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LOCAL FRACTAL INTERPOLATION ON UNBOUNDED DOMAINS

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Abstract We define fractal interpolation on unbounded domains for a certain class of topological spaces and construct local fractal functions. In addition, we derive some properties of these local fractal functions, consider their tensor products, and give conditions for local fractal functions on unbounded domains to be elements of Bochner–Lebesgue spaces.

Keywords: local iterated function system; attractor; fractal interpolation; Read–Bajraktarević operator; fractal function; Bochner–Lebesgue space; unbounded component; non-compact Hausdorff space

2010 *Mathematics subject classification:* Primary 28A80; 37C70; 41A05; 41A30; 42B35

1. Introduction

The concept of fractal interpolation was first introduced in [**4**], and was subsequently extended and investigated by numerous authors. (For an albeit incomplete list, refer to the references provided in [**14**, **17**].) The more geometrically inspired definition given in [**4**] was later replaced by a more analytic approach based on Read–Bajraktarević (RB) operators; see, for instance, [**10**, **13**, **14**, **17**]. In this paper, we follow this latter approach.

Fractal interpolation is usually defined on compact Hausdorff spaces X which translate to compact and thus bounded subsets when $X \subset \mathbb{R}^d$ is chosen. There are, however, situations where an unbounded domain may be warranted; one such scenario for $d := 1$ involves sampling on the positive half line $\mathbb{R}^+$ to describe the long-term asymptotic behaviour of a system.

One can obtain fractal interpolation on unbounded domains D of $\mathbb{R}$ in two ways. Firstly, one constructs a fractal interpolant f on a compact subset, say the unit interval I, and then defines the pullback $f \circ j$ of f, where j is a homeomorphism mapping D onto I. Or, secondly, one defines a (global) iterated function system (IFS) on unbounded domains of $\mathbb{R}$, which amounts to writing the domain for the fractal interpolant as the union of bounded domains plus one unbounded domain. Both methods require that the

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unbounded domain is partitioned into one unbounded component and a finite number of bounded components.

In order to have more flexibility in the construction, the recently rediscovered concept of a local IFS (see [5] for the original definition and, for instance, [7, 15, 16] for extensions and new results) can be used for fractal interpolation on unbounded domains. The more general structure of a local IFS allows the definition of mappings from proper subsets into a given compact subspace. The main focus of this paper lies in a construction of so-called local fractal functions on unbounded domains that is based on the structure of local IFSs.

The outline of this paper is as follows. After some preliminary comments in § 1 about univariate fractal interpolation on unbounded domains in $\mathbb{R}$, we briefly introduce in § 2 the concepts of the local IFS and local attractor. The next section provides the general set-up for the construction of local fractal functions on unbounded domains in a certain type of topological space X. Section 5 then deals with the construction itself using an RB operator acting on the Banach space of bounded functions over X. We also present a result that shows how Lagrange-type basis elements for these local fractal functions can be constructed. The tensor product of local fractal functions defined on unbounded domains is defined in § 6, and in § 7 we derive conditions for local fractal functions on unbounded domains to be elements of Bochner–Lebesgue spaces. Finally, we show in § 8 that the graph of a local fractal function on an unbounded domain is a local attractor of an associated local IFS.

Throughout, we use the following notation. The set of positive integers is denoted by $\mathbb{N} := \{1, 2, 3, \ldots\}$. For an $n \in \mathbb{N}$, we denote the initial segment $\{1, \ldots, n\}$ of $\mathbb{N}$ by $\mathbb{N}_n$. We write the closure of a set S as $\operatorname{cl} S$ and its interior as $\operatorname{int} S$. As usual, we define $x_+ := \max\{0, x\}$, $x \in \mathbb{R}$.

2. Preliminary remarks

Let us consider some of the different ways to extend fractal interpolation from a compact domain in $\mathbb{R}$ to an unbounded domain, say $\mathbb{R}_0^+ := [0, \infty)$. To this end, let f be a continuous fractal function supported on $I := [0, 1]$ generated by the iterates of the RB operator $\Phi : C_0(I) \to C_0(I)$,

$$\Phi h = g + \sum_{i=1}^{n} s_i \, h \circ u_i^{-1} \, \chi_{u_i(I)}, \tag{2.1}$$

where $g \in C_0(I) := \{v \in C(I) : v(0) = 0 = v(1)\}$ and $u_i : I \to u_i(I) =: I_i$ are homeomorphisms with $I = \bigcup_{i \in \mathbb{N}_n} I_i$ and $\operatorname{int} I_i \cap \operatorname{int} I_j = \emptyset$, $i \neq j$. The s_i are real numbers with modulus less than one. This class of RB operators is, for instance, investigated in [14].

2.1. Construction via pullbacks

Denote by $\overline{\mathbb{R}}_0^+ := \mathbb{R}_0^+ \cup \{\infty\}$ the extended real half-line, i.e., the Alexandroff compactification of $\mathbb{R}_0^+$. Any subset of $\overline{\mathbb{R}}_0^+$ which contains ∞ is called *unbounded*. Suppose that we are given an arbitrary but fixed homeomorphism $j : \overline{\mathbb{R}}_0^+ \to I$. We define the pullback of f by j, $f^* := f \circ j$, which is a continuous function from the unbounded domain $\overline{\mathbb{R}}_0^+ \to \mathbb{R}$.

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Furthermore, since f is the fixed point of (2.1) and j a homeomorphism, one obtains the following self-referential equation for f^*:

$$(f \circ j)(x) = (g \circ j)(x) + \sum_{i=1}^{n} s_i\,(f \circ j)((u_i \circ j)^{-1}(x))\,\chi_{(u_i \circ j)(\overline{\mathbb{R}}_0^+)}(x).$$

If we denote the pullbacks of g and u_i by j as g^* and u_i^*, respectively, the above equation can be rewritten as

$$f^* = g^* + \sum_{i=1}^{n} s_i\, f^* \circ (u_i^*)^{-1}\,\chi_{u_i^*(\overline{\mathbb{R}}_0^+)}. \tag{2.2}$$

In other words, the pullback f^* satisfies the same type of self-referential equation as f. Note that $(\overline{\mathbb{R}}_0^+, d)$ is a complete metric space, where the metric d is defined by $d := d_I(b(x), b(y)$ with d_I being any metric on I and $b : \overline{\mathbb{R}}_0^+ \to I$ any bijection.

The unbounded domain $\overline{\mathbb{R}}_0^+$ is partitioned into n subdomains R_i such that $j(R_i) = I_i$, $i \in \mathbb{N}_n$. However, there is only one subdomain R_k, $k \in \mathbb{N}_n$, which contains ∞ and is therefore unbounded; the remaining $n-1$ subdomains are bounded.

Define $g(\infty) := \lim_{x\to\infty} g(x)$, provided this limit exists. Notice that since $j(0), j(\infty) \in \partial I$ and $f \in C_0(I)$, we have that $f(\infty) = 0$.

Example 2.1. In Figure 1 below, we depict on the left-hand side a fractal function generated by the above RB operator with $g(x) := (\frac{1}{2} - |x - \frac{1}{2}|)_+$, $u_1(x) := \frac{x}{2}$, $u_2(x) := \frac{x+1}{2}$, and $s_1 := \frac{4}{5}$ and $s_2 := -\frac{3}{5}$. Choosing $j(x) := (x+1)^{-1}$, we display the pullback f^* of f by j on the right-hand side of Figure 1.

Note the slow convergence of the pullback f^* towards the asymptote $y = 0$, which reflects the slow convergence of j towards zero as $x \to \infty$.

The aforementioned example and an examination of (2.2) show that the asymptotic behaviour of the pullback f^* is completely determined by the asymptotics of the homeomorphism j.

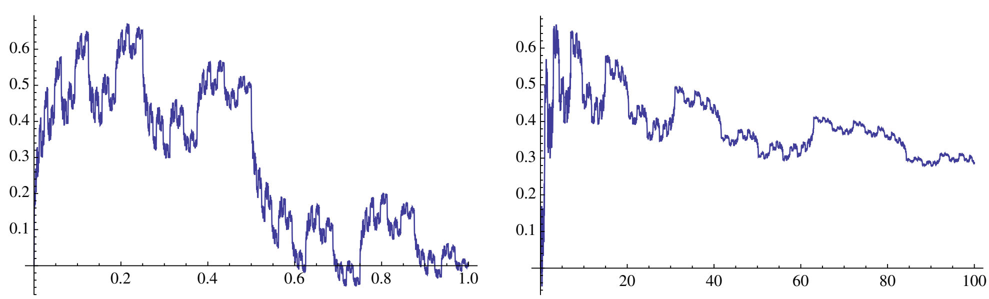

Figure 1. A fractal function supported on $[0, 1]$ (left) and its pullback supported on $\mathbb{R}_0^+$ (right).

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2.2. Construction via global IFSs

Recall that an IFS on a complete metric space (X, d) is a pair $(X, \mathcal{F})$, where $\mathcal{F}$ is collection of continuous functions $\{f_i : X \to X\}_{i\in\mathbb{N}_n}$. In the case where $\mathcal{F}$ consists entirely of contractions, the IFS $(X, \mathcal{F})$ is called *hyperbolic* or *contractive*.

It is known that contractive IFSs have a unique attractor $A \in \mathbb{H}(X)$, the hyperspace of non-empty compact subsets of X. This unique attractor is obtained as the fixed point of the set-valued mapping $\mathcal{F} : \mathbb{H}(X) \to \mathbb{H}(X)$ defined by

$$\mathcal{F}(S) := \bigcup_{i\in\mathbb{N}_n} f_i(S).$$

By a slight abuse of notation, we write $\mathcal{F}$ for the IFS $(X, \mathcal{F})$, its collection of functions $\{f_i : X \to X\}_{i\in\mathbb{N}_n}$, and the above set-valued operator.

For more details about IFSs and proofs, we refer the interested reader to the original papers [**3**, **11**] or the monographs [**2**, **17**].

Let us again consider a special case, namely, $X := \mathbb{R}_0^+$. To be even more specific, we only consider the following exemplary set-up, which nevertheless, reflects the general setting.

To this end, let $u_1 : \mathbb{R}_0^+ \to I$, $x \mapsto (2/\pi)\tan^{-1} x$, and $u_2 : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \setminus [0, 1)$, $x \mapsto x + 1$. Then $\mathbb{R}_0^+ = u_1(\mathbb{R}_0^+) \cup u_2(\mathbb{R}_0^+)$ and the RB operator (2.1) now reads

$$\begin{aligned}\Phi h &= g + s_1\, h \circ u_1^{-1}\, \chi_{u_1(\mathbb{R}_0^+)} + s_2\, h \circ u_2^{-1}\, \chi_{u_2(\mathbb{R}_0^+)}\\ &= g + s_1\, h \circ \tan\left(\frac{\pi}{2}\,\cdot\right) \chi_I + s_2\, h(\,\cdot\, + 1)\, \chi_{[1,\infty)},\end{aligned}$$

where $g \in C_1(\mathbb{R}_0^+) := \{v \in C(\mathbb{R}_0^+) : v(0) = 0 = \lim_{x\to\infty} v(x)\}$. In the case where $|s_1|$, $|s_2| < 1$, the fixed point of Φ is an element of $C_1(\mathbb{R}_0^+)$, i.e., a continuous fractal function defined on the unbounded domain $\mathbb{R}_0^+$. Note that as in the case of the construction by pullback, there is only one unbounded component, namely, $u_2(\mathbb{R}_0^+)$. (If there were n maps u_i, then these maps would define $n - 1$ bounded and one unbounded component.) The graph of such a continuous fractal function is displayed in Figure 2. The function g has been chosen as

$$g(x) := \begin{cases} \left|x - \dfrac{1}{2}\right| - \dfrac{1}{2}, & x \in [0, 2];\\[2ex] \dfrac{2}{x}, & x \geq 2,\end{cases}$$

and $s_1 := \frac{3}{4}$ and $s_2 := \frac{7}{10}$.

Notice that the rate of decay of f for large values of x is determined by that of g. For this example, we have $f \in \mathcal{O}(x^{-1})$ as $x \to \infty$.

Both procedures to extend fractal interpolation to unbounded domains result in having the unbounded domain partitioned into one unbounded component and $n - 1$ bounded components (in the case of n maps u_i and $\mathbb{R}_0^+$). If, for instance, $\mathbb{R}$ is used, there will be two unbounded components and $n - 2$ bounded components. In the latter case, one may map one unbounded component to the other, adding a little flexibility to the construction.

In the next section, we introduce the concept of a *local* IFS and then use it in § 4 to construct fractal functions on unbounded domains. As we will see, the locality of the IFS adds considerable flexibility to fractal interpolation on unbounded domains.

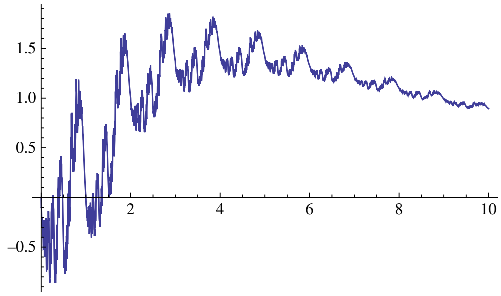

Figure 2. A continuous fractal function supported on $\mathbb{R}_0^+$.

3. Local iterated function systems

The concept of *local* IFSs is a generalization of an IFSs and was first introduced in [**5**] and reconsidered in [**7**]. Their properties have also been investigated in [**15**, **16**].

Definition 3.1. Suppose that $\{X_k : k \in \mathbb{N}_m\}$ is a family of non-empty subsets of a Hausdorff space X. Further assume that for each X_k there exists a continuous mapping $f_k : X_k \to X$, $k \in \mathbb{N}_m$. Then the pair $(X, \mathcal{F}_{\text{loc}})$, where $\mathcal{F}_{\text{loc}} := \{f_k : X_k \to X\}_{k\in\mathbb{N}_m}$, is called a *local IFS*.

Note that if each $X_k = X$, then Definition 3.1 coincides with the usual definition of a standard (global) IFS. However, the possibility of choosing the domain for each continuous mapping f_k different from the entire space X adds additional flexibility, as will be recognized in the sequel. Also notice that one may choose the same X_k as the domain for different mappings $f \in \mathcal{F}_{\text{loc}}$.

We can associate with a local IFS a set-valued operator $\mathcal{F}_{\text{loc}} : \mathbb{P}(X) \to \mathbb{P}(X)$, where $\mathbb{P}(X)$ denotes the power set of X, by setting

$$\mathcal{F}_{\text{loc}}(S) := \bigcup_{k\in\mathbb{N}_m} f_k(S \cap X_k). \tag{3.1}$$

By a slight abuse of notation, we use again the same symbol for a local IFS, its collection of functions, and its associated operator.

There exists an alternative definition for (3.1). For given functions f_k that are only defined on X_k, one could introduce set functions (also denoted by f_k) which are defined on $\mathbb{P}(X)$ via

$$f_k(S) := \begin{cases} f_k(S \cap X_k), & S \cap X_k \neq \emptyset; \\ \emptyset, & S \cap X_k = \emptyset, \end{cases} \qquad k \in \mathbb{N}_m.$$

On the left-hand side of the above equation, $f_k(S \cap X_k)$ is the set of values of the original f_k as in the previous definition. This extension of a given function f_k to sets S which include elements not in the domain of f_k basically just ignores those elements. In the following, we use this definition of the set functions f_k.

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Definition 3.2. A subset $A \in \mathbb{P}(X)$ is called a *local attractor* for the local IFS $(X, \mathcal{F}_{\mathrm{loc}})$ if

$$A = \mathcal{F}_{\mathrm{loc}}(A) = \bigcup_{k \in \mathbb{N}_m} f_k(A \cap X_k). \tag{3.2}$$

In (3.2) it is allowed that $A \cap X_k$ is the empty set. Thus, every local IFS has at least one local attractor, namely $A = \emptyset$. However, it may also have many distinct ones. In the latter case, if A_1 and A_2 are distinct local attractors, then $A_1 \cup A_2$ is also a local attractor. Hence, there exists a largest local attractor for $\mathcal{F}_{\mathrm{loc}}$, namely the union of all distinct local attractors. We refer to this largest local attractor as *the* local attractor of a local IFS $\mathcal{F}_{\mathrm{loc}}$. For more details about local attractors and their relation to the global attractor, the interested reader may consult [**7**, **15**].

4. General set-up for unbounded domains

Let X be a topological space and suppose $K \subset X$ is a compact subspace, i.e., an element of the hyperspace $\mathbb{K}(X)$ of all compact subsets of X. We denote the family of connected components of $X \setminus K$ by $\mathcal{C}(X \setminus K)$. An element $B \in \mathcal{C}(X \setminus K)$ is called *bounded* if its closure is compact, and *unbounded* otherwise. Define

$$\widehat{K} := X \setminus \bigcup \{U \in \mathcal{C}(X \setminus K) : U \text{ is unbounded}\}.$$

For the following, we require a result whose proof can be found in [**8**, Lemma 9].

Proposition 4.1. *Let X be a connected, non-compact, locally connected, locally compact Hausdorff space. Let $K \subset X$ be a compact subspace. Then $X \setminus K$ has only finitely many unbounded components and $\widehat{K}$ is compact.*

As an example of a topological space satisfying the conclusions of Proposition 4.1, we mention $X := \mathbb{R}^s$, $s \in \mathbb{N}$. We also note that the existence of unbounded components is connected to the existence of ends in topological spaces. The fact that $X := \mathbb{R}$ has two unbounded components relates to X having two ends $\pm\infty$. For more details, we refer the interested reader to [**9**, § 13.4].

We now list the assumptions for the remainder of this paper.

General set-up:

(i) X is a non-empty connected, non-compact, locally connected, locally compact Hausdorff space.

(ii) $K \subset X$ is a compact, connected subspace such that $\mathcal{C}(X \setminus K)$ contains no bounded components. Denote by $\mathcal{U}(X \setminus K) := \{U_i : i \in \mathbb{N}_n\}$, $n \in \mathbb{N}$, the finite collection of unbounded components of $X \setminus K$.

(iii) $\{K_j : j \in \mathbb{N}_m\}$ is a family of (not necessarily distinct) compact, connected subspaces of K. The collection of unbounded components of $X \setminus K_j$ is denoted by $\mathcal{U}_j(X \setminus K_j) := \{U_{j,k} : k = 1, \ldots, r_j\}$, $r_j \in \mathbb{N}$, $j \in \mathbb{N}_m$.

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(iv) $\mathcal{U}_n$ is an n-element subset of $\bigcup \mathcal{U}_j(X \setminus K_j)$. Let $\{V_1, \ldots, V_n\}$ be the n elements of $\mathcal{U}_n$.

(v) Let $\pi : \mathbb{N}_n \to \mathbb{N}_n$ be a fixed permutation.

(vi) For each $i \in \mathbb{N}_n$ and each $j \in \mathbb{N}_m$, let $u_i : V_i \to U_{\pi(i)}$ and $b_j : K_j \to K$ be homeomorphisms.

(vii) The family of homeomorphisms

$$\mathcal{H} := \{b_j : K_j \to K : j \in \mathbb{N}_m\} \cup \{u_i : V_i \to U_{\pi(i)} : i \in \mathbb{N}_n\} \tag{4.1}$$

is required to satisfy the following two conditions:

(P1) $K = \bigcup_{j=1}^m b_j(K_j)$ and $\forall\, j \neq j' \in \mathbb{N}_m : b_j(\operatorname{int} K_j) \cap b_{j'}(\operatorname{int} K_{j'}) = \emptyset$;

(P2) $X \setminus K = \bigcup_{i=1}^n u_i(V_i)$ and $\forall i \neq i' \in \mathbb{N}_n :\ u_i(\operatorname{int} V_i) \cap u_{i'}(\operatorname{int} V_{i'}) = \emptyset$.

Remark 4.2. In the case where $K = \emptyset$, $m = 1$, $V_1 = X$, and $\pi = \mathrm{id}$. The set of mappings $\{b_j\} = \emptyset$ and the family $\mathcal{H} = \{u_i : X \to X : i \in \mathbb{N}_n\}$ of homeomorphisms are only required to satisfy condition *(P2)*.

Remark 4.3. Note that the requirement on K implies that $\widehat{K} = K$. Also, notice that $\sum r_j \geq m$, since every K_j has at least one unbounded component.

Example 4.4. In every topological vector space of dimension ≥ 2, in particular, in every metric or normed space of dimension ≥ 2, the complement of a bounded set has exactly one unbounded component [**12**].

5. Local fractal functions on unbounded domains

In this section, we define local fractal functions on X. These extend in a natural way the (global) fractal interpolation functions first introduced in [**2**] and investigated in, for instance, [**10**, **13**, **14**, **17**]. An, albeit incomplete, list of references for local fractal functions is [**7**, **15**, **16**].

To this end, suppose that $(\mathsf{Y}, \|\cdot\|_{\mathsf{Y}})$ is a Banach space. Denote by $\mathsf{B}(X, \mathsf{Y})$ the set

$$\mathsf{B}(X, \mathsf{Y}) := \{f : X \to \mathsf{Y} : f \text{ is bounded}\}.$$

Recall that a function $f : X \to \mathsf{Y}$ is called bounded (with respect to $\|\cdot\|_{\mathsf{Y}}$) if there exists an $M > 0$ so that $\|f(x_1) - f(x_2)\|_{\mathsf{Y}} < M$ for all $x_1, x_2 \in \mathbb{X}$. Under the usual definition of addition and scalar multiplication of mappings, and endowed with the norm

$$\|f - g\| := \sup_{x \in X} \|f(x) - g(x)\|_{\mathsf{Y}},$$

$(\mathsf{B}(X, \mathsf{Y}), \|\cdot\|)$ becomes a Banach space.

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For $j \in \mathbb{N}_m$ and $i \in \mathbb{N}_n$, let $v_j : K_j \times \mathsf{Y} \to \mathsf{Y}$ and $w_i : V_i \times \mathsf{Y} \to \mathsf{Y}$ be mappings that are uniformly contractive in the second variable, i.e., there exist $\ell_1, \ell_2 \in [0,1)$ so that for all $y_1, y_2 \in \mathsf{Y}$

$$\|v_j(x,y_1) - v_j(x,y_2)\|_{\mathsf{Y}} \le \ell_1 \, \|y_1 - y_2\|_{\mathsf{Y}}, \quad \forall x \in K_j, \tag{5.1a}$$

$$\|w_i(x,y_1) - w_i(x,y_2)\|_{\mathsf{Y}} \le \ell_2 \, \|y_1 - y_2\|_{\mathsf{Y}}, \quad \forall x \in V_i. \tag{5.1b}$$

Define an *RB operator* $\Phi : B(X,\mathsf{Y}) \to \mathsf{Y}^X$ by

$$\begin{aligned} \Phi f(x) := &\sum_{j=1}^m v_j(b_j^{-1}(x), f_j \circ b_j^{-1}(x)) \, \chi_{b_j(K_j)}(x) \\ &+ \sum_{i=1}^n w_i((u_i^{-1}(x), f_i \circ u_i^{-1}(x)) \, \chi_{u_i(V_i)}(x), \end{aligned} \tag{5.2}$$

where $f_i := f|_{V_i}$ and $f_j := f|_{K_j}$ denote the restrictions of f to V_i and K_j, respectively, and χ_M denotes the characteristic function of a set M. Note that Φ is well defined, and since f is bounded and each v_j and w_i is contractive in its second variable, $\Phi f \in \mathsf{B}(X,\mathsf{Y})$.

Moreover, by (5.1a) and (5.1b), we obtain for all $f, g \in \mathsf{B}(X,\mathsf{Y})$ the following inequality:

$$\begin{aligned} \|(\Phi f - \Phi g\| &= \sup_{x\in X} \|\Phi f(x) - \Phi g(x)\|_{\mathsf{Y}} \\ &\le \sup_{x\in X} \|v(b_j^{-1}(x), f_j(u_j^{-1}(x))) - v(b_j^{-1}(x), g_j(b_j^{-1}(x)))\|_{\mathsf{Y}} \\ &\quad + \sup_{x\in X} \|w(u_i^{-1}(x), f_i(u_i^{-1}(x))) - w(u_i^{-1}(x), g_i(u_i^{-1}(x)))\|_{\mathsf{Y}} \\ &\le \ell_1 \sup_{x\in X} \|f_j \circ b_j^{-1}(x) - g_j \circ b_j^{-1}(x)\|_{\mathsf{Y}} \\ &\quad + \ell_2 \sup_{x\in X} \|f_i \circ u_i^{-1}(x) - g_i \circ u_i^{-1}(x)\|_{\mathsf{Y}} \\ &\le \max\{\ell_1, \ell_2\} \, \|f - g\|. \end{aligned} \tag{5.3}$$

Above, we set $v(x,y) := \sum_{j=1}^m v_j(x,y) \, \chi_{K_j}(x)$ and $w(x,y) := \sum_{i=1}^n w_i(x,y) \, \chi_{V_i}(x)$ to simplify notation.

These arguments lead immediately to the following theorem.

Theorem 5.1. *Let $(\mathsf{Y}, d_{\mathsf{Y}})$ be a Banach space and let X, $\{K_j\}$, $\{V_i\}$, and $\mathcal{H} := \{b_j : K_j \to K : j \in \mathbb{N}_m\} \cup \{u_i : V_i \to U_{\pi(i)} : i \in \mathbb{N}_n\}$ be as in the general set-up. Let the mappings $v_j : K_j \times \mathsf{Y} \to \mathsf{Y}$, $j \in \mathbb{N}_m$ and $w_i : V_i \times \mathsf{Y} \to \mathsf{Y}$, $i \in \mathbb{N}_n$ satisfy (5.1a) and (5.1b), respectively. Then the RB operator Φ defined by (5.2) is a contraction on $\mathsf{B}(X,\mathsf{Y})$. Its unique fixed point f satisfies the* self-referential equation

$$\begin{aligned} f(x) := &\sum_{j=1}^m v_j(b_j^{-1}(x), f_{\Phi,j} \circ b_j^{-1}(x)) \, \chi_{b_j(K_j)}(x) \\ &+ \sum_{i=1}^n w_i((u_i^{-1}(x), f_{\Phi,i} \circ u_i^{-1}(x)) \, \chi_{u_i(V_i)}(x), \end{aligned} \tag{5.4}$$

where $f_i := f|_{V_i}$ and $f_j := f|_{K_j}$.

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Proof. It follows directly from (5.3) that Φ is a contraction on the Banach space $\mathsf{B}(X, \mathsf{Y})$ and, by the Banach fixed point theorem, has a unique fixed point f in $\mathsf{B}(X, \mathsf{Y})$. The self-referential equation for f is a direct consequence of (5.2). □

We refer to this unique fixed point as a *bounded local fractal function with unbounded domain* X. Note that f depends on the form of Φ, i.e., the sets of functions $\{b_j : j \in \mathbb{N}_m\}$, $\{u_i : i \in \mathbb{N}_n\}$, $\{v_j : j \in \mathbb{N}_m\}$, and $\{w_i : i \in \mathbb{N}_n\}$. Unless necessary, we usually suppress these dependencies.

Next, we would like to consider special choices for the mappings v_j and w_i. For this purpose, suppose that $p_j \in \mathsf{B}(K_j, \mathsf{Y})$, $q_i \in \mathsf{B}(V_i, \mathsf{Y})$, and that $s_j : K_j \to \mathbb{R}$ and $t_i : V_i \to \mathbb{R}$ are bounded functions. Then, we define

$$v_j(x, y) := p_j(x) + s_j(x)\, y, \quad j \in \mathbb{N}_m, \tag{5.5}$$

$$w_i(x, y) := q_i(x) + t_i(x)\, y, \quad i \in \mathbb{N}_n. \tag{5.6}$$

The mappings v_j and w_i given by (5.5) and (5.6) satisfy conditions (5.1a) and (5.1b), respectively, provided that the functions s_j are bounded on K_j with bounds in $[0, 1)$ and the functions t_i are bounded on V_i also with bounds in $[0, 1)$. Then, for a fixed $x \in K_j$,

$$\begin{aligned}\|v_j(x, y_1) - v_j(x, y_2)\|_{\mathsf{Y}} = \|s_j(x)(y_1 - y_2)\|_{\mathsf{Y}} &\le \|s_j\|_{\infty, K_j}\, \|y_1 - y_2\|_{\mathsf{Y}} \\ &\le s\, \|y_1 - y_2\|_{\mathsf{Y}}.\end{aligned}$$

Here, we denoted the supremum norm with respect to K_j by $\|\cdot\|_{\infty, K_j}$, and set $s := \max\{\|s_j\|_{\infty, K_j} : j \in \mathbb{N}_m\}$. Similarly, we obtain for the w_i the estimate

$$\|w_i(x, y_1) - w_i(x, y_2)\|_{\mathsf{Y}} \le t\, \|y_1 - y_2\|_{\mathsf{Y}},$$

with $t := \max\{\|t_i\|_{\infty, V_i} : i \in \mathbb{N}_n\}$.

For *fixed* sets of mappings $\{p_j\}$, $\{q_i\}$ and functions $\{s_j\}$, $\{t_i\}$, the associated RB operator (5.2) now has the form

$$\begin{aligned}\Phi f = &\sum_{j=1}^{m} p_j \circ b_j^{-1}\, \chi_{b_j(K_j)} + \sum_{j=1}^{m} (s_j \circ b_j^{-1}) \cdot (f_j \circ b_j^{-1})\, \chi_{b_j(K_j)} \\ &+ \sum_{i=1}^{n} q_i \circ u_i^{-1}\, \chi_{u_i(V_i)} + \sum_{i=1}^{n} (t_i \circ u_i^{-1}) \cdot (f_i \circ u_i^{-1})\, \chi_{u_i(V_i)}\end{aligned}$$

or, equivalently,

$$\begin{aligned}\Phi f_j \circ b_j &= p_j + s_j \cdot f_j, \quad \text{on } K_j, \quad \forall\, j \in \mathbb{N}_m, \\ \Phi f_i \circ u_i &= q_i + t_i \cdot f_i, \quad \text{on } V_i, \quad \forall\, i \in \mathbb{N}_n.\end{aligned}$$

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To simplify notation, we set

$$\boldsymbol{p} := (p_1, \ldots, p_m) \in \mathsf{B}^m_{\mathsf{Y}} := \mathop{\times}_{j=1}^{m} \mathsf{B}(K_j, \mathsf{Y}),$$

$$\boldsymbol{q} := (q_1, \ldots, q_n) \in \mathsf{B}^n_{\mathsf{Y}} := \mathop{\times}_{i=1}^{n} \mathsf{B}(V_i, \mathsf{Y}),$$

$$\boldsymbol{s} := (s_1, \ldots, s_m) \in \mathsf{B}^m_{\mathbb{R}} := \mathop{\times}_{j=1}^{m} \mathsf{B}(K_j, \mathbb{R}),$$

$$\boldsymbol{t} := (t_1, \ldots, t_n) \in \mathsf{B}^n_{\mathbb{R}} := \mathop{\times}_{i=1}^{n} \mathsf{B}(V_i, \mathbb{R}).$$

Thus, we have in summary the following result.

Theorem 5.2. *Let* $(\mathsf{Y}, d_{\mathsf{Y}})$ *be a Banach space and let* X, $\{K_j\}$, $\{V_i\}$, *and* $\mathcal{H} := \{b_j : K_j \to K : j \in \mathbb{N}_m\} \cup \{u_i : V_i \to U_{\pi(i)} : i \in \mathbb{N}_n\}$ *be as in the general set-up. Let* $\boldsymbol{p} \in \mathsf{B}^m_{\mathsf{Y}}$, $\boldsymbol{q} \in \mathsf{B}^n_{\mathsf{Y}}$, $\boldsymbol{s} \in \mathsf{B}^m_{\mathbb{R}}$ *and* $\boldsymbol{t} \in \mathsf{B}^n_{\mathbb{R}}$.

Define a mapping $\Phi : \mathsf{B}^m_{\mathsf{Y}} \times \mathsf{B}^n_{\mathsf{Y}} \times \mathsf{B}^m_{\mathbb{R}} \times \mathsf{B}^n_{\mathbb{R}} \times \mathsf{B}(X, \mathsf{Y}) \to \mathsf{B}(X, \mathsf{Y})$ *by*

$$\begin{aligned}\Phi(\boldsymbol{p})(\boldsymbol{q})(\boldsymbol{s})(\boldsymbol{t})f = &\sum_{j=1}^{m} p_j \circ b_j^{-1}\, \chi_{b_j(K_j)} + \sum_{j=1}^{m} (s_j \circ b_j^{-1}) \cdot (f_j \circ b_j^{-1})\, \chi_{b_j(K_j)} \\ &+ \sum_{i=1}^{n} q_i \circ u_i^{-1}\, \chi_{u_i(V_i)} + \sum_{i=1}^{n} (t_i \circ u_i^{-1}) \cdot (f_i \circ u_i^{-1})\, \chi_{u_i(V_i)}. \qquad (5.7)\end{aligned}$$

If $\max\{\max\{\|s_j\|_{\infty,K_j} : j \in \mathbb{N}_m\}, \max\{\|t_i\|_{\infty,V_i} : i \in \mathbb{N}_m\}\} < 1$, *then the operator* $\Phi(\boldsymbol{p})(\boldsymbol{q})(\boldsymbol{s})(\boldsymbol{t})$ *is contractive on* $\mathsf{B}(X, \mathsf{Y})$ *and its unique fixed point* f *satisfies the self-referential equation*

$$\begin{aligned}f = &\sum_{j=1}^{m} p_j \circ b_j^{-1}\, \chi_{b_j(K_j)} + \sum_{j=1}^{m} (s_j \circ b_j^{-1}) \cdot (f_{\Phi,j} \circ b_j^{-1})\, \chi_{b_j(K_j)} \\ &+ \sum_{i=1}^{n} q_i \circ u_i^{-1}\, \chi_{u_i(V_i)} + \sum_{i=1}^{n} (t_i \circ u_i^{-1}) \cdot (f_{\Phi,i} \circ u_i^{-1})\, \chi_{u_i(V_i)} \qquad (5.8)\end{aligned}$$

or, equivalently,

$$\begin{aligned} f_j \circ b_j &= p_j + s_j \cdot f_j, \quad \text{on } K_j, \quad \forall\, j \in \mathbb{N}_m, \\ f_i \circ u_i &= q_i + t_i \cdot f_i, \quad \text{on } V_i, \quad \forall\, i \in \mathbb{N}_n, \end{aligned}$$

where $f_i := f|_{V_i}$ *and* $f_j := f|_{K_j}$.

Proof. The statements follow directly from the preceding arguments and Theorem 5.1. □

As above, we refer to f as a bounded local fractal function with unbounded domain X.

Remark 5.3. The local fractal function f generated by the operator Φ defined by (5.7) depends not only on the families of subsets $\{K_j : j \in \mathbb{N}_m\}$ and $\{V_i : i \in \mathbb{N}_n\}$, but

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also on the four tuples of bounded mappings $\boldsymbol{p} \in \mathsf{B}_{\mathsf{Y}}^m$, $\boldsymbol{q} \in \mathsf{B}_{\mathsf{Y}}^n$, $\boldsymbol{s} \in \mathsf{B}_{\mathbb{R}}^m$, and $\boldsymbol{t} \in \mathsf{B}_{\mathbb{R}}^n$. The fixed point f should therefore be written more precisely as $f(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{s}, \boldsymbol{t})$. However, for the sake of notational simplicity, we usually suppress this dependence for both f and Φ when not necessary.

The following result found in [10], and in more general form in [13], is the extension to the present setting of local fractal functions on unbounded domains.

Theorem 5.4. *Suppose that the tuples* $\boldsymbol{s}$ *and* $\boldsymbol{t}$ *are fixed. The mapping* $\Theta : \mathsf{B}_{\mathsf{Y}}^m \times \mathsf{B}_{\mathsf{Y}}^n \to \mathsf{B}(X, \mathsf{Y})$, $(\boldsymbol{p}, \boldsymbol{q}) \mapsto f(\boldsymbol{p}, \boldsymbol{q})$ *defines a linear isomorphism.*

Proof. Let $\alpha, \beta \in \mathbb{R}$, let $\boldsymbol{p}, \widetilde{\boldsymbol{p}} \in \mathsf{B}_{\mathsf{Y}}^m$, and $\boldsymbol{q}, \widetilde{\boldsymbol{q}} \in \mathsf{B}_{\mathsf{Y}}^n$.

Injectivity follows immediately from the fixed point equation (5.8) and the uniqueness of the fixed point: $(\boldsymbol{p}, \boldsymbol{q}) = (\widetilde{\boldsymbol{p}}, \widetilde{\boldsymbol{q}}) \iff f(\boldsymbol{p}, \boldsymbol{q}) = f(\widetilde{\boldsymbol{p}}, \widetilde{\boldsymbol{q}})$.

Linearity in $(\boldsymbol{p}, \boldsymbol{q})$ follows from (5.8), the uniqueness of the fixed point, and injectivity:

$$\begin{aligned} f(\alpha(\boldsymbol{p}, \boldsymbol{q}) + \beta(\widetilde{\boldsymbol{p}}, \widetilde{\boldsymbol{q}})) = &\sum_{j=1}^m (\alpha p_j + \beta \widetilde{p}_j) \circ b_j^{-1}\, \chi_{b_j(K_j)} + \sum_{i=1}^n (\alpha q_j + \beta \widetilde{q}_j) \circ u_i^{-1}\, \chi_{u_i(V_i)} \\ &+ \sum_{j=1}^m (s_j \circ b_j^{-1}) \cdot (f_{\Phi,j}(\alpha \boldsymbol{p} + \beta \widetilde{\boldsymbol{p}})(\alpha \boldsymbol{q} + \beta \widetilde{\boldsymbol{q}}) \circ b_j^{-1})\, \chi_{b_j(K_j)} \\ &+ \sum_{i=1}^n (t_i \circ u_i^{-1}) \cdot (f_{\Phi,i}(\alpha \boldsymbol{p} + \beta \widetilde{\boldsymbol{p}})(\alpha \boldsymbol{q} + \beta \widetilde{\boldsymbol{q}}) \circ u_i^{-1})\, \chi_{u_i(V_i)} \end{aligned}$$

and

$$\begin{aligned} \alpha f(\boldsymbol{p}, \boldsymbol{q}) + \beta f(\widetilde{\boldsymbol{p}}, \widetilde{\boldsymbol{q}}) = &\sum_{j=1}^m (\alpha p_j + \beta \widetilde{p}_j) \circ b_j^{-1}\, \chi_{b_j(K_j)} \\ &+ \sum_{i=1}^n (\gamma q_j + \delta \widetilde{q}_j) \circ u_i^{-1}\, \chi_{u_i(V_i)} \\ &+ \sum_{j=1}^m (s_j \circ b_j^{-1}) \cdot (\alpha f_{\Phi,j}(\boldsymbol{p})(\boldsymbol{q}) + \beta f(\widetilde{\boldsymbol{p}})(\widetilde{\boldsymbol{q}})) \circ b_j^{-1}\, \chi_{b_j(K_j)} \\ &+ \sum_{i=1}^n (t_i \circ u_i^{-1}) \cdot (\alpha f_{\Phi,j}(\boldsymbol{p})(\boldsymbol{q}) + \beta f(\widetilde{\boldsymbol{p}})(\widetilde{\boldsymbol{q}})) \circ u_i^{-1}\, \chi_{u_i(V_i)}. \end{aligned}$$

Hence, $f(\alpha(\boldsymbol{p}, \boldsymbol{q}) + \beta(\widetilde{\boldsymbol{p}}, \widetilde{\boldsymbol{q}})) = \alpha f(\boldsymbol{p}, \boldsymbol{q}) + \beta f(\widetilde{\boldsymbol{p}}, \widetilde{\boldsymbol{q}})$.

For surjectivity, we define $p_j := f \circ b_j - s_j \cdot f$, $j \in \mathbb{N}_m$ and $q_i := f \circ u_i - t_i \cdot f$, $i \in \mathbb{N}_n$. Since $f \in \mathsf{B}(X, \mathsf{Y})$, we have $\boldsymbol{p} \in \mathsf{B}_{\mathsf{Y}}^m$ and $\boldsymbol{q} \in \mathsf{B}_{\mathsf{Y}}^n$. Thus, $f(\boldsymbol{p}, \boldsymbol{q}) = f$. □

We denote the image of $\mathsf{B}_{\mathsf{Y}}^m \times \mathsf{B}_{\mathsf{Y}}^n$ under Θ by $\mathfrak{F}_{m,n}(X, \mathsf{Y})$ and remark that $\mathfrak{F}_{m,n}(X, \mathsf{Y})$ is an $\mathbb{R}$-vector space.

Consider now the special case $X := \mathbb{R} =: \mathsf{Y}$ and suppose that $\boldsymbol{p}$ and $\boldsymbol{q}$ are tuples of polynomials. Set $\operatorname{ord} \boldsymbol{p} := \sum_{j=1}^m \operatorname{ord} p_j$ and $\operatorname{ord} \boldsymbol{q} := \sum_{i=1}^n \operatorname{ord} q_i$, where ord denotes the

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order of a polynomial. Since each polynomial of order d is uniquely determined by d real values, there exits a canonical bijection between the set Π_d of polynomials of order d and $\mathbb{R}^d$. These observations imply the following corollary of Theorem 5.4.

Corollary 5.5. *Suppose that* $X := \mathbb{R} =: \mathsf{Y}$ *and that* $\boldsymbol{p} \in \bigtimes_{j=1}^{m} \Pi_{\mu_j}$ *and* $\boldsymbol{q} \in \bigtimes_{i=1}^{n} \Pi_{\nu_i}$. *Then there exists a linear isomorphism* $\iota : \mathbb{R}^{\operatorname{ord} \boldsymbol{p}} \times \mathbb{R}^{\operatorname{ord} \boldsymbol{q}} \to \mathfrak{F}_{m,n}(X, \mathsf{Y})$. *Moreover,* $\dim \mathfrak{F}_{m,n}(X, \mathsf{Y}) = \operatorname{ord} \boldsymbol{p} + \operatorname{ord} \boldsymbol{q}$.

We remark that in the case when $\mu_j := d$, $j \in \mathbb{N}_m$, and $\nu_i := e$, $i \in \mathbb{N}_n$, the sets $\bigtimes_{j=1}^{m} \Pi_d$ and $\bigtimes_{i=1}^{n} \Pi_e$, respectively, coincide with the set of all piecewise polynomials on $\bigcup_{j=1}^m (j, K_j)$ and $\bigcup_{i=1}^n V_i$, respectively.

The linear isomorphism $\iota : \mathbb{R}^{\operatorname{ord} \boldsymbol{p}} \times \mathbb{R}^{\operatorname{ord} \boldsymbol{q}} \to \mathfrak{F}_{m,n}(X, \mathsf{Y})$ allows the construction of a basis for $\mathfrak{F}_{m,n}(X, \mathsf{Y})$. To this end, choose in each K_j and V_i, respectively, $\operatorname{ord} p_j$ and $\operatorname{ord} q_i$ many points. Denote the sets of these points by $X^j := \{x^j_\kappa : \kappa \in \{1, \ldots, \operatorname{ord} p_j\}\}$, and $\Xi^i := \{\xi^i_\lambda : \lambda \in \{1, \ldots, \operatorname{ord} q_i\}$, respectively. Let

$$p_j = \sum_{\kappa=1}^{\operatorname{ord} p_j} p_j(x^j_\kappa)\, L^j_\kappa$$

be the Lagrange representation of p_j. Here, L^j_κ denotes the Lagrange interpolant. Similarly, we have

$$q_i = \sum_{\lambda=1}^{\operatorname{ord} q_i} q_i(\xi^i_\lambda)\, L^i_\lambda,$$

with the appropriate interpretation of the symbols. Then, Theorem 5.4 and Corollary 5.5 imply the following representation of a bounded local fractal function f in terms of its fractal Lagrange interpolants:

$$f = \sum_{j=1}^{m} \sum_{\kappa=1}^{\operatorname{ord} p_j} p_j(x^j_\kappa)\, \mathfrak{L}^j_\kappa + \sum_{i=1}^{n} \sum_{\lambda=1}^{\operatorname{ord} q_i} q_i(\xi^i_\kappa)\, \mathfrak{L}^i_\lambda,$$

where $\mathfrak{L}^*_\bullet$ denotes $j(L^*_\bullet, L^*_\bullet)$.

6. Tensor products of bounded local fractal functions with unbounded domains

In this section, we define the tensor product of bounded local fractal functions with unbounded domains, thus extending the previous construction to higher dimensions.

For this purpose, we follow the notation of the previous section, and assume that Y is a Banach space, and that X, $\widetilde{X}$, K, $\widetilde{K}$, $\{K_j\}$, $\{\widetilde{K}_j\}$, $\{V_i\}$, $\{\widetilde{V}_i\}$, $\mathcal{H} := \{b_j : K_j \to K : j \in \mathbb{N}_m\} \cup \{u_i : V_i \to U_{\pi(i)} : i \in \mathbb{N}_n\}$ and $\widetilde{\mathcal{H}} := \{\widetilde{b}_j : \widetilde{K}_j \to \widetilde{K} : j \in \mathbb{N}_m\} \cup \{\widetilde{u}_i : \widetilde{V}_i \to \widetilde{U}_{\widetilde{\pi}(i)} : i \in \mathbb{N}_n\}$ are as in the general set-up.

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Furthermore, we assume that $(\mathsf{Y}, \|\cdot\|_{\mathsf{Y}})$ is a *Banach algebra*, i.e., a Banach space that is also an associate algebra for which multiplication is continuous:

$$\|y_1 y_2\|_{\mathsf{Y}} \leq \|y_1\|_{\mathsf{Y}} \, \|y_2\|_{\mathsf{Y}}, \quad \forall \, y_1, y_2 \in \mathsf{Y}.$$

Let $f \in \mathsf{B}(X, \mathsf{Y})$ and $\widetilde{f} \in \mathsf{B}(\widetilde{X}, \mathsf{Y})$. The tensor product of f with $\widetilde{f}$, written $f \otimes \widetilde{f} : X \times \widetilde{X} \to \mathbb{Y}$, with values in Y is defined by

$$(f \otimes \widetilde{f})(x, \widetilde{x}) := f(x) \widetilde{f}(\widetilde{x}), \quad \forall \, (x, \widetilde{x}) \in X \times \widetilde{X}.$$

As f and $\widetilde{f}$ are bounded, the inequality

$$\|(f \otimes \widetilde{f})(x, \widetilde{x})\|_{\mathsf{Y}} = \|f(x) \widetilde{f}(\widetilde{x}\|_{\mathsf{Y}} \leq \|f(x)\|_{\mathsf{Y}} \, \|\widetilde{f}(\widetilde{x})\|_{\mathsf{Y}},$$

implies that $f \otimes \widetilde{f}$ is bounded. Under the usual addition and scalar multiplication of functions, the set

$$\mathsf{B}(X \times \widetilde{X}, \mathsf{Y}) := \{f \otimes \widetilde{f} : X \times \widetilde{X} \to \mathsf{Y} : f \otimes \widetilde{f} \text{ is bounded}\}$$

becomes a complete metric space when endowed with the metric

$$d(f \otimes \widetilde{f}, g \otimes \widetilde{g}) := \sup_{x \in X} \|f(x) - g(x)\|_{\mathsf{Y}} + \sup_{\widetilde{x} \in \widetilde{X}} \|\widetilde{f}(\widetilde{x}) - \widetilde{g}(\widetilde{x})\|_{\mathsf{Y}}.$$

Now let $\Phi : \mathsf{B}(X, \mathsf{Y}) \to \mathsf{B}(X, \mathsf{Y})$ and $\widetilde{\Phi} : \mathsf{B}(\widetilde{X}, \mathsf{Y}) \to \mathsf{B}(\widetilde{X}, \mathsf{Y})$ be contractive RB operators of the form (5.2). We define the tensor product of Φ with $\widetilde{\Phi}$ to be the RB operator $\Phi \otimes \widetilde{\Phi} : \mathsf{B}(X \times \widetilde{X}, \mathsf{Y}) \to \mathsf{B}(X \times \widetilde{X}, \mathsf{Y})$ given by

$$(\Phi \otimes \widetilde{\Phi})(f \otimes \widetilde{f}) := (\Phi f) \otimes (\widetilde{\Phi} \, \widetilde{f}).$$

It follows that $\Phi \otimes \widetilde{\Phi}$ maps bounded functions to bounded functions. Furthermore, $\Phi \otimes \widetilde{\Phi}$ is contractive on the complete metric space $(\mathsf{B}(X \times \widetilde{X}, \mathsf{Y}), d)$. To see this, note that

$$\begin{aligned}
&\sup_{x \in X} \|(\Phi f)(x) - (\Phi g)(x)\|_{\mathsf{Y}} + \sup_{\widetilde{x} \in \widetilde{X}} \|(\Phi \widetilde{f})(\widetilde{x}) - (\Phi \widetilde{g})(\widetilde{x})\|_{\mathsf{Y}} \\
&\quad \leq \ell \sup_{x \in X} \|f(x) - g(x)\|_{\mathsf{Y}} + \widetilde{\ell} \sup_{\widetilde{x} \in \widetilde{X}} \|\widetilde{f}(\widetilde{x}) - \widetilde{g}(\widetilde{x})\|_{\mathsf{Y}} \\
&\quad \leq \max\{\ell, \widetilde{\ell}\} \, d(f \otimes \widetilde{f}, g \otimes \widetilde{g}),
\end{aligned}$$

where we used (5.3) and denoted the uniform contractivity constant of $\widetilde{\Phi}$ by $\widetilde{\ell}$.

The unique fixed point of the RB operator $\Phi \otimes \widetilde{\Phi}$ will be called a *tensor product bounded local fractal function with unbounded domain* and its graph a *tensor product bounded local fractal surface over an unbounded domain*.

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7. Bochner–Lebesgue spaces $L^p(\mathbb{X}, \mathsf{Y})$

We may construct local fractal functions on spaces other than $B(X, \mathsf{Y})$. (See also [7, 15].) In this section, we derive conditions under which local fractal functions over unbounded domains are elements of the Bochner–Lebesgue spaces $L^p(\mathbb{X}, \mathsf{Y})$ for $p > 0$.

To this end, assume that X is a closed subspace of a Banach space X and that $\mathbb{X} := (X, \Sigma, \mu)$ is a measure space. Recall that the Bochner–Lebesgue space $L^p(\mathbb{X}, \mathsf{Y})$, $1 \le p \le \infty$, consists of all Bochner measurable functions $f : X \to \mathsf{Y}$ such that

$$\|f\|_{L^p(\mathbb{X},\mathsf{Y})} := \left(\int_X \|f(x)\|_{\mathsf{Y}}^p \,\mathrm{d}\mu(x)\right)^{1/p} < \infty, \quad 1 \le p < \infty,$$

and

$$\|f\|_{L^\infty(\mathbb{X},\mathsf{Y})} := \operatorname{ess\,sup}_{x\in X} \|f(x)\|_{\mathsf{Y}} < \infty, \quad p = \infty.$$

For $0 < p < 1$, the spaces $L^p(\mathbb{X}, \mathsf{Y})$ are defined as above, but instead of a norm, a metric is used to obtain completeness. More precisely, define $d_p : L^p(\mathbb{X}, \mathsf{Y}) \times L^p(\mathbb{X}, \mathsf{Y}) \to \mathbb{R}$ by

$$d_p(f, g) := \|f - g\|_{\mathsf{Y}}^p.$$

Then $(L^p(\mathbb{X}, \mathsf{Y}), d_p)$ becomes an F-space. (Note that the inequality $(a + b)^p \le a^p + b^p$ holds for all $a, b \ge 0$.) For more details, refer to [1, 18].

Theorem 7.1. *Let $(\mathsf{Y}, d_{\mathsf{Y}})$ be a Banach space and let X, $\{K_j\}$, $\{V_i\}$, and $\mathcal{H} := \{b_j : K_j \to K : j \in \mathbb{N}_m\} \cup \{u_i : V_i \to U_{\pi(i)} : i \in \mathbb{N}_n\}$ be as in the general set-up. Assume that $\mathbb{X} := (X, \Sigma, \mu)$ is a measure space and that the families $\{b_j\}$ and $\{u_i\}$ are μ-measurable diffeomorphisms. Further assume that $J_{b_j} := \sup\{\|Db_j^{-1}\|_{K_j}\} < \infty$ and $J_{u_i} := \sup\{\|Du_i^{-1}\|_{V_i}\} < \infty$, where D denotes the derivative. Suppose $\boldsymbol{p} \in \mathop{\times}_{j=1}^{m} L^p(K_j, \mathsf{Y})$, $\boldsymbol{q} \in \mathop{\times}_{i=1}^{n} L^p(V_i, \mathsf{Y})$, $\boldsymbol{s} \in \mathop{\times}_{j=1}^{m} L^p(K_j, \mathbb{R})$, and $\boldsymbol{t} \in \mathop{\times}_{i=1}^{n} L^p(V_i, \mathbb{R})$.*

The operator $\Phi : L^p(\mathbb{X}, \mathsf{Y}) \to \mathbb{R}^X$, $p \in (0, \infty]$, defined by (5.7) is well defined and maps $L^p(\mathbb{X}, \mathsf{Y})$ into itself. Moreover, if

$$\begin{cases} \displaystyle\sum_{j=1}^{m} J_{b_j}\, \|s_j\|_{L^p(K_j,\mathbb{R})}^p + \sum_{i=1}^{n} J_{u_i}\, \|t_i\|_{L^p(V_i,\mathbb{R})}^p < 1, & p \in (0, 1) \\ \displaystyle\left(\sum_{j=1}^{m} J_{b_j}\, \|s_j\|_{L^p(K_j,\mathbb{R})}^p + \sum_{i=1}^{n} J_{u_i}\, \|t_i\|_{L^p(V_i,\mathbb{R})}^p\right)^{1/p} < 1, & p \in [1, \infty) \\ \max\{\|s_j\|_{L^\infty(K_j,\mathbb{R})} : j \in \mathbb{N}_m\} + \max\{\|t_i\|_{L^\infty(V_i,\mathbb{R})} : i \in \mathbb{N}_n\} < 1, & p = \infty, \end{cases}$$

then Φ is contractive on $L^p(\mathbb{X}, \mathsf{Y})$. Its unique fixed point f is called a fractal function of class $L^p(\mathbb{X}, \mathsf{Y})$ on the unbounded domain X.

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Proof. Note that under the hypotheses on the functions p_j, q_i and s_j, t_i as well as the mappings b_j, u_i, Φf is well defined and an element of $L^p(\mathbb{X},\mathsf{Y})$. It remains to be shown that, under the stated conditions, Φ is contractive on $L^p(\mathbb{X},\mathsf{Y})$.

For this purpose, first consider the case $1 \le p < \infty$. If $g, h \in L^p(\mathbb{X},\mathsf{Y})$, then

$$\begin{aligned}
\|\Phi g - \Phi h\|^p_{L^p(\mathbb{X},\mathsf{Y})} &= \int\limits_X \|\Phi g(x) - \Phi h(x)\|^p_{\mathsf{Y}}\,\mathrm{d}\mu(x) \\
&\le \int\limits_X \left\| \sum_{j=1}^m (s_j \circ b_j^{-1})[(g_j \circ b_j^{-1}) - (h_j \circ b_j^{-1})]\,\chi_{b_j(K_j)} \right\|^p_{\mathsf{Y}} \mathrm{d}\mu \\
&\quad + \int\limits_X \left\| \sum_{i=1}^n (t_i \circ u_i^{-1})[(g_i \circ u_i^{-1}) - (h_i \circ u_i^{-1})]\,\chi_{u_i(V_i)} \right\|^p_{\mathsf{Y}} \mathrm{d}\mu \\
&\le \sum_{j=1}^m J_{b_j} \int\limits_{K_j} |s_j|^p_{\mathbb{R}}\,\|g_j - h_j\|^p_{\mathsf{Y}}\,\mathrm{d}\mu + \sum_{i=1}^n J_{u_i} \int\limits_{V_i} |t_i|^p_{\mathbb{R}}\,\|g_i - h_i\|^p_{\mathsf{Y}}\,\mathrm{d}\mu \\
&\le \left(\sum_{j=1}^m J_{b_j}\,\|s_j\|^p_{L^p(K_j,\mathbb{R})} + \sum_{i=1}^n J_{u_i}\,\|t_i\|^p_{L^p(V_i,\mathbb{R})} \right) \|g - h\|_{L^p(\mathbb{X},\mathsf{Y})}.
\end{aligned}$$

The case $0 < p < 1$ follows now in very much the same fashion. We again have after substitution and rearrangement

$$\begin{aligned}
d_p(\Phi g, \Phi h) &= \|\Phi g - \Phi h\|^p_{L^p(\mathbb{X},\mathsf{Y})} \\
&\le \sum_{j=1}^m J_{b_j} \int\limits_{K_j} |s_j|^p_{\mathbb{R}}\,\|g_j - h_j\|^p_{\mathsf{Y}}\,\mathrm{d}\mu + \sum_{i=1}^n J_{u_i} \int\limits_{V_i} |t_i|^p_{\mathbb{R}}\,\|g_i - h_i\|^p_{\mathsf{Y}}\,\mathrm{d}\mu \\
&\le \left(\sum_{j=1}^m J_{b_j}\,\|s_j\|^p_{L^p(K_j,\mathbb{R})} + \sum_{i=1}^n J_{u_i}\,\|t_i\|^p_{L^p(V_i,\mathbb{R})} \right) d_p(g, h).
\end{aligned}$$

Now let $p = \infty$. Then

$$\begin{aligned}
\|\Phi g - \Phi h\|_{L^\infty(\mathbb{X},\mathsf{Y})} &= \operatorname{ess\,sup}_{x \in X} \|\Phi g(x) - \Phi h(x)\|_{\mathsf{Y}} \\
&\le \operatorname{ess\,sup}_{x \in b_j(K_j)} \left\| \sum_{j=1}^m s_j \circ b_j^{-1} \cdot (g - h) \circ b_j^{-1}) \right\|_{\mathsf{Y}} \\
&\quad + \operatorname{ess\,sup}_{x \in u_i(V_i)} \left\| \sum_{i=1}^n t_i \circ u_i^{-1} \cdot (g - h) \circ u_i^{-1}) \right\|_{\mathsf{Y}} \\
&\le \big(\max\{\|s_j\|_{L^\infty(K_j,\mathbb{R})} : j \in \mathbb{N}_m\} + \max\{\|t_i\|_{L^\infty(V_i,\mathbb{R})} : i \in \mathbb{N}_n\} \big) \\
&\quad \times \|g - h\|_{L^\infty(\mathbb{X},\mathsf{Y})}.
\end{aligned}$$

These calculations prove the claims. □

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8. The local IFS associated with the RB operator

In this section, we associate with the RB operator (5.2) a local IFS and show that the graph of the unique fixed point of Φ is a local attractor of this local IFS.

To this end, let

$$X_\ell := \begin{cases} K_\ell, & \ell \in \{1, \ldots, m\}; \\ V_{\ell-m}, & \ell \in \{m+1, \ldots, m+n\}. \end{cases}$$

With the sets X_ℓ we associate continuous mappings $f_\ell : X_\ell \to X$ by setting

$$f_\ell := \begin{cases} b_\ell, & \ell \in \{1, \ldots, m\}; \\ u_{\ell-m}, & \ell \in \{m+1, \ldots, m+n\}. \end{cases}$$

In addition, define mappings $g_\ell : X_\ell \times Y \to Y$ by

$$g_\ell := \begin{cases} v_\ell, & \ell \in \{1, \ldots, m\}; \\ w_{\ell-m}, & \ell \in \{m+1, \ldots, m+n\}, \end{cases}$$

and $h_\ell : X_\ell \times Y \to X_\ell \times Y$ by

$$h_\ell(x, y) := (f_\ell(x), g_\ell(x, y)), \quad \ell \in \mathbb{N}_{m+n}.$$

Assume that the functions v_j and w_i are continuous as functions $X \to Y$. Then the mappings g_ℓ and therefore the mappings h_ℓ are continuous. We define $\mathcal{H}_{\text{loc}} := \{h_\ell : X_\ell \times Y \to X_\ell \times Y\}_{\ell \in \mathbb{N}_{m+n}}$.

Hence, the pair $(X \times Y, \mathcal{H}_{\text{loc}})$ is a local IFS. As $X \times Y$ is locally compact, the set-valued mapping $\mathcal{F}_{\text{loc}} : 2^{X\times Y} \to 2^{X\times Y}$, defined by

$$\mathcal{F}_{\text{loc}}(S) := \bigcup_{\ell \in \mathbb{N}_{m+n}} h_\ell(S \cap (X_\ell \times Y)),$$

is continuous [**6**, Theorem 1].

Proposition 8.1. *The graph G of the fixed point f of the RB operator (5.2) is an attractor of the local IFS $(X \times Y, \mathcal{H}_{\text{loc}})$.*

Proof. We have

$$\begin{aligned}
\mathcal{F}_{\text{loc}}(G) &= \bigcup_{\ell \in \mathbb{N}_{m+n}} h_\ell(G \cap (X_\ell \times Y)) = \bigcup_{\ell \in \mathbb{N}_{m+n}} h_\ell(\{(x, f(x)) : x \in X_\ell\}) \\
&= \bigcup_{\ell \in \mathbb{N}_{m+n}} \{(f_\ell(x), g_\ell(x, f(x))) : x \in X_\ell\} \\
&= \bigcup_{j \in \mathbb{N}_m} \{(b_j(x), v_j(x, f(x))) : x \in K_j\} \cup \bigcup_{i \in \mathbb{N}_n} \{(u_i(x), w_i(x, f(x))) : x \in V_i\} \\
&= \bigcup_{j \in \mathbb{N}_m} \{(b_j(x), f(b_j(x))) : x \in K_j\} \cup \bigcup_{i \in \mathbb{N}_n} \{(u_i(x), f(u_i(x))) : x \in V_i\}
\end{aligned}$$

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$$= \bigcup_{j\in\mathbb{N}_m} \{(x, f(x)) : x \in b_j(K_j)\} \cup \bigcup_{i\in\mathbb{N}_n} \{(x, f(x)) : x \in u_i(V_i)\}$$

$$= \bigcup_{\ell\in\mathbb{N}_{m+n}} \{(x, f(x)) : x \in f_\ell(X_\ell)\} = G. \qquad \square$$

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