorphisms of the underlying semi-graph of $\mathcal{G}^{\mathrm{disc}}$, the desired equality follows immediately from [10, Cor. 3.27(ii)]. This completes the proof of assertion (vi). ∎

REMARK 2.24.1. It is not clear to the authors at the time of writing whether or not one can remove the strictness assumption imposed in Theorem 2.24(v). Indeed, from the point of view of induction on n, the essential difficulty in removing this assumption may already be seen in the case of a non-E-strict tripod when $E^{\sharp} = n = 2$. From another point of view, this difficulty may be thought of as arising from the lack of an analogue for discrete topological fundamental groups of nth configuration spaces, when $n \geq 2$, of Corollary 2.17.

REMARK 2.24.2.

(i) In the notation of Theorem 2.24, let us observe that it follows from Corollary 2.19(i) that we have an equality

$$\mathrm{Aut}(\mathcal{G}^{\mathrm{disc}}) = \mathrm{Aut}(\mathcal{G}) \cap \mathrm{Out}^{\mathrm{C}}(\Pi^{\mathrm{disc}}) \ \ (\subseteq \mathrm{Out}^{\mathrm{C}}(\Pi))$$

of subgroups of $\mathrm{Out}^{\mathrm{C}}(\Pi)$ (cf. Corollary 2.20(i)). On the other hand, it is by no means clear whether or not the evident inclusion

$$\mathrm{Aut}(\mathcal{G}^{\mathrm{disc}})^{-} \subseteq \mathrm{Aut}(\mathcal{G}) \cap \mathrm{Out}^{\mathrm{C}}(\Pi^{\mathrm{disc}})^{-} \ \ (\subseteq \mathrm{Out}^{\mathrm{C}}(\Pi)) \qquad (*)$$

—where the superscript "$^{-}$'s" denote the closure in the profinite topology—is an equality in general. On the other hand, when $X^{\log}$ is *totally degenerate*, this *equality* is the content of Theorem 2.24(vi).

(ii) We continue to use the notation of (i). Write $\mathcal{M}_{\mathbb{Q}}$ for the moduli stack of hyperbolic curves of type (g,r) over $\mathbb{Q}$ and $\mathcal{C}_{\mathbb{Q}} \to \mathcal{M}_{\mathbb{Q}}$ for the tautological hyperbolic curve over $\mathcal{M}_{\mathbb{Q}}$. Thus, for appropriate choices of basepoints, if we write $\Pi_{\mathcal{C}} \stackrel{\mathrm{def}}{=} \pi_1(\mathcal{C}_{\mathbb{Q}})$, $\Pi_{\mathcal{M}} \stackrel{\mathrm{def}}{=} \pi_1(\mathcal{M}_{\mathbb{Q}})$ for the respective étale fundamental groups, then we obtain an exact sequence of profinite groups

$$1 \longrightarrow \Delta_{\mathcal{C}/\mathcal{M}} \longrightarrow \Pi_{\mathcal{C}} \longrightarrow \Pi_{\mathcal{M}} \longrightarrow 1$$

—where $\Delta_{\mathcal{C}/\mathcal{M}}$ is defined so as to render the sequence exact—as well as a natural outer representation

$$\rho_{\mathcal{M}} \colon \Pi_{\mathcal{M}} \longrightarrow \mathrm{Out}^{\mathrm{C}}(\Pi)$$

—where, by choosing appropriate basepoints, we identify Π with $\Delta_{\mathcal{C}/\mathcal{M}}$—and a natural outer surjection

$$\Pi_{\mathcal{M}} \twoheadrightarrow G_{\mathbb{Q}}$$

onto the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$ (cf. the discussion of [10, Rem. 3.19.1]). Write $G_{\mathbb{R}} \subseteq G_{\mathbb{Q}}$ for the decomposition group (which is well-defined up to $G_{\mathbb{Q}}$-conjugation) of

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the unique archimedean prime of $\mathbb{Q}$. In the spirit of [4]–[6], let us write

$$\Gamma \stackrel{\text{def}}{=} \mathrm{Out}^{\mathrm{C}}(\Pi^{\mathrm{disc}}) \ (\subseteq \mathrm{Out}^{\mathrm{C}}(\Pi)); \ \check{\Gamma} \stackrel{\text{def}}{=} \rho_{\mathcal{M}}(\Pi_{\mathcal{M}} \times_{G_{\mathbb{Q}}} G_{\mathbb{R}})$$

(cf. Corollary 2.20(i)). Thus, for appropriate choices of basepoints, $\check{\Gamma}$ is equal to the closure of Γ in $\mathrm{Out}^{\mathrm{C}}(\Pi)$. If σ is a simplex of the complex of profinite curves $L(\Pi)$ studied in [4]–[6] that arises from Π^{disc}, then the stabilizer in Γ of σ is denoted Γ_σ, while the stabilizer in $\check{\Gamma}$ of the image of σ in the profinite curve complex corresponding to $\check{\Gamma}$ is denoted $\check{\Gamma}_\sigma$. Then [6, Th. 4.2] (cf. also [4, Prop. 6.5]) asserts that

> The natural inclusion $\Gamma_\sigma^- \subseteq \check{\Gamma}_\sigma$ is, in fact, an equality.

Translated into the language of the present paper, this assertion corresponds precisely to the assertion that the inclusion $(*)$ considered in (i) is, in fact, an equality. In particular, Theorem 2.24(vi) corresponds, essentially, to a special case (i.e., the totally degenerate case) of [6, Th. 4.2]. At a more concrete level, when $\mathrm{Node}(\mathcal{G})^\sharp = 1$, and σ arises from a single simple closed curve that corresponds to the unique node e of $\mathcal{G}$, this assertion corresponds precisely to the assertion that

> the *profinite stabilizer* in $\check{\Gamma}$ of the Π-conjugacy class of nodal subgroups of Π determined by e *coincides* with the closure in $\check{\Gamma}$ of the *discrete stabilizer* in Γ of the Π^{disc}-conjugacy class of nodal subgroups of Π^{disc} determined by e

—cf. Theorem 3.3, Remark 3.3.1, and Corollary 3.4 in §3 below. As discussed in (i), this sort of assertion is *highly nontrivial*. That is to say, this sort of coincidence between a profinite stabilizer and the closure of a corresponding discrete stabilizer is, in fact, *false* in general, as the example given in (iv) below demonstrates. In particular, this sort of coincidence is by no means a consequence of superficial "general nonsense"-type considerations, but rather, when true (cf., e.g., the case treated in Theorem 2.24(vi)), a consequence of deep properties of the specific groups and specific spaces (on which these groups act) under consideration.

(iii) In closing, we observe that many of the results derived in [6] as a consequence of the assertion discussed in (ii) were, in fact, already obtained in earlier papers by the authors. Indeed, the faithfulness asserted in [6, Th. 7.7]—that is, the injectivity of the restriction of $\rho_{\mathcal{M}}$ to a section $G_F \hookrightarrow \Pi_{\mathcal{M}}$ arising from a hyperbolic curve of type (g,r) defined over a number field F—is a special case of [8, Th. C]. On the other hand, in [9, Th. D], a computation is given of the centralizer in $\mathrm{Out}^{\mathrm{C}}(\Pi)$ of an open subgroup of $\check{\Gamma}$. Thus, the computation of centers given in [6, Cor. 6.2] amounts to a special case of [9, Th. D]. Finally, [6, Cor. 7.6]—which may be regarded as the assertion that the inverse image via $\rho_{\mathcal{M}}$ of the centralizer of $\check{\Gamma}$ in $\mathrm{Out}^{\mathrm{C}}(\Pi)$ maps trivially to $G_{\mathbb{Q}}$—amounts to a concatenation of

- the computation of the centralizer given in [9, Th. D] with
- the fact, stated in [8, Cor. 6.4] that $\rho_{\mathcal{M}}^{-1}(\check{\Gamma})$ maps trivially to $G_{\mathbb{Q}}$.

(iv) Let $n \geq 3$ be an integer. Consider the natural conjugation action of the special linear group $\mathrm{SL}_n(\mathbb{Z})$ with coefficients $\in \mathbb{Z}$ on the module $\mathrm{M}_n(\mathbb{Z})$ of n by n matrices with coefficients $\in \mathbb{Z}$. Write $A \in \mathrm{M}_n(\mathbb{Z})$ for the diagonal matrix whose entries are given by the integers $1, \ldots, n$. Then one verifies immediately that the stabilizer

$$\mathrm{SL}_n(\mathbb{Z})_A$$

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of A, relative to the conjugacy action of $\mathrm{SL}_n(\mathbb{Z})$, is equal to the subgroup of diagonal matrices of $\mathrm{SL}_n(\mathbb{Z})$, hence isomorphic to the finite group given by a product of $n-1$ copies of the finite group of order two $\{\pm 1\}$. On the other hand, if one considers the action of the special linear group $\mathrm{SL}_n(\widehat{\mathbb{Z}})$ with coefficients $\in \widehat{\mathbb{Z}}$ on the module $\mathrm{M}_n(\widehat{\mathbb{Z}})$ of n by n matrices with coefficients $\in \widehat{\mathbb{Z}}$, then one verifies immediately that the stabilizer

$$\mathrm{SL}_n(\widehat{\mathbb{Z}})_A$$

of A, relative to the conjugacy action of $\mathrm{SL}_n(\widehat{\mathbb{Z}})$, is equal to the subgroup of diagonal matrices of $\mathrm{SL}_n(\widehat{\mathbb{Z}})$, hence isomorphic to a product of $n-1$ copies of $\widehat{\mathbb{Z}}^\times$, a group of uncountable cardinality. That is to say,

> The *profinite stabilizer* $\mathrm{SL}_n(\widehat{\mathbb{Z}})_A$ is *much larger* than the profinite completion of the *discrete stabilizer* $\mathrm{SL}_n(\mathbb{Z})_A$.

Here, we recall that since, as is well-known, the congruence subgroup problem has been resolved in the affirmative, in the case of $n \geq 3$, the topological group $\mathrm{SL}_n(\widehat{\mathbb{Z}})$ may be identified with the profinite completion of the group $\mathrm{SL}_n(\mathbb{Z})$. A similar example may be given in the case of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{Z})$.

Corollary 2.25 (Characterization of the archimedean local Galois groups in the global Galois image associated with a hyperbolic curve). *Let F be a number field (i.e., a finite extension of the field of rational numbers); $\mathfrak{p}$ an archimedean prime of F; $\overline{F}_\mathfrak{p}$ an algebraic closure of the $\mathfrak{p}$-adic completion $F_\mathfrak{p}$ of F (so $\overline{F}_\mathfrak{p}$ is isomorphic to $\mathbb{C}$); $\overline{F} \subseteq \overline{F}_\mathfrak{p}$ the algebraic closure of F in $\overline{F}_\mathfrak{p}$; $X_F^{\log}$ a smooth log curve over F. Write $G_\mathfrak{p} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p}) \subseteq G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}/F)$; $X_{\overline{F}}^{\log} \stackrel{\mathrm{def}}{=} X_F^{\log} \times_F \overline{F}$; $X_{F_\mathfrak{p}}^{\log} \stackrel{\mathrm{def}}{=} X_F^{\log} \times_F F_\mathfrak{p}$; $X_{\overline{F}_\mathfrak{p}}^{\log} \stackrel{\mathrm{def}}{=} X_F^{\log} \times_F \overline{F}_\mathfrak{p}$;*

$$\pi_1(X_{\overline{F}}^{\log})$$

for the log fundamental group of $X_{\overline{F}}^{\log}$;

$$\pi_1^{\mathrm{disc}}(X_{\overline{F}_\mathfrak{p}}^{\log})$$

for the (discrete) topological fundamental group of the analytic space associated with the interior of the log scheme $X_{\overline{F}_\mathfrak{p}}^{\log}$;

$$\pi_1^{\mathrm{disc}}(X_{\overline{F}_\mathfrak{p}}^{\log})^\wedge$$

for the profinite completion of $\pi_1^{\mathrm{disc}}(X_{\overline{F}_\mathfrak{p}}^{\log})$;

$$\rho_{X_F^{\log}} \colon G_F \longrightarrow \mathrm{Out}(\pi_1(X_{\overline{F}}^{\log}))$$

for the natural outer Galois action associated with $X_F^{\log}$;

$$\rho_{X_F^{\log},\mathfrak{p}}^{\mathrm{disc}} \colon G_\mathfrak{p} \longrightarrow \mathrm{Out}(\pi_1^{\mathrm{disc}}(X_{\overline{F}_\mathfrak{p}}^{\log}))$$

for the natural outer Galois action associated with $X_{F_\mathfrak{p}}^{\log}$. Thus, we have a natural outer isomorphism

$$\pi_1^{\mathrm{disc}}(X_{\overline{F}_\mathfrak{p}}^{\log})^\wedge \xrightarrow{\sim} \pi_1(X_{\overline{F}}^{\log}),$$

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which determines a natural injection

$$\mathrm{Out}(\pi_1^{\mathrm{disc}}(X^{\log}_{\overline{F}_{\mathfrak{p}}})) \hookrightarrow \mathrm{Out}(\pi_1(X^{\log}_{\overline{F}}))$$

(cf. Corollary 2.20(i)). Then the following hold:

(i) *We have a natural commutative diagram*

$$\begin{array}{ccc} G_{\mathfrak{p}} & \xrightarrow{\rho^{\mathrm{disc}}_{X_F^{\log},\mathfrak{p}}} & \mathrm{Out}(\pi_1^{\mathrm{disc}}(X^{\log}_{\overline{F}_{\mathfrak{p}}})) \\ \downarrow & & \downarrow \\ G_F & \xrightarrow{\rho_{X_F^{\log}}} & \mathrm{Out}(\pi_1(X^{\log}_{\overline{F}})) \end{array}$$

—where the vertical arrows are the natural inclusions, and all arrows are injective.

(ii) *The diagram of (i) is Cartesian, that is, if we regard the various groups involved as subgroups of* $\mathrm{Out}(\pi_1(X^{\log}_{\overline{F}}))$*, then we have an equality*

$$G_{\mathfrak{p}} = G_F \cap \mathrm{Out}(\pi_1^{\mathrm{disc}}(X^{\log}_{\overline{F}_{\mathfrak{p}}})).$$

Proof. Assertion (i) follows immediately from the injectivity of the lower horizontal arrow $\rho_{X_F^{\log}}$ (cf. [8, Th. C]), together with the various definitions involved.

Finally, we verify assertion (ii). Write $(X_{\overline{F}})_3^{\log}$ for the 3-rd log configuration space of $X^{\log}_{\overline{F}}$. Then it follows immediately from [8, Th. B] that the group $\mathrm{Out}^{\mathrm{FC}}(\pi_1((X_{\overline{F}})_3^{\log}))$ of FC-admissible outomorphisms of the log fundamental group $\pi_1((X_{\overline{F}})_3^{\log})$ of $(X_{\overline{F}})_3^{\log}$ may be regarded as a closed subgroup of $\mathrm{Out}(\pi_1(X^{\log}_{\overline{F}}))$. Moreover, it follows immediately from the various definitions involved that the respective images $\mathrm{Im}(\rho_{X_F^{\log}})$, $\mathrm{Im}(\rho^{\mathrm{disc}}_{X_F^{\log},\mathfrak{p}})$ of the natural outer Galois actions $\rho_{X_F^{\log}}$, $\rho^{\mathrm{disc}}_{X_F^{\log},\mathfrak{p}}$ associated with $X_F^{\log}$, $X_{F_{\mathfrak{p}}}^{\log}$ are contained in this closed subgroup $\mathrm{Out}^{\mathrm{FC}}(\pi_1((X_{\overline{F}})_3^{\log})) \subseteq \mathrm{Out}(\pi_1(X^{\log}_{\overline{F}}))$. Thus, to verify assertion (ii), one verifies immediately from Corollary 2.20(v) that it suffices to verify the equality

$$\mathrm{Im}(\rho^{\mathrm{disc}}_{X_F^{\log},\mathfrak{p}}) \;=\; \mathrm{Im}(\rho_{X_F^{\log}}) \cap \mathrm{Out}(\pi_1^{\mathrm{disc}}((X_{\overline{F}_{\mathfrak{p}}})_3^{\log}))$$

—where we write $(X_{\overline{F}_{\mathfrak{p}}})_3^{\log} \overset{\mathrm{def}}{=} (X_{\overline{F}})_3^{\log} \times_{\overline{F}} \overline{F}_{\mathfrak{p}}$ and $\pi_1^{\mathrm{disc}}((X_{\overline{F}_{\mathfrak{p}}})_3^{\log})$ for the (discrete) topological fundamental group of the analytic space associated with the interior of the log scheme $(X_{\overline{F}_{\mathfrak{p}}})_3^{\log}$. On the other hand, since the "$\rho_{X_F^{\log}}$" that occurs in the case where we take "$X_F^{\log}$" to be the smooth log curve associated with $\mathbb{P}^1_F \setminus \{0,1,\infty\}$ is injective (cf. assertion (i)), this equality follows immediately—by considering the images of the subgroups

$$\mathrm{Im}(\rho^{\mathrm{disc}}_{X_F^{\log},\mathfrak{p}}) \subseteq \mathrm{Im}(\rho_{X_F^{\log}}) \cap \mathrm{Out}(\pi_1^{\mathrm{disc}}((X_{\overline{F}_{\mathfrak{p}}})_3^{\log}))$$

of $\mathrm{Out}(\pi_1^{\mathrm{disc}}((X_{\overline{F}_{\mathfrak{p}}})_3^{\log}))$ via the (manifestly compatible!) tripod homomorphisms associated with $\pi_1^{\mathrm{disc}}((X_{\overline{F}_{\mathfrak{p}}})_3^{\log})$ (cf. Theorem 2.24(iv)) and $\pi_1((X_{\overline{F}})_3^{\log})$ (cf. [10, Th. 3.16(i) and (v)])—from [1, Th. 3.3.1]. This completes the proof of assertion (ii), hence also of Corollary 2.25. ∎

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REMARK 2.25.1. Corollary 2.25 is a generalization of [1, Th. 3.3.2] (cf. also the footnote of [1] following [1, Th. 3.3.2]). Although the proof given here of Corollary 2.25 is by no means the first proof of this result (cf. the discussion of this footnote of [1] following [1, Th. 3.3.2]; [8, Cor. 6.4]), it is of interest to note that this result may also be derived in the context of the theory of the present paper, that is, via an argument that parallels the proof given in [11] of [11, Th. B] in the p-adic case (for which no alternative proofs are known!).

§3. Canonical liftings of cycles

In the present section, we discuss certain canonical liftings of cycles (cf. Theorems 3.10 and 3.14 below). These canonical liftings are constructed in a fashion illustrated in Figure 1. This approach to constructing such canonical liftings was motivated (cf. Remark 3.10.1 below) by the arguments of [5], where these canonical liftings were applied, in the context of the congruence subgroup problem for hyperelliptic modular groups, to derive certain injectivity results (cf. [5, §2]), which may be regarded as special cases of [8, Th. B]. Unfortunately, however, the authors of the present paper were unable to follow in detail these arguments of [5], which appear to be based to a substantial extent on geometric intuition concerning the geometry of topological surfaces. Although, in the development of the present series of papers on combinatorial anabelian geometry, the authors were motivated by similar geometric intuition, the proofs of the results given in the present series of papers proceed by means of purely combinatorial and algebraic arguments concerning combinatorial (e.g., graphs) and group-theoretic (e.g., profinite fundamental groups) data that arise from a pointed stable curve. From the point of view of arithmetic geometry, the geometric intuition which underlies the topological arguments given in [5] involving objects such as topological Dehn twists is of an essentially archimedean nature, hence, in particular, is fundamentally incompatible, at least from the point of view of establishing a rigorous mathematical formulation, with the highly nonarchimedean properties of profinite fundamental groups, as studied in the present series of papers—cf. the discussion of [17, Rem. 1.5.1]. It was this state of affairs that motivated the authors to give, in the present section, a formulation of the constructions of [5, §2] in terms of the purely combinatorial and algebraic techniques developed in the present series of papers.

In the present section, let (g,r) be a pair of nonnegative integers such that $2g-2+r>0$; n a positive integer; Σ a set of prime numbers which is either equal to the entire set of prime numbers or of cardinality one; k an algebraically closed field of characteristic $\notin \Sigma$; $S^{\log} \stackrel{\text{def}}{=} \mathrm{Spec}(k)^{\log}$ the log scheme obtained by equipping $S \stackrel{\text{def}}{=} \mathrm{Spec}(k)$ with the log structure determined by the fs chart $\mathbb{N} \to k$ that maps $1 \mapsto 0$; $X^{\log} = X_1^{\log}$ a stable log curve of type (g,r) over $S^{\log}$. For each (possibly empty) subset $E \subseteq \{1,\ldots,n\}$, write

$$X_E^{\log}$$

for the $E^{\sharp}$-th log configuration space of the stable log curve $X^{\log}$ (cf. the discussion entitled "Curves" in [9, §0]), where we think of the factors as being labeled by the elements of $E \subseteq \{1,\ldots,n\}$;

$$\Pi_E$$

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for the maximal pro-Σ quotient of the kernel of the natural surjection $\pi_1(X_E^{\log}) \twoheadrightarrow \pi_1(S^{\log})$;

$$p_{E/E'}^{\log} \colon X_E^{\log} \to X_{E'}^{\log},\ p_{E/E'}^{\Pi} \colon \Pi_E \twoheadrightarrow \Pi_{E'},$$

$$\Pi_{E/E'} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(p_{E/E'}^{\Pi}) \subseteq \Pi_E,\ X_n^{\log} \stackrel{\mathrm{def}}{=} X_{\{1,\dots,n\}}^{\log},\ \Pi_n \stackrel{\mathrm{def}}{=} \Pi_{\{1,\dots,n\}},$$

$$p_{n/m}^{\log} \stackrel{\mathrm{def}}{=} p_{\{1,\dots,n\}/\{1,\dots,m\}}^{\log} \colon X_n^{\log} \longrightarrow X_m^{\log},$$

$$p_{n/m}^{\Pi} \stackrel{\mathrm{def}}{=} p_{\{1,\dots,n\}/\{1,\dots,m\}}^{\Pi} \colon \Pi_n \twoheadrightarrow \Pi_m,$$

$$\Pi_{n/m} \stackrel{\mathrm{def}}{=} \Pi_{\{1,\dots,n\}/\{1,\dots,m\}} \subseteq \Pi_n,$$

$$\mathcal{G},\ \mathbb{G},\ \Pi_{\mathcal{G}},\ \mathcal{G}_{i\in E,x},\ \Pi_{\mathcal{G}_{i\in E,x}}$$

for the objects defined in the discussion at the beginning of [10, §3]; [10, Def. 3.1]. In addition, we suppose that we have been given a pair of nonnegative integers $({}^Yg, {}^Yr)$ such that $2{}^Yg - 2 + {}^Yr > 0$ and a stable log curve $Y^{\log} = Y_1^{\log}$ of type $({}^Yg, {}^Yr)$ over $S^{\log}$. We shall use similar notation

$$Y_E^{\log},\ {}^Y\Pi_E,\ {}^Yp_{E/E'}^{\log} \colon Y_E^{\log} \to Y_{E'}^{\log},\ {}^Yp_{E/E'}^{\Pi} \colon {}^Y\Pi_E \twoheadrightarrow {}^Y\Pi_{E'},$$

$${}^Y\Pi_{E/E'} \stackrel{\mathrm{def}}{=} \mathrm{Ker}({}^Yp_{E/E'}^{\Pi}) \subseteq {}^Y\Pi_E,\ Y_n^{\log} \stackrel{\mathrm{def}}{=} Y_{\{1,\dots,n\}}^{\log},\ {}^Y\Pi_n \stackrel{\mathrm{def}}{=} {}^Y\Pi_{\{1,\dots,n\}},$$

$${}^Yp_{n/m}^{\log} \stackrel{\mathrm{def}}{=} {}^Yp_{\{1,\dots,n\}/\{1,\dots,m\}}^{\log} \colon Y_n^{\log} \longrightarrow Y_m^{\log},$$

$${}^Yp_{n/m}^{\Pi} \stackrel{\mathrm{def}}{=} {}^Yp_{\{1,\dots,n\}/\{1,\dots,m\}}^{\Pi} \colon {}^Y\Pi_n \twoheadrightarrow {}^Y\Pi_m,$$

$${}^Y\Pi_{n/m} \stackrel{\mathrm{def}}{=} {}^Y\Pi_{\{1,\dots,n\}/\{1,\dots,m\}} \subseteq {}^Y\Pi_n,$$

$${}^Y\mathcal{G},\ {}^Y\mathbb{G},\ \Pi_{{}^Y\mathcal{G}},\ {}^Y\mathcal{G}_{i\in E,y},\ \Pi_{{}^Y\mathcal{G}_{i\in E,y}}$$

for objects associated with the stable log curve $Y^{\log} = Y_1^{\log}$ to the notation introduced above for $X^{\log}$ (cf. the discussion at the beginning of [10, §3]]; [10, Def. 3.1]).

LEMMA 3.1 (Graphicity in the case of a single node). *In the notation of the discussion at the beginning of the present §3, suppose that* $\mathrm{Node}(\mathcal{G})^\sharp = \mathrm{Node}({}^Y\mathcal{G})^\sharp = 1$. *Write*

$$e \in \mathrm{Node}(\mathcal{G}) \quad (\text{resp. } {}^Ye \in \mathrm{Node}({}^Y\mathcal{G}))$$

for the unique node of $\mathcal{G}$ *(resp.* ${}^Y\mathcal{G}$*). Let* $\Pi_e \subseteq \Pi_1$ *(resp.* $\Pi_{{}^Ye} \subseteq {}^Y\Pi_1$*) be a nodal subgroup of* $\Pi_1 \xrightarrow{\sim} \Pi_{\mathcal{G}}$ *(resp.* ${}^Y\Pi_1 \xrightarrow{\sim} \Pi_{{}^Y\mathcal{G}}$*) associated with* $e \in \mathrm{Node}(\mathcal{G})$ *(resp.* ${}^Ye \in \mathrm{Node}({}^Y\mathcal{G})$*);* $e_2 \in X_2(k)$ *(resp.* ${}^Ye_2 \in Y_2(k)$*) a k-valued point of the underlying scheme* X_2 *(resp.* Y_2*) of the log scheme* $X_2^{\log}$ *(resp.* $Y_2^{\log}$*) that lies, relative to* $p_{2/1}^{\log}$ *(resp.* ${}^Yp_{2/1}^{\log}$*), over the k-valued point of X (resp. Y) determined by the node* $e \in \mathrm{Node}(\mathcal{G})$ *(resp.* ${}^Ye \in \mathrm{Node}({}^Y\mathcal{G})$*). Thus, we obtain an outer isomorphism*

$$\Pi_{2/1} \xrightarrow{\ \sim\ } \Pi_{\mathcal{G}_{2\in\{1,2\},e_2}} \quad (\text{resp. } {}^Y\Pi_{2/1} \xrightarrow{\sim} \Pi_{{}^Y\mathcal{G}_{2\in\{1,2\},{}^Ye_2}})$$

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(cf. [10, Def. 3.1(iii)]) that may be characterized, up to composition with elements of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{2\in\{1,2\},e_2}) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}_{2\in\{1,2\},e_2}})$ *(resp.* $\mathrm{Aut}^{|\mathrm{grph}|}({}^Y\mathcal{G}_{2\in\{1,2\},{}^Ye_2}) \subseteq \mathrm{Out}(\Pi_{{}^Y\mathcal{G}_{2\in\{1,2\},{}^Ye_2}})$*) (cf. [9, Def. 2.6(i)]; [10, Rem. 4.1.2]), as the group-theoretically cuspidal (cf. [18, Def. 1.4(iv)]) outer isomorphism such that the semi-graph of anabelioids structure on* $\mathcal{G}_{2\in\{1,2\},e_2}$ *(resp.* ${}^Y\mathcal{G}_{2\in\{1,2\},{}^Ye_2}$*) is the semi-graph of anabelioids structure determined (cf. [8, Th. A) by the resulting composite outer representation*

$$\Pi_e \hookrightarrow \Pi_1 \to \mathrm{Out}(\Pi_{2/1}) \xrightarrow{\sim} \mathrm{Out}(\Pi_{\mathcal{G}_{2\in\{1,2\},e_2}})$$

$$(\text{resp. } \Pi_{{}^Ye} \hookrightarrow {}^Y\Pi_1 \to \mathrm{Out}({}^Y\Pi_{2/1}) \xrightarrow{\sim} \mathrm{Out}(\Pi_{{}^Y\mathcal{G}_{2\in\{1,2\},{}^Ye_2}}))$$

—where the second arrow is the outer action determined by the exact sequence $1 \to \Pi_{2/1} \to \Pi_2 \to \Pi_1 \to 1$ *(resp.* $1 \to {}^Y\Pi_{2/1} \to {}^Y\Pi_2 \to {}^Y\Pi_1 \to 1$*)—in a fashion compatible with the restriction* $\Pi_{2/1} \twoheadrightarrow \Pi_{\{2\}}$ *(resp.* ${}^Y\Pi_{2/1} \twoheadrightarrow {}^Y\Pi_{\{2\}}$*) of* $p^{\Pi}_{\{1,2\}/\{2\}}$ *(resp.* ${}^Yp^{\Pi}_{\{1,2\}/\{2\}}$*) to* $\Pi_{2/1} \subseteq \Pi_2$ *(resp.* ${}^Y\Pi_{2/1} \subseteq {}^Y\Pi_2$*) and the given outer isomorphisms* $\Pi_{\{2\}} \xrightarrow{\sim} \Pi_1 \xrightarrow{\sim} \Pi_{\mathcal{G}}$ *(resp.* ${}^Y\Pi_{\{2\}} \xrightarrow{\sim} {}^Y\Pi_1 \xrightarrow{\sim} {}^Y\Pi_{\mathcal{G}}$*). Let*

$$v \in \mathrm{Vert}(\mathcal{G}_{2\in\{1,2\},e_2}) \quad (\text{resp. } {}^Yv \in \mathrm{Vert}({}^Y\mathcal{G}_{2\in\{1,2\},{}^Ye_2}))$$

be the $\{1,2\}$*-tripod (cf. [10, Def. 3.1(v)]) that arises from* $e \in \mathrm{Node}(\mathcal{G})$ *(resp.* ${}^Ye \in \mathrm{Node}({}^Y\mathcal{G})$*) (cf. [10, Def. 3.7(i)]);* $\Pi_v \subseteq \Pi_{\mathcal{G}_{2\in\{1,2\},e_2}} \xleftarrow{\sim} \Pi_{2/1}$ *(resp.* $\Pi_{{}^Yv} \subseteq \Pi_{{}^Y\mathcal{G}_{2\in\{1,2\},{}^Ye_2}} \xleftarrow{\sim} {}^Y\Pi_{2/1}$*) a* $\{1,2\}$*-tripod in* Π_2 *(resp.* ${}^Y\Pi_2$*) associated with the tripod* v *(resp.* ${}^Yv$*) (cf. [10, Def. 3.3(i)]);*

$$\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{{}^Y\mathcal{G}}$$

an outer isomorphism of profinite groups. Suppose that the following conditions are satisfied:

(a) *The outer isomorphism* α *is group-theoretically nodal (cf. [8, Def. 1.12]), that is, determines a bijection of the set of* $\Pi_{\mathcal{G}}$*-conjugates of* $\Pi_e \subseteq \Pi_{\mathcal{G}}$ *and the set of* $\Pi_{{}^Y\mathcal{G}}$*-conjugates of* $\Pi_{{}^Ye} \subseteq \Pi_{{}^Y\mathcal{G}}$*.*

(b) *The outer isomorphism* α *is* 2*-cuspidalizable (cf. [10, Def. 3.20]), that is, the outer isomorphism*

$$\Pi_1 \xrightarrow{\sim} \Pi_{\mathcal{G}} \overset{\alpha}{\xrightarrow{\sim}} \Pi_{{}^Y\mathcal{G}} \xleftarrow{\sim} {}^Y\Pi_1$$

arises from a (uniquely determined, up to permutation of the 2 *factors—cf. [8, Th. B) PFC-admissible (cf. [9, Def. 1.4(iii)]) outer isomorphism* $\Pi_2 \xrightarrow{\sim} {}^Y\Pi_2$*. (In particular, the outer isomorphism* α *is group-theoretically cuspidal.)*

Then the following hold:

(i) *There exists a PFC-admissible isomorphism* $\widetilde{\alpha}_2\colon \Pi_2 \xrightarrow{\sim} {}^Y\Pi_2$ *that lifts* α *such that the composite*

$$\Pi_{\mathcal{G}_{2\in\{1,2\},e_2}} \xleftarrow{\sim} \Pi_{2/1} \xrightarrow{\sim} {}^Y\Pi_{2/1} \xrightarrow{\sim} \Pi_{{}^Y\mathcal{G}_{2\in\{1,2\},{}^Ye_2}}$$

—where the second arrow is the restriction of $\widetilde{\alpha}_2$*—is graphic (cf. [18, Def. 1.4(i)]).*

(ii) *The outer isomorphism* $\alpha_2\colon \Pi_2 \xrightarrow{\sim} {}^Y\Pi_2$ *determined by the isomorphism* $\widetilde{\alpha}_2$ *of (i) induces a bijection between the set of* Π_2*-conjugates of* $\Pi_v \subseteq \Pi_2$ *and the set of* ${}^Y\Pi_2$*-conjugates of* $\Pi_{{}^Yv} \subseteq {}^Y\Pi_2$*. Moreover, if we think of* Π_v*,* $\Pi_{{}^Yv}$ *as the respective (pro-*Σ*) fundamental groups of* $\mathcal{G}_{2\in\{1,2\},e_2}|_v$*,* ${}^Y\mathcal{G}_{2\in\{1,2\},{}^Ye_2}|_{{}^Yv}$ *(cf. [9, Def. 2.1(iii)]; [9, Rem. 2.1.1]), then*

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the induced outer isomorphism $\Pi_v \xrightarrow{\sim} \Pi_{^Y v}$ *(cf. [10, Th. 3.16(i)]) is group-theoretically cuspidal.*

(iii) *The outer isomorphism* α *is graphic.*

Proof. In light of conditions (a) and (b), assertion (i) follows immediately from [8, Th. A] (cf. also our assumption that $\mathrm{Node}(\mathcal{G})^\sharp = \mathrm{Node}(^Y\mathcal{G})^\sharp = 1$, which implies that the outer representation $\Pi_e \to \mathrm{Out}(\Pi_{\mathcal{G}_{2\in\{1,2\},e_2}})$ [resp. $\Pi_{^Y e} \to \mathrm{Out}(\Pi_{^Y\mathcal{G}_{2\in\{1,2\},^Y e_2}})$] is nodally nondegenerate!). Next, let us observe that the $\Pi_{\mathcal{G}_{2\in\{1,2\},e_2}}$- (resp. $\Pi_{^Y\mathcal{G}_{2\in\{1,2\},^Y e_2}}$-) conjugacy class of $\Pi_v \subseteq \Pi_{\mathcal{G}_{2\in\{1,2\},e_2}}$ (resp. $\Pi_{^Y v} \subseteq \Pi_{^Y\mathcal{G}_{2\in\{1,2\},^Y e_2}}$) may be characterized as the unique $\Pi_{\mathcal{G}_{2\in\{1,2\},e_2}}$- (resp. $\Pi_{^Y\mathcal{G}_{2\in\{1,2\},^Y e_2}}$-) conjugacy class of verticial subgroups that fails to map injectively via the surjection $\Pi_{2/1} \twoheadrightarrow \Pi_{\{2\}}$ (resp. $^Y\Pi_{2/1} \twoheadrightarrow {}^Y\Pi_{\{2\}}$). Now assertion (ii) follows immediately from assertion (i). Assertion (iii) follows immediately—in light of [20, Prop. 1.2(iii)]—from assertions (i) and (ii), together with the various definitions involved. This completes the proof of Lemma 3.1. ∎

Before proceeding, we pause to observe that Lemma 3.1 may be applied to obtain an alternative proof of a slightly weaker version of Theorem 3.3 below, as follows.

PROPOSITION 3.2 (Graphicity of group-theoretically nodal 2-cuspidalizable outer isomorphisms). *In the notation of the discussion at the beginning of the present §3, let*

$$\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{^Y\mathcal{G}}$$

be an outer isomorphism of profinite groups. Suppose that the following conditions are satisfied:

(a) *The outer isomorphism* α *is group-theoretically nodal (cf. [8, Def. 1.12]).*

(b) *The outer isomorphism* α *is* 2*-cuspidalizable (cf. [10, Def. 3.20]), that is, the outer isomorphism*

$$\Pi_1 \xrightarrow{\sim} \Pi_{\mathcal{G}} \overset{\alpha}{\xrightarrow{\sim}} \Pi_{^Y\mathcal{G}} \xleftarrow{\sim} {}^Y\Pi_1$$

arises from a (uniquely determined, up to permutation of the 2 *factors—cf. [8, Th. B]) PFC-admissible (cf. [9, Def. 1.4(iii)]) outer isomorphism* $\Pi_2 \xrightarrow{\sim} {}^Y\Pi_2$*. (In particular, the outer isomorphism* α *is group-theoretically cuspidal—cf. [18, Def. 1.4(iv)].)*

Then the outer isomorphism α *is graphic (cf. [18, Def. 1.4(i)]).*

Proof. Let us first observe that it follows from condition (a), together with [18, Prop. 1.2(i)] that α determines a bijection $\mathrm{Node}(\mathcal{G}) \xrightarrow{\sim} \mathrm{Node}(^Y\mathcal{G})$, so $\mathrm{Node}(\mathcal{G})^\sharp = \mathrm{Node}(^Y\mathcal{G})^\sharp$. We verify Proposition 3.2 by induction on $\mathrm{Node}(\mathcal{G})^\sharp = \mathrm{Node}(^Y\mathcal{G})^\sharp$. If $\mathrm{Node}(\mathcal{G}) = \mathrm{Node}(^Y\mathcal{G}) = \emptyset$, then Proposition 3.2 is immediate. Thus, we may assume without loss of generality that $\mathrm{Node}(\mathcal{G})$, $\mathrm{Node}(^Y\mathcal{G}) \neq \emptyset$. Let $e \in \mathrm{Node}(\mathcal{G})$. Write $^Y e \in \mathrm{Node}(^Y\mathcal{G})$ for the node of $^Y\mathcal{G}$ that corresponds, via α, to e. Write $\mathcal{G}_{\rightsquigarrow\{e\}}$ (resp. $^Y\mathcal{G}_{\rightsquigarrow\{^Y e\}}$) for the generization of $\mathcal{G}$ (resp. $^Y\mathcal{G}$) with respect to $\{e\} \subseteq \mathrm{Node}(\mathcal{G})$ (resp. $\{^Y e\} \subseteq \mathrm{Node}(^Y\mathcal{G})$) (cf. [9, Def. 2.8]); β for the composite outer isomorphism

$$\Pi_{\mathcal{G}_{\rightsquigarrow\{e\}}} \overset{\Phi_{\mathcal{G}_{\rightsquigarrow\{e\}}}}{\xrightarrow{\sim}} \Pi_{\mathcal{G}} \overset{\alpha}{\xrightarrow{\sim}} \Pi_{^Y\mathcal{G}} \overset{\Phi^{-1}_{^Y\mathcal{G}_{\rightsquigarrow\{^Y e\}}}}{\xrightarrow{\sim}} \Pi_{^Y\mathcal{G}_{\rightsquigarrow\{^Y e\}}}$$

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(cf. [9, Def. 2.10]); $v_0 \in \mathrm{Vert}(\mathcal{G}_{\rightsquigarrow\{e\}})$ (resp. ${}^Y v_0 \in \mathrm{Vert}({}^Y\mathcal{G}_{\rightsquigarrow\{{}^Y e\}})$) for the (uniquely determined) vertex of the generization $\mathcal{G}_{\rightsquigarrow\{e\}}$ (resp. ${}^Y\mathcal{G}_{\rightsquigarrow\{{}^Y e\}}$) that does not arise from a vertex of $\mathrm{Vert}(\mathcal{G})$ (resp. $\mathrm{Vert}({}^Y\mathcal{G})$). Let $\Pi_{v_0} \subseteq \Pi_{\mathcal{G}_{\rightsquigarrow\{e\}}}$ (resp. $\Pi_{{}^Y v_0} \subseteq \Pi_{{}^Y\mathcal{G}_{\rightsquigarrow\{{}^Y e\}}}$) be a verticial subgroup associated with $v_0 \in \mathrm{Vert}(\mathcal{G}_{\rightsquigarrow\{e\}})$ (resp. ${}^Y v_0 \in \mathrm{Vert}({}^Y\mathcal{G}_{\rightsquigarrow\{{}^Y e\}})$); $\Pi_e \subseteq \Pi_{v_0}$ (resp. $\Pi_{{}^Y e} \subseteq \Pi_{{}^Y v_0}$) a subgroup that maps to a nodal subgroup associated with e in $\Pi_{\mathcal{G}}$ (resp. to ${}^Y e$ in $\Pi_{{}^Y\mathcal{G}}$). Thus, it follows immediately from [8, Lem. 1.9(i) and (ii)] (cf. also [8, Lem. 1.5]; condition (2) of [9, Prop. 2.9(i)]) that Π_{v_0} (resp. $\Pi_{{}^Y v_0}$) may be characterized as the unique verticial subgroup of $\Pi_{\mathcal{G}_{\rightsquigarrow\{e\}}}$ (resp. $\Pi_{{}^Y\mathcal{G}_{\rightsquigarrow\{{}^Y e\}}}$) that contains Π_e (resp. $\Pi_{{}^Y e}$).

Next, let us observe that, by applying the induction hypothesis to β, we conclude that β is graphic. Thus, it follows immediately—in light of [18, Prop. 1.5(ii)]—from the definition of the generizations under consideration (cf. condition (3) of [9, Prop. 2.9(i)]) that, to complete the verification of Proposition 3.2, it suffices to verify that the following assertion holds:

> Claim 3.2.A: Let $H \subseteq \Pi_{v_0} \subseteq \Pi_{\mathcal{G}_{\rightsquigarrow\{e\}}}$ be a closed subgroup of Π_{v_0} whose image in $\Pi_{\mathcal{G}}$ is a verticial subgroup. Then the image of H via the composite
>
> $$\Pi_{\mathcal{G}_{\rightsquigarrow\{e\}}} \overset{\beta}{\underset{\sim}{\longrightarrow}} \Pi_{{}^Y\mathcal{G}_{\rightsquigarrow\{{}^Y e\}}} \overset{\Phi_{{}^Y\mathcal{G}_{\rightsquigarrow\{{}^Y e\}}}}{\underset{\sim}{\longrightarrow}} \Pi_{{}^Y\mathcal{G}}$$
>
> is a verticial subgroup.

To verify Claim 3.2.A, let us observe that since β is graphic, it follows immediately from the above characterization of Π_{v_0}, $\Pi_{{}^Y v_0}$ that β maps Π_{v_0} bijectively onto a $\Pi_{{}^Y\mathcal{G}_{\rightsquigarrow\{{}^Y e\}}}$-conjugate of $\Pi_{{}^Y v_0}$. Thus, it follows immediately from condition (b), together with the evident isomorphism (i.e., as opposed to outomorphism—cf. [10, Rem. 4.14.1]) version of [10, Lem. 4.8(i) and (ii)] that, in the notation of [10, Def. 4.3], the outer isomorphism $\Pi_2 \overset{\sim}{\to} {}^Y\Pi_2$ of condition (b) induces compatible outer isomorphisms $(\Pi_{v_0})_2 \overset{\sim}{\to} (\Pi_{{}^Y v_0})_2$, $\Pi_{v_0} \overset{\sim}{\to} \Pi_{{}^Y v_0}$. In particular, by applying Lemma 3.1(iii) to these outer isomorphisms, one concludes that Claim 3.2.A holds, as desired. This completes the proof of Proposition 3.2. ∎

THEOREM 3.3 (Graphicity of profinite outer isomorphisms). *Let Σ_0 be a nonempty set of prime numbers; $\mathcal{H}$, $\mathcal{J}$ semi-graphs of anabelioids of pro-Σ_0 PSC-type; $\Pi_{\mathcal{H}}$, $\Pi_{\mathcal{J}}$ the (pro-Σ_0) fundamental groups of $\mathcal{H}$, $\mathcal{J}$, respectively;*

$$\alpha\colon \Pi_{\mathcal{H}} \overset{\sim}{\longrightarrow} \Pi_{\mathcal{J}}$$

an outer isomorphism of profinite groups. Then the following conditions are equivalent:

(i) *The outer isomorphism α is graphic (cf. [18, Def. 1.4(i)]).*
(ii) *The outer isomorphism α is group-theoretically verticial and group-theoretically cuspidal (cf. [18, Def. 1.4(iv)]).*
(iii) *The outer isomorphism α is group-theoretically nodal (cf. [8, Def. 1.12]) and group-theoretically cuspidal.*

Proof. The implication (i) ⇒ (ii) (resp. (ii) ⇒ (iii)) follows from the various definitions involved (resp. [8, Lem. 1.9(i)]). Thus, it suffices to verify the implication (iii) ⇒ (i). Suppose that condition (iii) holds. Then, to verify the graphicity of α, it follows from [18, Th. 1.6(ii)] that it suffices to verify that α is graphically filtration-preserving (cf. [18, Def. 1.4(iii)]). In particular, by replacing $\Pi_{\mathcal{H}}$, $\Pi_{\mathcal{J}}$ by suitable open subgroups of $\Pi_{\mathcal{H}}$, $\Pi_{\mathcal{J}}$, it suffices to verify that α determines isomorphisms

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$$\Pi^{\text{ab-edge}}_{\mathcal{H}} \xrightarrow{\sim} \Pi^{\text{ab-edge}}_{\mathcal{J}},\ \Pi^{\text{ab-vert}}_{\mathcal{H}} \xrightarrow{\sim} \Pi^{\text{ab-vert}}_{\mathcal{J}},$$

where we write "$\Pi^{\text{ab-edge}}_{(-)}$," "$\Pi^{\text{ab-vert}}_{(-)}$" for the closed subgroups of the abelianization "$\Pi^{\text{ab}}_{(-)}$" of "$\Pi_{(-)}$" topologically generated by the images of the edge-like, verticial subgroups of "$\Pi_{(-)}$." Here, we may assume without loss of generality that $\mathcal{H}$ and $\mathcal{J}$ are sturdy, hence admit compactifications (cf. [18, Rems. 1.1.5 and 1.1.6]). Now the assertion concerning "$\Pi^{\text{ab-edge}}_{(-)}$" follows immediately from condition (iii). On the other hand, the assertion concerning "$\Pi^{\text{ab-vert}}_{(-)}$" follows immediately from the duality discussed in [18, Prop. 1.3] applied to the compactifications of $\mathcal{H}$, $\mathcal{J}$, together with condition (iii). This completes the proof of Theorem 3.3. $\square$

Remark 3.3.1. Here, we observe that results such as [6, Cor. 6.1]; [6, Cor. 6.4(ii)]; [6, Th. 6.6] amount, when translated into the language of the present paper, to a special case of the result obtained by concatenating the equivalence (i) $\Leftrightarrow$ (iii) of Theorem 3.3, with the computation of the normalizer given in [9, Th. 5.14(iii)] (i.e., in essence, [18, Cor. 2.7(iii) and (iv)]). Moreover, the proof given above of this equivalence (i) $\Leftrightarrow$ (iii) of Theorem 3.3 is, essentially, a restatement of various results from the theory of [18]. That is to say, although the statements of these results that occur in the present series of papers and in [6] are formulated and arranged in a somewhat different way, the essential mathematical content that underlies these results is, in fact, entirely identical; moreover, this state of affairs is by no means a coincidence. Indeed, this mathematical content is given in [18] as [18, Prop. 1.3]; [18, Prop. 2.6]. In [6], this mathematical content is given as [6, Lem. 5.11] (and the surrounding discussion), which, in fact, was related to the author of [6] by the senior author of the present paper in the context of an explanation of the theory of [18].

Corollary 3.4 (Graphicity of discrete outer isomorphisms). *Let $\mathcal{H}$, $\mathcal{J}$ be semi-graphs of temperoids of HSD-type (cf. Definition 2.3(iii)); $\Pi_{\mathcal{H}}$, $\Pi_{\mathcal{J}}$ the fundamental groups of $\mathcal{H}$, $\mathcal{J}$, respectively (cf. Proposition 2.5(i));*

$$\alpha\colon \Pi_{\mathcal{H}} \xrightarrow{\sim} \Pi_{\mathcal{J}}$$

an outer isomorphism. Then the following conditions are equivalent:

(i) *The outer isomorphism α is graphic (cf. Definition 2.7(ii)).*
(ii) *The outer isomorphism α is group-theoretically verticial and group-theoretically cuspidal (cf. Definition 2.7(i)).*
(iii) *The outer isomorphism α is group-theoretically nodal and group-theoretically cuspidal (cf. Definition 2.7(i)).*

Proof. This follows immediately from Theorem 3.3, together with Corollary 2.19(i). $\square$

Definition 3.5. Let $({}^{Y}\mathcal{G}, S \subseteq \mathrm{Node}({}^{Y}\mathcal{G}), \phi\colon {}^{Y}\mathcal{G}_{\rightsquigarrow S} \xrightarrow{\sim} \mathcal{G})$ be a degeneration structure on $\mathcal{G}$ (cf. [10, Def. 3.23(i)]) and $e \in S$.

(i) We shall say that a closed subgroup $J \subseteq \Pi_1$ of Π_1 is a *cycle-subgroup* of Π_1 (with respect to $({}^{Y}\mathcal{G}, S \subseteq \mathrm{Node}({}^{Y}\mathcal{G}), \phi\colon {}^{Y}\mathcal{G}_{\rightsquigarrow S} \xrightarrow{\sim} \mathcal{G})$, associated with $e \in S$) if J is contained in the Π_1-conjugacy class of closed subgroups of Π_1 obtained by forming the image of a nodal subgroup of $\Pi_{{}^{Y}\mathcal{G}}$ associated with e via the composite of outer isomorphisms

$$\Pi_{{}^{Y}\mathcal{G}} \overset{\Phi^{-1}_{{}^{Y}\mathcal{G}_{\rightsquigarrow S}}}{\xrightarrow{\sim}} \Pi_{{}^{Y}\mathcal{G}_{\rightsquigarrow S}} \xrightarrow{\sim} \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_1$$

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—where the first arrow is the inverse of the specialization outer isomorphism $\Phi_{{}^Y\mathcal{G}_{\rightsquigarrow S}}$ (cf. [9, Def. 2.10]), the second arrow is the graphic outer isomorphism $\Pi_{{}^Y\mathcal{G}_{\rightsquigarrow S}} \xrightarrow{\sim} \Pi_{\mathcal{G}}$ induced by ϕ, and the third arrow is the natural outer isomorphism $\Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_1$ of [10, Def. 3.1(ii)] (cf. the left-hand portion of Figure 1).

(ii) Let n be a positive integer. Then we shall say that a cycle-subgroup of Π_1 is *n-cuspidalizable* if it is a cycle-subgroup of Π_1 with respect to some n-cuspidalizable degeneration structure on $\mathcal{G}$ (cf. [10, Def. 3.23(v)]).

REMARK 3.5.1. Let $J \subseteq \Pi_1$ be a cycle-subgroup of Π_1 with respect to a degeneration structure $({}^Y\mathcal{G}, S \subseteq \mathrm{Node}({}^Y\mathcal{G}), \phi\colon {}^Y\mathcal{G}_{\rightsquigarrow S} \xrightarrow{\sim} \mathcal{G})$, associated with a node $e \in S$. Then it follows immediately from [18, Prop. 1.2(i)] that the node e of ${}^Y\mathcal{G}$ is uniquely determined by the subgroup $J \subseteq \Pi_1$ and the degeneration structure $({}^Y\mathcal{G}, S \subseteq \mathrm{Node}({}^Y\mathcal{G}), \phi\colon {}^Y\mathcal{G}_{\rightsquigarrow S} \xrightarrow{\sim} \mathcal{G})$.

DEFINITION 3.6. Let $J \subseteq \Pi_1$ be a 2-cuspidalizable cycle-subgroup of Π_1 (cf. Definition 3.5(i) and (ii)).

(i) It follows immediately from the various definitions involved that we have data as follows:

  (a) a 2-cuspidalizable degeneration structure $({}^Y\mathcal{G}, S \subseteq \mathrm{Node}({}^Y\mathcal{G}),\ \phi\colon {}^Y\mathcal{G}_{\rightsquigarrow S} \xrightarrow{\sim} \mathcal{G})$ on $\mathcal{G}$ (cf. [10, Def. 3.23(i) and (v)]),
  (b) an isomorphism ${}^Y\Pi_1 \xrightarrow{\sim} \Pi_1$ that is compatible with the composite of the display of Definition 3.5(i) (cf. also [10, Def. 3.1(ii)]) in the case where we take the "$({}^Y\mathcal{G}, S \subseteq \mathrm{Node}({}^Y\mathcal{G}), \phi\colon {}^Y\mathcal{G}_{\rightsquigarrow S} \xrightarrow{\sim} \mathcal{G})$" of Definition 3.5 to be the degeneration structure of (a),
  (c) a PFC-admissible isomorphism ${}^Y\Pi_2 \xrightarrow{\sim} \Pi_2$ that lifts the isomorphism of (b), and
  (d) a nodal subgroup $\Pi_e \subseteq {}^Y\Pi_1$ of ${}^Y\Pi_1$ associated with a (uniquely determined—cf. Remark 3.5.1) node e of ${}^Y\mathcal{G}$

such that the image of the nodal subgroup $\Pi_e \subseteq {}^Y\Pi_1$ of (d) via the isomorphism ${}^Y\Pi_1 \xrightarrow{\sim} \Pi_1$ of (b) coincides with $J \subseteq \Pi_1$. We shall say that a closed subgroup $T \subseteq \Pi_{2/1}$ of $\Pi_{2/1}$ is a *tripodal subgroup* associated with J if T coincides—relative to some choice of data (a), (b), (c), (d) as above (but cf. also Remark 3.6.1!)—with the image, via the lifting ${}^Y\Pi_2 \xrightarrow{\sim} \Pi_2$ of (c), of some $\{1,2\}$-tripod in ${}^Y\Pi_{2/1} \subseteq {}^Y\Pi_2$ (cf. [10, Def. 3.3(i)]) arising from e (cf. [10, Def. 3.7(i)]), and, moreover, the centralizer $Z_{\Pi_2}(T)$ maps bijectively, via $p^{\Pi}_{2/1}\colon \Pi_2 \twoheadrightarrow \Pi_1$, onto $J \subseteq \Pi_1$ (cf. [10, Lem. 3.11(iv) and (vii)]).

(ii) Let $T \subseteq \Pi_{2/1}$ be a tripodal subgroup associated with J (cf. (i)). Then we shall refer to a closed subgroup of T that arises from a nodal (resp. cuspidal) subgroup contained in the $\{1,2\}$-tripod in ${}^Y\Pi_{2/1} \subseteq {}^Y\Pi_2$ of (i) as a *lifting cycle-subgroup* (resp. *distinguished cuspidal subgroup*) of T (cf. the right-hand portion of Figure 1).

REMARK 3.6.1. Note that, in the situation of Definition 3.6, (i), it follows immediately from Lemma 3.1(ii) (i.e., by considering the generization of ${}^Y\mathcal{G}$ with respect to $\mathrm{Node}({}^Y\mathcal{G}) \setminus \{e\}$—cf. [9, Def. 2.8]), together with the computation of the centralizer given in [10, Lem. 3.11(vii)], and the commensurable terminality of $J \subseteq \Pi_1$ (cf. [18, Prop. 1.2(ii)]), that the $\Pi_{2/1}$-conjugacy class of a tripodal subgroup T is completely determined by the cycle-subgroup $J \subseteq \Pi_1$.

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REMARK 3.6.2.

(i) Suppose that we are in the situation of Definition 3.5(i). Recall the module $\Lambda_{\mathcal{G}}$, that is, the cyclotome associated with $\mathcal{G}$, defined in [9, Def. 3.8(i)]. Thus, as an abstract module, $\Lambda_{\mathcal{G}}$ is isomorphic to the pro-Σ completion $\widehat{\mathbb{Z}}^{\Sigma}$ of $\mathbb{Z}$. Recall, furthermore, from [9, Cor. 3.9(v) and (vi)] that one may construct a natural, functorial $\{\pm\}$-orbit of isomorphisms

$$\Pi_e \xrightarrow{\sim} \Lambda_{{}^Y\mathcal{G}}$$

—where $\Pi_e \subseteq {}^Y\Pi_1 \xrightarrow{\sim} \Pi_{{}^Y\mathcal{G}}$ (cf. [10, Def. 3.1(ii)]) denotes a nodal subgroup associated with e. Thus, by applying the natural, functorial (outer) isomorphisms $\Lambda_{{}^Y\mathcal{G}} \xrightarrow{\sim} \Lambda_{{}^Y\mathcal{G}_{\rightsquigarrow S}}$ (cf. [9, Cor. 3.9(i)]) and $\Phi^{-1}_{{}^Y\mathcal{G}_{\rightsquigarrow S}} : \Pi_{{}^Y\mathcal{G}} \xrightarrow{\sim} \Pi_{{}^Y\mathcal{G}_{\rightsquigarrow S}}$ (cf. [9, Def. 2.10]), together with the (outer) isomorphisms $\Lambda_{{}^Y\mathcal{G}_{\rightsquigarrow S}} \xrightarrow{\sim} \Lambda_{\mathcal{G}}$ and $\Pi_{{}^Y\mathcal{G}_{\rightsquigarrow S}} \xrightarrow{\sim} \Pi_{\mathcal{G}}$ induced by ϕ, we obtain a natural $\{\pm\}$-orbit of isomorphisms

$$J \xrightarrow{\sim} \Lambda_{\mathcal{G}}$$

associated with the cycle-subgroup $J \subseteq \Pi_1$. Note that this $\{\pm\}$-orbit of isomorphisms is functorial with respect to automorphisms α of Π_1 such that $\alpha(J) = J$, and, moreover, the outer automorphism of $\Pi_{\mathcal{G}}$ obtained by forming the conjugate of α by the natural outer isomorphism $\Pi_1 \xrightarrow{\sim} \Pi_{\mathcal{G}}$ is graphic (cf. the equivalence (i) $\Leftrightarrow$ (iii) of Theorem 3.3). In this context, it is natural to refer to either of the two isomorphisms in this $\{\pm\}$-orbit as an *orientation on the cycle-subgroup J*.

(ii) Now suppose that we are in the situation of Definition 3.6(i) and (ii). Then let us observe that the natural outer surjection ${}^Y\Pi_{2/1} \twoheadrightarrow {}^Y\Pi_{\{2\}} \xrightarrow{\sim} {}^Y\Pi_1$ determined by ${}^Yp^{\Pi}_{\{1,2\}/\{2\}}$ induces a natural isomorphism

$$\Lambda_{{}^Y\mathcal{G}_{2\in\{1,2\},e_2}} \xrightarrow{\sim} \Lambda_{{}^Y\mathcal{G}}$$

(cf. [9, Cor. 3.9(ii)]), where we write $e_2 \in Y_2(k)$ for a k-valued point of Y_2 that lies, relative to ${}^Yp^{\log}_{2/1}$, over the k-valued point of Y determined by the node e. Write v for the vertex of ${}^Y\mathcal{G}_{2\in\{1,2\},e_2}$ that gives rise to the tripodal subgroup $T \subseteq \Pi_{2/1}$. Thus, we have a natural isomorphism

$$\Lambda_v \xrightarrow{\sim} \Lambda_{{}^Y\mathcal{G}_{2\in\{1,2\},e_2}}$$

(cf. [9, Cor. 3.9(ii)]). Now suppose that e^* is a node of ${}^Y\mathcal{G}_{2\in\{1,2\},e_2}$ that abuts to v and, moreover, gives rise to a lifting cycle-subgroup $J^* \subseteq T$ of the tripodal subgroup T. Thus, one verifies immediately that the natural outer surjection $\Pi_{2/1} \twoheadrightarrow \Pi_{\{2\}} \xrightarrow{\sim} \Pi_1$ determined by $p^{\Pi}_{\{1,2\}/\{2\}}$ induces a natural isomorphism $J^* \xrightarrow{\sim} J$ (cf. [10, Lem. 3.6(iv)]). Let $\Pi_{e^*} \subseteq \Pi_{{}^Y\mathcal{G}_{2\in\{1,2\},e_2}}$ be a nodal subgroup associated with e^*. Then the (unique!) branch of e^* that abuts to v determines a natural isomorphism

$$\Pi_{e^*} \xrightarrow{\sim} \Lambda_v$$

(cf. [9, Cor. 3.9(v)]). Thus, by composing the isomorphisms of the last three displays with the isomorphism $\Lambda_{{}^Y\mathcal{G}} \xrightarrow{\sim} \Lambda_{{}^Y\mathcal{G}_{\rightsquigarrow S}} \xrightarrow{\sim} \Lambda_{\mathcal{G}}$ discussed in (i) and the inverse of the tautological isomorphism $\Pi_{e^*} \xrightarrow{\sim} J^*$, we obtain a natural isomorphism

$$J^* \xrightarrow{\sim} \Lambda_{\mathcal{G}}$$

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associated with the lifting cycle-subgroup $J^* \subseteq T$. Note that this natural isomorphism is functorial with respect to FC-admissible automorphisms α_2 of Π_2 such that $\alpha_2(J^*) = J^*$, $\alpha_2(T) = T$, and, moreover, the outer automorphism of $\Pi_{\mathcal{G}}$ obtained by forming the conjugate, by the natural outer isomorphism $\Pi_1 \xrightarrow{\sim} \Pi_{\mathcal{G}}$, of the outer automorphism of Π_1 determined by α_2 is graphic (cf. the equivalence (i) $\Leftrightarrow$ (iii) of Theorem 3.3; [10, Lem. 3.11(vii)]). Finally, one verifies immediately from the construction of the isomorphisms of [9, Cor. 3.9(v)] that if one composes this isomorphism $J^* \xrightarrow{\sim} \Lambda_{\mathcal{G}}$ with the inverse of the natural isomorphism $J^* \xrightarrow{\sim} J$ discussed above, then the resulting isomorphism $J \xrightarrow{\sim} \Lambda_{\mathcal{G}}$ is an orientation on the cycle-subgroup J, in the sense of the discussion of (i), and, moreover, that, if we define an *orientation on the tripodal subgroup* T to be a choice of a T-conjugacy class of lifting cycle-subgroups of T, then the resulting assignment

$$\Big\{\text{orientations on } T\Big\} \longrightarrow \Big\{\text{orientations on } J\Big\}$$

is a bijection [between sets of cardinality 2].

Lemma 3.7. (Induced outomorphisms of tripods). *In the situation of Lemma 3.1, suppose that $X^{\log} = Y^{\log}$. Write $c \in \mathrm{Cusp}(\mathcal{G}_{2\in\{1,2\},e_2})$ for the cusp arising from the diagonal divisor in $X \times_k X$. Let $\Pi_c \subseteq \Pi_{\mathcal{G}_{2\in\{1,2\},e_2}}$ be a cuspidal subgroup of $\Pi_{\mathcal{G}_{2\in\{1,2\},e_2}}$ associated with c. Write*

$$\alpha_v \stackrel{\text{def}}{=} \mathfrak{T}_{\Pi_v}(\alpha_2) \in \mathrm{Out}(\Pi_v)$$

(cf. Lemma 3.1(ii); [10, Th. 3.16(i)]) for the result of applying the tripod homomorphism $\mathfrak{T}_{\Pi_v}$ to α_2. (Thus, it follows immediately from Lemma 3.1(ii) that $\alpha_v \in \mathrm{Out}^{\mathrm{C}}(\Pi_v)$.) Suppose, moreover, that the following condition is satisfied:

(c) *The cuspidal subgroup $\Pi_c \subseteq \Pi_{\mathcal{G}_{2\in\{1,2\},e_2}} \xleftarrow{\sim} \Pi_{2/1}$ is contained in Π_v.*

Then the following hold:

(i) *Since Π_v may be regarded as the "Π_1" that occurs in the case where we take "$X^{\log}$" to be the smooth log curve associated with $\mathbb{P}^1_k \setminus \{0,1,\infty\}$ (cf. [10, Rem. 3.3.1]), there exists a uniquely determined outomorphism*

$$\iota \in \mathrm{Out}(\Pi_v)$$

of Π_v that arises from an automorphism of $\mathbb{P}^1_k \setminus \{0,1,\infty\}$ over k and induces a nontrivial automorphism of the set $\mathcal{N}(v)$. Write

$$|\alpha_v| \stackrel{\text{def}}{=} \alpha_v \in \mathrm{Out}(\Pi_v) \quad (\text{resp. } |\alpha_v| \stackrel{\text{def}}{=} \iota \circ \alpha_v \in \mathrm{Out}(\Pi_v))$$

if $\alpha_v \in \mathrm{Out}^{\mathrm{C}}(\Pi_v)^{\mathrm{cusp}}$ (resp. $\notin \mathrm{Out}^{\mathrm{C}}(\Pi_v)^{\mathrm{cusp}}$) (cf. [10, Def. 3.4(i)]). Then it holds that $|\alpha_v| \in \mathrm{Out}^{\mathrm{C}}(\Pi_v)^{\mathrm{cusp}}$.

(ii) *Let $\Pi_{\mathrm{tpd}} \subseteq \Pi_3$ be a central $\{1,2,3\}$-tripod of Π_3 (cf. [10, Defs. 3.3(i) and 3.7(ii)]). Then every geometric (cf. [10, Def. 3.4(ii)]) outer isomorphism $\Pi_{\mathrm{tpd}} \xrightarrow{\sim} \Pi_v$ satisfies the following condition: let $\beta \in \mathrm{Out}(\Pi_1) \xrightarrow{\sim} \mathrm{Out}(\Pi_{\mathcal{G}})$ be an outomorphism of $\Pi_1 \xrightarrow{\sim} \Pi_{\mathcal{G}}$ that is group-theoretically nodal and 3-cuspidalizable, that is, $\beta \in \mathrm{Out}(\Pi_1)$ arises from a(n) (uniquely determined—cf. [8, Th. B]) FC-admissible outomorphism $\beta_3 \in \mathrm{Out}^{\mathrm{FC}}(\Pi_3)$.*

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Then the image

$$\mathfrak{T}_{\Pi_{\rm tpd}}(\beta_3) \in \mathrm{Out}(\Pi_{\rm tpd})$$

(cf. [10, Def. 3.19]) coincides—relative to the outer isomorphism $\Pi_{\rm tpd} \xrightarrow{\sim} \Pi_v$ *under consideration—with*

$$|\beta_v| \in \mathrm{Out}(\Pi_v)$$

(cf. (i)), where we write $\beta_v \stackrel{\rm def}{=} \mathfrak{T}_{\Pi_v}(\beta_3) \in \mathrm{Out}(\Pi_v)$*. In particular, it holds that* $|\beta_v| \in \mathrm{Out}^{\rm C}(\Pi_v)^{\Delta+}$ *(cf. [10, Def. 3.4(i)]).*

Proof. Assertion (i) follows immediately from the various definitions involved. Next, we verify assertion (ii). Let us first observe that the inclusion $|\beta_v| \in \mathrm{Out}^{\rm C}(\Pi_v)^{\Delta}$ follows immediately from the coincidence of $\mathfrak{T}_{\Pi_{\rm tpd}}(\beta_3)$ with $|\beta_v|$, relative to some specific geometric outer isomorphism $\Pi_{\rm tpd} \xrightarrow{\sim} \Pi_v$, together with the second displayed equality of [10, Th. 3.16(v)]. The inclusion $|\beta_v| \in \mathrm{Out}^{\rm C}(\Pi_v)^{\Delta+}$ then follows from [10, Lem. 3.5]; [10, Th. 3.17(i)] (applied in the case where we take the "(Π_2, T, T')" of [10, Th. 3.17(i)] to be $(\Pi_{3/1}, \Pi_v, \Pi_{\rm tpd})$). Moreover, it follows immediately from the various definitions involved that the inclusion $|\beta_v| \in \mathrm{Out}^{\rm C}(\Pi_v)^{\Delta}$ allows one to conclude that the coincidence of $\mathfrak{T}_{\Pi_{\rm tpd}}(\beta_3)$ with $|\beta_v|$, relative to some specific geometric outer isomorphism $\Pi_{\rm tpd} \xrightarrow{\sim} \Pi_v$, implies the coincidence of $\mathfrak{T}_{\Pi_{\rm tpd}}(\beta_3)$ with $|\beta_v|$, relative to an arbitrary geometric outer isomorphism $\Pi_{\rm tpd} \xrightarrow{\sim} \Pi_v$. Thus, to complete the verification of assertion (ii), it suffices to verify the coincidence of $\mathfrak{T}_{\Pi_{\rm tpd}}(\beta_3)$ with $|\beta_v|$, relative to the specific geometric outer isomorphism $\Pi_{\rm tpd} \xrightarrow{\sim} \Pi_v$ whose existence is guaranteed by [10, Th. 3.18(ii)]. In the following discussion, we fix this specific geometric outer isomorphism $\Pi_{\rm tpd} \xrightarrow{\sim} \Pi_v$.

Next, let us observe that if $\beta_v = |\beta_v|$, that is, $\beta_v \in \mathrm{Out}^{\rm C}(\Pi_v)^{\rm cusp}$, then it follows immediately from [10, Ths. 3.16(v) and 3.18(ii)] that $\mathfrak{T}_{\Pi_{\rm tpd}}(\beta_3) \in \mathrm{Out}(\Pi_{\rm tpd})$ coincides with $|\beta_v| \in \mathrm{Out}(\Pi_v)$. Thus, to complete the verification of assertion (ii), we may assume without loss of generality that $\beta_v \neq |\beta_v|$, that is, that $\beta_v \notin \mathrm{Out}^{\rm C}(\Pi_v)^{\rm cusp}$. Then let us observe that collections of data consisting of smooth log curves that (by gluing at prescribed cusps) give rise to a stable log curve whose associated semi-graph of anabelioids (of pro-Σ PSC-type) is isomorphic to $\mathcal{G}$ may be parametrized by a smooth, connected moduli stack. Thus, one verifies easily that, by considering the étale fundamental groupoid of this moduli stack, together with a suitable scheme-theoretic automorphism of order 2 of a collection of data parametrized by this moduli stack, one obtains a 3-cuspidalizable automorphism $\xi \in \mathrm{Aut}(\mathcal{G})$ $(\hookrightarrow \mathrm{Out}(\Pi_{\mathcal{G}}))$ of $\mathcal{G}$ such that ξ_v (i.e., the "α_v" that occurs in the case where we take "α" to be ξ) coincides with ι. Thus, by applying the portion of assertion (ii) that has already been verified to $\xi \circ \beta$, we conclude that, to complete the verification of assertion (ii), it suffices to verify that $\mathfrak{T}_{\Pi_{\rm tpd}}(\xi_3) = 1$. On the other hand, this follows immediately from the fact that ξ was assumed to arise from a scheme-theoretic automorphism (cf. also [10, Th. 3.16(v)]). This completes the proof of assertion (ii) and hence of Lemma 3.7. ∎

DEFINITION 3.8. Let $J \subseteq \Pi_1$ be a 2-cuspidalizable cycle-subgroup (cf. Definition 3.5(i) and (ii)); let us fix associated data as in Definition 3.6(i.a)–(i.d). Relative to these data, suppose that $T \subseteq \Pi_{2/1}$ is a tripodal subgroup associated with $J \subseteq \Pi_1$ (cf. Definition 3.6(i)), and that $I \subseteq T$ is a distinguished cuspidal subgroup of T (cf. Definition 3.6(ii)). Note that these data, together with the log scheme structure of $Y^{\log}$, allow one to speak of

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geometric (cf. [10, Def. 3.4(ii)]) outomorphisms of T. Then one verifies easily that there exists a uniquely determined nontrivial geometric outomorphism of T that preserves the T-conjugacy class of I. Thus, since I is commensurably terminal in T (cf. [18, Prop. 1.2(ii)]), there exists a uniquely determined I-conjugacy class of automorphisms of T that lifts this outomorphism and preserves $I \subseteq T$. We shall refer to this I-conjugacy class of automorphisms of T as the *cycle symmetry associated with* I.

Before proceeding, we pause to observe the following interesting "alternative formulation" of the essential content of Lemma 3.7(ii).

LEMMA 3.9 (Geometricity of conjugates of geometric outer isomorphisms). *Suppose that we are in the situation of [10, Th. 3.18(ii)], that is, $n \geq 3$, and T (resp. T') is an E- (resp. E'-) tripod of Π_n for some subset $E \subseteq \{1,\dots,n\}$ (resp. $E' \subseteq \{1,\dots,n\}$). Let $\phi\colon T \xrightarrow{\sim} T'$ be a geometric (cf. [10, Def. 3.4(ii)]) outer isomorphism. Then, for every $\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T,T' : \{|C|\}]$, the composite of outer isomorphisms*

$$T \overset{\mathfrak{T}_T(\alpha)}{\underset{\sim}{\longrightarrow}} T \overset{\phi}{\underset{\sim}{\longrightarrow}} T' \overset{\mathfrak{T}_{T'}(\alpha)^{-1}}{\underset{\sim}{\longrightarrow}} T'$$

(cf. [10, Th. 3.16(i)]) is equal to ϕ.

Proof. Let us first observe that the validity of Lemma 3.9 for some specific geometric outer isomorphism "ϕ" follows formally from the commutative diagram of [10, Th. 3.18(ii)]. Thus, the validity of Lemma 3.9 for an arbitrary geometric outer isomorphism "ϕ" follows immediately from the equality of the first display of [10, Th. 3.18(i)], that is, the fact that $\mathfrak{T}_T(\alpha)$ commutes with arbitrary geometric outomorphisms of T. This completes the proof of Lemma 3.9. ∎

REMARK 3.9.1. One verifies immediately that a similar argument to the argument applied in the proof of Lemma 3.9 yields evident analogues of Lemma 3.9 in the respective situations of [10, Th. 3.17(i) and (ii)].

THEOREM 3.10 (Canonical liftings of cycles). *In the notation of the discussion at the beginning of the present §3, let $I \subseteq \Pi_{2/1} \subseteq \Pi_2$ be a cuspidal inertia group associated with the diagonal cusp of a fiber of $p^{\log}_{2/1}$; $\Pi_{\mathrm{tpd}} \subseteq \Pi_3$ a 3-central $\{1,2,3\}$-tripod of Π_3 (cf. [10, Def. 3.7(ii)]); $I_{\mathrm{tpd}} \subseteq \Pi_{\mathrm{tpd}}$ a cuspidal subgroup of Π_{tpd} that does not arise from a cusp of a fiber of $p^{\log}_{3/2}$; J^*_{tpd}, $J^{**}_{\mathrm{tpd}} \subseteq \Pi_{\mathrm{tpd}}$ cuspidal subgroups of Π_{tpd} such that I_{tpd}, J^*_{tpd}, and J^{**}_{tpd} determine three distinct Π_{tpd}-conjugacy classes of closed subgroups of Π_{tpd}. (Note that one verifies immediately from the various definitions involved that such cuspidal subgroups I_{tpd}, J^*_{tpd}, and J^{**}_{tpd} always exist.) For positive integers $n \geq 2$, $m \leq n$, and $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)$ (cf. [20, Def. 1.1(ii)]), write*

$$\alpha_m \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_m)$$

for the automorphism of Π_m determined by α;

$$\mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I) \subseteq \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)$$

for the subgroup consisting of $\beta \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)$ such that $\beta_2(I) = I$;

$$\mathrm{Aut}^{\mathrm{FC}}(\Pi_n)^{\mathrm{G}} \subseteq \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)$$

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for the subgroup consisting of $\beta \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)$ *such that the image of* β *via the composite* $\mathrm{Aut}^{\mathrm{FC}}(\Pi_n) \twoheadrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1) \to \mathrm{Out}(\Pi_{\mathcal{G}})$*—where the second arrow is the natural injection of [8, Th. B] and the third arrow is the homomorphism induced by the natural outer isomorphism* $\Pi_1 \xrightarrow{\sim} \Pi_{\mathcal{G}}$*—is graphic (cf. [18, Def. 1.4(i)]);*

$$\mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I) \cap \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)^{\mathrm{G}};$$

$$\mathrm{Cycle}^n(\Pi_1)$$

for the set of n-cuspidalizable cycle-subgroups of Π_1 *(cf. Definition 3.5(i) and (ii));*

$$\mathrm{Tpd}_I(\Pi_{2/1})$$

for the set of closed subgroups $T \subseteq \Pi_{2/1}$ *such that* T *is a tripodal subgroup associated with some* 2*-cuspidalizable cycle-subgroup of* Π_1 *(cf. Definition 3.6(i)), and, moreover,* I *is a distinguished cuspidal subgroup (cf. Definition 3.6(ii)) of* T*. Then the following hold:*

(i) *Let* $n \geq 2$ *be a positive integer,* $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}}$*,* $J \in \mathrm{Cycle}^n(\Pi_1)$*, and* $T \in \mathrm{Tpd}_I(\Pi_{2/1})$*. Then it holds that*

$$\alpha_1(J) \in \mathrm{Cycle}^n(\Pi_1), \quad \alpha_2(T) \in \mathrm{Tpd}_I(\Pi_{2/1}).$$

Thus, $\mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}}$ *acts naturally on* $\mathrm{Cycle}^n(\Pi_1)$*,* $\mathrm{Tpd}_I(\Pi_{2/1})$*.*

(ii) *Let* $n \geq 2$ *be a positive integer. Then there exists a unique* $\mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}}$*-equivariant (cf. (i)) map*

$$\mathfrak{C}_I \colon \mathrm{Cycle}^n(\Pi_1) \longrightarrow \mathrm{Tpd}_I(\Pi_{2/1})$$

such that, for every $J \in \mathrm{Cycle}^n(\Pi_1)$*,* $\mathfrak{C}_I(J)$ *is a tripodal subgroup associated with* J*. Moreover, for every* $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}}$ *and* $J \in \mathrm{Cycle}^n(\Pi_1)$*, the isomorphism* $\mathfrak{C}_I(J) \xrightarrow{\sim} \mathfrak{C}_I(\alpha_1(J))$ *induced by* α_2 *maps every lifting cycle-subgroup (cf. Definition 3.6(ii)) of* $\mathfrak{C}_I(J)$ *bijectively onto a lifting cycle-subgroup of* $\mathfrak{C}_I(\alpha_1(J))$*.*

(iii) *Let* $n \geq 3$ *be a positive integer. Then there exists an assignment*

$$\mathrm{Cycle}^n(\Pi_1) \ni J \mapsto \mathfrak{syn}_{I,J}$$

—where $\mathfrak{syn}_{I,J}$ *denotes an I-conjugacy class of isomorphisms* $\Pi_{\mathrm{tpd}} \xrightarrow{\sim} \mathfrak{C}_I(J)$*—such that*

(a) $\mathfrak{syn}_{I,J}$ *maps* I_{tpd} *bijectively onto* I*,*

(b) $\mathfrak{syn}_{I,J}$ *maps the subgroups* J^*_{tpd}*,* J^{**}_{tpd} *bijectively onto lifting cycle-subgroups of* $\mathfrak{C}_I(J)$*, and*

(c) *for* $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}}$*, the diagram (of* I_{tpd}*-, I-conjugacy classes of isomorphisms)*

$$\begin{tikzcd} \Pi_{\mathrm{tpd}} \arrow[r] \arrow[d, "\mathfrak{syn}_{I,J}"'] & \Pi_{\mathrm{tpd}} \arrow[d, "\mathfrak{syn}_{I,\alpha_1(J)}"] \\ \mathfrak{C}_I(J) \arrow[r] & \mathfrak{C}_I(\alpha_1(J)) \end{tikzcd}$$

—where the upper horizontal arrow is the (uniquely determined—cf. the commensurable terminality of I_{tpd} *in* Π_{tpd} *discussed in [18, Prop. 1.2(ii)])* I_{tpd}*-conjugacy class of automorphisms of* Π_{tpd} *that lifts* $\mathfrak{T}_{\Pi_{\mathrm{tpd}}}(\alpha)$ *(cf. [10, Def. 3.19]) and preserves* I_{tpd}*; the lower horizontal arrow is the I-conjugacy class of isomorphisms*

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induced by α_2 *(cf. (ii))—commutes up to possible composition with the cycle symmetry of* $\mathfrak{C}_I(\alpha_1(J))$ *associated with* I *(cf. Definition 3.8).*

Finally, the assignment

$$J \mapsto \mathfrak{syn}_{I,J}$$

is uniquely determined, up to possible composition with cycle symmetries, by these conditions (a), (b), and (c).

(iv) *Let* $n \geq 3$ *be a positive integer,* $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}}$*, and* $J \in \mathrm{Cycle}^n(\Pi_1)$*. Suppose that one of the following conditions is satisfied:*

(a) *The FC-admissible outomorphism of* Π_3 *determined by* α_3 *is* $\in \mathrm{Out}^{\mathrm{FC}}(\Pi_3)^{\mathrm{geo}}$ *(cf. [10, Def. 3.19]).*
(b) $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$.
(c) $n \geq 4$.

Then there exists an automorphism $\beta \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}}$ *such that the FC-admissible outomorphism of* Π_3 *determined by* β_3 *is contained in* $\mathrm{Out}^{\mathrm{FC}}(\Pi_3)^{\mathrm{geo}}$*, and, moreover,* $\alpha_1(J) = \beta_1(J)$*. Finally, the diagram (of* I_{tpd}*-, I-conjugacy classes of isomorphisms)*

$$\begin{array}{ccc} \Pi_{\mathrm{tpd}} & = & \Pi_{\mathrm{tpd}} \\ {\scriptstyle \mathfrak{syn}_{I,J}}\downarrow & & \downarrow{\scriptstyle \mathfrak{syn}_{I,\alpha_1(J)} = \mathfrak{syn}_{I,\beta_1(J)}} \\ \mathfrak{C}_I(J) & \longrightarrow & \mathfrak{C}_I(\alpha_1(J)) = \mathfrak{C}_I(\beta_1(J)) \end{array}$$

—where the lower horizontal arrow is the isomorphism induced by β_2 *(cf. (ii))—commutes up to possible composition with the cycle symmetry of* $\mathfrak{C}_I(\alpha_1(J)) = \mathfrak{C}_I(\beta_1(J))$ *associated with I.*

Proof. Assertion (i) follows immediately from the various definitions involved. Next, we verify assertion (ii). The initial portion of assertion (ii) follows immediately from the discussion of Remark 3.6.1, together with the fact that T is uniquely determined among its $\Pi_{2/1}$-conjugates by the condition $I \subseteq T$ (cf. [18, Prop. 1.5(i)]). The final portion of assertion (ii) follows immediately from Lemma 3.1(ii) (i.e., by considering a suitable generization operation, as in the discussion of Remark 3.6.1). This completes the proof of assertion (ii).

Next, we verify assertion (iii). Let us fix data

$$({}^Y\mathcal{G}, S \subseteq \mathrm{Node}({}^Y\mathcal{G}), \phi\colon {}^Y\mathcal{G}_{\rightsquigarrow S} \xrightarrow{\sim} \mathcal{G});\ {}^Y\Pi_1 \xrightarrow{\sim} \Pi_1;$$

$${}^Y\Pi_2 \xrightarrow{\sim} \Pi_2;\ \Pi_e \subseteq {}^Y\Pi_1$$

for $J \in \mathrm{Cycle}^n(\Pi_1)$ as in Definition 3.6(i.a)–(i.d), and let ${}^YT \subseteq {}^Y\Pi_{2/1}$ be a $\{1,2\}$-tripod as in the discussion of Definition 3.6(i). Let ${}^Y\Pi_{\mathrm{tpd}} \subseteq {}^Y\Pi_3$ be a 3-central tripod of ${}^Y\Pi_3$. Here, we note that since $J \in \mathrm{Cycle}^n(\Pi_1)$, and $n \geq 3$, it follows that the above isomorphism ${}^Y\Pi_2 \xrightarrow{\sim} \Pi_2$ lifts to a PFC-admissible isomorphism ${}^Y\Pi_3 \xrightarrow{\sim} \Pi_3$ that maps ${}^Y\Pi_{\mathrm{tpd}}$ to a Π_3-conjugate of Π_{tpd} (cf. [8, Th. B]; [10, Th. 3.16(v)]; [10, Rem 4.14.1]).

Now one verifies immediately that, to verify the existence portion of assertion (iii), by applying a suitable generization operation as in the discussion of Remark 3.6.1, we may assume without loss of generality that $\mathrm{Node}({}^Y\mathcal{G})^\sharp = 1$ (an assumption that will be invoked when we apply Lemma 3.7 in the argument to follow). Then, by considering the geometric

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(hence, in particular, C-admissible) outer isomorphism of [10, Th. 3.18(ii)] in the case where we take the "(T,T')" of [10, Th. 3.18(ii)] to be $({}^{Y}\Pi_{\mathrm{tpd}}, {}^{Y}T)$, we obtain an outer isomorphism $\Pi_{\mathrm{tpd}} \xrightarrow{\sim} \mathfrak{C}_I(J)$. Moreover, by considering the composite of this outer isomorphism with a suitable geometric outomorphism of Π_{tpd}, we may assume without loss of generality that this outer isomorphism $\Pi_{\mathrm{tpd}} \xrightarrow{\sim} \mathfrak{C}_I(J)$ maps the Π_{tpd}-conjugacy class of I_{tpd} to the $\mathfrak{C}_I(J)$-conjugacy class of I. Thus, since I is commensurably terminal in $\mathfrak{C}_I(J)$ (cf. [18, Prop. 1.2(ii)]), we obtain a uniquely determined I-conjugacy class of isomorphisms $\mathfrak{syn}_{I,J}\colon \Pi_{\mathrm{tpd}} \xrightarrow{\sim} \mathfrak{C}_I(J)$ that lifts the outer isomorphism just discussed and satisfies condition (a). On the other hand, one verifies immediately from the various definitions involved that $\mathfrak{syn}_{I,J}$ also satisfies condition (b).

Next, we verify that $\mathfrak{syn}_{I,J}$ satisfies condition (c). To this end, let us observe that it follows immediately from the various definitions involved (cf. also our assumption that $\mathrm{Node}({}^{Y}\mathcal{G})^{\sharp} = 1$), that $\alpha_1(J)$ admits data as in Definition 3.6 (i.a)–(i.d) such that

- the portion of these data that correspond to the data of Definition 3.6(i.a) and (i.d) is of the form

$$({}^{Y}\mathcal{G}, S \subseteq \mathrm{Node}({}^{Y}\mathcal{G}), \psi\colon {}^{Y}\mathcal{G}_{\rightsquigarrow S} \xrightarrow{\sim} \mathcal{G});\ \Pi_e \subseteq {}^{Y}\Pi_1$$

  for some isomorphism $\psi\colon {}^{Y}\mathcal{G}_{\rightsquigarrow S} \xrightarrow{\sim} \mathcal{G}$, and, moreover,
- the composite

$${}^{Y}\Pi_2 \xrightarrow{\sim} \Pi_2 \xrightarrow[\sim]{\alpha_2} \Pi_2 \xleftarrow{\sim} {}^{Y}\Pi_2$$

  —where the first (resp. third) arrow is the isomorphism arising from the data (cf. Definition 3.6(i.c)) for J (resp. $\alpha_1(J)) \in \mathrm{Cycle}^n(\Pi_1)$ under consideration— is the identity automorphism.

Thus, to verify the assertion that $\mathfrak{syn}_{I,J}$ satisfies condition (c), it suffices to verify that the I-conjugacy class of isomorphisms "$\mathfrak{syn}_{I,J}\colon \Pi_{\mathrm{tpd}} \xrightarrow{\sim} \mathfrak{C}_I(J)$" constructed above from a fixed choice of data as in Definition 3.6(i.a)–(i.d) does not depend on this choice of data. On the other hand, this follows immediately from Lemma 3.7(ii) (cf. our assumption that $\mathrm{Node}({}^{Y}\mathcal{G})^{\sharp} = 1$).

Finally, we consider the final portion of assertion (iii) concerning uniqueness. To this end, we observe that, by considering the case where ${}^{Y}\mathcal{G}$, as well as each of the branches of the underlying semi-graph of ${}^{Y}\mathcal{G}$, is defined over a number field F, it follows immediately, by considering automorphisms $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}}$ that arise from scheme theory, that given any element $\gamma \in \mathrm{Out}(\Pi_{\mathrm{tpd}})$ that arises from an element of the absolute Galois group of F, there exists an $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n, I)^{\mathrm{G}}$ such that $\alpha(J) = J$ and $\mathfrak{T}_{\Pi_{\mathrm{tpd}}}(\alpha) = \gamma$. Thus, the uniqueness under consideration follows immediately from the geometricity of elements of $\mathrm{Out}(\Pi_{\mathrm{tpd}})$ that commute with the image of the absolute Galois group of F, that is, in other words, from the Grothendieck Conjecture for tripods over number fields (cf. [28, Th. 0.3]; [13, Th. A]). This completes the proof of assertion (iii).

Finally, we verify assertion (iv). If condition (a) is satisfied, then, by taking the "β" of assertion (iv) to be α, we conclude that assertion (iv) follows immediately from assertion (iii), together with the definition of $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{geo}}$. Next, let us observe that, by applying assertion (iv) in the case where condition (a) is satisfied, we conclude that, to verify assertion

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(iv) in the case where either (b) or (c) is satisfied, it suffices to verify that the following assertion holds:

Claim 3.10.A: Write

$$\mathrm{Out}(\Pi_1 \supseteq J) \ \subseteq \ \mathrm{Out}(\Pi_1)$$

for the subgroup of $\mathrm{Out}(\Pi_1)$ consisting of outomorphisms of Π_1 that preserve the Π_1-conjugacy class of J and

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{G}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)^{\mathrm{G}}/\mathrm{Inn}(\Pi_n) \ \subseteq \ \mathrm{Out}^{\mathrm{FC}}(\Pi_n).$$

Then every element of the image of the natural injection

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{G}} \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$$

(cf. [8, Th. B]) may be written as a product of an element of the image of the natural injection $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{geo}} \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ and an element of $\mathrm{Out}(\Pi_1 \supseteq J)^{\mathrm{G}} \stackrel{\mathrm{def}}{=} \mathrm{Out}(\Pi_1 \supseteq J) \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_1)^{\mathrm{G}}$.

To verify Claim 3.10.A, write $\mathrm{Out}^{\mathrm{FC}}(\Pi_n, J)^{\mathrm{G}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{G}}$ for the subgroup of $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{G}}$ obtained by forming the inverse image of the closed subgroup $\mathrm{Out}(\Pi_1 \supseteq J) \subseteq \mathrm{Out}(\Pi_1)$ via the natural injection $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{G}} \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$. Then one verifies immediately, by considering the exact sequence

$$1 \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{geo}} \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \xrightarrow{\mathfrak{T}_{\Pi_{\mathrm{tpd}}}} \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{tpd}})^{\Delta+} \longrightarrow 1$$

(cf. conditions (b) and (c); [10, Def. 3.19]; [10, Cor. 4.15]), that, to verify Claim 3.10.A, it suffices to verify that the following assertion holds:

Claim 3.10.B: The composite

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n, J)^{\mathrm{G}} \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \stackrel{\mathfrak{T}_{\Pi_{\mathrm{tpd}}}}{\twoheadrightarrow} \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{tpd}})^{\Delta+}$$

is surjective.

To verify Claim 3.10.B, let $({}^{Y}\mathcal{G}, S \subseteq \mathrm{Node}({}^{Y}\mathcal{G}), \phi\colon {}^{Y}\mathcal{G}_{\rightsquigarrow S} \xrightarrow{\sim} \mathcal{G})$ be an n-cuspidalizable degeneration structure on $\mathcal{G}$ with respect to which J is a cycle-subgroup such that ${}^{Y}\mathcal{G}$ is totally degenerate (cf. [9, Def. 2.3(iv)]). (One verifies immediately that such a degeneration structure always exists.) Now let us identify $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ with $\mathrm{Out}^{\mathrm{FC}}({}^{Y}\Pi_n)$ via a(n) (uniquely determined, up to permutation of the n factors—cf. [8, Th. B]) PFC-admissible (cf. [9, Def. 1.4(iii)]) outer isomorphism $\Pi_n \xrightarrow{\sim} {}^{Y}\Pi_n$ that is compatible with the outomorphism of the display of Definition 3.5(i) (cf. [10, Prop. 3.24(i)]). Then it follows immediately from the various definitions involved that the closed subgroup $\mathrm{Out}^{\mathrm{FC}}({}^{Y}\Pi_n)^{\mathrm{brch}} \subseteq \mathrm{Out}^{\mathrm{FC}}({}^{Y}\Pi_n)$ (cf. [10, Def. 4.6(i)]) is contained in the closed subgroup $\mathrm{Out}^{\mathrm{FC}}(\Pi_n, J)^{\mathrm{G}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$. On the other hand, it follows immediately from the proof of [10, Cor. 4.15] that the composite

$$\mathrm{Out}^{\mathrm{FC}}({}^{Y}\Pi_n)^{\mathrm{brch}} \hookrightarrow \mathrm{Out}^{\mathrm{FC}}({}^{Y}\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \stackrel{\mathfrak{T}_{\Pi_{\mathrm{tpd}}}}{\twoheadrightarrow} \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{tpd}})^{\Delta+}$$

is surjective. This completes the proof of Claim 3.10.B, hence also of assertion (iv) in the case where either (b) or (c) is satisfied. $\square$

Remark 3.10.1.

(i) The content of Theorem 3.10(iv) may be regarded, that is, by considering the various lifting cycle-subgroups involved, as a formulation of the construction of the two sections discussed in [5, Prop. 2.7] (which plays an essential role in the proof of [5, Th. 2.4]), in terms of the purely combinatorial and algebraic techniques developed in the present series of papers.

(ii) In this context, we observe in passing that (one verifies immediately that) for arbitrary nonnegative integers g, r such that

- $3g-3+r>0$, and, moreover,
- if $g=0$, then r is even,

there exists a stable log curve of type (g,r) which admits an automorphism that is linear over the base scheme under consideration and fixes a node of the stable log curve, but switches the branches of this node. Thus, by considering the resulting automorphism of the associated semi-graph of anabelioids of pro-Σ PSC-type, one concludes that the diagrams of Theorem 3.10(iii) and (iv) fail to commute, in general, if one does not allow for the possibility of composition with a cycle symmetry. This situation contrasts with the situation discussed in [5, Prop. 2.7], where two independent sections are obtained, by considering orientations on the various cycles involved.

(iii) The orientation-theoretic portion of [5, Prop. 2.7] referred to in (ii) above may be interpreted, from the point of view of the present paper, as a lifting "$\mathfrak{C}_I^{\pm}$" of the map $\mathfrak{C}_I$ of Theorem 3.10(ii) as follows. In the notation of Theorem 3.10, let us write

- $\mathrm{Cycle}^n(\Pi_1)^{\pm}$ for the set of pairs consisting of a cycle-subgroup $J\in\mathrm{Cycle}^n(\Pi_1)$ and an *orientation* on J (cf. Remark 3.6.2(i));
- $\mathrm{Tpd}_I(\Pi_{2/1})^{\pm}$ for the set of pairs consisting of a tripodal subgroup $T\in\mathrm{Tpd}_I(\Pi_{2/1})$ and an *orientation* on T (cf. Remark 3.6.2(ii)).

Thus, one has natural surjections $\mathrm{Cycle}^n(\Pi_1)^{\pm}\twoheadrightarrow\mathrm{Cycle}^n(\Pi_1)$, $\mathrm{Tpd}_I(\Pi_{2/1})^{\pm}\twoheadrightarrow\mathrm{Tpd}_I(\Pi_{2/1})$, which may be regarded as torsors over the group $\{\pm1\}$. Moreover, one verifies immediately from the functoriality of the various isomorphisms that appeared in the constructions of Remark 3.6.2(i) and (ii) that the action (cf. Theorem 3.10(i)) of $\mathrm{Aut}^{\mathrm{FC}}(\Pi_n,I)^{\mathrm{G}}$ on the sets $\mathrm{Cycle}^n(\Pi_1)$, $\mathrm{Tpd}_I(\Pi_{2/1})$ lifts naturally to an action of $\mathrm{Aut}^{\mathrm{FC}}(\Pi_n,I)^{\mathrm{G}}$ on the sets $\mathrm{Cycle}^n(\Pi_1)^{\pm}$, $\mathrm{Tpd}_I(\Pi_{2/1})^{\pm}$. Thus, the inverse of the bijective correspondence of the final display of Remark 3.6.2(ii) determines a natural $\mathrm{Aut}^{\mathrm{FC}}(\Pi_n,I)^{\mathrm{G}}$-*equivariant lifting*

$$\mathfrak{C}_I^{\pm}\colon \mathrm{Cycle}^n(\Pi_1)^{\pm}\longrightarrow\mathrm{Tpd}_I(\Pi_{2/1})^{\pm}$$

of the map $\mathfrak{C}_I$ of Theorem 3.10(ii). (Thus, the $\mathrm{Aut}^{\mathrm{FC}}(\Pi_n,I)^{\mathrm{G}}$-equivariance of $\mathfrak{C}_I^{\pm}$ implies, in particular, that $\mathfrak{C}_I^{\pm}$ does not factor through the natural surjection $\mathrm{Cycle}^n(\Pi_1)^{\pm}\twoheadrightarrow\mathrm{Cycle}^n(\Pi_1)$.) Moreover, if $n\geq 3$, and one regards the Π_{tpd}-conjugacy class of cuspidal subgroups of Π_{tpd} determined by J^*_{tpd} as being "positive," then it follows immediately from the definition of $\mathrm{Tpd}_I(\Pi_{2/1})^{\pm}$ that this lifting $\mathfrak{C}_I^{\pm}$ naturally determines an assignment

$$\mathrm{Cycle}^n(\Pi_1)^{\pm}\ni J^{\pm}\ \mapsto\ \mathfrak{syn}^{\pm}_{I,J^{\pm}}$$

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—where $J^{\pm} \mapsto J \in \mathrm{Cycle}^n(\Pi_1)$, and $\mathfrak{syn}^{\pm}_{I,J^{\pm}}$ denotes an I-conjugacy class of isomorphisms $\Pi_{\mathrm{tpd}} \xrightarrow{\sim} \mathfrak{C}_I(J)$ that coincides, up to possible composition with a cycle symmetry, with the I-conjugacy class of isomorphisms $\mathfrak{syn}_{I,J}$ of Theorem 3.10(iii)—such that if, in the diagram (of I_{tpd}-, I-conjugacy classes of isomorphisms) in the display of Theorem 3.10(iii.c), one replaces "$\mathfrak{syn}$" by "$\mathfrak{syn}^{\pm}$," then the diagram *commutes*, that is, even if one does not allow for possible composition with cycle symmetries.

DEFINITION 3.11. Suppose that $\Sigma = \mathfrak{Primes}$, and that $k = \mathbb{C}$, that is, that we are in the situation of Definition 2.22. We shall apply the notational conventions established in Definition 2.22. Moreover, we shall use similar notation

$$\mathfrak{Y}_E \overset{\mathrm{def}}{=} (Y_E^{\log})_{\mathrm{an}}(\mathbb{C})|_s,\ {}^{Y}\Pi_E^{\mathrm{disc}} \overset{\mathrm{def}}{=} \pi_1(\mathfrak{Y}_E),\ \mathfrak{Y}_n \overset{\mathrm{def}}{=} \mathfrak{Y}_{\{1,\dots,n\}},\ \mathfrak{Y} \overset{\mathrm{def}}{=} \mathfrak{Y}_1,$$

$${}^{Y}\Pi_n^{\mathrm{disc}} \overset{\mathrm{def}}{=} {}^{Y}\Pi_{\{1,\dots,n\}}^{\mathrm{disc}},\ {}^{Y}p_{E/E'}^{\mathrm{an}} \colon \mathfrak{Y}_E \to \mathfrak{Y}_{E'},\ {}^{Y}p_{E/E'}^{\Pi^{\mathrm{disc}}} \colon {}^{Y}\Pi_E^{\mathrm{disc}} \twoheadrightarrow {}^{Y}\Pi_{E'}^{\mathrm{disc}},$$

$${}^{Y}\Pi_{E/E'}^{\mathrm{disc}} \overset{\mathrm{def}}{=} \mathrm{Ker}({}^{Y}p_{E/E'}^{\Pi^{\mathrm{disc}}}) \subseteq {}^{Y}\Pi_E^{\mathrm{disc}},$$

$${}^{Y}p_{n/m}^{\mathrm{an}} \overset{\mathrm{def}}{=} {}^{Y}p_{\{1,\dots,n\}/\{1,\dots,m\}}^{\mathrm{an}} \colon \mathfrak{Y}_n \longrightarrow \mathfrak{Y}_m,$$

$${}^{Y}p_{n/m}^{\Pi^{\mathrm{disc}}} \overset{\mathrm{def}}{=} {}^{Y}p_{\{1,\dots,n\}/\{1,\dots,m\}}^{\Pi^{\mathrm{disc}}} \colon {}^{Y}\Pi_n^{\mathrm{disc}} \twoheadrightarrow {}^{Y}\Pi_m^{\mathrm{disc}},$$

$${}^{Y}\Pi_{n/m}^{\mathrm{disc}} \overset{\mathrm{def}}{=} {}^{Y}\Pi_{\{1,\dots,n\}/\{1,\dots,m\}}^{\mathrm{disc}} \subseteq {}^{Y}\Pi_n^{\mathrm{disc}},\ {}^{Y}\widehat{\Pi}_{(-)}^{\mathrm{disc}},$$

$${}^{Y}\mathcal{G}^{\mathrm{disc}},\ {}^{Y}\mathcal{G}_{i\in E,y}^{\mathrm{disc}},\ \Pi_{{}^{Y}\mathcal{G}^{\mathrm{disc}}},\ \Pi_{{}^{Y}\mathcal{G}_{i\in E,y}^{\mathrm{disc}}}$$

for objects associated with the stable log curve $Y^{\log} = Y_1^{\log}$ to the notation introduced in Definitions 2.22 and 2.23.

DEFINITION 3.12. Let $\mathcal{J}$ be a semi-graph of temperoids of HSD-type (cf. Definition 2.3(iii)). Then we shall refer to a triple

$$(\mathcal{H}, S \subseteq \mathrm{Node}(\mathcal{H}), \phi \colon \mathcal{H}_{\rightsquigarrow S} \xrightarrow{\sim} \mathcal{J})$$

(cf. Definition 2.9) consisting of a semi-graph of temperoids of HSD-type $\mathcal{H}$, a subset $S \subseteq \mathrm{Node}(\mathcal{H})$, and an isomorphism $\phi \colon \mathcal{H}_{\rightsquigarrow S} \xrightarrow{\sim} \mathcal{J}$ of semi-graphs of temperoids of HSD-type as a *degeneration structure* on $\mathcal{J}$ (cf. [10, Def. 3.23(i)]).

DEFINITION 3.13. In the situation of Definition 3.11:

(i) Let $({}^{Y}\mathcal{G}^{\mathrm{disc}}, S \subseteq \mathrm{Node}({}^{Y}\mathcal{G}^{\mathrm{disc}}), \phi \colon {}^{Y}\mathcal{G}^{\mathrm{disc}}_{\rightsquigarrow S} \xrightarrow{\sim} \mathcal{G}^{\mathrm{disc}})$ be a degeneration structure on $\mathcal{G}^{\mathrm{disc}}$ (cf. Definition 3.12), $e \in S$, and $J \subseteq \Pi_1^{\mathrm{disc}}$ a subgroup of Π_1^{disc}. Then we shall say that $J \subseteq \Pi_1^{\mathrm{disc}}$ is a *cycle-subgroup* of Π_1^{disc} (with respect to $[{}^{Y}\mathcal{G}^{\mathrm{disc}}, S \subseteq \mathrm{Node}({}^{Y}\mathcal{G}^{\mathrm{disc}}), \phi \colon {}^{Y}\mathcal{G}^{\mathrm{disc}}_{\rightsquigarrow S} \xrightarrow{\sim} \mathcal{G}^{\mathrm{disc}}]$, associated with $e \in S$) if J is contained in the Π_1^{disc}-conjugacy class of subgroups of Π_1^{disc} obtained by forming the image of a nodal subgroup of $\Pi_{{}^{Y}\mathcal{G}^{\mathrm{disc}}}$ associated with e via the composite of outer isomorphisms

$$\Pi_{{}^{Y}\mathcal{G}^{\mathrm{disc}}} \xrightarrow[\sim]{\Phi^{-1}_{{}^{Y}\mathcal{G}^{\mathrm{disc}}_{\rightsquigarrow S}}} \Pi_{{}^{Y}\mathcal{G}^{\mathrm{disc}}_{\rightsquigarrow S}} \xrightarrow{\sim} \Pi_{\mathcal{G}^{\mathrm{disc}}} \xrightarrow{\sim} \Pi_1^{\mathrm{disc}}$$

—where the first arrow is the inverse of the specialization outer isomorphism $\Phi_{{}^{Y}\mathcal{G}^{\mathrm{disc}}_{\rightsquigarrow S}}$ (cf. Proposition 2.10), the second arrow is the graphic (cf. Definition 2.7(ii)) outer

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isomorphism $\Pi_{{}^Y\mathcal{G}^{\mathrm{disc}}_{\rightsquigarrow S}} \xrightarrow{\sim} \Pi_{\mathcal{G}^{\mathrm{disc}}}$ induced by ϕ, and the third arrow is the natural outer isomorphism $\Pi_{\mathcal{G}^{\mathrm{disc}}} \xrightarrow{\sim} \Pi_1^{\mathrm{disc}}$ of (the second to last display of) Definition 2.23(i) (cf. the left-hand portion of Figure 1).

(ii) Let $J \subseteq \Pi_1^{\mathrm{disc}}$ be a cycle-subgroup of Π_1^{disc} (cf. (i)). Thus, we have

(a) a degeneration structure $({}^Y\mathcal{G}^{\mathrm{disc}}, S \subseteq \mathrm{Node}({}^Y\mathcal{G}^{\mathrm{disc}}), \phi\colon {}^Y\mathcal{G}^{\mathrm{disc}}_{\rightsquigarrow S} \xrightarrow{\sim} \mathcal{G}^{\mathrm{disc}})$ on $\mathcal{G}^{\mathrm{disc}}$ (cf. Definition 3.12),

(b) an isomorphism ${}^Y\Pi_1^{\mathrm{disc}} \xrightarrow{\sim} \Pi_1^{\mathrm{disc}}$ that is compatible with the composite of the display of (i) (cf. also [the second to last display of] Definition 2.23(i)) in the case where we take the "$({}^Y\mathcal{G}^{\mathrm{disc}}, S \subseteq \mathrm{Node}({}^Y\mathcal{G}^{\mathrm{disc}}), \phi\colon {}^Y\mathcal{G}^{\mathrm{disc}}_{\rightsquigarrow S} \xrightarrow{\sim} \mathcal{G}^{\mathrm{disc}})$" of (i) to be the degeneration structure of (a),

(c) an isomorphism ${}^Y\Pi_2^{\mathrm{disc}} \xrightarrow{\sim} \Pi_2^{\mathrm{disc}}$ that lifts (cf. Corollary 2.20(v)) the isomorphism of (b) and, moreover, determines a PFC-admissible isomorphism between the respective profinite completions, and

(d) a nodal subgroup $\Pi_e \subseteq {}^Y\Pi_1^{\mathrm{disc}}$ of ${}^Y\Pi_1^{\mathrm{disc}}$ associated with a (uniquely determined—cf. Corollary 2.18(iii)) node e of ${}^Y\mathcal{G}^{\mathrm{disc}}$

such that the image of the nodal subgroup $\Pi_e \subseteq {}^Y\Pi_1^{\mathrm{disc}}$ of (d) via the isomorphism ${}^Y\Pi_1^{\mathrm{disc}} \xrightarrow{\sim} \Pi_1^{\mathrm{disc}}$ of (b) coincides with $J \subseteq \Pi_1^{\mathrm{disc}}$. We shall say that a subgroup $T \subseteq \Pi_{2/1}^{\mathrm{disc}}$ of $\Pi_{2/1}^{\mathrm{disc}}$ is a *tripodal subgroup* associated with J if T coincides—relative to some choice of data (a), (b), (c), (d) as above (but cf. also Remark 3.6.1 and Corollary 2.19(i)!)—with the image, via the lifting ${}^Y\Pi_2^{\mathrm{disc}} \xrightarrow{\sim} \Pi_2^{\mathrm{disc}}$ of (c), of some $\{1,2\}$-tripod in ${}^Y\Pi_{2/1}^{\mathrm{disc}} \subseteq {}^Y\Pi_2^{\mathrm{disc}}$ (cf. Definition 2.23(ii)) arising from e (cf. Definition 2.23(iii); [10, Def. 3.7(i)]), and, moreover, the centralizer $Z_{\Pi_2^{\mathrm{disc}}}(T)$ maps bijectively, via $p_{2/1}^{\Pi^{\mathrm{disc}}}: \Pi_2^{\mathrm{disc}} \twoheadrightarrow \Pi_1^{\mathrm{disc}}$, onto $J \subseteq \Pi_1^{\mathrm{disc}}$ (cf. Corollary 2.17(i); [10, Lem. 3.11(iv) and (vii)]).

(iii) Let $J \subseteq \Pi_1^{\mathrm{disc}}$ be a cycle-subgroup of Π_1^{disc} (cf. (i)) and $T \subseteq \Pi_{2/1}^{\mathrm{disc}}$ a tripodal subgroup associated with J (cf. (ii)). Then we shall refer to a subgroup of T that arises from a nodal (resp. cuspidal) subgroup contained in the $\{1,2\}$-tripod in ${}^Y\Pi_{2/1}^{\mathrm{disc}} \subseteq {}^Y\Pi_2^{\mathrm{disc}}$ of (ii) as a *lifting cycle-subgroup* (resp. *distinguished cuspidal subgroup*) of T (cf. the right-hand portion of Figure 1).

(iv) Let $J \subseteq \Pi_1^{\mathrm{disc}}$ be a cycle-subgroup (cf. (i)); $T \subseteq \Pi_{2/1}^{\mathrm{disc}}$ a tripodal subgroup associated with J (cf. (ii)); $I \subseteq T$ a distinguished cuspidal subgroup of T (cf. (iii)). Then it follows immediately from the various definitions involved, together with Theorem 2.24(i) that there exists a unique outomorphism ι of T such that the induced outomorphism of the profinite completion $\widehat{T}$ of T coincides with the outomorphism of $\widehat{T}$ determined by the cycle symmetry of $\widehat{T}$ associated with the profinite completion $\widehat{I}$ of I (cf. Definition 3.8). Moreover, since I is commensurably terminal in T (cf. Corollary 2.18(v)), it follows immediately from Corollary 2.17(ii) that there exists a uniquely determined I-conjugacy class of automorphisms of T that lifts ι and preserves $I \subseteq T$. We shall refer to this I-conjugacy class of automorphisms of T as the *cycle symmetry of T associated with I.*

THEOREM 3.14 (Discrete version of canonical liftings of cycles). *In the notation of Definition 3.11, let $I \subseteq \Pi_{2/1}^{\mathrm{disc}} \subseteq \Pi_2^{\mathrm{disc}}$ be a cuspidal inertia group associated with the diagonal cusp of a fiber of $p_{2/1}^{\mathrm{an}}$; $\Pi_{\mathrm{tpd}} \subseteq \Pi_3^{\mathrm{disc}}$ a 3-central $\{1,2,3\}$-tripod of Π_3^{disc} (cf. Definition 2.23(ii) and (iii)); $I_{\mathrm{tpd}} \subseteq \Pi_{\mathrm{tpd}}$ a cuspidal subgroup of Π_{tpd} that does not arise from a cusp of a*

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fiber of $p^{\mathrm{an}}_{3/2}$*;* J^{*}_{tpd}*,* $J^{**}_{\mathrm{tpd}} \subseteq \Pi_{\mathrm{tpd}}$ *cuspidal subgroups of* Π_{tpd} *such that* I_{tpd}*,* J^{*}_{tpd}*, and* J^{**}_{tpd} *determine three distinct* Π_{tpd}*-conjugacy classes of subgroups of* Π_{tpd}*. (Note that one verifies immediately from the various definitions involved that such cuspidal subgroups* I_{tpd}*,* J^{*}_{tpd}*, and* J^{**}_{tpd} *always exist.) For* $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_2)$ *(cf. the notational conventions introduced in the statement of Corollary 2.20), write*

$$\alpha_1 \in \mathrm{Aut}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_1)$$

for the automorphism of Π^{disc}_1 *determined by* α*;*

$$\mathrm{Aut}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_2, I) \subseteq \mathrm{Aut}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_2)$$

for the subgroup consisting of $\beta \in \mathrm{Aut}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_2)$ *such that* $\beta(I) = I$*;*

$$\mathrm{Aut}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_2)^{\mathrm{G}} \subseteq \mathrm{Aut}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_2)$$

for the subgroup consisting of $\beta \in \mathrm{Aut}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_2)$ *such that the image of* β *via the composite* $\mathrm{Aut}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_2) \twoheadrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_2) \overset{\sim}{\to} \mathrm{Out}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_1) \to \mathrm{Out}(\Pi_{\mathcal{G}^{\mathrm{disc}}})$*—where the second arrow is the natural bijection of Corollary 2.20(v), and the third arrow is the homomorphism induced by the natural outer isomorphism* $\Pi^{\mathrm{disc}}_1 \overset{\sim}{\to} \Pi_{\mathcal{G}^{\mathrm{disc}}}$*—is graphic (cf. Definition 2.7(ii));*

$$\mathrm{Aut}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_2, I)^{\mathrm{G}} \overset{\mathrm{def}}{=} \mathrm{Aut}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_2, I) \cap \mathrm{Aut}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_2)^{\mathrm{G}};$$

$$\mathrm{Cycle}(\Pi^{\mathrm{disc}}_1)$$

for the set of cycle-subgroups of Π^{disc}_1 *(cf. Definition 3.13(i));*

$$\mathrm{Tpd}_I(\Pi^{\mathrm{disc}}_{2/1})$$

for the set of subgroups $T \subseteq \Pi^{\mathrm{disc}}_{2/1}$ *such that* T *is a tripodal subgroup associated with some cycle-subgroup of* Π^{disc}_1 *(cf. Definition 3.13(ii)), and, moreover,* I *is a distinguished cuspidal subgroup (cf. Definition 3.13(iii)) of* T*. Then the following hold:*

(i) *Let* $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_2, I)^{\mathrm{G}}$*,* $J \in \mathrm{Cycle}(\Pi^{\mathrm{disc}}_1)$*, and* $T \in \mathrm{Tpd}_I(\Pi^{\mathrm{disc}}_{2/1})$*. Then it holds that*

$$\alpha_1(J) \in \mathrm{Cycle}(\Pi^{\mathrm{disc}}_1), \quad \alpha(T) \in \mathrm{Tpd}_I(\Pi^{\mathrm{disc}}_{2/1}).$$

Thus, $\mathrm{Aut}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_2, I)^{\mathrm{G}}$ *acts naturally on* $\mathrm{Cycle}(\Pi^{\mathrm{disc}}_1)$*,* $\mathrm{Tpd}_I(\Pi^{\mathrm{disc}}_{2/1})$*.*

(ii) *There exists a unique* $\mathrm{Aut}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_2, I)^{\mathrm{G}}$*-equivariant (cf. (i)) map*

$$\mathfrak{C}_I \colon \mathrm{Cycle}(\Pi^{\mathrm{disc}}_1) \longrightarrow \mathrm{Tpd}_I(\Pi^{\mathrm{disc}}_{2/1})$$

such that, for every $J \in \mathrm{Cycle}(\Pi^{\mathrm{disc}}_1)$*,* $\mathfrak{C}_I(J)$ *is a tripodal subgroup associated with* J*. Moreover, for every* $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi^{\mathrm{disc}}_2, I)^{\mathrm{G}}$ *and* $J \in \mathrm{Cycle}(\Pi^{\mathrm{disc}}_1)$*, the isomorphism* $\mathfrak{C}_I(J) \overset{\sim}{\to} \mathfrak{C}_I(\alpha_1(J))$ *induced by* α *maps every lifting cycle-subgroup (cf. Definition 3.13(iii)) of* $\mathfrak{C}_I(J)$ *bijectively onto a lifting cycle-subgroup of* $\mathfrak{C}_I(\alpha_1(J))$*.*

(iii) *There exists an assignment*

$$\mathrm{Cycle}(\Pi^{\mathrm{disc}}_1) \ni J \mapsto \mathfrak{syn}_{I,J}$$

—where $\mathfrak{syn}_{I,J}$ *denotes an* I*-conjugacy class of isomorphisms* $\Pi_{\mathrm{tpd}} \overset{\sim}{\to} \mathfrak{C}_I(J)$*—such that*

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(a) $\mathfrak{syn}_{I,J}$ *maps* I_{tpd} *bijectively onto* I *in a fashion that is compatible with the natural isomorphism* $I_{\mathrm{tpd}} \xrightarrow{\sim} I$ *induced by the projection* $p^{\Pi^{\mathrm{disc}}}_{\{1,2,3\}/\{1,3\}} \colon \Pi_3^{\mathrm{disc}} \twoheadrightarrow \Pi_{\{1,3\}}^{\mathrm{disc}}$ *and the natural outer isomorphism* $\Pi_{\{1,3\}}^{\mathrm{disc}} \xrightarrow{\sim} \Pi_{\{1,2\}}^{\mathrm{disc}}$ *obtained by switching the labels "2" and "3" (cf. Corollary 2.17(ii); Corollary 2.18(v); [10, Lem. 3.6(iv)]),*

(b) $\mathfrak{syn}_{I,J}$ *maps the subgroups* J^*_{tpd}, J^{**}_{tpd} *bijectively onto lifting cycle-subgroups of* $\mathfrak{C}_I(J)$, *and*

(c) *for* $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_2^{\mathrm{disc}}, I)^{\mathrm{G}}$, *the diagram (of* I_{tpd}*-, I-conjugacy classes of isomorphisms)*

$$\begin{tikzcd} \Pi_{\mathrm{tpd}} \arrow[r] \arrow[d, "\mathfrak{syn}_{I,J}"'] & \Pi_{\mathrm{tpd}} \arrow[d, "\mathfrak{syn}_{I,\alpha_1(J)}"] \\ \mathfrak{C}_I(J) \arrow[r] & \mathfrak{C}_I(\alpha_1(J)) \end{tikzcd}$$

—where the upper horizontal arrow is the (uniquely determined—cf. the commensurable terminality of I_{tpd} *of* Π_{tpd} *discussed in Corollary 2.18(v))* I_{tpd}*-conjugacy class of automorphisms of* Π_{tpd} *that lifts* $\mathfrak{T}_{\Pi_{\mathrm{tpd}}}(\alpha)$ *(cf. Corollary 2.20(v); Theorem 2.24(iv)) and preserves* I_{tpd}*; the lower horizontal arrow is the I-conjugacy class of isomorphisms induced by* α *(cf. (ii))—commutes up to possible composition with the cycle symmetry of* $\mathfrak{C}_I(\alpha_1(J))$ *associated with I (cf. Definition 3.13(iv)).*

Finally, the assignment

$$J \mapsto \mathfrak{syn}_{I,J}$$

is uniquely determined, up to possible composition with cycle symmetries, by these conditions (a), (b), and (c).

(iv) *Let* $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_2^{\mathrm{disc}}, I)^{\mathrm{G}}$ *and* $J \in \mathrm{Cycle}(\Pi_1)$. *Then there exists an automorphism* $\beta \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_2^{\mathrm{disc}}, I)^{\mathrm{G}}$ *such that* $\mathfrak{T}_{\Pi_{\mathrm{tpd}}}(\beta)$ *(cf. Corollary 2.20(v); Theorem 2.24(iv)) is trivial, and, moreover,* $\alpha_1(J) = \beta_1(J)$. *Finally, the diagram (of* I_{tpd}*-, I-conjugacy classes of isomorphisms)*

$$\begin{tikzcd} \Pi_{\mathrm{tpd}} \arrow[r, equal] \arrow[d, "\mathfrak{syn}_{I,J}"'] & \Pi_{\mathrm{tpd}} \arrow[d, "\mathfrak{syn}_{I,\alpha_1(J)} = \mathfrak{syn}_{I,\beta_1(J)}"] \\ \mathfrak{C}_I(J) \arrow[r] & \mathfrak{C}_I(\alpha_1(J)) = \mathfrak{C}_I(\beta_1(J)) \end{tikzcd}$$

—where the lower horizontal arrow is the isomorphism induced by β *(cf. (ii))—commutes up to possible composition with the cycle symmetry of* $\mathfrak{C}_I(\alpha_1(J)) = \mathfrak{C}_I(\beta_1(J))$ *associated with I.*

Proof. Assertion (i) follows from the various definitions involved. Assertion (ii) follows immediately from the evident discrete version (cf. Corollaries 2.17(ii); 2.19(i)) of the argument involving Remark 3.6.1 that was given in the proof of Theorem 3.10,(ii). The existence portion of assertion (iii) follows, in light of Corollaries 2.17(ii); 2.20(i) and (v), from a similar argument to the argument applied in the proof of the existence portion of Theorem 3.10(iii) (cf. also the fact that the "$\mathfrak{syn}_{I,J}$" of Theorem 3.10(iii) was constructed from a suitable geometric outer isomorphism). The uniqueness portion of assertion (iii) follows from the compatibility portion of condition (a), together with the computation of discrete outomorphism groups given in Theorem 2.24(ii). Assertion (iv) follows immediately

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from assertion (iii), together with a similar argument to the argument applied in the proof of the surjectivity portion of Theorem 2.24(iv) (cf. the argument given in the proof of Theorem 3.10(iv)). This completes the proof of Theorem 3.14. ∎

REMARK 3.14.1. One verifies immediately that the discrete constructions of Theorem 3.14(i)–(iv) are compatible, in an evident sense, with the pro-Σ constructions of Theorem 3.10(i)–(iv). We leave the routine details to the reader.

REMARK 3.14.2. One verifies immediately that remarks analogous to Remarks 3.6.2 and 3.10.1 in the profinite case may be made in the discrete situation treated in Theorem 3.14. In this context, we observe that the theory of the "modules of local orientations Λ" developed in [9, §3] admits a straightforward discrete analogue, which may be applied to conclude that the "orientation isomorphisms $J \xrightarrow{\sim} \Lambda_{\mathcal{G}}$" of Remark 3.6.2(i) are compatible with the natural discrete structures on the domain and codomain. Alternatively, in the discrete case, relative to the notation of Definition 2.2(iii), one may think of these modules "Λ" as the $\mathbb{Z}$-duals of the second relative singular cohomology modules (with $\mathbb{Z}$-coefficients)

$$H^2(U_X, \partial U_X; \mathbb{Z})$$

—cf. the discussion of orientations in [9, Introduction]. Then the discrete version of the key isomorphisms (cf. the constructions of Remark 3.6.2) of [9, Cor. 3.9(v) and (vi)] may be obtained by considering the connecting homomorphism (from first to second cohomology modules) in the long exact cohomology sequence associated with the pair $(U_X, \partial U_X)$. We leave the routine details to the reader.

Appendix. Explicit limit seminorms associated with sequences of toric surfaces

In the proof of Corollary 1.15(ii), we considered sequences of discrete valuations that arose from vertices or edges of the dual semi-graphs associated with the geometric special fibers of a tower of coverings of stable log curves and, in particular, observed that the convergence of a suitable subsequence of such a sequence follows immediately from the general theory of Berkovich spaces. In the present appendix, we reexamine this convergence phenomenon from a more elementary and explicit—albeit logically unnecessary, from the point of view of proving Corollary 1.15(ii)!—point of view that only requires a knowledge of elementary facts concerning log regular log schemes, that is, without applying the terminology and notions (e.g., of "Stone–Čech compactifications") that frequently appear in the general theory of Berkovich spaces (cf. the proof of [2, Th. 1.2.1]). In particular, we discuss the notion of a "stratum" of a "toric surface" (cf. Definition A.1 below), which generalizes the notion of a vertex or edge of the dual graph of the special fiber of a stable curve over a complete discrete valuation ring. We observe that such a stratum determines a discrete valuation (cf. Definition A.4) and consider, at a quite explicit level, the limit of a suitable subsequence of a given sequence of such discrete valuations (cf. Theorem A.7 below). The material presented in this appendix is quite elementary and "well-known," but we chose to include it in the present paper since we were unable to find a suitable reference that discusses this material from a similar point of view.

In the present appendix, let R be a complete discrete valuation ring. Write K for the field of fractions of R and $\mathcal{S}^{\log}$ for the log scheme obtained by equipping $\mathcal{S} \stackrel{\text{def}}{=} \mathrm{Spec}(R)$ with the log structure determined by the unique closed point of $\mathcal{S}$.

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DEFINITION A.1.

(i) We shall refer to an fs log scheme $\mathcal{X}^{\log}$ over $\mathcal{S}^{\log}$ as a *toric surface over* $\mathcal{S}^{\log}$ if the following conditions are satisfied:

  (a) The underlying scheme $\mathcal{X}$ of $\mathcal{X}^{\log}$ is of finite type, flat, and of pure relative dimension one (i.e., every irreducible component of every fiber of the underlying morphism of schemes $\mathcal{X} \to \mathcal{S}$ is of dimension one) over $\mathcal{S}$.
  (b) The fs log scheme $\mathcal{X}^{\log}$ is log regular.
  (c) The interior (cf., e.g., [23, Def. 5.1(i)]) of the log scheme $\mathcal{X}^{\log}$ is equal to the open subscheme $\mathcal{X} \times_R K \subseteq \mathcal{X}$.

  Given two toric surface s over $\mathcal{S}^{\log}$, there is an evident notion of *isomorphism of toric surfaces over* $\mathcal{S}^{\log}$.

(ii) Let $\mathcal{X}^{\log}$ be a toric surface over $\mathcal{S}^{\log}$ (cf. (i)) and n a nonnegative integer. Write $\mathcal{X}^{[n]} \subseteq \mathcal{X}$ for the *n-interior* of $\mathcal{X}^{\log}$ (cf. [23, Def. 5.1(i)]) and $\mathcal{X}^{[-1]} \subseteq \mathcal{X}$ for the empty subscheme. Then we shall refer to a connected component of $\mathcal{X}^{[n]} \setminus \mathcal{X}^{[n-1]}$ as an *n-stratum* of $\mathcal{X}^{\log}$. We shall write

$$\mathrm{Str}^n(\mathcal{X}^{\log})$$

  for the set of n-strata of $\mathcal{X}^{\log}$ (so $\mathrm{Str}^n(\mathcal{X}^{\log}) = \emptyset$ if $n \geq 3$) and

$$\mathrm{Str}(\mathcal{X}^{\log}) \overset{\mathrm{def}}{=} \mathrm{Str}^1(\mathcal{X}^{\log}) \sqcup \mathrm{Str}^2(\mathcal{X}^{\log}).$$

DEFINITION A.2. Let I be a totally ordered set that is isomorphic to $\mathbb{N}$ (equipped with its usual ordering). In particular, it makes sense to speak of *"limits $i \to \infty$"* of collections of objects indexed by $i \in I$, as well as to speak of the "*next largest element*" $i+1 \in I$ associated with a given element $i \in I$. Then we shall refer to a sequence of fs log schemes

$$\cdots \longrightarrow \mathcal{X}_{i+1}^{\log} \longrightarrow \mathcal{X}_i^{\log} \longrightarrow \cdots$$

—where i ranges over the elements of I —over $\mathcal{S}^{\log}$ (indexed by I) as a *sequence of toric surface s over* $\mathcal{S}^{\log}$ if, for each $i \in I$, $\mathcal{X}_i^{\log}$ is a toric surface over $\mathcal{S}^{\log}$ (cf. Definition A.1(i)), and, moreover, the morphism $\mathcal{X}_{i+1}^{\log} \to \mathcal{X}_i^{\log}$ is dominant. Observe that the horizontal arrows of the above diagram determine (by considering the induced maps of generic points of strata) a sequence of maps of sets

$$\cdots \longrightarrow \mathrm{Str}(\mathcal{X}_{i+1}^{\log}) \longrightarrow \mathrm{Str}(\mathcal{X}_i^{\log}) \longrightarrow \cdots.$$

Finally, given two sequences of toric surface s over $\mathcal{S}^{\log}$, there is an evident notion of *isomorphism of sequences of toric surface s over* $\mathcal{S}^{\log}$.

DEFINITION A.3. Let $\mathcal{X}^{\log}$ be a toric surface over $\mathcal{S}^{\log}$ and A a strict henselization of $\mathcal{X}$ at (the closed point determined by) $z \in \mathrm{Str}^2(\mathcal{X}^{\log})$ (cf. Definition A.1(i) and (ii)). Write F for the field of fractions of A; k for the residue field of A; $\mathfrak{m}_A$ for the maximal ideal of A; $\mathcal{X}_z \overset{\mathrm{def}}{=} \mathrm{Spec}(A)$; $\mathcal{M}_{\mathcal{X}}$ for the sheaf of monoids on $\mathcal{X}$ that defines the log structure of $\mathcal{X}^{\log}$; M for the fiber of $\mathcal{M}_{\mathcal{X}}/\mathcal{O}_{\mathcal{X}}^{\times}$ at the maximal ideal of A;

$$Q \overset{\mathrm{def}}{=} \mathrm{Hom}(M, \mathbb{Q}_{\geq 0}) \subseteq P \overset{\mathrm{def}}{=} \mathrm{Hom}(M, \mathbb{R}_{\geq 0}) \subseteq V \overset{\mathrm{def}}{=} \mathrm{Hom}(M, \mathbb{R})$$

—where we write $\mathbb{Q}_{\geq 0}$, $\mathbb{R}_{\geq 0}$ for the respective submonoids determined by the nonnegative elements of the (additive groups) $\mathbb{Q}$, $\mathbb{R}$ and "$\mathrm{Hom}(M, -)$" for the monoid consisting of

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homomorphisms of monoids from M to "$(-)$." Thus, one verifies easily that V is equipped with a natural structure of two-dimensional vector space over $\mathbb{R}$. In the following, we shall use the superscript "gp" to denote the *groupification* of any of the monoids of the above discussion.

(i) We shall say that a submonoid $L \subseteq P$ of P is a *P-ray* if L is the $\mathbb{R}_{\geq 0}$-orbit of some nonzero element of P, relative to the natural (multiplicative) action of $\mathbb{R}_{\geq 0}$ on P.

(ii) We shall say that a P-ray $L \subseteq P$ (cf. (i)) is *rational* (resp. *irrational*) if $L \cap Q \neq \{0\}$ (resp. $L \cap Q = \{0\}$).

(iii) Let $L \subseteq P$ be a rational P-ray (cf. (i) and (ii)). Then we shall write $v_L \colon F^\times \to \mathbb{Q} \subseteq \mathbb{R}$ for the discrete valuation associated with the irreducible component of the blowup of $\mathcal{X}_z$ associated with $L \subseteq P$, normalized so as to map each prime element π_R of $R \subseteq F$ to $1 \in \mathbb{Q}$. That is to say, if $\lambda \in L$ (which, by a slight abuse of notation, we regard as a homomorphism $M^{\mathrm{gp}} \to \mathbb{R}$) maps $\pi_R \mapsto 1 \in \mathbb{Q}$ (so $\lambda \in L \cap Q$), and $f \in F$ lies in the $A^\times$-orbit determined by $m \in M^{\mathrm{gp}}$, then

$$v_L(f) \ = \ \lambda(m) \ \in \ \mathbb{Q}.$$

Here, we observe that (one verifies easily that) the submonoid $M_L \stackrel{\mathrm{def}}{=} \lambda^{-1}(\mathbb{Q}_{\geq 0}) \subseteq M^{\mathrm{gp}}$ is isomorphic to $\mathbb{Z} \times \mathbb{N}$. In particular, if we denote by $F_L \subseteq F$ the set of $f \in F$ that lie in the $A^\times$-orbits determined by $m \in M_L$ and write $A_L \subseteq F$ for the A-subalgebra generated by $f \in F_L$, then the "blowup of $\mathcal{X}_z$ associated with L" referred to above may be described explicitly as

$$\mathcal{X}_L \ \stackrel{\mathrm{def}}{=} \ \mathrm{Spec}(A_L) \ \longrightarrow \ \mathcal{X}_z.$$

Indeed, if we write $\mathfrak{p}_L \subseteq A_L$ for the ideal generated by the set of $f \in F$ that lie in the $A^\times$-orbits determined by the noninvertible elements $m \in M_L$, then it follows immediately from the simple structure of the monoid $\mathbb{Z} \times \mathbb{N}$ that $\mathfrak{p}_L$ is the prime ideal of height one in A_L that corresponds to the discrete valuation v_L, and that the k-algebra $A_L/\mathfrak{p}_L$ is isomorphic to $k[U, U^{-1}]$, where U is an indeterminate.

(iv) Write $\mathcal{M}_\mathcal{S}$ for the sheaf of monoids on $\mathcal{S}$ that defines the log structure of $\mathcal{S}^{\log}$; M_R for the fiber of $\mathcal{M}_\mathcal{S}/\mathcal{O}_\mathcal{S}^\times$ at the unique closed point of $\mathcal{S}$; $V_R \stackrel{\mathrm{def}}{=} \mathrm{Hom}(M_R, \mathbb{R})$. Then one verifies easily that V_R is a one-dimensional vector space over $\mathbb{R}$, and that the morphism $\mathcal{X}^{\log} \to \mathcal{S}^{\log}$ determines an $\mathbb{R}$-linear surjection $V \twoheadrightarrow V_R$. Let e_α, $e_\beta \in P$ be such that $\mathbb{R}_{\geq 0} \cdot e_\alpha + \mathbb{R}_{\geq 0} \cdot e_\beta = P$, and, moreover, the images of e_α, e_β in V_R coincide. (Note that the existence of such elements e_α, $e_\beta \in P$ follows, for example, from [14, Prop. 1.7].) Then we shall refer to the (necessarily rational—cf. (ii)) P-ray $\mathbb{R}_{\geq 0} \cdot (e_\alpha + e_\beta) \subseteq P$ (cf. (i)) as the *midpoint P-ray* at $z \in \mathrm{Str}^2(\mathcal{X}^{\log})$. Here, we note that one verifies easily that the P-ray $\mathbb{R}_{\geq 0} \cdot (e_\alpha + e_\beta)$ does not depend on the choice of the pair (e_α, e_β).

(v) We shall refer to a valuation $w \colon F^\times \to \mathbb{R}$ as *admissible* if w dominates A and maps each prime element π_R of $R \subseteq F$ to $1 \in \mathbb{R}$. Let w be an admissible valuation. Then, by restricting w to the elements $f \in F$ that lie in the $A^\times$-orbits determined by $m \in M$, one obtains a nonzero homomorphism of monoids $M \to \mathbb{R}_{\geq 0}$, that is, an element of P. We shall refer to the P-ray L_w determined by this element of P as the *P-ray associated with the admissible valuation w*. Thus, if L_w is rational (cf. (ii)), then it follows immediately from the definitions that, in the notation of (iii), the valuation of A determined by w extends to a valuation of $A_{L_w} (\supseteq A)$.

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REMARK A.3.1. In the notation of Definition A.3, the usual topology on the real vector space V naturally determines a topology on the subspace $P \subseteq V$, as well as on the set of P-rays (i.e., which may be regarded as the complement of the "zero element" in the quotient space $P/\mathbb{R}_{\geq 0}$). Moreover, one verifies easily that, if e_α and e_β are as in Definition A.3(iv), then the assignment

$$\mathbb{R} \supseteq [0,1] \ni \gamma \mapsto \mathbb{R}_{\geq 0} \cdot (\gamma \cdot e_\alpha + (1-\gamma) \cdot e_\beta)$$

determines a homeomorphism of the closed interval $[0,1] \subseteq \mathbb{R}$ onto the resulting topological space of P-rays, and that the subset of rational P-rays is dense in the space of P-rays. In particular, it makes sense to speak of *non-extremal* (resp. *extremal*) P-rays, that is, P-rays that lie (resp. do not lie) in the interior—that is, relative to the homeomorphism just discussed, the open interval $(0,1) \subseteq [0,1]$ (resp. the endpoints $\{0,1\} \subseteq [0,1]$)—of the space of P-rays. Finally, we observe that the two extremal P-rays are rational, and that a rational P-ray is non-extremal if and only if its associated discrete valuation (cf. Definition A.3(iii)) is admissible (cf. Definition A.3(v)).

DEFINITION A.4. Let $\mathcal{X}^{\log}$ be a toric surface over $\mathcal{S}^{\log}$; $z \in \mathrm{Str}(\mathcal{X}^{\log})$ (cf. Definition A.1 (i) and (ii)). Write F for the residue field of the generic point of the irreducible component of $\mathcal{X}$ on which (the subset of $\mathcal{X}$ determined by) $z \in \mathrm{Str}(\mathcal{X}^{\log})$ lies. Then one may associate with $z \in \mathrm{Str}(\mathcal{X}^{\log})$ a collection of *distinguished valuations* on F, as well as a uniquely determined *canonical valuation* on F, as follows:

(i) If z is a 1-stratum, then we take both the unique *distinguished valuation* and the *canonical valuation* associated with z to be the discrete valuation

$$F^\times \longrightarrow \mathbb{Q} \subseteq \mathbb{R}$$

associated with the prime of height 1 determined by z, normalized so as to map each prime element π_R of $R \subseteq F$ to $1 \in \mathbb{Q}$.

(ii) If z is a 2-stratum, then we take the collection of *distinguished valuations* associated with z to be the discrete valuations

$$F^\times \longrightarrow \mathbb{Q} \subseteq \mathbb{R}$$

determined by the restrictions of the discrete valuations associated with the rational P-rays (cf. Definition A.3(iii)). We take the *canonical valuation* associated with z to be the discrete valuation determined by the restriction of the discrete valuation associated with the midpoint P-ray at z (cf. Definition A.3(iii) and (iv)).

Here, we note that the construction from z of either the collection of distinguished valuations or the uniquely determined canonical valuation is functorial with respect to arbitrary isomorphisms of pairs $(\mathcal{X}^{\log}, z)$ (i.e., pairs consisting of a toric surface over $\mathcal{S}^{\log}$ and an element of "$\mathrm{Str}(-)$" of the toric surface).

REMARK A.4.1. One verifies immediately that the (noncuspidal) valuations of the discussion preceding Corollary 1.15 correspond precisely to the canonical valuations of Definition A.4.

LEMMA A.5 (Valuations associated with irrational rays). *In the notation of Definition A.3, let $L \subseteq P$ be an irrational P-ray (cf. Definition A.3(i) and (ii)), $\{L_i\}_{i=1}^{\infty}$ a sequence of P-rays such that $L = \lim_{i\to\infty} L_i$ (cf. Remark A.3.1), and $\{w_i\}_{i=1}^{\infty}$ a sequence of admissible*

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valuations such that, for each positive integer i, L_i is the P-ray associated with w_i (cf. Definition A.3(v)). Then there exists an admissible valuation (cf. Definition A.3(v))

$$v_L \colon F^\times \longrightarrow \mathbb{R}$$

which satisfies the following conditions:

(a) *The P-ray associated with v_L (cf. Definition A.3(v)) is equal to L.*
(b) *For each $f \in F^\times$, it holds that*

$$v_L(f) \ = \ \lim_{i\to\infty} \ w_i(f).$$

(c) *If $\lambda \in L$ maps a prime element π_R of R to $1 \in \mathbb{R}$, J is a nonempty finite set, $\{m_j\}_{j\in J}$ is a collection of distinct elements of M^{gp}, and $\{f_j\}_{j\in J}$ is a collection of elements of F such that f_j lies in the $A^\times$-orbit determined by m_j, then*

$$v_L(\sum_{j\in J} f_j) \ = \ \min_{j\in J} \ \lambda(m_j) \ \in \ \mathbb{R}.$$

Moreover, this valuation v_L is the unique admissible valuation (i.e., in the sense of Definition A.3(v)) that satisfies condition (a). In particular, v_L depends only on the P-ray $L \subseteq P$, that is, is independent of the choice of the sequences $\{L_i\}_{i=1}^\infty$ and $\{w_i\}_{i=1}^\infty$.

Proof. One may define a map $v_L \colon F^\times \to \mathbb{R}$ by applying the formula in the display of condition (c) in the case where $J^\sharp = 1$. Then one verifies easily that this map v_L is a homomorphism (with respect to the multiplicative structure of $F^\times$) and satisfies condition (b). Next, let us observe that since (we have assumed that) L is irrational, the map $M^{\mathrm{gp}} \to \mathbb{R}$ determined by $\lambda \in L$ is injective. Thus, it follows from condition (b), together with the fact that each of the w_i's is a valuation, that the map v_L satisfies condition (c), which implies that the map v_L is a (necessarily admissible) valuation on F. Moreover, it follows immediately from the definition of v_L that v_L satisfies condition (a). This completes the proof of Lemma A.5. ∎

Lemma A.6 (Convergence of midpoints of closed intervals). *Let*

$$\cdots \subseteq [a_{i+1}, b_{i+1}] \subseteq [a_i, b_i] \subseteq [a_{i-1}, b_{i-1}] \subseteq \cdots \subseteq [a_0, b_0] \stackrel{\mathrm{def}}{=} [0,1] \subseteq \mathbb{R}$$

—where i ranges over the nonnegative integers—be a sequence of inclusions of nonempty closed intervals in $[0,1]$. For each i, write c_i for the midpoint of the closed interval $[a_i, b_i]$, that is, $c_i \stackrel{\mathrm{def}}{=} (a_i + b_i)/2 \in [a_i, b_i]$. Then the sequence of midpoints $\{c_i\}_{i=1}^\infty$ converges.

Proof. This follows immediately from the (easily verified) fact that the sequences $\{a_i\}_{i=1}^\infty$, $\{b_i\}_{i=1}^\infty$ converge. ∎

Theorem A.7 (Explicit limit seminorms associated with sequences of toric surfaces). *Let R be a complete discrete valuation ring and I a totally ordered set that is isomorphic to $\mathbb{N}$ (equipped with its usual ordering). Write K for the field of fractions of R and $\mathcal{S}^{\log}$ for the log scheme obtained by equipping $\mathcal{S} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(R)$ with the log structure determined by the unique closed point of $\mathcal{S}$. Let*

$$\cdots \longrightarrow \mathcal{X}_{i+1}^{\log} \longrightarrow \mathcal{X}_i^{\log} \longrightarrow \cdots$$

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be a sequence of toric surface s over $\mathcal{S}^{\log}$ indexed by I (cf. Definition A.2) and

$$\{z_i\}_{i\in I} \ \in \ \varprojlim_{i\in I} \ \mathrm{Str}(\mathcal{X}_i^{\log})$$

(cf. Definitions A.1 (ii) and A.2). Then, after possibly replacing I by a suitable cofinal subset of I, there exist sequences

$$\{v_i\colon F_i^\times \to \mathbb{R}\}_{i\in I}, \quad \{v_{z_i}\}_{i\in I}$$

—where, for each $i\in I$, F_i denotes the residue field of some point $x_i \in \mathcal{X}_i \times_R K$; $v_i\colon F_i^\times \to \mathbb{R}$ is a valuation; v_{z_i} is a distinguished valuation associated with z_i (cf. Definition A.4)—such that

(a) *v_i maps each prime element of $R \subseteq F_i$ to $1\in\mathbb{R}$ (which thus implies that v_i dominates R);*
(b) *the x_i's and v_i's are compatible (in the evident sense) with respect to the upper horizontal arrows $\mathcal{X}_{i+1}^{\log} \to \mathcal{X}_i^{\log}$ of the above diagram;*
(c) *for every nonzero rational function f on the irreducible component of $\mathcal{X}_i$ containing x_i that is regular at x_i, hence determines an element $\overline{f}\in F_i$ (cf. Remark A.7.1 below), it holds that*

$$v_i(\overline{f}) \ = \ \lim_{j\to\infty} v_{z_j}(f)$$

(cf. Definition A.4)—where j ranges over the elements of I that are $\geq i$, and we regard v_i as a map defined on F_i by sending $F_i \ni 0 \mapsto +\infty$.

Finally, these sequences of valuations $\{v_i\}_{i\in I}$, $\{v_{z_i}\}_{i\in I}$ may be constructed in a way that is functorial (in the evident sense) with respect to isomorphisms of pairs consisting of a sequence of toric surfaces over $\mathcal{S}^{\log}$ and a compatible collection of strata (i.e., “$\{z_i\}_{i\in I}$”).

Proof. Until further notice, we take, for each $i\in I$, v_{z_i} to be the canonical valuation associated with z_i (cf. Definition A.4). Next, let us observe that one verifies easily that we may assume without loss of generality, by replacing I by a suitable cofinal subset of I, that there exists an element $n\in\{1,2\}$ such that every member of $\{z_i\}$ is an n-stratum, that is, one of the following conditions is satisfied:

(1) Every member of $\{z_i\}$ is a 1-stratum.
(2) Every member of $\{z_i\}$ is a 2-stratum.

First, we consider Theorem A.7 in the case where condition (1) is satisfied. For each $i\in I$, write $\mathcal{Z}_i \subseteq \mathcal{X}_i$ for the reduced closed subscheme of $\mathcal{X}_i$ whose underlying closed subset $[\subseteq \mathcal{X}_i]$ is the closure of the subset of $\mathcal{X}$ determined by the 1-stratum z_i. Then let us observe that if, after possibly replacing I by a suitable cofinal subset of I, it holds that, for each $i\in I$, the composite $\mathcal{Z}_{i+1} \hookrightarrow \mathcal{X}_{i+1} \to \mathcal{X}_i$ is quasi-finite, then the system consisting of the v_{z_i}'s (cf. Definition A.4(i)) already yields a system of valuations $\{v_i\}_{i\in I}$ as desired. Thus, we may assume without loss of generality, by replacing I by a suitable cofinal subset of I, that, for each $i\in I$, the composite $\mathcal{Z}_{i+1} \hookrightarrow \mathcal{X}_{i+1} \to \mathcal{X}_i$ is not quasi-finite, that is, that the image of this composite is a closed point $y_i \in \mathcal{X}_i$ of $\mathcal{X}_i$. Here, we observe that since we are operating under the assumption that condition (1) is satisfied, it follows from the fact that $z_{i+1} \mapsto z_i$ that y_i necessarily lies in the regular locus of $\mathcal{X}_i$.

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For each $i \in I$, write B_i for the local ring of $\mathcal{X}_i$ at $y_i \in \mathcal{X}_i$, E_i for the field of fractions of B_i, and $v_{z_i}\colon E_i^\times \to \mathbb{R}$ for the discrete valuation defined in Definition A.4(i). Thus, one verifies immediately that the morphisms

$$\cdots \to \mathcal{X}_{i+1} \to \mathcal{X}_i \to \cdots$$

induce compatible chains of injections

$$\cdots \hookrightarrow B_i \hookrightarrow B_{i+1} \hookrightarrow \cdots,$$

$$\cdots \hookrightarrow E_i \hookrightarrow E_{i+1} \hookrightarrow \cdots.$$

Moreover, if π_R is a prime element of R, then the discrete valuation v_{z_i} may be interpreted as the discrete valuation of B_i determined by the unique height one prime of B_i that contains π_R. In particular, since B_i is regular, hence a unique factorization domain, one verifies immediately—by considering the extent to which positive powers of an element $f \in B_i$ are divisible, in B_i or in B_{i+1}, by positive powers of π_R—that, for each $i \in I$ and $f \in B_i$, it holds that

$$(0 \leq)\ v_{z_i}(f) \leq v_{z_{i+1}}(f). \tag{$*$}$$

For each $i \in I$, write

$$\mathfrak{p}_i \stackrel{\text{def}}{=} \{ f \in B_i \mid \lim_{j\to\infty} v_{z_j}(f) = +\infty \} \subseteq B_i.$$

Then since each v_{z_j} is a (discrete) valuation, one verifies immediately that $\mathfrak{p}_i \subseteq B_i$ is a prime ideal of B_i. Moreover, since $\pi_R \notin \mathfrak{p}_i$, we conclude that the ideal $\mathfrak{p}_i$ is not maximal, that is, that the height of $\mathfrak{p}_i$ is $\in \{0,1\}$. Next, let us observe that if, after possibly replacing I by a suitable cofinal subset of I, it holds that, for each $i \in I$, the prime ideal $\mathfrak{p}_i$ is of height 1, then it follows immediately that $\mathfrak{p}_i$ determines a closed point x_i of the generic fiber of $\mathcal{X}_i$, and that, if we write F_i for the residue field of $\mathcal{X}_i$ at x_i and $v_i\colon F_i^\times \to \mathbb{R}$ for the uniquely determined (since F_i is a finite extension of K) discrete valuation on F_i that extends the given discrete valuation on K and maps $\pi_R \mapsto 1 \in \mathbb{R}$, then the limit $\lim_{j\to\infty} v_{z_j}(-)$ (cf. $(*)$) determines a valuation on $F_i = (B_i)_{\mathfrak{p}_i}/\mathfrak{p}_i(B_i)_{\mathfrak{p}_i}$ that necessarily coincides (since F_i is a finite extension of K) with v_i; in particular, one obtains a system of valuations $\{v_i\}_{i\in I}$ as desired.

Thus, we may assume without loss of generality, by replacing I by a suitable cofinal subset of I, that, for each $i \in I$, the prime ideal $\mathfrak{p}_i$ is of height 0, that is, $\mathfrak{p}_i = \{0\}$, hence determines a generic point x_i of some irreducible component of $\mathcal{X}_i$ such that E_i may be naturally identified with the residue field F_i of $\mathcal{X}_i$ at x_i. But this implies that, for $f \in E_i^\times = F_i^\times$, the quantity

$$v_i(f) \stackrel{\text{def}}{=} \lim_{j\to\infty} v_{z_j}(f) \in \mathbb{R}$$

is well-defined (cf. $(*)$). Moreover, one verifies immediately that this definition of v_i determines a valuation on $E_i = F_i$. In particular, one obtains a system of valuations $\{v_i\}_{i\in I}$ as desired. This completes the proof of Theorem A.7 in the case where condition (1) is satisfied.

Next, we consider Theorem A.7 in the case where condition (2) is satisfied. For each $i \in I$, write Q_i, P_i, V_i for the objects "Q," "P," "V" defined in Definition A.3 in the case where

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we take the data "$(\mathcal{X}^{\log}, z \in \mathrm{Str}^2(\mathcal{X}^{\log}))$" in Definition A.3 to be $(\mathcal{X}_i^{\log}, z_i \in \mathrm{Str}^2(\mathcal{X}_i^{\log}))$. Then one verifies easily that the morphism $\mathcal{X}_{i+1}^{\log} \to \mathcal{X}_i^{\log}$ determines a nontrivial $\mathbb{R}$-linear map $V_{i+1} \to V_i$ that maps Q_{i+1}, $P_{i+1} \subseteq V_{i+1}$ into Q_i, $P_i \subseteq V_i$, respectively.

Next, let us observe that if, after possibly replacing I by a suitable cofinal subset of I, it holds that, for each $i \in I$, the $\mathbb{R}$-linear map $V_{i+1} \to V_i$ is of rank one, that is, the image of $P_{i+1} \subseteq V_{i+1}$ in V_i is a rational P_i-ray L_i (cf. Definition A.3(i) and (ii)), then we may assume without loss of generality, by taking v_{z_i} to be the distinguished valuation associated with the rational P_i-ray L_i (cf. Definition A.4(ii); Remark A.7.2 below) and then replacing the pair $(\mathcal{X}_i, z_i)$ by the pair consisting of the blow-up of $\mathcal{X}_i$ and the 1-stratum of this blow-up determined by L_i (cf. the discussion of Definition A.3(iii)), that condition (1) is satisfied. Thus, we may assume without loss of generality, by replacing I by a suitable cofinal subset of I, that, for each $i \in I$, the $\mathbb{R}$-linear map $V_{i+1} \to V_i$ is of rank $\neq 1$, hence (cf. the existence of the $\mathbb{R}$-linear surjection "$V \twoheadrightarrow V_R$" of Definition A.3(iv)) of rank two, that is, an isomorphism.

Since the $\mathbb{R}$-linear map $V_{i+1} \to V_i$ is an isomorphism, it follows immediately from Lemma A.6, together with Remark A.3.1, that, for each $i \in I$, the sequence consisting of the images in P_i of the midpoint P_j-rays (cf. Definition A.3(iv)), where j ranges over the elements of I such that $j \geq i$, converges to a (not necessarily rational) P_i-ray $L_{i,\infty} \subseteq P_i$. If, after possibly replacing I by a suitable cofinal subset of I, it holds that, for each $i \in I$, the P_i-ray $L_{i,\infty}$ is rational, then we may assume without loss of generality, by taking v_{z_i} to be the distinguished valuation associated with the rational P_i-ray $L_{i,\infty}$(cf. Definition A.4(ii); Remark A.7.2 below) and then replacing the pair $(\mathcal{X}_i, z_i)$ by the pair consisting of the blow-up of $\mathcal{X}_i$ and the 1-stratum of this blow-up determined by $L_{i,\infty}$ (cf. the discussion of Definition A.3(iii)), that condition (1) is satisfied. Thus, it remains to consider the case in which we may assume without loss of generality, by replacing I by a suitable cofinal subset of I, that, for each $i \in I$, the P_i-ray $L_{i,\infty}$ is irrational. Then the system consisting of the valuations $v_{L_{i,\infty}}$'s of Lemma A.5 yields a system of valuations $\{v_i\}_{i\in I}$ as desired. This completes the proof of Theorem A.7. ∎

REMARK A.7.1. In the situation of Theorem A.7, for $I \ni j \geq i$, write z_j^i for the irreducible locally closed subset of $\mathcal{X}_i$ determined by the image of the stratum z_j in $\mathcal{X}_i$. Thus, $z_{j'}^i \subseteq z_j^i$ for all $j' \geq j$, and one verifies immediately that the intersection

$$z_\infty^i \stackrel{\text{def}}{=} \bigcap_{j \geq i} z_j^i$$

is nonempty. Moreover, it follows immediately from the constructions discussed in the proof of Theorem A.7 that if $\xi_i \in z_\infty^i$, then any element f of the local ring $\mathcal{O}_{\mathcal{X}_i,\xi_i}$ of $\mathcal{X}_i$ at ξ_i determines a rational function on the irreducible component of $\mathcal{X}_i$ containing x_i that is regular at x_i (cf. Theorem A.7(c)).

REMARK A.7.2. Although, in certain cases (cf. the final portion of the proof of Theorem A.7), the distinguished valuation v_{z_i} in the statement of Theorem A.7 is not necessarily canonical, the system of valuations $\{v_i\}_{i\in I}$ obtained in Theorem A.7 is nevertheless sufficient (cf. the functoriality discussed in the final portion of Theorem A.7) to derive the conclusion of Corollary 1.15(ii), that is, without applying the theory of [2].

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