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Geometric aspects of self-adjoint Sturm–Liouville problems

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In this paper we use U(2), the group of 2×2 unitary matrices, to parametrize the space of all self-adjoint boundary conditions for a fixed Sturm–Liouville equation on the interval $[0, 1]$. The adjoint action of U(2) on itself naturally leads to a refined classification of self-adjoint boundary conditions – each adjoint orbit is a subclass of these boundary conditions. We give explicit parametrizations of those adjoint orbits of principal type, i.e. orbits diffeomorphic to the 2-sphere S^2, and investigate the behaviour of the nth eigenvalue λ_n as a function on such orbits.

Keywords: regular Sturm-Liouville problem; adjoint orbit; eigenvalue; space of self-adjoint boundary conditions

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1. Introduction

Unbounded self-adjoint (SA, for brevity) operators are very important objects in mathematical physics. In quantum mechanics, an observable is represented by an SA operator, rather than a symmetric one. In perturbative quantum field theory, when calculating the contribution of a one-loop graph, one should obtain the (regularized) determinant of a differential operator, but before that a suitable SA extension should be chosen first. However, generally, there may be too many SA extensions – different SA boundary conditions represent different SA extensions. For example, consider the classical Sturm–Liouville (SL) equation on $J = [0, 1]$:

$$ly := -(py')' + qy = \lambda y, \quad 0 < p \in C^1(J),\ q \in C(J). \tag{1.1}$$

Then the set $\mathcal{U}$ of all complex SA boundary conditions can be divided into two mutually exclusive subsets. The first, called separated, includes boundary conditions of the form

$$\left.\begin{aligned} y(0)\cos\alpha - (py')(0)\sin\alpha = 0,\\ y(1)\cos\beta - (py')(1)\sin\beta = 0,\end{aligned}\right\} \tag{1.2}$$

where $\alpha \in [0, \pi)$, $\beta \in (0, \pi]$. The second, called coupled, includes boundary conditions of the form

$$\begin{pmatrix} y(1) \\ (py')(1) \end{pmatrix} = e^{i\varphi} K \begin{pmatrix} y(0) \\ (py')(0) \end{pmatrix}, \tag{1.3}$$

where

$$K \in \mathrm{SL}(2,\mathbb{R}) =: \left\{K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix};\ k_{ij} \in \mathbb{R},\ \det K = 1\right\}$$

and $\varphi \in [0, 2\pi)$.

It is well known that the eigenvalues of the above SL problem consisting of (1.1) and an SA boundary condition are bounded from below and can be ordered to form a non-decreasing sequence

$$-\infty < \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$$

that approaches ∞ such that the number of times an eigenvalue can appear is equal to its (geometric) multiplicity.

In this paper we mainly put emphasis on the structure of $\mathcal{U}$ and assume that $p \equiv 1$ and $q \in C(J)$ for brevity, though the main body of results here holds for more general p, q.

It is of interest to consider these λ_n as functions on $\mathcal{U}$ and explore how they change when the boundary condition varies. It is already clear that λ_n are not continuous on $\mathcal{U}$ equipped with the natural topology [5]. However, when restricted on certain subsets $S \subset \mathcal{U}$, λ_n may have nice properties. For example, if, on S, λ_0 is bounded from below, then all these λ_n are continuous on S. This is called the *continuity principle* in [5], and we shall use frequently in what follows.

An abstract theorem of von Neumann implies that $\mathcal{U}$ is *globally* parametrized by U(2), the unitary group in complex dimension 2, but in the mathematical literature on SL problems, in terms of boundary data, $\mathcal{U}$ is often viewed as a set of equivalence classes of matrices, for example, as a submanifold of the Grassmanian of two-dimensional subspaces in $\mathbb{C}^4$. In this context, (1.2) and (1.3) are in fact preferred representatives of these classes, and the underlying group U(2) cannot be seen directly in this manner. Recently it was found [1] that there is, more or less, a *canonical* way to identify $\mathcal{U}$ with U(2),[1] which is the starting point of our paper.

As a smooth 4-manifold, U(2) is very special. It is a compact Lie group and has a rich geometry. In this paper, however, we mainly consider one aspect of this geometry and its interplay with SL problems: U(2) acts on itself by conjugation, i.e. $g \cdot u = gug^{-1}$ for $g, u \in \mathrm{U}(2)$. Orbits of this action are called adjoint orbits, each characterized by its eigenvalues (matrices in an orbit all have the same eigenvalues). *Topologically*, these orbits are divided into two types, those consisting of a single point (the two eigenvalues are the same), and those diffeomorpic to the 2-sphere S^2 (the two eigenvalues are different). We shall mainly explore the behaviour of the λ_n as functions on these spheres in the latter case.

Note that in this paper, for brevity, by eigenvalues of a boundary condition A (represented by a matrix) we always mean eigenvalues of the associated boundary-value problem, while eigenvalues of A refer to eigenvalues of the matrix A.

The paper is organized as follows.

We divide § 2 into two subsections. In the first, we discuss the structure of U(2) as the space of SA boundary conditions. We identify several subsets of U(2), parametrize them and show how these parametrizations are related to the ones

[1] This way of parametrizing SA extensions by U(2) is already known in the context of boundary triples; see, for example, [8, ch. 14].

given in (1.2) and (1.3). In the second, we give a refined classification of SA boundary conditions in terms of adjoint orbits, and parametrize orbits of *principal type* – those diffeomorphic to S^2.

We devote § 3 to briefly investigating the so-called *characteristic curve* Γ, which is of great importance when one considers all SA boundary conditions together. To the best of our knowledge, this curve was first investigated in [5]. The behaviour of Γ is complicated and we add hardly any new insight into it. We only rewrite it out in our context and write down *the characteristic equation* in terms of it (theorem 3.1). The advantage is that this equation is canonical and valid for all SA boundary conditions. At the end of this section, we make the observation that Γ has no point in common with almost all adjoint orbits of principal type. This will imply that the situation considered in § 4 is general.

In § 4, we investigate the behaviour of λ_n as a function on an adjoint orbit $\mathcal{O}$ of principal type. We show that λ_n is continuous on $\mathcal{O}$. If, furthermore, Γ has no point in common with $\mathcal{O}$, then λ_n is a real analytic function on $\mathcal{O}$ and has exactly two critical points. If $[a_n, b_n]$ is the range of λ_n, then these $a_n, b_n, n \in \mathbb{N}$, are *precisely* the zeros of a certain real analytic function, and $a_n < b_n < a_{n+1} < b_{n+1}$ (theorems 4.3, 4.7 and 4.9). There are two ways to regard eigenvalues of SL problems. On the one hand, eigenvalues are roots of the characteristic equation. On the other hand, eigenvalues can also be characterized in terms of quadratic forms using the *min–max principle*. To obtain our results, we freely switch our viewpoint between the two if it is convenient. At the end of this section, we investigate the shape of the level subset of λ in $\mathcal{O}$.

Finally, § 5 can be seen as a complement to § 4. We consider λ_n as a function on the diagonal of the torus in U(2). We show that the range of λ_n on U(2) is in fact already determined by its restriction on the diagonal (theorem 5.1).

In the final section we outline how our main results can be generalized to a wider context.

2. The space of SA boundary conditions and adjoint orbits

2.1. The space of SA boundary conditions

Let l_0 and l_1 respectively be the minimal and the maximal operators associated with l. Von Neumann's abstract theory [8, ch. 13] implies that the set $\mathcal{U}$ of all SA extensions of l_0 is parametrized by unitary transforms from $\ker(l_1 - \mathrm{i}\mathbb{I})$ to $\ker(l_1 + \mathrm{i}\mathbb{I})$. Since the spaces are both two dimensional, *topologically* $\mathcal{U}$ is just U(2). In this description, however, there is no canonical way to identify $\mathcal{U}$ with U(2), because to realize such a parametrization a distinguished transform should be chosen.

Recently, an explicit and canonical way of expressing SA boundary conditions in terms of elements of U(2) was found [1]. Let y be a function in the Sobolev space $W_{2,2}(J)$ and let[2]

$$\psi := \begin{pmatrix} y(0) \\ y(1) \end{pmatrix}, \qquad \dot{\psi} := \begin{pmatrix} \dot{y}(0) \\ \dot{y}(1) \end{pmatrix}.$$

[2] Here we use $\dot{y}$ to denote the outward unit normal derivative of y. So $\dot{y}(0) = -y'(0)$ and $\dot{y}(1) = y'(1)$.

An SA boundary condition then takes the form

$$\mathrm{i}(I+U)\dot{\psi} = (I-U)\psi, \tag{2.1}$$

where I is the 2×2 identity matrix. This way we shall identify $\mathcal{U}$ with U(2). For the details of (2.1) and even its generalization, we refer the interested reader to [1].

Before proceeding further, we recall a description of U(2). Any element g of U(2) can be decomposed into two factors,

$$g = \sqrt{\det g}\cdot (g/\sqrt{\det g}),$$

where $\sqrt{\det g} \in \mathrm{U}(1)$ is a square root of $\det g$, and $g/\sqrt{\det g} \in \mathrm{SU}(2)$, i.e. with determinant 1. Since there are two square roots of $\det g$, U(2) is the quotient of $\mathrm{U}(1)\times \mathrm{SU}(2)$ under the natural action of $\mathbb{Z}_2$. This result is often written as $\mathrm{U}(2) = \mathrm{U}(1)\times_{\mathbb{Z}_2}\mathrm{SU}(2)$. We denote the corresponding quotient map by P.

It is natural to classify all SA boundary conditions into two mutually exclusive subclasses according to whether $\det(I+U)$ equals 0 or not. Define

$$\mathcal{U}_0 = \{U\in\mathcal{U} \mid \det(I+U)=0\}, \qquad \mathcal{U}_1 = \{U\in\mathcal{U} \mid \det(I+U)\neq 0\}.$$

$\mathcal{U}_1$ is certainly open and dense in U(2). If $U\in\mathcal{U}_1$, then

$$A := -\mathrm{i}(I+U)^{-1}(I-U) \tag{2.2}$$

is actually a Hermitian matrix and precisely the Cayley transform of U. In terms of A, the boundary condition (2.1) can then be rewritten as

$$\dot{\psi} = A\psi. \tag{2.3}$$

Note that since A and U are in one-to-one correspondence in $\mathcal{U}_1$, A can also be viewed as the coordinate in the chart $\mathcal{U}_1\subset\mathcal{U}$ (so topologically $\mathcal{U}_1\simeq\mathbb{R}^4$). Let y_1, y_2 be the solutions of (1.1) satisfying

$$y_1(0)=1,\quad y_1'(0)=0,\qquad y_2(0)=0,\quad y_2'(0)=1.$$

If

$$A = \begin{pmatrix} a & b\\ \bar{b} & c\end{pmatrix},$$

where a, c are real numbers and b is complex, then the characteristic equation is[3]

$$\Delta(\lambda) = \det\left[\begin{pmatrix} 0 & -1\\ \dot{y}_1 & \dot{y}_2\end{pmatrix} - \begin{pmatrix} a & b\\ \bar{b} & c\end{pmatrix}\begin{pmatrix} 1 & 0\\ y_1 & y_2\end{pmatrix}\right] = 0,$$

i.e.

$$-a\dot{y}_2 + (ac-|b|^2)y_2 - cy_1 + \dot{y}_1 - 2\operatorname{Re} b = 0. \tag{2.4}$$

Let us consider the structure of $\mathcal{U}_0$. If $U\in\mathcal{U}_0$, we can set

$$U = \mathrm{e}^{\mathrm{i}\theta}\begin{pmatrix} a & b\\ -\bar{b} & \bar{a}\end{pmatrix},$$

[3] For brevity, in a characteristic equation we shall always write $y_{1/2}$ instead of $y_{1/2}(1,\lambda)$.

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where $\theta \in [0, \pi]$ and

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{SU}(2).$$

Let $a = r\mathrm{e}^{\mathrm{i}\omega}$, $r \in [0, 1]$, $\omega \in [0, 2\pi)$. Then one can find that

$$\mathrm{e}^{\mathrm{i}\theta} = -r\cos\omega + \mathrm{i}\sqrt{1 - r^2\cos^2\omega}. \tag{2.5}$$

So U is completely determined by its factor in SU(2). But $\pm I \in \mathrm{SU}(2)$ determine the same $U = -I$. This argument shows that $\mathcal{U}_0$ is topologically the 3-sphere S^3 with two points glued together.[4] A general element of $\mathcal{U}_0$ is of the form

$$\mathrm{e}^{\mathrm{i}\theta}\begin{pmatrix} r\mathrm{e}^{\mathrm{i}\omega} & \sqrt{1-r^2}\mathrm{e}^{\mathrm{i}\gamma} \\ -\sqrt{1-r^2}\mathrm{e}^{-\mathrm{i}\gamma} & r\mathrm{e}^{-\mathrm{i}\omega} \end{pmatrix},$$

where θ is given by (2.5) and $r \in [0, 1]$, $\omega, \gamma \in [0, 2\pi)$.

There is another interesting subset $\mathcal{U}^{\mathbb{R}} \subset \mathcal{U}$ consisting of all real SA boundary conditions. As for the shape of $\mathcal{U}^{\mathbb{R}}$ in U(2), we have the following proposition.

Proposition 2.1. *$\mathcal{U}^{\mathbb{R}} = \{U \in \mathrm{U}(2) \mid U = U^{\mathrm{T}}\}$, where the superscript* "T" *denotes the transpose of a matrix. Furthermore, with* $\mathrm{U}(2)$ *viewed as* $\mathrm{U}(1) \times_{\mathbb{Z}_2} \mathrm{SU}(2)$, *$\mathcal{U}^{\mathbb{R}}$ is topologically just $P(S^1 \times S^2)$ (for the precise meaning, see the proof).*

Proof. A real SA boundary condition is precisely one whose complex conjugate represents the same boundary condition except that ψ, $\dot{\psi}$ are replaced by $\bar{\psi}$, $\dot{\bar{\psi}}$. The complex conjugate of (2.1) is

$$\mathrm{i}(I + \bar{U})\dot{\bar{\psi}} = -(I - \bar{U})\bar{\psi}.$$

It can be rewritten as

$$\mathrm{i}\bar{U}(I + \bar{U}^{-1})\dot{\bar{\psi}} = \bar{U}(I - \bar{U}^{-1})\bar{\psi},$$

i.e.

$$\mathrm{i}(I + \bar{U}^{-1})\dot{\bar{\psi}} = (I - \bar{U}^{-1})\bar{\psi}.$$

Then that the boundary condition is real precisely means $U = \bar{U}^{-1}$, which is precisely $U = U^{\mathrm{T}}$. Let

$$U = \mathrm{e}^{\mathrm{i}\theta}\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

Then $U = U^{\mathrm{T}}$ precisely means that b is purely imaginary or zero. This observation immediately leads to the conclusion that $\mathcal{U}^{\mathbb{R}} = P(S^1 \times S^2)$. □

Remark. In [6, theorem 3.3], there is also a description of the space of all real SA boundary conditions. However, the global picture is more clear here.

Let us now see how the boundary conditions (1.2), (1.3) look in U(2). It is easy to see that the separated boundary conditions correspond to those U of diagonal form, forming a Cartan subgroup H of U(2), topologically a 2-torus.

For the coupled case, two subcases should be distinguished: $k_{12} \neq 0$ and $k_{12} = 0$.

[4] Topologically, $\mathrm{SU}(2) \simeq S^3$.

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Proposition 2.2. *If in the coupled case* $k_{12} \neq 0$*, then the corresponding* U *lies in* $\mathcal{U}_1$ *and the associated Hermitian matrix is*

$$A(\mathrm{e}^{\mathrm{i}\varphi}K) = \frac{1}{k_{12}} \begin{pmatrix} k_{11} & -\mathrm{e}^{-\mathrm{i}\varphi} \\ -\mathrm{e}^{\mathrm{i}\varphi} & k_{22} \end{pmatrix}.$$

Proof. If $k_{12} \neq 0$, the boundary condition (1.3) can be rewritten as

$$\dot{\psi} = \begin{pmatrix} k_{11}/k_{12} & -\mathrm{e}^{-\mathrm{i}\varphi}/k_{12} \\ -\mathrm{e}^{\mathrm{i}\varphi}/k_{12} & k_{22}/k_{12} \end{pmatrix} \psi.$$

Comparing this with (2.3), we come to the conclusion. □

It is not hard to see that if $k_{12} = 0$, the corresponding boundary condition cannot be rewritten as (2.3), and so the corresponding $U(\mathrm{e}^{\mathrm{i}\varphi}K) \in \mathcal{U}_0$. However, from the above proposition we can obtain a unified expression of $U(\mathrm{e}^{\mathrm{i}\varphi}K)$ no matter whether $k_{12} = 0$ or not.

Proposition 2.3. *For the coupled boundary condition (1.3), the corresponding element* $U(\mathrm{e}^{\mathrm{i}\varphi}K) \in \mathrm{U}(2)$ *is*

$$\frac{1}{k_{12} - k_{21} + \mathrm{i}(k_{11} + k_{22})} \begin{pmatrix} k_{12} + k_{21} + \mathrm{i}(k_{22} - k_{11}) & 2\mathrm{i}\mathrm{e}^{-\mathrm{i}\varphi} \\ 2\mathrm{i}\mathrm{e}^{\mathrm{i}\varphi} & k_{12} + k_{21} - \mathrm{i}(k_{22} - k_{11}) \end{pmatrix},$$

and it has -1 *as its eigenvalue if and only if* $k_{12} = 0$.

Proof. If $k_{12} \neq 0$, then, from proposition 2.2, $U(\mathrm{e}^{\mathrm{i}\varphi}K) \in \mathcal{U}_1$ and

$$U(\mathrm{e}^{\mathrm{i}\varphi}K) = [I - \mathrm{i}A(\mathrm{e}^{\mathrm{i}\varphi}K)][I + \mathrm{i}A(\mathrm{e}^{\mathrm{i}\varphi}K)]^{-1}$$

due to (2.2). This leads to the expression, as required. Obviously, this expression extends smoothly to the $k_{12} = 0$ case.

Note that $\det[I + U(\mathrm{e}^{\mathrm{i}\varphi}K)] = 0$ is equivalent to

$$k_{12}[k_{12} - k_{21} + \mathrm{i}(k_{22} + k_{11})] = 0,$$

which holds if and only if $k_{12} = 0$. □

For the case in which $k_{12} = 0$, we also have the following proposition.

Proposition 2.4. *If in the coupled case* $k_{12} = 0$*, then in terms of* r*,* ω*,* γ *the matrix* $\mathrm{e}^{\mathrm{i}\varphi}K$ *is determined by (without loss of generality, we set* $k_{11} > 0$*)*

$$\begin{aligned} k_{11} &= \frac{\sqrt{1 - r^2\cos^2\omega} + r\sin\omega}{\sqrt{1 - r^2}}, \\ k_{21} &= \frac{-2r\cos\omega}{\sqrt{1 - r^2}}, \\ \mathrm{e}^{\mathrm{i}\varphi} &= \mathrm{e}^{-\mathrm{i}(\gamma + \pi/2)}. \end{aligned}$$

Proof. In terms of r, ω, γ the associated boundary condition is

$$y(1) = \frac{\sqrt{1-r^2\cos^2\omega} + r\sin\omega}{\sqrt{1-r^2}} \mathrm{e}^{-\mathrm{i}(\gamma+\pi/2)} y(0), \tag{2.6}$$

$$y'(1) = -\frac{2r\cos\omega}{\sqrt{1-r^2}} \mathrm{e}^{-\mathrm{i}(\gamma+\pi/2)} y(0) + \frac{\sqrt{1-r^2\cos^2\omega} - r\sin\omega}{\sqrt{1-r^2}} \mathrm{e}^{-\mathrm{i}(\gamma+\pi/2)} y'(0). \tag{2.7}$$

Comparing this with (1.3), we get the conclusion. □

$\mathcal{U}_0$ has been investigated in another way in the literature. From the above discussion, it is easy to see that $\mathcal{U}_0$ is actually the set $\mathcal{J}^{\mathbb{C}}$ in [5].

Remark. No matter whether $k_{12} = 0$ or not, the eigenvalues of $U(\mathrm{e}^{\mathrm{i}\varphi}K)$ are independent of φ. So for a fixed K, $U(\mathrm{e}^{\mathrm{i}\varphi}K)$, $\varphi \in [0, 2\pi)$, all lie in the same adjoint orbit, tracing out a circle. There is a beautiful inequality among eigenvalues of SL problems when the boundary condition varies only on this circle [4].

2.2. Adjoint orbits

Let $\mathrm{H} \subset \mathrm{U}(2)$ be the Cartan subgroup as in the last section, and let $W(\cong \mathbb{Z}_2)$ be the corresponding Weyl group. Then the quotient H/W is a two-dimensional manifold with boundary – in fact, it is topologically the famous Möbius strip. H/W can be viewed as the space of adjoint orbits in $\mathrm{U}(2)$, with each interior point representing an adjoint orbit of principal type and with each point on the boundary representing an adjoint orbit consisting of a single matrix. In this sense, a generic adjoint orbit is diffeomorphic to S^2. Let $\Pi\colon \mathrm{U}(2) \to \mathrm{H}/W$ be the quotient map. We refer the reader to [9] for the basics of compact Lie groups.

Since both $\mathcal{U}_0$ and $\mathcal{U}_1$ are invariant under the adjoint action, an adjoint orbit would lie entirely in either $\mathcal{U}_0$ or $\mathcal{U}_1$. This, of course, leads to a more refined classification of SA boundary conditions – each adjoint orbit represents a subclass. In this section we mainly consider orbits of principal type. These are in fact real analytic two-dimensional manifolds.

By (2.2), $\mathcal{U}_1$ is diffeomorphic to the space $\mathcal{M}$ of 2×2 Hermitian matrices, and the adjoint action of $\mathrm{U}(2)$ on $\mathcal{U}_1$ corresponds to the one on $\mathcal{M}$. This way we can identify adjoint orbits in $\mathcal{U}_1$ with adjoint orbits in $\mathcal{M}$. An adjoint orbit $\mathcal{O} \subset \mathcal{M}$ is characterized by its eigenvalues $\zeta_1 > \zeta_2$. Let $\mu = \frac{1}{2}(\zeta_1 + \zeta_2)$, $\nu = \frac{1}{2}(\zeta_1 - \zeta_2)$ and denote the adjoint orbit by $\mathcal{O}_{\mu,\nu}$.

Proposition 2.5. *A general element in $\mathcal{O}_{\mu,\nu}$ is of the form*

$$A = \begin{pmatrix} \mu - \nu\cos 2\theta & \nu\sin 2\theta \mathrm{e}^{-\mathrm{i}\gamma} \\ \nu\sin 2\theta \mathrm{e}^{\mathrm{i}\gamma} & \mu + \nu\cos 2\theta \end{pmatrix}, \quad \gamma \in [0, 2\pi),\ \theta \in [0, \tfrac{1}{2}\pi].$$

Proof. $\mathcal{O}_{\mu,\nu}$ is the adjoint orbit through

$$\begin{pmatrix} \mu - \nu & 0 \\ 0 & \mu + \nu \end{pmatrix}.$$

Then each element in $\mathcal{O}_{\mu,\nu}$ can be represented by

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} \mu - \nu & 0 \\ 0 & \mu + \nu \end{pmatrix} \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}^{-1},$$

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for some

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{SU}(2).$$

We can even further require that $a \geqslant 0$. Setting $a = \cos\theta$, $\theta \in [0, \frac{1}{2}\pi]$, and $b = \sin\theta \mathrm{e}^{-\mathrm{i}\gamma}$ then leads to the representation. □

Remark. From proposition 2.2 we can see that γ essentially contains the same geometric content as φ in (1.3). Note that $\theta = 0, \frac{1}{2}\pi$ actually correspond to the only two separated boundary conditions in $\mathcal{O}_{\mu,\nu}$. It is easy to find that in $\mathcal{O}_{\mu,\nu}$ real boundary conditions lie precisely on the circle formed by the two semicircles $\gamma = 0$ and $\gamma = \pi$. It will soon be clear that this is a general property of orbits of principal type.

An adjoint orbit in $\mathcal{U}_0$ is determined by the other eigenvalue $\mathrm{e}^{\mathrm{i}\chi}$ ($\chi \in [0, \pi) \cup (\pi, 2\pi)$) not equal to -1. We shall denote the orbit by $\mathcal{O}_\chi$. Since, in this case,

$$\operatorname{tr} U = \mathrm{e}^{\mathrm{i}\theta}(a + \bar{a}) = -1 + \mathrm{e}^{\mathrm{i}\chi},$$

we find that $\operatorname{Re} a = \sin\frac{1}{2}\chi$ if $\chi \in [0, \pi)$, and $\operatorname{Re} a = -\sin\frac{1}{2}\chi$ if $\chi \in (\pi, 2\pi)$.

Proposition 2.6. *For $\chi \in [0, \pi)$, a general element of $\mathcal{O}_\chi$ is of the form*

$$U = \mathrm{i}\mathrm{e}^{\mathrm{i}\chi/2} \begin{pmatrix} \sin\frac{1}{2}\chi + \mathrm{i}t & \sqrt{\cos^2\frac{1}{2}\chi - t^2}\,\mathrm{e}^{\mathrm{i}\gamma} \\ -\sqrt{\cos^2\frac{1}{2}\chi - t^2}\,\mathrm{e}^{-\mathrm{i}\gamma} & \sin\frac{1}{2}\chi - \mathrm{i}t \end{pmatrix},$$

where $t \in [-\cos\frac{1}{2}\chi, \cos\frac{1}{2}\chi]$ and $\gamma \in [0, 2\pi)$.

For $\chi \in (\pi, 2\pi)$, a general element of $\mathcal{O}_\chi$ is of the form

$$U = -\mathrm{i}\mathrm{e}^{\mathrm{i}\chi/2} \begin{pmatrix} -\sin\frac{1}{2}\chi + \mathrm{i}t & \sqrt{\cos^2\frac{1}{2}\chi - t^2}\,\mathrm{e}^{\mathrm{i}\gamma} \\ -\sqrt{\cos^2\frac{1}{2}\chi - t^2}\,\mathrm{e}^{-\mathrm{i}\gamma} & -\sin\frac{1}{2}\chi - \mathrm{i}t \end{pmatrix},$$

where $t \in [\cos\frac{1}{2}\chi, -\cos\frac{1}{2}\chi]$ and $\gamma \in [0, 2\pi)$.

Proof. By (2.5),

$$\mathrm{e}^{\mathrm{i}\theta} = -\sin\tfrac{1}{2}\chi + \mathrm{i}\cos\tfrac{1}{2}\chi = \mathrm{i}\mathrm{e}^{\mathrm{i}\chi/2}$$

if $\chi \in [0, \pi)$, and

$$\mathrm{e}^{\mathrm{i}\theta} = \sin\tfrac{1}{2}\chi - \mathrm{i}\cos\tfrac{1}{2}\chi = -\mathrm{i}\mathrm{e}^{\mathrm{i}\chi/2}$$

if $\chi \in (\pi, 2\pi)$. The conclusion then easily follows. □

3. The characteristic curve

The characteristic curve $\Gamma\colon \mathbb{R} \to \mathrm{U}(2)$ is a parametrized curve, the image of which consists of all SA boundary conditions having a double eigenvalue. This curve contains all information concerning eigenvalues of SA boundary conditions (of course, if one puts eigenfunctions aside) [6].

From (2.1), it is easy to find that Γ is of the form[5]

$$\Gamma(\lambda) = \frac{1}{y_2 - \dot{y}_1 + \mathrm{i}\dot{y}_2 + \mathrm{i}y_1}\begin{pmatrix} y_2 + \dot{y}_1 + \mathrm{i}\dot{y}_2 - \mathrm{i}y_1 & 2\mathrm{i} \\ 2\mathrm{i} & y_2 + \dot{y}_1 - \mathrm{i}\dot{y}_2 + \mathrm{i}y_1 \end{pmatrix},$$

where $\lambda \in \mathbb{R}$. The image of Γ is completely included in $\mathcal{U}^{\mathbb{R}}$. $\Pi \circ \Gamma$ is a curve in H/W, which we call the *induced curve* of Γ, and is characterized by the two eigenvalues of $\Gamma(\lambda)$, say,

$$\kappa_{\pm}(\lambda) = \frac{y_2 + \dot{y}_1 \pm \mathrm{i}\sqrt{4 + (\dot{y}_2 - y_1)^2}}{y_2 - \dot{y}_1 + \mathrm{i}\dot{y}_2 + \mathrm{i}y_1}.$$

PROPOSITION 3.1. *In terms of $\Gamma(\lambda)$, the characteristic equation for an SA boundary condition U can be written in the form*

$$\det(U - \Gamma(\lambda)) = 0. \tag{3.1}$$

The subset $S_\lambda \subset \mathcal{U}$ of boundary conditions with λ as an eigenvalue is diffeomorphic to $\mathcal{U}_0$.

Proof. U(2) acts on itself by left translation and S_λ can be represented as $-\Gamma(\lambda)\mathcal{U}_0$. By (3.1), S_λ is diffeomorphic to $\mathcal{U}_0$, i.e. a 3-sphere with two points glued together. This observation was already noted in [6], but in more complicated language. □

COROLLARY 3.2. *The matrix $\Gamma(\lambda)$ has -1 as an eigenvalue if and only if $\lambda = \lambda_n^{\mathrm{D}}$ for some $n \in \mathbb{N}$. Therefore, the characteristic curve Γ intersects $\mathcal{U}_0$ countably infinite times.*

Proof. This is obvious. □

REMARK. However, the above result does not mean that Γ has infinitely many intersection points with $\mathcal{U}_0$. In addition, -1 can be replaced by $\mathrm{e}^{\mathrm{i}\theta}I$, $\theta \in [0, 2\pi)$, and a similar result holds.

EXAMPLE 3.3. Let $q \equiv 0$. Then, for $\lambda > 0$,

$$y_1(x,\lambda) = \cos\sqrt{\lambda}x, \qquad y_2(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}.$$

The two eigenvalues of $\Gamma(\lambda)$ are

$$\kappa_{\pm}(\lambda) = \frac{(1/\sqrt{\lambda} - \sqrt{\lambda})\sin\sqrt{\lambda} \pm 2\mathrm{i}}{(1/\sqrt{\lambda} + \sqrt{\lambda})\sin\sqrt{\lambda} + 2\mathrm{i}\cos\sqrt{\lambda}}.$$

For the Dirichlet boundary condition, $\lambda_n^{\mathrm{D}} = (n+1)^2\pi^2$, $\kappa_{\pm}(\lambda_n^{\mathrm{D}}) = \pm(-1)^{n+1}$. So in this case, the intersection points of Γ and $\mathcal{U}_0$ all lie in the orbit through

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[5] An eigenfunction should be of the form $ay_1(x) + by_2(x)$ for constants a, b. Then (2.1) reduces to an equation of the form $(U - \Gamma(\lambda))\binom{a}{b} = 0$, which implies that the coefficient matrix $U - \Gamma(\lambda)$ should be zero for a double eigenvalue.

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In fact, there are only two such points, i.e.

$$\pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since the space H/W parametrizing all adjoint orbits is of dimension 2, and the induced curve of Γ is analytic and, of course, of dimension 1, the characteristic curve Γ would not intersect a generic adjoint orbit of principal type.

4. λ_n as functions on adjoint orbits of principal type

In this section, by adjoint orbits we will always mean those of principal type. We mainly consider adjoint orbits that have no point in common with the characteristic curve Γ. From the previous section, we know a generic adjoint orbit is of this kind. We denote by λ_n^{N} the nth eigenvalue of the Neumann boundary condition.

For the orbit $\mathcal{O}_{\mu,\nu}$, by (2.4) the corresponding characteristic equation is

$$(\mu - \nu\cos 2\theta)\dot{y}_2 + (\nu^2 - \mu^2)y_2 + (\mu + \nu\cos 2\theta)y_1 - \dot{y}_1 = -2\nu\sin 2\theta\cos\gamma. \tag{4.1}$$

Lemma 4.1. *Let λ_n^+ and λ_n^- be the nth eigenvalues of the boundary conditions*

$$\dot{y}(0) = (\mu - \nu)y(0), \qquad \dot{y}(1) = (\mu - \nu)y(1)$$

and

$$\dot{y}(0) = (\mu + \nu)y(0), \qquad \dot{y}(1) = (\mu + \nu)y(1)$$

respectively. Then the function λ_n on $\mathcal{O}_{\mu,\nu}$ satisfies

$$\lambda_n^- \leqslant \lambda_n \leqslant \lambda_n^+.$$

In particular, by the continuity principle, λ_n is continuous on $\mathcal{O}_{\mu,\nu}$.

Proof. For $A \in \mathcal{O}_{\mu,\nu}$, the associated quadratic form is

$$Q(y) = \int_0^1 |y'|^2\,\mathrm{d}x + \int_0^1 q(x)|y|^2\,\mathrm{d}x - \psi^\dagger A\psi, \quad y \in H^1,$$

where H^1 is the Sobolev space $W_{1,2}(J)$.

Note that

$$\psi^\dagger A\psi = \mu|\psi|^2 + (|y(1)|^2 - |y(0)|^2)\nu\cos 2\theta + 2\nu\,\mathrm{Re}[\bar{y}(0)y(1)\mathrm{e}^{-\mathrm{i}\gamma}]\sin 2\theta.$$

By the inequality $2|ab| \leqslant |a|^2 + |b|^2$, we have

$$2\,\mathrm{Re}[\bar{y}(0)y(1)\mathrm{e}^{-\mathrm{i}\gamma}]\sin 2\theta \leqslant (1 + \cos 2\theta)|y(0)|^2 + (1 - \cos 2\theta)|y(1)|^2$$

and

$$2\,\mathrm{Re}[\bar{y}(0)y(1)\mathrm{e}^{-\mathrm{i}\gamma}]\sin 2\theta \geqslant -(1 - \cos 2\theta)|y(0)|^2 - (1 + \cos 2\theta)|y(1)|^2.$$

Therefore, we come to the estimation

$$(\mu - \nu)|\psi|^2 \leqslant \psi^\dagger A\psi \leqslant (\mu + \nu)|\psi|^2.$$

The conclusion then follows from the variational characterization of $\lambda_n(A)$ – the min–max principle. □

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Remark. The boundedness of λ_n on $\mathcal{O}_{\mu,\nu}$ is actually a conclusion of [5] that λ_n is continuous on $\mathcal{U}_1$, together with the fact that S^2 is compact. Conversely, minor modification of the proof of lemma 4.1 gives another proof that λ_n is continuous on $\mathcal{U}_1$.

Proposition 4.2. *λ_n is a continuous function on $\mathcal{U}_1$.*

Proof. For any given $A_0 \in \mathcal{U}_1$, let $\mathcal{O}_{\mu_0,\nu_0}$ be the orbit through A_0 (we allow ν_0 to be 0 here). Then, for $\delta > 0$, the set

$$V_\delta = \bigcup_{\mu+\nu<\mu_0+\nu_0+\delta} \mathcal{O}_{\mu,\nu}$$

is an open neighbourhood of A_0. Note that, for $A \in V_\delta$,

$$(\mu_0 + \nu_0 + \delta)|\psi|^2 \geqslant \psi^\dagger A\psi.$$

The min–max principle implies that λ_0 is bounded from below on V_δ, and thus λ_n is continuous on V_δ and, in particular, continuous at A_0. □

Theorem 4.3. *Assume that $\mathcal{O}_{\mu,\nu}$ has no point in common with Γ. Then, for each n, λ_n as a function on $\mathcal{O}_{\mu,\nu}$ is real analytic, and has exactly two critical points. Let $[a_n, b_n]$ be the range of λ_n on $\mathcal{O}_{\mu,\nu}$. Then, for each n,*

$$a_n < b_n < a_{n+1} < b_{n+1}.$$

These a_n, b_n, $n = 0, 1, 2, \dots$, are exactly roots of

$$\nu^2(\dot{y}_2 - y_1)^2 + 4\nu^2 = [\mu(\dot{y}_2 + y_1) + (\nu^2 - \mu^2)y_2 - \dot{y}_1]^2. \tag{4.2}$$

Proof. Denote the left-hand side of (4.1) by $D(\lambda, p)$, viewed as a function on $\mathbb{R} \times \mathcal{O}_{\mu,\nu}$. Since $\mathcal{O}_{\mu,\nu}$ has no point in common with Γ, $(\partial D/\partial\lambda)|_{(\lambda_n(p),p)} \neq 0$ for any $p \in \mathcal{O}_{\mu,\nu}$. In addition, $D(\lambda, p)$ and the right-hand side of (4.1) are real analytic functions on $\mathbb{R} \times \mathcal{O}_{\mu,\nu}$. So, by the implicit function theorem, λ_n is a real analytic function on $\mathcal{O}_{\mu,\nu}$.

It is easy to find that for a critical point p, we must have $\sin\gamma = 0$. This implies that all critical points must lie on the circle C_0 formed by the two semicircles $\gamma = 0$ and $\gamma = \pi$. So, to find all critical points of λ_n on $\mathcal{O}_{\mu,\nu}$, we only need to find all critical points of λ_n on C_0. Now consider the characteristic equation restricted on C_0, i.e.

$$\nu(\dot{y}_2 - y_1)\cos 2\theta - 2\nu \sin 2\theta = \mu(\dot{y}_2 + y_1) + (\nu^2 - \mu^2)y_2 - \dot{y}_1, \tag{4.3}$$

where $\theta \in (-\frac{1}{2}\pi, \frac{1}{2}\pi]$. For a given $\lambda \in \mathbb{R}$, there are at most two values of θ satisfying the above equation. It is an elementary calculation to show that λ_n has no degenerate critical point. These together imply that there are at most two critical points of λ_n as a function on C_0. Since C_0 is compact, we know that there are *precisely* two critical points, one the maximizer and the other the minimizer.

Any critical value κ of λ_n must satisfy (4.2). Conversely, it is not hard to find that any root κ of (4.2) must be a critical value of some λ_n. By the uniqueness of minimizer and maximizer, $\kappa = a_n$ or b_n.

If $a_{n+1} = \lambda_{n+1}(p_0)$ for some $p_0 \in C_0$, then

$$a_{n+1} > \lambda_n(p_0) \geqslant a_n.$$

We only need to check that $a_{n+1} > b_n$. If this is not the case, then $a_{n+1} \in (a_n, b_n]$ and there is another point $p_1 \in C_0$ such that

$$\lambda_n(p_1) = a_{n+1} = \lambda_{n+1}(p_0).$$

If $p_1 = p_0$, this means that a_{n+1} is a double eigenvalue of the boundary condition p_0, contradicting that Γ has no point in common with $\mathcal{O}_{\mu,\nu}$; if $p_1 \neq p_0$, then, for $\lambda = a_{n+1}$, at least two different values of θ satisfy (4.3). This contradicts the fact that a_{n+1} is the unique minimum of λ_{n+1}. The proof is then complete. □

Example 4.4. Let $q \equiv 0$. Then for $\lambda > 0$ (4.3) is

$$-2\nu \sin 2\theta = 2\mu \cos\sqrt{\lambda} + (\nu^2 - \mu^2)\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} + \sqrt{\lambda}\cos\sqrt{\lambda}.$$

The common critical points of all λ_n such that $\lambda_n > 0$ are $\theta = \pm\frac{\pi}{4}$. Equation (4.2) is now

$$2\mu \cos\sqrt{\lambda} + (\nu^2 - \mu^2)\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} + \sqrt{\lambda}\cos\sqrt{\lambda} = \pm 2\nu.$$

If $\mu = \nu$, then the above equation obtains the more accessible form

$$\cos\sqrt{\lambda} = \pm\frac{2\nu}{2\nu + \sqrt{\lambda}}.$$

From [5, theorem 3.73], we can derive that nearly all points in $\mathcal{U}_0$ are discontinuity points of λ_n as a function on U(2). This, of course, does not exclude the possibility that λ_n is continuous on adjoint orbits lying in $\mathcal{U}_0$.

For the orbit $\mathcal{O}_\chi$ with $\chi \in [0, \pi)$, the associated characteristic equation is

$$(-\cos\tfrac{1}{2}\chi + t)\dot{y}_2 + (-\cos\tfrac{1}{2}\chi - t)y_1 - 2\sin\tfrac{1}{2}\chi y_2 = 2\cos(\gamma + \tfrac{1}{2}\pi)\sqrt{\cos^2\tfrac{1}{2}\chi - t^2}.$$

Lemma 4.5. *On the orbit $\mathcal{O}_\chi$ with $\chi \in [0, \pi)$,*

$$\lambda_n \geqslant \lambda_n^{\mathrm{N}}, \quad n = 0, 1, 2, \ldots.$$

In particular, by the continuity principle, λ_n is continuous on $\mathcal{O}_\chi$.

Proof. If $t \neq \cos\frac{1}{2}\chi$, the associated quadratic form is

$$Q_1(y) = \int_0^1 |y'|^2\,\mathrm{d}x + \int_0^1 q(x)|y|^2\,\mathrm{d}x + \frac{2\sin\frac{1}{2}\chi}{\cos\frac{1}{2}\chi - t}|y(0)|^2, \quad y \in H^1_{\gamma,t},$$

where

$$H^1_{\gamma,t} = \left\{ y \in H^1 \;\middle|\; y(1) = \mathrm{e}^{-\mathrm{i}(\gamma+\pi/2)}\sqrt{\frac{\cos\frac{1}{2}\chi + t}{\cos\frac{1}{2}\chi - t}}y(0) \right\} \subset H^1.$$

Note that in this case, by the min–max principle,

$$\lambda_n = \min_{S_{n+1}\subset H^1_{\gamma,t}} \max_{y\in S_{n+1}-\{0\}} \frac{Q_1(y)}{\|y\|^2},$$

where S_{n+1} ranges over all $(n+1)$-dimensional subspaces of $H^1_{\gamma,t}$. Since

$$\min_{S_{n+1}\subset H^1_{\gamma,t}} \max_{y\in S_{n+1}-\{0\}} \frac{Q_1(y)}{\|y\|^2} \geqslant \min_{S_{n+1}\subset H^1_{\gamma,t}} \max_{y\in S_{n+1}-\{0\}} \frac{Q_0(y)}{\|y\|^2},$$

where $Q_0(y) = \int_0^1 |y'|^2\,\mathrm{d}x + \int_0^1 q(x)|y|^2\,\mathrm{d}x$, and

$$\lambda_n^{\mathrm{N}} = \min_{S_{n+1}\subset H^1} \max_{y\in S_{n+1}-\{0\}} \frac{Q_0(y)}{\|y\|^2},$$

where S_{n+1} ranges over all $(n+1)$-dimensional subspaces of H^1, we must have

$$\lambda_n \geqslant \lambda_n^{\mathrm{N}}.$$

If $t = \cos\frac{1}{2}\chi$, the associated quadratic form is

$$Q_2(y) = \int_0^1 |y'|^2\,\mathrm{d}x + \int_0^1 q(x)|y|^2\,\mathrm{d}x + \tan\tfrac{1}{2}\chi|y(1)|^2, \quad y\in H^1,\ y(0)=0.$$

A similar argument then leads to the inequality $\lambda_n \geqslant \lambda_n^{\mathrm{N}}$. □

Corollary 4.6. *Let $\Omega := \bigcup_{\chi\in[0,\pi)} \mathcal{O}_\chi$. Then λ_n are continuous functions on Ω.*

Proof. Note that λ_0^{N} is independent of the orbit parameter χ. The inequality in lemma 4.5 holds uniformly on Ω. By the continuity principle, the conclusion follows. □

Theorem 4.7. *Assume that $\mathcal{O}_\chi$ with $\chi\in[0,\pi)$ has no point in common with Γ. Then, on $\mathcal{O}_\chi$, λ_n is real analytic, and has exactly two critical points. Let $[a_n, b_n]$ be the range of λ_n on $\mathcal{O}_\chi$. Then, for each n,*

$$a_n < b_n < a_{n+1} < b_{n+1}.$$

These a_n, b_n, $n = 0, 1, 2, \dots$, are exactly roots of

$$(y_1 - \dot{y}_2)^2 + 4 = (\dot{y}_2 + y_1 + 2y_2\tan\tfrac{1}{2}\chi)^2. \tag{4.4}$$

Proof. Let $t = \cos\frac{1}{2}\chi\sin\tau$, $\tau\in[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. Then the characteristic equation becomes

$$2\cos(\gamma + \tfrac{1}{2}\pi)\cos\tau + (1-\sin\tau)\dot{y}_2 + (1+\sin\tau)y_1 + 2y_2\tan\tfrac{1}{2}\chi = 0.$$

Then the argument in the proof of theorem 4.3 still holds. We omit the details here. □

For the orbit $\mathcal{O}_\chi$ with $\chi\in(\pi, 2\pi)$, the corresponding characteristic equation is

$$(\cos\tfrac{1}{2}\chi + t)\dot{y}_2 + (\cos\tfrac{1}{2}\chi - t)y_1 + 2\sin\tfrac{1}{2}\chi y_2 = 2\cos(\gamma + \tfrac{1}{2}\pi)\sqrt{\cos^2\tfrac{1}{2}\chi - t^2}. \tag{4.5}$$

Lemma 4.8. *On the orbit $\mathcal{O}_\chi$ with $\chi\in(\pi, 2\pi)$, λ_0 is bounded from below. In particular, by the continuity principle, λ_n is continuous on $\mathcal{O}_\chi$.*

Proof. We only need to prove that for λ sufficiently negative (4.5) cannot hold for any γ and t. For this purpose, we should use the following estimations for $\lambda = -s^2$ ($s > 0$) sufficiently negative:

$$y_1(1,\lambda) = \cosh s + O\left(\frac{\mathrm{e}^s}{s}\right), \qquad y_2(1,\lambda) = \frac{\sinh s}{s} + O\left(\frac{\mathrm{e}^s}{s^2}\right),$$
$$\dot{y}_2(1,\lambda) = \cosh s + O\left(\frac{\mathrm{e}^s}{s}\right).$$

These results are not hard to obtain from [7, theorems 1.2.1 and 1.2.2, ch. 1]. Note that unlike in the previous situation, the continuity of q is used to obtain these estimations.

Divide the left-hand side of (4.5) by $\cosh s$. Then for s sufficiently large, the result is less than $\cos\frac{1}{2}\chi$. Divide the right-hand side of (4.5) by $\cosh s$. Then for s sufficiently large, the result is greater than $\cos\frac{1}{2}\chi$. This is exactly what we want. □

Remark. For $t < -\cos\frac{1}{2}\chi$, the associated quadratic form is

$$Q_1(y) = \int_0^1 |y'|^2\,\mathrm{d}x + \int_0^1 q(x)|y|^2\,\mathrm{d}x + \frac{2\sin\frac{1}{2}\chi}{\cos\frac{1}{2}\chi + t}|y(0)|^2, \quad y \in H^1_{\gamma,t},$$

where

$$H^1_{\gamma,t} = \left\{ y \in H^1 \;\middle|\; y(1) = \mathrm{e}^{-\mathrm{i}(\gamma+\pi/2)}\sqrt{\frac{-\cos\frac{1}{2}\chi + t}{-\cos\frac{1}{2}\chi - t}}\,y(0) \right\} \subset H^1.$$

The argument in the proof of lemma 4.5 fails to hold. The situation is similar for $t = -\cos\frac{1}{2}\chi$. This is the reason that we have turned to the several estimations in the proof of lemma 4.8.

Theorem 4.9. *Assume that $\mathcal{O}_\chi$ with $\chi \in (\pi, 2\pi)$ has no point in common with Γ. Then, on $\mathcal{O}_\chi$, λ_n is real analytic and has exactly two critical points. Let $[a_n, b_n]$ be the range of λ_n on $\mathcal{O}_\chi$. Then, for each n,*

$$a_n < b_n < a_{n+1} < b_{n+1}.$$

These a_n, b_n, $n = 0, 1, 2, \ldots,$ are exactly roots of

$$(y_1 - \dot{y}_2)^2 + 4 = (\dot{y}_2 + y_1 + 2y_2 \tan\tfrac{1}{2}\chi)^2. \tag{4.6}$$

Proof. The proof is similar to that of theorem 4.7 and we omit the details. □

Remark. In [4] Eastham *et al.* obtained a general inequality among eigenvalues of different coupled boundary conditions. In fact, these boundary conditions lie on the circle parametrized by γ in our adjoint orbit $\mathcal{O}$. In [2] this inequality was re-derived via variational characterization of the eigenvalues. To a certain extent, our inequality $a_n < b_n < a_{n+1} < b_{n+1}$ can be viewed as an extension in this direction – we consider an adjoint orbit rather than a circle in it.

Example 4.10. Let $q \equiv 0$. Then, for $\lambda > 0$, the equation in theorem 4.7 or theorem 4.9 is

$$\cos\sqrt{\lambda} + \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\tan\tfrac{1}{2}\chi = \pm 1.$$

The critical points are $t = 0$ and $\gamma = 0$ or π.

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To conclude this section we will find out the level set Λ^κ in an adjoint orbit $\mathcal{O}$ consisting of boundary conditions with κ as an eigenvalue.

THEOREM 4.11. *Let $\mathcal{O}$ be an adjoint orbit and let $p_0 \in \mathcal{O}$. If $\lambda_n(p_0) = \kappa$ for some n, then the level set Λ^κ is a set either consisting of a single point or diffeomorphic to a circle.*

Proof. Let $\zeta_1 \neq \zeta_2$ be the two eigenvalues of $\mathcal{O} \subset \mathrm{U}(2)$, and let $\varrho_1 \neq \varrho_2$ be the two eigenvalues of $\Gamma(\kappa)$. The general element of $\mathcal{O}$ is of the form

$$U(x,\gamma) = \begin{pmatrix} \zeta_1 x + \zeta_2(1-x) & (\zeta_2-\zeta_1)\sqrt{x(1-x)}\mathrm{e}^{\mathrm{i}\gamma} \\ (\zeta_2-\zeta_1)\sqrt{x(1-x)}\mathrm{e}^{-\mathrm{i}\gamma} & \zeta_1(1-x)+\zeta_2 x \end{pmatrix},$$

where $x \in [0,1]$, $\gamma \in [0, 2\pi)$. The level set $\Lambda^\kappa \subset \mathcal{O}$ is characterized by the equation $\det(U(x,\gamma) - \Gamma(\kappa)) = 0$. Since we are only interested in the shape of Λ^κ, we can safely set

$$\Gamma(\kappa) = \begin{pmatrix} \varrho_1 & 0 \\ 0 & \varrho_2 \end{pmatrix}.$$

This implies that

$$x = -\frac{(\zeta_2-\varrho_1)(\zeta_1-\varrho_2)}{(\varrho_1-\varrho_2)(\zeta_2-\zeta_1)},$$
$$1-x = \frac{(\zeta_1-\varrho_1)(\zeta_2-\varrho_2)}{(\varrho_1-\varrho_2)(\zeta_2-\zeta_1)}.$$

If at least one of ζ_1, ζ_2 coincides with ϱ_1 or ϱ_2, then x or $1-x$ equal 0 and Λ^κ consists only of the point

$$p = \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \zeta_2 & 0 \\ 0 & \zeta_1 \end{pmatrix}.$$

If ζ_1, ζ_2 are both different from ϱ_1 and ϱ_2, then both x and $1-x$ are non-zero and determined by these values. It follows that Λ^κ is diffeomorphic to S^1, parametrized by γ. □

REMARK. If $\mathcal{O}$ has no point in common with Γ, then, from our previous result, Λ^κ is actually the level-κ set of λ_n. If $\kappa = a_n$ or b_n, then Λ^κ consists of the minimizer or maximizer of λ_n. For other values of κ, Λ^κ are all diffeomorphic to S^1.

5. λ_n as functions on the boundary circle of H/W

In the previous sections we have mainly analysed the behaviour of the λ_n as functions on generic adjoint orbits represented by interior points in H/W. However, attention should also be paid to points on the boundary circle $\partial(\mathrm{H}/W)$ – important boundary conditions, such as the Dirichlet, Neumann and Robin boundary conditions, lie on this circle. In this section we shall consider the λ_n as functions on $\partial(\mathrm{H}/W)$. Note that $\partial(\mathrm{H}/W)$ can be naturally viewed as the diagonal circle S^{d} of H, consisting of matrices of the form $\mathrm{e}^{\mathrm{i}\theta}I$.

It is known that the range of λ_n on $\mathrm{U}(2)$ is the same as that of λ_n on H [5], and the range is closely related to the eigenvalues of the Dirichlet boundary condition.

Since H is two dimensional, it is possible to determine this range by restricting λ_n on S^{d}. Let $\Lambda_{n,\kappa} \subset \mathrm{H}$ be the nth level-κ curve on H, i.e. the subset of boundary conditions in H whose nth eigenvalue is κ; see [5]. Then we have the following theorem.

THEOREM 5.1. *The range of λ_n on* U(2) *is the same as that of λ_n on S^{d}. More precisely, S^{d} intersects $\Lambda_{n,\kappa}$ at a unique point.*

Proof. The proof is based on [3, theorem 2.2], where in fact the level curve $\Lambda_{n,\kappa} \subset \mathrm{H}$ is characterized.

In [3], boundary conditions in H are written in the form of (1.2). It is easy to find that the diagonal of H corresponds to α, β satisfying $\alpha + \beta = \pi$. The level curve $\Lambda_{n,\kappa} \subset \mathrm{H}$ can be written as

$$\{(\alpha, \beta) \in [0, \pi) \times (0, \pi] \mid \alpha = f(\beta),\ \beta \in J_0\},$$

where the precise form of the interval $J_0 \subset (0, \pi]$ depends on whether $\kappa > \lambda_{n-1}^{\mathrm{D}}$ or not, and the function f is strictly increasing on J_0. So if S^{d} intersects $\Lambda_{n,\kappa}$, the intersection point is unique. As for the existence of the intersection point, it can be derived easily from the argument of [3]; see figure 1 therein. □

REMARK. It should be pointed out that in [3] there is another 'diagonal' $\mathcal{C}$ in H (see (1.10) in [3]), which is different from ours. $\mathcal{C}$ corresponds to α, β satisfying $\alpha = \beta$, rather than $\alpha + \beta = \pi$. Theorem 5.1 does not hold when S^{d} is replaced by $\mathcal{C}$.

COROLLARY 5.2. *If S^{d} is parametrized by $\beta \in (0, \pi]$ (so $\alpha = \pi - \beta$), then, for each n, λ_n as a function of β is strictly increasing and continuous.*

Proof. That λ_n is continuous can be derived from [2, lemma 2.1]. For $\beta \in (0, \pi)$, the associated quadratic form is

$$Q_\beta(y) = \int_0^1 |y'|^2 \, \mathrm{d}x + \int_0^1 q(x)|y|^2 \, \mathrm{d}x - \cot \beta |\psi|^2, \quad y \in H^1,$$

and the strict monotonicity of λ_n is a conclusion of the min–max principle and theorem 5.1. □

6. Conclusion

In this brief section, we address generalizing the previous results to more general coefficients p and q in (1.1), i.e.

$$1/p,\ q \text{ are Lebesgue integrable on } (0,1) \text{ and } p > 0 \text{ almost everywhere on } [0,1].$$

Our previous investigation was based on the formulation of SA boundary conditions in the form of (2.1), the several characteristic equations, and the min–max characterization of eigenvalues. It can be seen that, for more general p, q as mentioned above, the form of the SA boundary conditions (see [1]) and the several characteristic equations are preserved in the sense that only $\dot{y}$ should be replaced by $p\dot{y}$. The min–max principle also holds in the more general context except that the Sobolev space H^1 should be replaced by a weighted Sobolev space H^1_p whose norm is given by

$$|y|_p^2 = \int_0^1 [p(x)|y'(x)|^2 + |y(x)|^2] \, \mathrm{d}x,$$

and $Q_0(y)$ should be replaced by $\int_0^1 p(x)|y'(x)|^2 \, \mathrm{d}x + \int_0^1 q(x)|y(x)|^2 \, \mathrm{d}x$. Note that the boundary contribution in the quadratic form $Q(y)$ is completely determined by the boundary condition U.

The work in §§ 2 and 3 only depended on the particular form of the boundary conditions, and so results there could be generalized directly. In § 4, when proving that λ_n are real analytic, besides the characteristic equations and the min–max principle, one should also use the additional fact that $y_{1/2}(1, \lambda)$, $\dot{y}_{1/2}(1, \lambda)$ are analytic functions of λ. The corresponding property still holds in the general case; see [10, theorem 2.5.3]. So the discussion in §§ 4 and 5 extends to the more general setting except for lemma 4.8 and theorem 4.9 (These two are based on some estimations that involve the continuity of q and $p \equiv 1$).

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