Immediate smoothing and global solutions for initial data in $L^1 \times W^{1,2}$ in a Keller–Segel system with logistic terms in 2D

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This paper deals with the logistic Keller–Segel model

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \kappa u - \mu u^2, \\ v_t = \Delta v - v + u \end{cases}$$

in bounded two-dimensional domains (with homogeneous Neumann boundary conditions and for parameters $\chi, \kappa \in \mathbb{R}$ and $\mu > 0$), and shows that any nonnegative initial data $(u_0, v_0) \in L^1 \times W^{1,2}$ lead to global solutions that are smooth in $\bar{\Omega} \times (0, \infty)$.

1. Introduction

Chemotaxis systems [16, 13, 3] are mainly known for their ability to produce singularities in the form of blow-up (see [15, 12, 21] or the recent survey [19] for an overview), that is, initially smooth solutions cease to exist and thereby lose their regularity—for example, converging to certain multiples of Dirac measures plus an L^1-function as time tends to some finite blow-up time, cf. [11, 22]. This paper is concerned with the opposite setting: given initial data (u_0, v_0) that are significantly less regular than smooth functions (we will strive for L^1 in the first component), we will ask whether the system

$$u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \kappa u - \mu u^2 \quad \text{in } \Omega \times (0, \infty), \tag{1.1a}$$

$$v_t = \Delta v - v + u \quad \text{in } \Omega \times (0, \infty), \tag{1.1b}$$

$$\partial_\nu u = 0 = \partial_\nu v \quad \text{on } \partial\Omega \times (0, \infty), \tag{1.1c}$$

$$u(\cdot, 0) = u_0, \ v(\cdot, 0) = v_0 \quad \text{in } \Omega, \tag{1.1d}$$

posed in a bounded domain $\Omega \subset \mathbb{R}^2$, admits solutions that immediately become classically smooth (and answer this question affirmatively, see theorem 1.1).

System (1.1), combining logistic source terms as the simplest form of population dynamics and chemotaxis—that is, the partially directed movement in response to a chemical signal —, arises in different contexts in mathematical biology, ranging from the study of cancer [2] to ecology [1]. From a modelling perspective, it seems important to understand all effects emerging from the interplay of taxis and population growth terms, so as to guide (and serve as solid foundation for) the creation of further and more refined models.

In the classical Keller–Segel system, that is (1.1) with $\kappa = \mu = 0$ and $\chi > 0$, in one-dimensional (1D) domains global bounded solutions exist for any reasonably regular initial data and the result is similar in the two-dimensional (2D) setting, as long as the initial mass is sufficiently small in the sense that $\int_\Omega u_0 < 4\pi/\chi$. Contrastingly, in 2D domains for any $m > 4\pi/\chi$ and in 3D for any $m > 0$, one can find smooth initial data u_0 with $\int_\Omega u_0 = m$ such that the corresponding solution blows up in finite time (cf. §2.1 of the survey [19] and results cited therein, in particular [14]).

A typical assumption on the initial data—apart from their nonnegativity—is that they be continuous (see, e.g. [3, lemma 3.1]). This requirement is necessary if one wishes to obtain classical solutions under any definition involving their continuity up to time 0, inclusively. Other than that, weaker assumptions can suffice: the system has been studied, for example, with initial data in certain Besov spaces, leading to solutions that are continuous as functions with values in said spaces.

Even more extensive results concerning local solvability were achieved by Biler in [4] already: for 2D domains, the admissible class of initial data covers all of $L^1(\Omega)$ and even some measures beyond; these solutions are mild solutions and their initial data are attained as weak-*-limit in the space $\mathcal{M}(\Omega)$ of finite signed measures. The widest range of initial data for higher-dimensional settings is treated in [39]: non-negative $L^1(\Omega)$-data lead to global generalized solutions (for the price of a solution concept that is based on a complicated integral inequality for a combination of both solution components and that involves rather weak regularity information).

However, the solutions arising in these works may (and for some initial data: do; compare also the above-mentioned results on blow-up even of classical solutions) still undergo blow-up.

On the other hand, logistic terms (i.e. $\kappa \in \mathbb{R}, \mu > 0$ in (1.1)), are known to have a regularizing effect on chemotaxis systems. For example, in 2D domains, (1.1) admits global classical, bounded solutions (see [25] and [38] or, for the analogous parabolic–elliptic system, [26]); similarly, in higher-dimensional domains large values of μ ensure the global existence of classical solutions ([31], also [23] or [37]). In the case of small $\mu > 0$ (and initial data in $L^2(\Omega) \times W^{1,2}(\Omega)$) at least weak solutions exist globally [18], and, moreover, become smooth after some (possibly large) time if κ is small, Ω is three-dimensional and convex [18], see also [28, 29]. Nevertheless, weaker degradation (at least in parabolic–elliptic relatives of (1.1), cf. [32, 33]) or spatial inhomogeneity of appropriate form in the zero-order terms, [9], can still facilitate blow-up.

Systems like (1.1) are covered by the general existence result of [36] even if $+\kappa u - \mu u^2$ is replaced by a weaker absorption $+\kappa u - \mu u^\alpha$ with, merely, $\alpha >$

$(2n + 4)/(n + 4)$: for nonnegative $(u_0, v_0) \in L^1(\Omega) \times L^2(\Omega)$, generalized solutions exist.

However, in order to allow for its wide range of applicability (including said sub-quadratic terms and not restricted to chemotaxis systems), the solution concept pursued in [**36**] is very weak: the first component, u, for instance, is supposed to satisfy the weak form of a differential inequality for $\ln(1 + u)$ and to belong to $L^\infty_{loc}([0, \infty); L^1(\Omega)) \cap L^\alpha_{loc}(\overline{\Omega} \times [0, \infty))$. Under more restrictive conditions on α and v_0—the borderline case in $n = 2$, in fact, coinciding with the case considered in the current paper—, [**36**] then goes on to assert more regular solutions, so that u additionally belongs to $L^1_{loc}([0, \infty); W^{1,1}(\Omega))$ and solves the equation in the standard weak sense.

The results on global existence of classical solutions reported above make us suspect that at least in the prototypical system with logistic population dynamics and a 2D domain, even classical solutions should be achievable, notwithstanding initial data merely belonging to a space of less regular than continuous functions. If $\kappa = \mu = 0$ and the domain is 1D (one-dimensionality is a way to prevent blow-up in the Keller–Segel system), the system can cope with initial data from $(C(\overline{\Omega}))^* \times L^2(\Omega)$ and turn them into classical solutions [**35**].

For a simplified parabolic–elliptic variant of (1.1) (with (1.1b) reading $0 = \Delta v - m(t) + u$, $m(t) = \int_\Omega u(\cdot, t)$) in a radially symmetric setting, classical solutions were found for initial data lying below, essentially, a multiple of $\exp(\lambda |x|^{-\alpha})$ with some $\lambda > 0$ and $\alpha < n$ [**34**].

For the possibly non-radial setting in the fully parabolic system (1.1), we will assume that the initial data comply with the requirements that

$$u_0 \in L^1(\Omega), \quad u_0 \geqslant 0, \qquad v_0 \in W^{1,2}(\Omega), \quad v_0 \geqslant 0 \qquad (1.2)$$

and show that then solutions result, which are immediately smooth (in the sense that they belong to $C^{2,1}(\overline{\Omega} \times (0, \infty))$) and continuous as $L^1(\Omega)$-valued functions (in the weak topology of $L^1(\Omega)$) up to the initial time, i.e. belong to $C_w([0, \infty); L^1(\Omega)) = C([0, \infty); (L^1(\Omega) \text{with weak topology}))$:

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, $\chi \in \mathbb{R}$, $\kappa \in \mathbb{R}$, $\mu > 0$, and assume that u_0, v_0 satisfy* (1.2). *Then there is*

$$(u, v) \in \left(C^{2,1}(\overline{\Omega} \times (0, \infty)) \cap C_w([0, \infty); L^1(\Omega)) \right)^2$$

solving (1.1).

REMARK 1.2. While the theorem covers both positive and negative values of χ, the relevant case is that of attractive chemotaxis ($\chi > 0$)—for chemo-repulsion ($\chi < 0$) one may probably expect such an immediate smoothing effect (and global solutions) even without a logistic source. For investigations in this direction, in a parabolic–elliptic system, the reader is referred to the upcoming work [**10**] by Heihoff.

The proof will proceed by approximation of these solutions by solutions to problems where global classical solvability is known, which in particular involves approximation of the initial data.

Given u_0 and v_0 as in (1.2) and $q_0 > 2$, for every $\varepsilon > 0$ we introduce $u_{0\varepsilon}$ and $v_{0\varepsilon}$ such that the following hold true:

$$u_{0\varepsilon} \in C(\overline{\Omega}), \quad v_{0\varepsilon} \in W^{1,q_0}(\Omega), \tag{1.3a}$$

$$u_{0\varepsilon} \geqslant 0, \qquad v_{0\varepsilon} \geqslant 0 \quad \text{in } \Omega, \tag{1.3b}$$

$$u_{0\varepsilon} \to u_0 \quad \text{in } L^1(\Omega) \quad \text{as } \varepsilon \to 0, \tag{1.3c}$$

$$v_{0\varepsilon} \to v_0 \quad \text{in } W^{1,2}(\Omega) \quad \text{as } \varepsilon \to 0, \tag{1.3d}$$

$$\|u_{0\varepsilon}\|_{L^1(\Omega)} \leqslant 2\|u_0\|_{L^1(\Omega)} \quad \text{for all } \varepsilon > 0, \tag{1.3e}$$

$$\|v_{0\varepsilon}\|_{W^{1,2}(\Omega)} \leqslant 2\|v_0\|_{W^{1,2}(\Omega)} \quad \text{for all } \varepsilon > 0, \tag{1.3f}$$

and deal with solutions of the initial boundary value problem

$$u_{\varepsilon t} = \Delta u_\varepsilon - \chi \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) + \kappa u_\varepsilon - \mu u_\varepsilon^2 \qquad \text{in } \Omega \times (0, \infty), \tag{1.4a}$$

$$v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon} \qquad \text{in } \Omega \times (0, \infty), \tag{1.4b}$$

$$\partial_\nu u_\varepsilon = 0 = \partial_\nu v_\varepsilon \qquad \text{on } \partial\Omega \times (0, \infty), \tag{1.4c}$$

$$u_\varepsilon(\cdot, 0) = u_{0\varepsilon}, \ v_\varepsilon(\cdot, 0) = v_{0\varepsilon} \qquad \text{in } \Omega. \tag{1.4d}$$

REMARK 1.3. The approximation used here coincides with that in [**36**, §4]. In (1.4b), we could, instead, replace $u_\varepsilon/(1 + \varepsilon u_\varepsilon)$ by u_ε. Then global existence of solutions (cf. lemma 2.1) might be less obvious, but can, nevertheless, be gained from [**23**]. Otherwise, the proofs below remain essentially unaffected.

For the proof of smoothness, it will suffice to deduce boundedness of the solutions in some $L^p(\Omega)$ space with sufficiently large p, rendering classical parabolic regularity theory applicable (cf. lemmata 4.5 to 4.7). The process of obtaining such boundedness information will be based on a study of $\partial_t \int_\Omega u^2$, which due to $\int_\Omega u_0^2$ possibly being infinite cannot be extended to time-intervals reaching zero, but thanks to the superlinear degradation term nevertheless provides estimates on intervals bounded away from the initial time (see lemma 4.4).

What remains is the somewhat separate investigation of the solution near time $t = 0$, which has to ensure that the initial data are attained in a reasonable way. This is where the main focus of this paper lies. A basic idea concerning continuity at $t = 0$ would be to look for a uniform bound of $u_\varepsilon(\cdot, t)$ on some interval $[0, t_0)$ and for $\varepsilon > 0$, observe that this bound can be transferred to $u(\cdot, t)$, $t \in [0, t_0)$, via the approximation procedure, hope for a corresponding compactness result (maybe in some weak topology) and conclude for every sequence $(t_k)_{k \in \mathbb{N}}$, $t_k \to 0$ as $k \to \infty$ that $u(\cdot, t_k)$ converges, at least along a subsequence.

It seems, however, that the derivation of these *a priori* bounds cannot be based on time derivatives of functionals like $\int_\Omega u^2(\cdot, t)$ (as discussed above); even for $\int_\Omega u \log u$—a term that is part of the helpful and often-employed Lyapunov functional for the Keller–Segel system—the regularity of u_0 in (1.2) is clearly insufficient.

However, as long as $u_0 \in L^1(\Omega)$, there is some function Φ of superlinear growth such that also $\Phi(u_0)$ belongs to $L^1(\Omega)$. We will use this together with a computation

of $\partial_t \int_\Omega \Phi(u(\cdot, t))$ to finally achieve weak compactness of $\{u(\cdot, t) \mid t \in [0, t_0)\}$ in $L^1(\Omega)$. We will dedicate §3 to this strategy, while §3.4, in particular, ensures that the limit as $t \to 0$ coincides with u_0. Concerning the choice of Φ (or rather the properties of Φ that will be used in the proof), see also the discussion at the beginning of §3.

Notation. We will assume the smooth, bounded domain $\Omega \subset \mathbb{R}^2$, the parameters $\kappa \in \mathbb{R}$, $\chi \in \mathbb{R}$, $\mu > 0$, initial data (u_0, v_0) as in (1.2), the exponent $q_0 > 2$ and one family $\{(u_{0\varepsilon}, v_{0\varepsilon}) | \varepsilon > 0\}$ obeying (1.3) to be fixed throughout the paper. Keep in mind that this, in particular, means that all constants mentioned below may depend on these objects. Constants denoted by C_i with capital letter C are unique within the paper (usually introduced in an earlier lemma), constants c_i denoted by lowercase c are local to each proof. For $\varepsilon > 0$, $(u_\varepsilon, v_\varepsilon)$ will always denote the corresponding solution of (1.4).

2. Existence and first *apriori* estimates

LEMMA 2.1. *For every $\varepsilon > 0$, (1.4) has a unique classical nonnegative solution $(u_\varepsilon, v_\varepsilon)$ with*

$$u_\varepsilon \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))$$

$$v_\varepsilon \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \cap L^\infty_{loc}([0, \infty); W^{1,q_0}(\Omega)).$$

Proof. Local existence and uniqueness are obtained as usual (see [**3**, lemma 3.1]); boundedness, and hence globality, results from the obvious boundedness of the force term $+u_\varepsilon/(1 + \varepsilon u_\varepsilon)$ in the second equation (for fixed $\varepsilon > 0$). □

LEMMA 2.2. *There is $C_1 > 0$ such that for every $\varepsilon > 0$ and every $t \in [0, \infty)$,*

$$\int_\Omega u_\varepsilon(\cdot, t) \leqslant C_1.$$

Proof. According to the Cauchy–Schwarz inequality, $(\int_\Omega u_\varepsilon(\cdot, t))^2 \leqslant |\Omega| \int_\Omega u_\varepsilon^2(\cdot, t)$ for all $t \in (0, \infty)$ and all $\varepsilon > 0$. By integration of (1.4a), the functions defined by $y_\varepsilon(t) := \int_\Omega u_\varepsilon(\cdot, t)$, $t \in (0, \infty)$, for $\varepsilon > 0$ therefore satisfy

$$y_\varepsilon'(t) \leqslant \kappa y_\varepsilon(t) - \frac{\mu}{|\Omega|}(y_\varepsilon(t))^2 \qquad \text{for all } t \in (0, \infty),$$

as well as $y_\varepsilon(0) \leqslant 2\|u_0\|_{L^1(\Omega)}$ by (1.3e), and their boundedness follows from an ODE comparison argument. □

LEMMA 2.3. *There is $C_2 > 0$ such that for every $\varepsilon > 0$ and every $t \in (0, \infty)$,*

$$\int_0^t \int_\Omega u_\varepsilon^2 \leqslant C_2(1 + t). \tag{2.1}$$

Proof. Integrating (1.4a) over $\Omega \times (0, t)$ we see that

$$\mu \int_0^t \int_\Omega u_\varepsilon^2 \leqslant \kappa \int_0^t \int_\Omega u_\varepsilon + \int_\Omega u_{0\varepsilon} - \int_\Omega u_\varepsilon(\cdot, t) \qquad \text{for every } t > 0 \text{ and } \varepsilon > 0,$$

which due to lemma 2.2 and (1.3e) yields (2.1) with $C_2 := (1/\mu) \max\{C_1\kappa, 2\|u_0\|_{L^1(\Omega)}\}$. □

LEMMA 2.4. *There is $C_3 > 0$ such that for every $\varepsilon > 0$ and every $t \in [0, \infty)$,*

$$\int_\Omega v_\varepsilon^2(\cdot, t) \leqslant C_3(1+t) \quad and \quad \int_0^t \int_\Omega v_\varepsilon^2 \leqslant C_3(1+t). \qquad (2.2)$$

Proof. Testing (1.4b) by v_ε and applying Young's inequality, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega v_\varepsilon^2 + \int_\Omega v_\varepsilon^2 = -2 \int_\Omega |\nabla v_\varepsilon|^2 - \int_\Omega v_\varepsilon^2 + 2 \int_\Omega \frac{u_\varepsilon v_\varepsilon}{1 + \varepsilon u_\varepsilon} \leqslant \int_\Omega u_\varepsilon^2 \qquad \text{on } (0, \infty),$$

so that (2.2) follows from lemma 2.3 and (1.3f) with $C_3 := C_2 + 4\|v_0\|_{W^{1,2}(\Omega)}^2$. $\quad\square$

LEMMA 2.5. *There are $C_4 > 0$ and $C_5 > 0$ such that for every $\varepsilon > 0$ and every $t \in (0, \infty)$,*

$$\int_\Omega |\nabla v_\varepsilon(\cdot, t)|^2 \leqslant C_4(1+t), \qquad \int_0^t \int_\Omega |\Delta v_\varepsilon|^2 \leqslant C_5(1+t). \qquad (2.3)$$

Proof. On account of Young's inequality, an integration of (1.4b) over Ω results in

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega |\Delta v_\varepsilon|^2 = - \int_\Omega |\Delta v_\varepsilon|^2 - 2 \int_\Omega |\nabla v_\varepsilon|^2 + 2 \int_\Omega \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon} \Delta v_\varepsilon \leqslant \int_\Omega u_\varepsilon^2$$

on $(0, \infty)$ for every $\varepsilon > 0$. Setting $C_4 := C_5 := C_2 + 4\|v_0\|_{W^{1,2}(\Omega)}^2$, we can rely on lemma 2.3 and (1.3f) to derive (2.3). $\quad\square$

LEMMA 2.6. *There is $C_6 > 0$ such that*

$$\int_0^t \int_\Omega v_{\varepsilon t}^2 \leqslant C_6(1+t)$$

for every $t > 0$ and $\varepsilon > 0$.

Proof. By (1.4b), for every $t > 0$ and $\varepsilon > 0$ we have

$$\int_0^t \int_\Omega v_{\varepsilon t}^2 \leqslant 3 \int_0^t \int_\Omega |\Delta v_\varepsilon|^2 + 3 \int_0^t \int_\Omega v_\varepsilon^2 + 3 \int_0^t \int_\Omega u_\varepsilon^2 \leqslant C_6(1+t)$$

if we rely on lemmata 2.5, 2.4 and 2.3 and set $C_6 := 3(C_5 + C_3 + C_2)$. $\quad\square$

LEMMA 2.7. *There is $C_7 > 0$ such that for every $\varepsilon > 0$ and every $t \in (0, \infty)$,*

$$\int_0^t \int_\Omega |\nabla v_\varepsilon|^4 \leqslant C_7(1+t)^3. \qquad (2.4)$$

Proof. We take $c_1 > 0$ and $c_2 > 0$ from a version of the Gagliardo–Nirenberg inequality ([**24**, p. 126] combined with [**8**, theorem I.19.1]) stating that

$$\|\nabla\varphi\|_{L^4(\Omega)}^4 \leqslant c_1\|\Delta\varphi\|_{L^2(\Omega)}^2\|\nabla\varphi\|_{L^2(\Omega)}^2 + c_2\|\nabla\varphi\|_{L^2(\Omega)}^4$$

for all $\varphi \in W^{2,2}(\Omega)$ with $\partial_\nu\varphi = 0$ on $\partial\Omega$. Then, due to lemma 2.5,

$$\int_0^t\int_\Omega |\nabla v_\varepsilon|^4 \leqslant c_1\int_0^t\|\Delta v_\varepsilon\|_{L^2(\Omega)}^2\|\nabla v_\varepsilon\|_{L^2(\Omega)}^2 + c_2\int_0^t\|\nabla v_\varepsilon\|_{L^2(\Omega)}^4$$

$$\leqslant c_1C_4(1+t)\int_0^t\|\Delta v_\varepsilon\|_{L^2(\Omega)}^2 + c_2C_4^2\int_0^t(1+t)^2$$

$$\leqslant c_1C_4C_5(1+t)^2 + c_2C_4^2t(1+t)^2$$

holds for every $\varepsilon > 0$ and $t \in (0,\infty)$, and (2.4) follows with $C_7 := c_1C_4C_5 + c_2C_4^2$. □

3. Initial regularity

As stated in the Introduction, the source for all regularity statements at the initial time will be the study of $\partial_t\int_\Omega \Phi(u_\varepsilon)$ for a well-chosen function Φ. Of Φ we will require the following:

- finiteness of $\int_\Omega \Phi(u_0)$ (or, more precisely: boundedness of $\int_\Omega \Phi(u_{0\varepsilon})$), so that there is a chance for ε-independent bounds on $\sup_t\int_\Omega \Phi(u_\varepsilon(\cdot,t))$,

- superlinear growth, in order to conclude some weak $L^1(\Omega)$ compactness; the condition can be stated as $(1/x)\Phi(x) \to \infty$ or, for differentiable functions, as $\Phi'(x) \to \infty$ as $x \to \infty$,

- differentiability—ideally $\Phi \in C^2$—enabling any reasoning relying on derivatives, along with nonnegativity of Φ, Φ' and Φ'',

- $\Phi''(x) \leqslant 1/x$, so as to facilitate estimates dealing with the cross-diffusive term during the computation of $\partial_t\int_\Omega \Phi(u_\varepsilon)$.

The first two of these are a product of de la Vallée Poussin's theorem, also the third is often stated as part of it. Finding a proof in the literature which covers the differentiability seems more difficult, however. As our subsequent reasoning crucially relies on it, we will recall a proof in the short §3.1. The key idea for the last of our requirements is to diminish Φ by setting $\Phi''_{new}(x) = \min\{1/x, \Phi''_{old}(x)\}$. This would ensure $\Phi_{new} \leqslant \Phi_{old}$ and thus preserve the first requirement; it could, however, violate the superlinear growth condition: even if $f_1 \notin L^1(\mathbb{R}_+)$ and $f_2 \notin L^1(\mathbb{R}_+)$, it may happen that their pointwise minimum $\min\{f_1, f_2\}$ lies in $L^1(\mathbb{R}_+)$. We will therefore, roughly, try to set $\Phi''_{new}(x) = 1/x$ whenever $\Phi''_{old}(x) \geqslant 1/x$ but in case of $\Phi''_{old}(x) < 1/x$ will also use $\Phi''_{new}(x) = 1/x$, until $\int_0^x \Phi''_{new}(y)\,\mathrm{d}y$ has 'caught up' with $\int_0^x \Phi''_{old}(y)\,\mathrm{d}y$, and only in case of $\Phi''_{old}(x) < 1/x$ and $\Phi'_{new}(x) = \Phi'_{old}(x)$ set $\Phi''_{new}(x) = \Phi''_{old}(x)$. Technically, we use a smoothed version of this idea and obtain the modified function Φ as solution of a fixed point problem to be dealt with in §3.2.

These preparations completed, in lemma 3.4 we will find a suitable Φ and use it to ensure equiintegrability of $\{u_\varepsilon(\cdot, t) \mid \varepsilon > 0, t \in [0, 1)\}$ in lemma 3.5.

Identification of possible limits of $u(\cdot, t)$ as $t \searrow 0$ will be the task of §3.4, which will once more rely on the outcome of lemma 3.5.

3.1. de la Vallée Poussin's theorem: the source of the existence of Φ

The following proof is very close to the original by de la Vallée Poussin [5, remarque 23], although a small modification is necessary to account for smoothness (and convexity).

LEMMA 3.1. *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a measurable set with finite measure and let $\{u_\alpha\}_\alpha$ be a family of nonnegative functions $u_\alpha \colon \Omega \to [0, \infty]$ that is equiintegrable (also 'uniformly integrable') in the sense that*

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall \alpha \, \forall M \subset \Omega : M \text{ measurable with } |M| < \delta \implies \int_M u_\alpha < \varepsilon. \quad (3.1)$$

Then there is a smooth, monotone, convex function $\Phi \colon [0, \infty) \to [0, \infty)$ such that $\Phi(0) = 0$, $\lim_{x \to \infty} \Phi(x)/x = \infty$ and $\sup_\alpha \int_\Omega \Phi(u_\alpha) < \infty$.

Proof. Firstly, (3.1) implies the existence of some $c_1 > 0$ such that $\int_\Omega u_\alpha \leqslant c_1$ for all α. We let $(\varepsilon_k)_{k \in \mathbb{N}} \subset (0, 1)$ be a decreasing sequence such that $\sum_{k=1}^\infty \varepsilon_k < \infty$ and set $\varepsilon_{-1} := \varepsilon_0 := 1$. We furthermore introduce an increasing sequence $(M_k)_{k \in \mathbb{N}}$ such that for every $k \in \mathbb{N}$

$$\int_{\{u_\alpha \geqslant M_k\}} u_\alpha < \varepsilon_k^2 \qquad \text{for all } \alpha$$

and let $M_0 := 0$. (Such a sequence exists by (3.1) and because $|\{u_\alpha \geqslant M_k\}| \leqslant (1/M_k) \sup_\alpha \int_\Omega u_\alpha \leqslant c_1/M_k$ for every α and for every $k \in \mathbb{N}$.)

With a function $\zeta \in C^\infty([0, 1])$ such that $\zeta^{(k)}(0) = \zeta^{(k)}(1) = 0$ for all $k \in \mathbb{N}$, $\zeta(0) = 0$, $\zeta(1) = 1$ and $\zeta' \geqslant 0$ in $(0, 1)$, we define a function $\phi \colon [0, \infty) \to [0, \infty)$ by setting

$$\phi(x) := \left(1 - \zeta\left(\frac{M_{k+1} - x}{M_{k+1} - M_k}\right)\right) \frac{1}{\varepsilon_k} + \zeta\left(\frac{M_{k+1} - x}{M_{k+1} - M_k}\right) \frac{1}{\varepsilon_{k-1}}$$

for $x \in [M_k, M_{k+1}]$, $k \in \mathbb{N}_0$, and note that

$$\phi'(x) = \frac{1}{M_{k+1} - M_k} \left(\frac{1}{\varepsilon_k} - \frac{1}{\varepsilon_{k-1}}\right) \zeta'\left(\frac{M_{k+1} - x}{M_{k+1} - M_k}\right) \geqslant 0$$

and that $\phi \in C^\infty([0, \infty))$. Introducing $\Phi(x) := \int_0^x \phi(y) \, \mathrm{d}y$, $x \in [0, \infty)$, we observe that Φ is smooth, monotone and convex and $\Phi(x) \leqslant x\phi(x)$ for all $x > 0$. Therefore,

for every α,

$$
\begin{aligned}
\int_\Omega \Phi(u_\alpha) = \sum_{k=0}^\infty \int_{\{M_k \leqslant u_\alpha \leqslant M_{k+1}\}} \Phi(u_\alpha) &\leqslant \sum_{k=0}^\infty \int_{\{M_k \leqslant u_\alpha \leqslant M_{k+1}\}} u_\alpha \phi(u_\alpha) \\
&= \int_\Omega u_\alpha + \sum_{k \in \mathbb{N}} \frac{1}{\varepsilon_k} \int_{\{M_k \leqslant u_\alpha\}} u_\alpha \leqslant c_1 + \sum_{k \in \mathbb{N}} \varepsilon_k < \infty.
\end{aligned}
$$

Finally,

$$
\lim_{x \to \infty} \frac{\Phi(x)}{x} = \lim_{x \to \infty} \phi(x) = \lim_{k \to \infty} \frac{1}{\varepsilon_k} = \infty. \qquad \square
$$

3.2. Adjusting the second derivative of Φ

As we wish to obtain a modified variant of Φ complying with all of the conditions mentioned at the beginning of §3 from an application of Schauder's fixed point theorem, let us isolate a technical prerequisite in the following lemma.

LEMMA 3.2. *Let $f \in L^1_{loc}([0,\infty))$ be nonnegative. The set*

$$
X_f := \{h \in L^1_{loc}([0,\infty)) \mid 0 \leqslant h \leqslant f \quad a.e.\} \tag{3.2}
$$

is a convex subset of $L^1_{loc}([0,\infty))$ which is compact in the weak topology of each of the spaces $L^1((0,n))$, $n \in \mathbb{N}$.

Proof. By the Eberlein–Shmulyan theorem [7, theorem V.6.1], it is sufficient to check for sequential weak compactness. Let $(h_k)_{k \in \mathbb{N}}$ be a sequence in X_f. Given $n \in \mathbb{N}$ we observe that $\{\|h_k\|_{L^1((0,n))}\}_{k \in \mathbb{N}}$ is bounded (by $\|f\|_{L^1((0,n))}$) and that

$$
\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall M \subset (0,n) : \; M \text{ measurable with } |M| < \delta \implies \int_M f < \varepsilon,
$$

which due to $0 \leqslant h_k \leqslant f$ for all $k \in \mathbb{N}$ implies

$$
\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall k \in \mathbb{N} \; \forall M \subset (0,n) : \; M \text{ measurable with } |M| < \delta \implies \int_M h_k < \varepsilon.
$$

The sequence $\{h_k\}_{k \in \mathbb{N}}$ therefore is equiintegrable on $(0,n)$ and by the Dunford–Pettis theorem ([7, theorem IV.8.9]) there is a subsequence which converges weakly in $L^1((0,n))$ to some $h \in L^1((0,n))$, where $h \leqslant f$ a.e. on $(0,n)$. Convexity of the set is obvious. $\qquad \square$

We will, in lemma 3.4, obtain Φ (or rather Φ'') from an application of the following lemma to $f = \Phi''$ with Φ from lemma 3.1 and $g(x) = 1/x$.

LEMMA 3.3. *Let $f \in C([0,\infty))$ and $g \in C((0,\infty))$ be nonnegative functions and such that $\lim_{x \to 0^+} g(x) \in [0,\infty]$ exists and $f \notin L^1((0,\infty))$ as well as $g \notin L^1((1,\infty))$. Then there is $h \in C([0,\infty))$ such that*

$$
0 \leqslant h(x) \leqslant g(x) \quad \text{and} \quad \int_0^x h(y)\,\mathrm{d}y \leqslant \int_0^x f(y)\,\mathrm{d}y \tag{3.3}
$$

for all $x > 0$ and $h \notin L^1((0,\infty))$.

Proof. Without loss of generality, we may assume that $g \in C([0, \infty))$. (For if $g \geqslant f$, the choice $h := f$ suffices to conclude, and if otherwise $g \notin L^1_{loc}([0, \infty))$, then $x_0 := \inf\{x > 0 \mid f(x) = g(x)\}$ is positive and redefining $g(x) := f(x)$ for $x \in (0, x_0)$ will ensure $g \in C([0, \infty))$ and will not affect the construction in (3.4).)

We pick a cut-off function $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi \equiv 0$ on $[-1/2, \infty)$, $\varphi \equiv 1$ on $(-\infty, -1]$ and $\varphi' \leqslant 0$ in $\mathbb{R}$ and note that φ is Lipschitz continuous with a Lipschitz constant that we will denote by L_φ. For $h \in L^1_{loc}([0, \infty))$ we then define $\Phi(h)$ by

$$
\Phi(h)(x) := \begin{cases} g(x), & \text{if } f(x) \geqslant g(x), \\ \varphi\left(\int_0^x (h(y) - f(y))\, \mathrm{d}y\right) g(x) \\ \quad + \left(1 - \varphi\left(\int_0^x (h(y) - f(y))\, \mathrm{d}y\right)\right) f(x), & \text{if } f(x) < g(x) \end{cases} \tag{3.4}
$$

for $x \in (0, \infty)$. Apparently, $\Phi(h) \in C([0, \infty))$. Moreover, for $h_1, h_2 \in L^1_{loc}([0, \infty))$ and $x \in (0, \infty)$, we have

$$
\begin{aligned}
&|(\Phi(h_1) - \Phi(h_2))(x)| \\
&= \begin{cases} 0 & \text{if } f(x) \geqslant g(x), \\ |g(x) - f(x)| \left(\varphi(\int_0^x h_1 - f) - \varphi(\int_0^x h_2 - f)\right) & \text{if } f(x) < g(x) \end{cases} \\
&\leqslant |g(x) - f(x)| L_\varphi \left|\int_0^x (h_1 - h_2)\right| \leqslant |g(x) - f(x)| L_\varphi \left|\int_0^x (h_1 - h_2)\right|. \quad (3.5)
\end{aligned}
$$

We show that Φ is sequentially continuous as map from $L^1_{loc}([0, \infty))$ with weak topology to $L^1_{loc}([0, \infty))$ with its usual topology. Let $(h_{1,k})_{k \in \mathbb{N}} \subset L^1_{loc}([0, \infty))$ be a sequence which weakly converges to $h_2 \in L^1_{loc}([0, \infty))$. Let $n \in \mathbb{N}$. Then, for every $x \in (0, n)$, $\int_0^x (h_{1,k} - h_2) \to 0$ as $k \to \infty$. Moreover, there is $c_1 > 0$ such that $\left|\int_0^x (h_{1,k} - h_2)\right| \leqslant \|h_{1,k}\|_{L^1((0,n))} + \|h_2\|_{L^1((0,n))} \leqslant c_1 \in L^1((0, n))$ for every $x \in (0, n)$ and $k \in \mathbb{N}$. An integration of (3.5) and Lebesgue's theorem therefore show that

$$
\|\Phi(h_{1,k}) - \Phi(h_2)\|_{L^1((0,n))} \leqslant \|f - g\|_{C([0,n])} L_\varphi \int_0^n \left|\int_0^x (h_{1,k} - h_2)\right| \mathrm{d}x \to 0
$$

as $k \to \infty$ and thus the desired sequential continuity. Moreover, by definition (3.4), $0 \leqslant \Phi(h)(x) \leqslant \max\{f, g\}$ for every $h \in L^1_{loc}([0, \infty))$, or, in the notation from (3.2), $\Phi(h) \in X_{\max\{f,g\}}$. In order to use the compactness statement from lemma 3.2, let us introduce $\Phi_1 : X_{\max\{f,g\}} \to X_{\max\{f,g\}}$, $\Phi_1(h)(x) = \Phi(h)(x)$, $x \in (0, 1)$, $h \in X_{\max\{f,g\}}$, where we consider $X_{\max\{f,g\}}$ as a subset of the space $L^1((0, 1))$ with its weak topology.

The set $X_{\max\{f,g\}}$ is convex and compact (cf. lemma 3.2) and Φ_1 maps $X_{\max\{f,g\}}$ into itself and is continuous (for the step from sequential continuity as shown above to continuity we rely on weak compactness of the set and the resulting metrizability of its weak topology, cf. [7, theorem V.6.3]). Schauder's theorem (in the form of, e.g. [7, theorem V.10.5]) ensures the existence of a fixed point $h_1 \in L^1((0, 1))$. If $h_n \in L^1((0, n))$ has been constructed for some $n \in \mathbb{N}$, we let $\Phi_{n+1} : X_{\max\{f,g\}} \subset$

$L^1((0, n+1)) \to X_{\max\{f,g\}}$,

$$\Phi_{n+1}(h)(x) = \begin{cases} h_n(x), & x \in (0, n), \\ \Phi(h)(x), & x \in [n, n+1), \end{cases}$$

for $h \in L^1((0, n+1))$, and by the same reasoning as before obtain a fixed point h_{n+1} of Φ_{n+1}. We note that $h_{n_2}|_{(0,n_1)} = h_{n_1}$ for every $n_2 \geqslant n_1$ and that $\Phi(h_n) = h_n$ on $(0, n)$ for $n \in \mathbb{N}$. Defining $h(x) := h_n(x)$ for $x \in (n-1, n)$, we obtain some $h \in L^1_{loc}([0, \infty))$ such that $h = \Phi(h) \in C([0, \infty))$. The construction in (3.4) ensures that $h \leqslant g$. In order to see the second part of (3.3), we define $\psi(x) := \int_0^x (h(y) - f(y)) \, dy$ and observe that due to continuity of h and f, $\psi \in C^1([0, \infty))$. We let $x_0 := \inf\{x \in [0, \infty) \mid \psi(x) > 0\}$ (assuming that this set were non-empty), so that $\psi(x_0) = 0$. Moreover, by this definition as an infimum and by continuity of ψ, we then could find $x_1 > x_0$ such that $\psi(x_1) > 0$ and that $\psi(x) > -1/2$ (and hence $\varphi(\psi(x)) = 0$) for all $x \in (x_0, x_1)$. For any $x \in (x_0, x_1)$ we would then conclude from (3.4) that $h(x) = g(x) \leqslant f(x)$ or $h(x) = \varphi(\psi(x))g(x) + (1 - \varphi(\psi(x)))f(x) = f(x)$, which would lead to the contradiction $0 < \psi(x_1) = \psi(x_0) + \int_{x_0}^{x_1} (h(y) - f(y)) \, dy \leqslant \psi(x_0) + 0 = 0$.

Finally, in order to show $h \notin L^1((0, \infty))$, we assume that $h \in L^1((0, \infty))$ and from nonnegativity of h could infer the existence of $c_2 > 0$ such that $\int_0^x h(y) \, dy \leqslant c_2$ for all $x > 0$. As $\int_0^x f(y) \, dy \to \infty$ as $x \to \infty$ (due to $0 \leqslant f \notin L^1((0, \infty))$), there would be $x_2 \in [0, \infty)$ such that $\int_0^x f(y) \, dy \geqslant c_2 + 1 > \int_0^x h(y) dy + 1$ for all $x > x_2$. Therefore, for every $x > x_2$, $\varphi(\int_0^x h_1 - f) = 1$, and hence $h(x) = g(x)$ according to (3.4). Thus, $\int_0^\infty h(y) \, dy \geqslant \int_{x_2}^\infty h(y) \, dy = \int_{x_2}^\infty g(y) \, dy = \infty$, contradicting the assumption $h \in L^1((0, \infty))$. □

3.3. Equiintegrability

LEMMA 3.4. *There is a function* $\Phi \colon [0, \infty) \to [0, \infty)$ *such that*

$$\Phi \text{ is twice differentiable, monotone increasing and convex} \tag{3.6}$$

$$\frac{\Phi(x)}{x} \to \infty \quad and \quad \Phi'(x) \to \infty \qquad as \ x \to \infty \tag{3.7}$$

$$\Phi''(x) \leqslant \frac{1}{x} \qquad for \ all \ x > 0 \tag{3.8}$$

and such that with some $C_8 > 0$

$$\int_\Omega \Phi(u_{0\varepsilon}) \leqslant C_8 \tag{3.9}$$

holds for every $\varepsilon > 0$.

Proof. Due to (1.3c), $\{u_{0\varepsilon} \mid \varepsilon > 0\}$ is equiintegrable. By de la Vallée Poussin's lemma (lemma 3.1), there is a function $\Psi \colon [0, \infty) \to [0, \infty)$ satisfying (3.6), (3.7) and (3.9) (with Φ replaced by Ψ). Noting that $\Psi'' \notin L^1((0, \infty))$, we apply lemma 3.3 with $f = \Psi''$ and $g(x) = 1/x$, obtaining some function h as described there. We then define $\Phi(x) = \int_0^x \left(\Psi'(0) + \int_0^y h(z) dz \right) dy$ for $x \geqslant 0$. Then (3.6) is satisfied; due to $h \notin L^1((0, \infty))$, (3.7) holds true; the first part of (3.3) ensures (3.8), whereas the

second shows that $\Phi'(x) \leqslant \Psi'(x)$ and consequently $\Phi(x) \leqslant \Psi(x)$ for all $x \geqslant 0$ and thus (3.9). $\qquad\square$

LEMMA 3.5. *With Φ as introduced in lemma 3.4, for every $T > 0$ there is $C_9 = C_9(T) > 0$ satisfying*

$$\int_\Omega \Phi(u_\varepsilon(\cdot,t)) \leqslant C_9 \quad and \quad \int_0^T \int_\Omega \Phi'(u_\varepsilon)u_\varepsilon^2 \leqslant C_9$$

for every $\varepsilon > 0$ and $t \in [0,T]$.

Proof. Integration by parts and Young's inequality let us conclude from (1.4a) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \Phi(u_\varepsilon) = -\int_\Omega \Phi''(u_\varepsilon)|\nabla u_\varepsilon|^2 + \chi \int_\Omega \Phi''(u_\varepsilon)u_\varepsilon \nabla v_\varepsilon \cdot \nabla u_\varepsilon$$
$$+ \kappa \int_\Omega \Phi'(u_\varepsilon)u_\varepsilon - \mu \int_\Omega \Phi'(u_\varepsilon)u_\varepsilon^2$$
$$\leqslant \chi^2 \int_\Omega \Phi''(u_\varepsilon)u_\varepsilon^2|\nabla v_\varepsilon|^2 + \kappa \int_\Omega \Phi'(u_\varepsilon)u_\varepsilon - \mu \int_\Omega \Phi'(u_\varepsilon)u_\varepsilon^2$$
$$\leqslant \int_\Omega (\Phi''(u_\varepsilon))^2 u_\varepsilon^4 + \chi^4 \int_\Omega |\nabla v_\varepsilon|^4 + \kappa \int_\Omega \Phi'(u_\varepsilon)u_\varepsilon - \mu \int_\Omega \Phi'(u_\varepsilon)u_\varepsilon^2$$
$$\tag{3.10}$$

holds on $(0,T)$. Aided by (3.6), we choose $c_{\kappa,\mu} > 0$ such that

$$(\kappa x - \mu x^2)\Phi'(x) \leqslant c_{\kappa,\mu} - \frac{\mu}{2}x^2\Phi'(x) \qquad \text{for all } x \geqslant 0$$

and, relying on (3.7), $c_{\Phi,\mu} > 0$ such that

$$\left(1 - \frac{\mu}{4}\Phi'(x)\right)x^2 \leqslant c_{\Phi,\mu} \qquad \text{for all } x \geqslant 0,$$

and use these definitions together with (3.8) to turn (3.10) into

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \Phi(u_\varepsilon) \leqslant \int_\Omega u_\varepsilon^2 + \chi^4 \int_\Omega |\nabla v_\varepsilon|^4 + \int_\Omega c_{\kappa,\mu} - \frac{\mu}{2} \int_\Omega \Phi'(u_\varepsilon)u_\varepsilon^2$$
$$\leqslant (c_{\kappa,\mu} + c_{\Phi,\mu})|\Omega| - \frac{\mu}{4} \int_\Omega \Phi'(u_\varepsilon)u_\varepsilon^2 + \chi^4 \int_\Omega |\nabla v_\varepsilon|^4 \qquad \text{on } (0,T).$$

Integration with respect to time results in the claim, due to lemma 2.7 and the bound (3.9) on the value at time 0 from lemma 3.4. $\qquad\square$

3.4. Identification of the limit as $t \searrow 0$. Estimates for $u_{\varepsilon t}$

The equiintegrability statement contained in lemma 3.5 (or resulting from this lemma in the limit $\varepsilon \to 0$, cf. proof of lemma 5.2) is sufficient to conclude that for every sequence $(t_k)_{k\in\mathbb{N}}$ with $t_k \to 0$ there is a subsequence $(t_{k_j})_{j\in\mathbb{N}}$ such that $u(\cdot, t_{k_j})$ weakly converges in $L^1(\Omega)$. This does not capture the value of $\lim_j u(\cdot, t_{k_j})$

and would still admit different accumulation points of $(u(\cdot, t_k))_{k \in \mathbb{N}}$. To make sure that this limit coincides with u_0, we turn our attention to estimates for $u_{\varepsilon t}$.

LEMMA 3.6. *There is $C_{10} > 0$ such that for every $\varepsilon > 0$ and $t \in (0, \infty)$,*

$$\|u_{\varepsilon t}(\cdot, t)\|_{(W_0^{3,3}(\Omega))^*} \leqslant C_{10}(1 + t) + C_{10}\|u_\varepsilon(\cdot, t)\|_{L^2(\Omega)}^2.$$

Proof. For any $\varepsilon > 0$ and $t \in (0, \infty)$, the norm of $u_{\varepsilon t}(\cdot, t)$ in $(W_0^{3,3})^*$ can be computed as

$$\|u_{\varepsilon t}(\cdot, t)\|_{(W_0^{3,3}(\Omega))^*} = \sup\left\{\left|\int_\Omega u_{\varepsilon t}\varphi\right|; \ \varphi \in C_c^\infty(\Omega), \|\varphi\|_{W_0^{3,3}(\Omega)} \leqslant 1\right\}.$$

Due to the embedding $W_0^{3,3}(\Omega) \hookrightarrow W^{2,\infty}(\Omega)$, there is $c_1 > 0$ such that for any such φ, $\|\varphi\|_{W^{2,\infty}(\Omega)} \leqslant c_1$, and accordingly

$$\left|\int_\Omega u_{\varepsilon t}\varphi\right| = \left|\int_\Omega \Delta u_\varepsilon \varphi - \chi \int_\Omega \nabla \cdot (u_\varepsilon \nabla v_\varepsilon)\varphi + \kappa \int_\Omega u_\varepsilon \varphi - \mu \int_\Omega u_\varepsilon^2 \varphi\right|$$

$$= \left|\int_\Omega u_\varepsilon \Delta \varphi - \chi \int_\Omega u_\varepsilon \nabla v_\varepsilon \cdot \nabla \varphi + \kappa \int_\Omega u_\varepsilon \varphi - \mu \int_\Omega u_\varepsilon^2 \varphi\right|$$

$$\leqslant c_1\|u_\varepsilon\|_{L^1(\Omega)} + c_1|\chi|\|u_\varepsilon \nabla v_\varepsilon\|_{L^1(\Omega)} + c_1|\kappa|\|u_\varepsilon\|_{L^1(\Omega)} + c_1\mu\|u_\varepsilon\|_{L^2(\Omega)}^2$$

$$\leqslant c_1(1 + |\kappa|)\|u_\varepsilon\|_{L^1(\Omega)} + c_1|\chi|\|\nabla v_\varepsilon\|_{L^2(\Omega)}^2 + c_1(|\chi| + \mu)\|u_\varepsilon\|_{L^2(\Omega)}^2$$

$$\leqslant c_1(1 + |\kappa|)C_1 + c_1|\chi|C_4(1 + t)$$

$$\quad + c_1(|\chi| + \mu)\|u_\varepsilon\|_{L^2(\Omega)}^2$$

for all $\varepsilon > 0, t \in (0, \infty)$,

where we have successively employed integration by parts, Hölder's and Young's inequality and lemmata 2.2 and 2.5. $\qquad\square$

For controlling the last summand in the estimate of lemma 3.6, we rely on lemma 3.5.

LEMMA 3.7. *For every $T > 0$, the family*

$$\{[0, T] \ni t \mapsto \|u_\varepsilon(\cdot, t)\|_{L^2(\Omega)}^2 | \varepsilon > 0\}$$

of functions on $[0, T]$ is equiintegrable.

Proof. Introducing

$$F(x) := x\Phi'\left(\sqrt{\frac{x}{|\Omega|}}\right), \qquad x > 0,$$

with Φ from lemma 3.4, we know that

$$F'(x) = \Phi'\left(\sqrt{x/|\Omega|}\right) + \frac{1}{2}\sqrt{x/|\Omega|}\,\Phi''\left(\sqrt{x/|\Omega|}\right) \geqslant \Phi'\left(\sqrt{x/|\Omega|}\right) \quad \text{for all } x > 0,$$

and hence $F'(x) \to \infty$ as $x \to \infty$. In other words, for every $l > 0$ there is $x_l > 0$ such that $F' > l$ on (x_l, ∞). We define

$$\Psi := \operatorname{conv} F$$

to be the convex hull of F (cf. e.g. [**27**, 5.16]) and observe that Ψ lies above each of the convex functions $x \mapsto l(x - x_l)_+$, so that $\Psi(x)/x \to \infty$ as $x \to \infty$. Its convexity allows us to invoke Jensen's inequality and estimate

$$\int_0^T \Psi(\|u_\varepsilon\|_{L^2(\Omega)}^2) \, \mathrm{d}t = \int_0^T \Psi\left(\int_\Omega |\Omega| u_\varepsilon^2 \frac{\mathrm{d}x}{|\Omega|}\right) \mathrm{d}t \leqslant \int_0^T \int_\Omega \Psi(|\Omega| u_\varepsilon^2) \frac{\mathrm{d}x}{|\Omega|}$$

$$\leqslant \int_0^T \int_\Omega F\left(|\Omega| u_\varepsilon^2\right) \frac{\mathrm{d}x}{|\Omega|} = \int_0^T \int_\Omega |\Omega| u_\varepsilon^2 \Phi'(u_\varepsilon) \frac{\mathrm{d}x}{|\Omega|}$$

$$= \int_0^T \int_\Omega u_\varepsilon^2 \Phi'(u_\varepsilon)$$

for every $\varepsilon > 0$. The final integral is bounded by lemma 3.5, and the resulting boundedness of the integral of a superlinear function of $\|u_\varepsilon\|_{L^2(\Omega)}^2$ ensures the latter's equiintegrability, as claimed. $\qquad \square$

LEMMA 3.8. *For every $T > 0$, the family*

$$\{[0, T] \ni t \mapsto \|u_{\varepsilon t}(\cdot, t)\|_{(W_0^{3,3}(\Omega))^*} | \varepsilon > 0\}$$

of functions on $[0, T]$ is equiintegrable.

Proof. In light of lemma 3.6, this is a simple consequence of lemma 3.7. $\qquad \square$

This equiintegrability statement can be turned into the following assertion on $\lim_{(\varepsilon,t)\to 0} u_\varepsilon(\cdot, t)$:

LEMMA 3.9. *For every $\eta > 0$ there are $t_0 > 0$ and $\varepsilon_0 > 0$ such that for every $t \in (0, t_0)$ and every $\varepsilon \in (0, \varepsilon_0)$,*

$$\|u_\varepsilon(\cdot, t) - u_0\|_{(W_0^{3,3}(\Omega))^*} \leqslant \eta.$$

Proof. Let $\eta > 0$. In accordance with (1.3c) and aided by the embedding $L^1(\Omega) \hookrightarrow (W_0^{3,3}(\Omega))^*$, we can find $\varepsilon_0 > 0$ such that

$$\text{for all } \varepsilon \in (0, \varepsilon_0): \qquad \|u_{0\varepsilon} - u_0\|_{(W_0^{3,3}(\Omega))^*} \leqslant \frac{\eta}{2}.$$

Relying on the outcome of lemma 3.8, we moreover choose $t_0 > 0$ such that for all sets $M \subset [0, 1]$ with $|M| \leqslant t_0$ and all $\varepsilon > 0$,

$$\int_M \|u_{\varepsilon t}\|_{(W_0^{3,3}(\Omega))^*} \leqslant \frac{\eta}{2}.$$

For every $t \in (0, t_0)$ and every $\varepsilon \in (0, \varepsilon_0)$, we then have

$$\|u_\varepsilon(\cdot, t) - u_0\|_{(W_0^{3,3}(\Omega))^*} \leqslant \|u_{0\varepsilon} - u_0\|_{(W_0^{3,3}(\Omega))^*} + \|u_\varepsilon(\cdot, t) - u_{0\varepsilon}\|_{(W_0^{3,3}(\Omega))^*}$$

$$\leqslant \frac{\eta}{2} + \int_{(0,t)} \|u_{\varepsilon t}\|_{(W_0^{3,3}(\Omega))^*} \leqslant \eta. \qquad \square$$

4. Regularity outside $t = 0$. Time-local bounds

Relying on the spatio-temporal bounds on $|\nabla v_\varepsilon|^4$ from lemma 2.7, we do not base the first stepping stone of our estimates on the functional $(1/2) \int_\Omega u_\varepsilon^2 + (1/4) \int_\Omega |\nabla v_\varepsilon|^4$ (cf. [**3**, lemma 3.3]), but on $(1/2) \int_\Omega u_\varepsilon^2$ alone. The approach adheres to a classical way to obtain estimates in parabolic PDEs (cf. [**17**, p. 140]), but is less used in a chemotaxis context—the reason often lying in lack of *a priori* information alike lemma 2.7. It avoids having to deal with boundary integrals involving $\partial_\nu |\nabla v_\varepsilon|^2$ and thus may be of interest in some related models. Let us note that it would, instead, have been possible to adapt the reasoning of [**3**, lemma 3.3] (resulting in a proof of similar length and simplicity).

On the space

$$V := L_{loc}^2((0,\infty); W^{1,2}(\Omega)) \cap C((0,\infty); L^2(\Omega))$$

we introduce the seminorms

$$\|\varphi\|_{V,t_1,t_2} = \sup_{t \in (t_1,t_2)} \|\varphi(t)\|_{L^2(\Omega)} + \|\nabla\varphi\|_{L^2(\Omega \times (t_1,t_2))}, \quad t_2 > t_1 > 0$$

and can derive the following inequality (cf. [**17**, p. 75]):

LEMMA 4.1. *There is $C_{11} > 0$ such that for every $t_1 > 0$ and $t_2 > t_1$*

$$\|\varphi\|_{L^4(\Omega \times (t_1,t_2))}^4 \leqslant C_{11}(1 + t_2)\|\varphi\|_{V,t_1,t_2}^4$$

holds for every $\varphi \in V$.

Proof. We take $c_1 > 0$ and $c_2 > 0$ from the Gagliardo–Nirenberg inequality

$$\|\varphi\|_{L^4(\Omega)}^4 \leqslant c_1 \|\nabla\varphi\|_{L^2(\Omega)}^2 \|\varphi\|_{L^2(\Omega)}^2 + c_2 \|\varphi\|_{L^2(\Omega)}^4 \qquad \text{for all } \varphi \in W^{1,2}(\Omega).$$

Then for every $\varphi \in V$, $t_2 > t_1 > 0$,

$$\|\varphi\|_{L^4(\Omega \times (t_1,t_2))}^4 \leqslant c_1 \int_{t_1}^{t_2} \|\nabla\varphi\|_{L^2(\Omega)}^2 \|\varphi\|_{L^2(\Omega)}^2 + c_2 \int_{t_1}^{t_2} \|\varphi\|_{L^2(\Omega)}^4$$

$$\leqslant c_1 \|\varphi\|_{V,t_1,t_2}^2 \int_{t_1}^{t_2} \|\nabla\varphi\|_{L^2(\Omega)}^2 + c_2(t_2 - t_1)\|\varphi\|_{V,t_1,t_2}^4$$

$$\leqslant (c_1 + c_2(t_2 - t_1))\|\varphi\|_{V,t_1,t_2}^4. \qquad \square$$

LEMMA 4.2. *For every $T > 0$ and $\tau \in (0, T)$ there is $C_{12}(\tau, T) > 0$ such that for every $\varepsilon > 0$,*

$$\|u_\varepsilon\|_{V,\tau,T} \leqslant C_{12}(\tau, T).$$

Proof. We fix $T > 0$ and $\tau \in (0, T)$. For any $\varepsilon > 0$ testing (1.4a) by u_ε, we see that with $c_1 = (1/|\Omega|) \sup_{x \geqslant 0} (\kappa x^2 - (\mu/2)x^3)$ and after an application of Young's inequality,

$$\frac{1}{2} \frac{d}{dt} \int_\Omega u_\varepsilon^2 \leqslant -\frac{1}{2} \int_\Omega |\nabla u_\varepsilon|^2 + \frac{\chi^2}{2} \int_\Omega u_\varepsilon^2 |\nabla v_\varepsilon|^2 + c_1 - \frac{\mu}{2} \int_\Omega u_\varepsilon^3 \quad \text{in } (0, T).$$

Integration with respect to time, Hölder's inequality and lemma 4.1 show that hence for every $\varepsilon > 0$, $t_1 > 0$, $t_2 \in (t_1, T]$ and $t \in (t_1, t_2)$,

$$\max \left\{ \int_\Omega u_\varepsilon^2(\cdot, t), \int_{t_1}^t \int_\Omega |\nabla u_\varepsilon|^2 \right\} + \mu \int_{t_1}^t \int_\Omega u_\varepsilon^3$$

$$\leqslant \int_\Omega u_\varepsilon^2(\cdot, t_1) + \chi^2 \int_{t_1}^t \int_\Omega u_\varepsilon^2 |\nabla v_\varepsilon|^2 + 2c_1(t - t_1)$$

$$\leqslant \int_\Omega u_\varepsilon^2(\cdot, t_1) + \chi^2 \left(\int_{t_1}^{t_2} \int_\Omega u_\varepsilon^4 \right)^{1/2} \left(\int_{t_1}^{t_2} \int_\Omega |\nabla v_\varepsilon|^4 \right)^{1/2} + 2c_1(t_2 - t_1)$$

$$\leqslant \int_\Omega u_\varepsilon^2(\cdot, t_1) + \chi^2 \sqrt{C_{11}(1 + t_2)} \|u_\varepsilon\|_{V, t_1, t_2}^2 \left(\int_{t_1}^{t_2} \int_\Omega |\nabla v_\varepsilon|^4 \right)^{1/2} + 2c_1 t_2.$$

We let $c_2 = 16 c_1 T$ and $c_3 = 4\chi^2 \sqrt{C_{11}(1 + T)}$ and take the supremum over $t \in (t_1, t_2)$, ending up with

$$\|u_\varepsilon\|_{V, t_1, t_2}^2 \leqslant 2\|u_\varepsilon(\cdot, t_1)\|_{L^2(\Omega)}^2 + \frac{c_2}{4} + \frac{c_3}{2} \|\nabla v_\varepsilon\|_{L^4(\Omega \times (t_1, t_2))}^2 \|u_\varepsilon\|_{V, t_1, t_2}^2 \tag{4.1}$$

for every $\varepsilon > 0$, $t_1 > 0$ and $t_2 \in (t_1, T)$. Aided by lemma 2.3 we pick $t_0 = t_0(\varepsilon) \in (\tau/2, \tau)$ such that

$$\int_\Omega u_\varepsilon^2(\cdot, t_0) \leqslant \frac{2}{\tau} C_2(1 + \tau) =: c_4.$$

Given $t_k = t_k(\varepsilon) \in (0, T]$, we choose $t_{k+1} = t_{k+1}(\varepsilon) \in (t_k, T)$ such that

$$\int_{t_k}^{t_{k+1}} \int_\Omega |\nabla v_\varepsilon|^4 = \frac{1}{c_3^2} \tag{4.2}$$

if such t_{k+1} exists, or $t_{k+1} = T$ otherwise. Then $\int_{t_0}^{t_k} \int_\Omega |\nabla v_\varepsilon|^4 = k/c_3^2$ if $t_k < T$, which means that for $k > c_5 := 1 + c_3^2 C_7(1 + T) > c_3^2 \int_0^T \int_\Omega |\nabla v_\varepsilon|^4$ (the last inequality herein draws upon lemma 2.7) clearly $t_k = T$. Moreover, (4.1) for $t_1 = t_k$ and $t_2 = t_{k+1}$ for any $\varepsilon > 0$ turns into

$$\|u_\varepsilon\|_{V, t_k, t_{k+1}}^2 \leqslant 4\|u_\varepsilon(\cdot, t_k)\|_{L^2(\Omega)}^2 + \frac{c_2}{2} \quad \text{for } k \in \mathbb{N}. \tag{4.3}$$

We now introduce $y_{\varepsilon, k} := \|u_\varepsilon\|_{V, t_0, t_k}^2$ for $k \in \mathbb{N}$ (and $y_{\varepsilon, 0} = \int_\Omega u_\varepsilon^2(\cdot, t_0)$) and see that by (4.3)

$$y_{\varepsilon, k+1} \leqslant \left(\sqrt{y_{\varepsilon, k}} + \|u_\varepsilon\|_{V, t_k, t_{k+1}} \right)^2 \leqslant 2y_{\varepsilon, k} + 2\|u_\varepsilon\|_{V, t_k, t_{k+1}}^2 \leqslant 2y_{\varepsilon, k} + 8y_{\varepsilon, k} + c_2,$$

so that $y_{\varepsilon,k} \leqslant 10^k y_{\varepsilon,0} + \sum_{l=0}^{k-1} 10^l c_2$ and thus $y_{\varepsilon,k} \leqslant 10^{c_5} c_4 + ((10^{c_5} - 1)/9)c_2 =: c_6$ for every $k \in \mathbb{N}$. In particular, for any $\varepsilon > 0$,

$$\|u_\varepsilon\|_{V,\tau,T} \leqslant \sqrt{c_6}. \qquad \square$$

LEMMA 4.3. *Let $q \in [1,\infty)$. For every $T > 0$ and $\tau \in (0,T)$ there is $C_{13}(q,\tau,T) > 0$ such that*

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \leqslant C_{13}(q,\tau,T) \qquad \text{for every } \varepsilon > 0 \text{ and } t \in (\tau,T).$$

Proof. Without loss of generality we assume $q \geqslant 4$. Because of lemma 2.7, for every $\varepsilon > 0$ we can find $t_0 = t_0(\varepsilon) \in (\tau/3, \tau/2)$ such that

$$\int_\Omega |\nabla v_\varepsilon(\cdot, t_0(\varepsilon))|^4 \leqslant \frac{2C_7(1+\tau)^3}{\tau} =: c_1.$$

With $c_2 > 0$ being the constant in the semigroup estimate [**30**, lemma 1.3] we then obtain for every $\varepsilon > 0$ and every $t \in (\tau,T) \subset (t_0(\varepsilon),T)$ that

$$\begin{aligned}
\|\nabla v_\varepsilon(\cdot,t)\|_{L^q(\Omega)} &\leqslant \|\nabla e^{(t-t_0)(\Delta-1)} v_\varepsilon(\cdot,t_0)\|_{L^q(\Omega)} \\
&\quad + c_2 \int_{t_0}^t \|\nabla e^{(t-s)(\Delta-1)} u_\varepsilon(\cdot,s)\|_{L^q(\Omega)} ds \\
&\leqslant c_2 e^{-(t-t_0)} \left(1 + (t-t_0)^{-1/4+1/q}\right) \|\nabla v_\varepsilon(t_0)\|_{L^4(\Omega)} \\
&\quad + c_2 \int_{t_0}^t \left(1 + (t-s)^{-1/2-(1/2-1/q)}\right) e^{-(t-s)} \|u_\varepsilon(\cdot,s)\|_{L^2(\Omega)} ds \\
&\leqslant \sqrt[4]{c_1} c_2 \left(1 + (\tau/2)^{1/q-1/4}\right) + c_2 c_3 C_{12}\left(\frac{\tau}{3},T\right)
\end{aligned}$$

holds for every $t \in (\tau,T)$ and every $\varepsilon > 0$, where we have set $c_3 = \int_0^\infty (1 + \sigma^{-1+1/q}) e^{-\sigma} d\sigma$ and relied on lemma 4.2. $\square$

LEMMA 4.4. *Let $p \in \mathbb{N}$. For every $T > 0$ and $\tau \in (0,T)$ there are $C_{14}(p,\tau,T) > 0$ and $C_{15}(p,\tau,T) > 0$ such that*

$$\int_\Omega u_\varepsilon^p(\cdot,t) \leqslant C_{14}(p,\tau,T) \quad \text{and} \quad \int_\tau^T \int_\Omega u_\varepsilon^{p+1} \leqslant C_{15}(p,\tau,T)$$

hold for every $\varepsilon > 0$ and every $t \in (\tau,T)$.

Proof. For $p = 1$, this is true by lemmata 2.2 and 2.3. Assuming that the claim holds for some $p - 1 \in \mathbb{N}$ and given $T > 0$ and $\tau \in (0,T)$, for every $\varepsilon > 0$ we can find $t_0 = t_0(\varepsilon) \in (\tau/2,\tau)$ such that $\int_\Omega u_\varepsilon^p(t_0) \leqslant (2/\tau)C_{15}(p-1,\tau/2,\tau) =: c_{1,p}$. Moreover, with $c_{2,p} := (1/|\Omega|) \sup\{\kappa x^p - (\mu/3)x^{p+1} \mid x \geqslant 0\}$ and by Young's inequality

then

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_\varepsilon^p = -(p-1)\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 + (p-1)\chi\int_\Omega u_\varepsilon^{p-1}\nabla u_\varepsilon \cdot \nabla v_\varepsilon$$

$$+ \kappa\int_\Omega u_\varepsilon^p - \mu\int_\Omega u_\varepsilon^{p+1}$$

$$\leqslant \frac{\mu}{3}\int_\Omega u_\varepsilon^{p+1} + \left(\frac{3}{\mu}\right)^p (p-1)^{p+1}\chi^{2p+2}\int_\Omega |\nabla v_\varepsilon|^{2(p+1)}$$

$$+ c_{2,p} - \frac{2\mu}{3}\int_\Omega u_\varepsilon^{p+1} \quad \text{in } (t_0, T),$$

so that with $c_{3,p} = p(3/\mu)^p(p+1)^{p+1}\chi^{2p+2}$ we obtain

$$\int_\Omega u_\varepsilon^p(\cdot, t) + \frac{p\mu}{3}\int_\tau^t\int_\Omega u_\varepsilon^{p+1} \leqslant \int_\Omega u_\varepsilon^p(\cdot, t_0) + pc_{2,p}\left(T - \frac{\tau}{2}\right) + c_{3,p}\int_{t_0}^t\int_\Omega |\nabla v_\varepsilon|^{2(p+1)}$$

$$\leqslant c_{1,p} + pc_{2,p}T + c_{3,p}\left(C_{13}(2p+2, \tau/2, T)\right)^{2p+2}$$

$$=: C_{14}(p, \tau, T)$$

for every $\varepsilon > 0$ and every $t \in (\tau, T]$, and finally set $C_{15}(p, \tau, T) := (3/p\mu)C_{14}(p, \tau, T)$. □

LEMMA 4.5. *For every $T > 0$ and $\tau \in (0, T)$ there is $C_{16}(\tau, T) > 0$ such that for every $\varepsilon > 0$ and every $t \in (\tau, T)$,*

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leqslant C_{16}(\tau, T).$$

Proof. With $c_1 > 0$ taken from semigroup estimates (cf. [**30**, lemma 1.3]) and $c_2 := \sup_{x\geqslant 0}(\kappa x - \mu x^2)$,

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leqslant c_1(1 + (t - t_0)^{-1})\|u_\varepsilon(\cdot, t_0)\|_{L^1(\Omega)} + c_2(t - t_0)$$

$$+ c_1|\chi|\int_{t_0}^t \left(1 + (t - s)^{-1/2-1/4}\right)\|u_\varepsilon(\cdot, s)\|_{L^8(\Omega)}\|\nabla v_\varepsilon(\cdot, s)\|_{L^8(\Omega)}\mathrm{d}s$$

holds for every $t_0 > 0$ and $t > t_0$, and we may conclude by lemmata 2.2, 4.3 and 4.4. □

LEMMA 4.6. *For every $T > 0$ and $\tau \in (0, T)$ there are $\alpha > 0$ and $C_{17}(\tau, T) > 0$ such that, for all $\varepsilon > 0$,*

$$\|u_\varepsilon\|_{C^{\alpha,\alpha/2}(\overline{\Omega}\times[\tau,T])} \leqslant C_{17}(\tau, T).$$

Proof. On account of lemmata 4.5 and 4.3, another classical parabolic regularity result (see e.g. [**6**, theorem 4]) is applicable and readily yields these Hölder bounds. □

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LEMMA 4.7. *For every $T > 0$ and $\tau \in (0, T)$ there are $\alpha > 0$, $C_{18}(\tau, T) > 0$ and $C_{19}(\tau, T) > 0$ such that, for all $\varepsilon > 0$,*

$$\|v_\varepsilon\|_{C^{2+\alpha, 1+\alpha/2}(\overline{\Omega} \times [\tau, T])} \leqslant C_{18}(\tau, T),$$

$$\|u_\varepsilon\|_{C^{2+\alpha, 1+\alpha/2}(\overline{\Omega} \times [\tau, T])} \leqslant C_{19}(\tau, T).$$

Proof. Classical Schauder estimates for parabolic equations (like [20, theorem 4.31]—or, more precisely, a time-local version thereof, i.e. said theorem is applied to ζv_ε or ζu_ε, where ζ is a smooth temporal cutoff function with $\zeta = 0$ at $t = 0$) relying on the outcome of lemma 4.6 can firstly be applied to v_ε and then to u_ε, resulting in the above statement. □

5. Proof of theorem 1.1: convergence and continuity at zero

LEMMA 5.1. *There are functions $u, v \colon \Omega \times [0, \infty) \to [0, \infty)$ and a sequence $(\varepsilon_j)_{j \in \mathbb{N}}$ such that $\varepsilon_j \to 0$ as $j \to \infty$ and*

$$u_{0\varepsilon} \to u(\cdot, 0) \qquad in \ L^1(\Omega), \tag{5.1}$$

$$u_\varepsilon \to u \qquad in \ C^{2,1}(\overline{\Omega} \times (0, \infty)), \tag{5.2}$$

$$v_{0\varepsilon} \to v(\cdot, 0) \qquad in \ W^{1,2}(\Omega) \tag{5.3}$$

$$v_\varepsilon \to v \qquad in \ C^{2,1}(\overline{\Omega} \times (0, \infty)). \tag{5.4}$$

as $\varepsilon = \varepsilon_j \searrow 0$.

Proof. While convergence at $t = 0$, that is, (5.1) and (5.3), has been prescribed in (1.3c) and (1.3d), (5.2) and (5.4) along a suitable sequence are easily obtained from lemma 4.7. □

LEMMA 5.2. *The function u from lemma 5.1 belongs to $C_w([0, \infty); L^1(\Omega))$ with $u(\cdot, 0) = u_0$.*

Proof. In light of (5.2) we only have to show continuity of u at $t = 0$. Let us assume there were one sequence $(t_k)_{k \in \mathbb{N}} \searrow 0$ such that $u(\cdot, t_{k_j}) \not\to u(\cdot, 0) = u_0$ in the sense of weak $L^1(\Omega)$-convergence for every subsequence $(t_{k_j})_{j \in \mathbb{N}}$ of $(t_k)_{k \in \mathbb{N}}$. Combining (5.2) with lemma 3.5 we then could see that $\int_\Omega \Phi(u(\cdot, t_k)) \leqslant C_9$ for all $k \in \mathbb{N}$ and hence $\{u(\cdot, t_k) | k \in \mathbb{N}\}$ would be equiintegrable and we could find a subsequence $(t_{k_l})_{l \in \mathbb{N}}$ of $(t_k)_{k \in \mathbb{N}}$ such that $u(\cdot, t_{k_l})$ converged to some $\tilde{u} \in L^1(\Omega)$ as $l \to \infty$, weakly in $L^1(\Omega)$, by the Dunford–Pettis theorem ([7, theorem IV.8.9]). But lemma 3.9 together with (5.2) would ensure that $\tilde{u} = u_0$, in contradiction to the choice of $(t_k)_{k \in \mathbb{N}}$. □

LEMMA 5.3. *The function v from lemma 5.1 is continuous at $t = 0$ as a function with values in $L^2(\Omega)$ (with respect to the norm topology) and as function with values in $W^{1,2}(\Omega)$ equipped with the weak topology and satisfies $v(\cdot, 0) = v_0$.*

Proof. If this were not the case, we could find a sequence $(t_k)_{k \in \mathbb{N}}$ such that $t_k \to 0$ and none of the subsequences of $(v(\cdot, t_k))_{k \in \mathbb{N}}$ converged to v_0 (in $L^2(\Omega)$ or weakly in

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$W^{1,2}(\Omega)$). The bounds in lemmata 2.4 and 2.5 together with (5.4), however, would ensure boundedness of $(v(t_k))_{k \in \mathbb{N}}$ in $W^{1,2}(\Omega)$ and hence convergence to some $\tilde{v}$ along a subsequence, in both $L^2(\Omega)$ and weakly in $W^{1,2}(\Omega)$. Because in

$$\|v(t_k) - v_0\|_{L^2(\Omega)} \leqslant \|v(t_k) - v_\varepsilon(t_k)\|_{L^2(\Omega)} + \|v_\varepsilon(t_k) - v_{0\varepsilon}\|_{L^2(\Omega)} + \|v_{0\varepsilon} - v_0\|_{L^2(\Omega)} \tag{5.5}$$

we could use Hölder's inequality and lemma 2.6 to estimate

$$\|v_\varepsilon(t_k) - v_{0\varepsilon}\|_{L^2(\Omega)} \leqslant \sqrt{t_k} \left(\int_0^{t_k} \|v_{\varepsilon t}\|_{L^2(\Omega)} \right)^2 \leqslant \sqrt{t_k} \sqrt{C_6(1 + t_k)} \quad \text{for every } k \in \mathbb{N},$$

independently of $\varepsilon > 0$, (1.3d) and (5.4) in conjunction with (5.5) would finally serve to identify $\tilde{v} = v_0$, in contradiction to the choice of $(t_k)_{k \in \mathbb{N}}$. $\qquad\square$

Proof of theorem 1.1. By lemma 5.1, along a suitable subsequence, the approximate solutions provided by lemma 2.1 converge to some pair (u, v) of functions which belong to $C^{2,1}(\overline{\Omega} \times (0, \infty))$ and solve (1.1a), (1.1b) and (1.1c) by (5.2) and (5.4). That they solve (1.1d) and belong to $C_w([0, \infty); L^1(\Omega))$ follows from lemma 5.2 for u and lemma 5.3 in case of v (where the statement is even slightly stronger). $\qquad\square$

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