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OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR DELAY EQUATIONS

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1. **Introduction.** It is the purpose of this paper to establish oscillation criteria for second order nonlinear differential equations with retarded argument. Specifically, we consider the equation

$$y''+f(t, y(t), y(g(t))) = 0 \tag{1.1}$$

where $f \in C[0, +\infty)\times R^2$, $g \in C[0, +\infty)$, and

$$0 < g(t) \leq t, \qquad t > 0, \qquad \lim_{t\to\infty} g(t) = +\infty \tag{1.2}$$

We shall restrict attention to solutions of (1.1) which exist on some ray $[T, +\infty)$. A solution of (1.1) is called oscillatory if it has no largest zero. For a general discussion of existence and uniqueness properties of equations with retarded argument, the reader is referred to El'sgol'ts [1]. Equation (1.1) is considered by Gollwitzer [2] in the form

$$y''+p(t)y_\tau(t)^\gamma = 0 \tag{1.3}$$

where $y_\tau(t)^\gamma \equiv \{y(t-\tau(t))\}^\gamma$ $0<\gamma\neq 1$, and γ is the quotient of odd integers. It is assumed also that $p(t)$ is continuous and eventually nonnegative on $[T, +\infty)$, and that the delay $\tau(t)$ satisfies

$$0 < \tau(t) \leq M, \qquad t \geq T, \tag{1.4}$$

where M is some positive constant.

Oscillation criteria for (1.3) as well as (1.1) may be found in [2], [3], [4], [5], [6], [7], and [8]. As a typical result, Gollwitzer [2] has shown that all solutions of (1.3) with $\gamma>1$ are oscillatory if, and only if,

$$\int^\infty tp(t)\, dt = +\infty. \tag{1.5}$$

If $\tau(t)\equiv 0$ and one restricts attention to continuable solutions of the resulting equation

$$y''+p(t)y^\gamma = 0 \tag{1.6}$$

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then (1.5) is the well-known Atkinson criterion for oscillation of solutions of (1.6), (cf. [7]). The literature concerning the oscillatory behavior of (1.6) and its generalizations is quite extensive, a survey of which may be found in [8]. In fact, the results in the references cited above for equation (1.1) have demonstrated that oscillation criteria for (1) when $g(t)\equiv t$ remain unchanged if one assumes condition (1.4) (i.e., $t-M\leq g(t)<t$ for some constant $M>0$).

Our technique will depend on the fact that, under appropriate conditions on f, solutions of (1.1) are solutions of related ordinary differential inequalities involving the retarded argument. We then may apply the theory of second order differential inequalities (see [12], for example) and thereby obtain oscillation criteria for the original equation (1.1). It will be clear that this technique extends to more general equations than (1.1) in which the function f involves several retardations. As corollaries, we obtain and extend several of the oscillation criteria in [2]–[6].

2. We begin this section with a preliminary lemma.

LEMMA 2.1. *Let* $g(t)$ *satisfy* (1.2) *and assume* $y(t)\in C^{(2)}[T,+\infty)$ *satisfies*

$$y(t)>0,\qquad y'(t)>0,\qquad y''(t)\leq 0\quad \text{on}\quad [T,+\infty). \tag{2.1}$$

Then for each $0<k<1$ *there is a* $T_k\geq T$ *such that*

$$y(g(t))\geq ky(t)\frac{g(t)}{t},\qquad t\geq T_k. \tag{2.2}$$

Proof. It suffices to consider only those t for which $g(t)<t$. Then we have for $t>g(t)\geq T$, $y(t)-y(g(t))\leq y'(g(t))(t-g(t))$ by the mean value theorem and the monotone properties of y'. Hence,

$$\frac{y(t)}{y(g(t))}\leq 1+\frac{y'(g(t))}{y(g(t))}(t-g(t)),\qquad t>g(t)\geq T. \tag{2.3}$$

Also, $y(g(t))\geq y(T)+y'(g(t))(g(t)-T)$ so that for any $0<k<1$ there is a $T_k\geq T$ with

$$\frac{y(g(t))}{y'(g(t))}\geq kg(t),\qquad t\geq T_k \tag{2.4}$$

Hence, using (2.4) in (2.3) we obtain

$$\frac{y(t)}{y(g(t))}\leq\frac{t+(k-1)g(t)}{kg(t)}\leq\frac{t}{kg(t)},\qquad t\geq T_k \tag{2.5}$$

which is (2.2).

Using Lemma 2.1, it is possible to establish several oscillation criteria for (1.1). We begin with a general result for the case when (1.1) is linear of the form

$$y''+p(t)y(g(t))=0 \tag{2.6}$$

where $p(t) \in C[0, +\infty)$.

THEOREM 2.2. *Let* $\mu(t) \equiv g(t)/t$, $p(t) \geq 0$ *for* $t > 0$. *Assume that the equation*

$$y'' + \lambda\mu(t)p(t)y = 0 \tag{2.7}$$

is oscillatory on $[0, +\infty)$ *for some* $0 < \lambda < 1$. *Then all solutions of* (2.6) *are oscillatory.*

Proof. If not, we may assume that $u(t)$ is a nonoscillatory solution of (2.6) with $u(t) > 0$ on $[T, +\infty)$, (since $-u(t)$ is a solution also). By (2.6) we may assume also that $u(g(t)) > 0$ for $t \geq T$ so that $u''(t) \leq 0$ on $[T, +\infty)$. Hence, $u'(t)$ decreases to a limit which must be nonnegative since $u(t)$ is nonoscillatory. In fact, we must have $u'(t) > 0$ on $[T, +\infty)$ for if $u'(t_0) = 0$ for some $t_0 > T$, then $u'(t) \equiv 0$ on $[t_0, +\infty)$ so that from (2.6) we have $p(t) \equiv 0$ on $[t_0, +\infty)$ contradicting the assumption that (2.7) is oscillatory. Hence, applying Lemma 2.1 we see that for $\lambda < \alpha < 1$ there is a $T_\alpha \geq T$ with

$$u''(t) + \alpha\mu(t)p(t)u(t) \leq 0, \qquad t \geq T_\alpha. \tag{2.8}$$

Letting $r(t) = u'(t)/u(t)$ in Theorem 7.2 of [9, p. 362], we conclude by the Sturm comparison theorem that (2.7) is nonoscillatory. This contradiction proves the theorem.

REMARK 2.3. It follows that any oscillation criterion for the second order linear equation $y'' + p(t)y = 0$, where $p(t) \geq 0$ on $[0, +\infty)$, may be immediately extended to an oscillation criterion for (2.6). The analogue of Theorem 2.1 for the case when several retardations are involved is clear. In the case when $t - M \leq g(t) \leq t$ for some $M > 0$ and all large t, $\mu(t) = g(t)/t$ is asymptotic to 1 so that all solutions of (2.6) are oscillatory if

$$y'' + \lambda p(t)y = 0 \tag{2.9}$$

is oscillatory for some $0 < \lambda < 1$. In particular, we have

COROLLARY 2.4. *All solutions of* (2.6) *are oscillatory in case either of the following holds:*

(i) $\int^\infty t^{\alpha-1}g(t)p(t)\,dt = +\infty$ *for some* $0 < \alpha < 1$

(ii) $\liminf_{t\to\infty} t\int_t^\infty \mu(t)p(t)\,dt > \frac{1}{4}$

Proof. If (i) holds, then all solutions of (2.7) are oscillatory for all $\lambda > 0$ (see [10]; also [9, p. 368, Example 7.8]). If (ii) holds, then all solutions of (2.7) are oscillatory for $1 - \lambda > 0$ sufficiently small (see [11]).

COROLLARY 2.5. *All bounded solutions of* (2.6) *are oscillatory in case*

$$\int^\infty g(t)p(t)\,dt = +\infty \tag{2.10}$$

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Proof. Condition (2.10) implies that all bounded solutions of (2.7) are oscillatory for any $\lambda>0$ (see [11], for example). Now if $u(t)$ is a bounded nonoscillatory solution of (2.6) with $u(t)>0$ and $u(g(t))>0$, $t\geq T$, then (2.8) holds for $t\geq T_\alpha\geq T$, $0<\alpha<1$. Applying Theorem 7.4 of [12] (with $\alpha(t)\equiv u(T)\leq u(t)\equiv\beta(t)$), we conclude the existence of a solution $y(t)$ of (2.7) (with $\alpha=\lambda$) satisfying $0<u(T)\leq y(t)\leq u(t)$ on $[T_\alpha, +\infty)$. This is a contradiction and proves the theorem.

REMARK 2.6. Corollary 2.4 (i) improves and extends a result of Bradley [3, Theorem 1].

3. In this section we shall consider equation (1.1) and obtain some improvements of results in [2]–[6]. Throughout this section we shall assume $f(t, u, v)$ satisfies

$$f(t, u, v) = -f(t, -u, -v) \quad \text{all} \quad t, u, v \tag{3.1}$$

and

$$f(t, u, v) > 0, \qquad u, v > 0 \quad \text{and all} \quad t \geq 0 \tag{3.2}$$

and that for each fixed t and $u>0$, $f(t, u, v)$ is nondecreasing in v for $v>0$ and that for each fixed t and $v>0$, $f(t, u, v)$ is nondecreasing in u for $u>0$.

THEOREM 3.1. *All bounded solutions of* (1.1) *are oscillatory in case*

$$\left| \int^\infty tf(t, \alpha, \alpha\mu(t))\, dt \right| = +\infty \tag{3.3}$$

for all $\alpha\neq 0$, *where* $\mu(t)$ *is as in Theorem* 2.2.

Proof. If not, let $u(t)$ be a bounded nonoscillatory solution of (1.1) which we may assume satisfies

$$u(t) > 0, \qquad u(g(t)) > 0, \qquad u'(t) > 0, \qquad u''(t) \leq 0, \qquad t \geq T \tag{3.4}$$

Hence, by Lemma 2.1, for $0<k<1$, we have

$$u''(t)+f(t, u(t), k\mu(t)u(t)) \leq 0, \qquad t \geq T_k \tag{3.5}$$

by the monotonicity assumption on f. Applying Theorem 7.4 of [12] with $\alpha(t)\equiv u(T_k)\leq u(t)\equiv\beta(t)$ we obtain the existence of a solution $y(t)$ of

$$y''+f(t, y, k\mu(t)y) = 0 \tag{3.6}$$

with $u(T_k)\leq y(t)\leq u(t)$ on $[T_k, \infty)$. But then by Theorem 3 of [8] it follows that

$$\left| \int^\infty tf(t, \alpha, k\mu(t)\alpha)\, dt \right| < +\infty$$

for some constant $\alpha\neq 0$. This contradicts (3.3) by the monotonicity assumption of f.

The next theorem shows the converse of Theorem 3.1 is true under an additional assumption.

THEOREM 3.2. *Assume that the following condition holds:*

$$\liminf_{t\to\infty} \mu(t) \geq \rho > 0 \quad \textit{for some} \quad \rho > 0. \tag{3.7}$$

Then (1.1) *has a bounded nonoscillatory solution if, and only if,*

$$\left| \int^{\infty} tf(t, \alpha, \alpha)\, dt \right| < +\infty \tag{3.8}$$

for some $\alpha \neq 0$.

Proof. Theorem 3.1, (3.7), and the monotonicity assumptions show the necessity of (3.8). If (3.8) holds, assume to be specific that $\alpha > 0$ and let $0 < \beta < \alpha$. Choose $T > 0$ so that

$$\int_T^{\infty} sf(s, \beta, \beta)\, ds < \beta/2$$

Then defining

$$\begin{aligned} y_0(t) &\equiv \beta, \qquad t \geq 0 \\ y_{n+1}(t) &= \beta - \int_T^{\infty} (s-T) f(s, y_n(s), y_n(g(s)))\, ds, \qquad t < T \\ &= \beta - \int_t^{\infty} (s-t) f(s, y_n(s), y_n(g(s)))\, ds, \qquad t \geq T \end{aligned}$$

it follows by induction that $\beta/2 \leq y_n(t) \leq \beta$, $t \geq T$ and all $n \geq 0$. Furthermore the sequence $\{y_n'(t)\}_{n=0}^{\infty}$ is bounded uniformly on $t \geq T$. Therefore, the Ascoli-Arzela theorem along with a standard diagonalization argument yields a subsequence of $\{y_n(t)\}_{n=0}^{\infty}$ which converges, uniformly on compact subintervals of $[T, +\infty)$, to a solution $y(t)$ of (1.1) satisfying $\beta/2 \leq y(t) \leq \beta$, $t \geq T$. This proves the theorem.

We shall note later (see Remark 3.5 below) that the converse of Theorem 3.1 is not true. To extend Theorems 3.1 and 3.2 to unbounded solutions, let $\phi(u)$ be a nondecreasing continuous function of u satisfying $u\phi(u) > 0$, $u \neq 0$ with

$$\int_{\pm 1}^{\pm\infty} \frac{du}{\phi(u)} < +\infty.$$

We will say that $f(t, u, v)$ satisfies condition (A) provided there exists a $c \neq 0$ and $0 < \alpha < 1$ such that

$$\liminf_{|u|\to+\infty} \frac{f(t, u, \alpha\mu(t)u)}{\phi(u)} \geq k\, |f(t, c, \alpha\mu(t)c)| \tag{3.9}$$

for some positive constant k and all $t \geq T$.

We may now prove the following result:

THEOREM 3.3. *Assume f satisfies condition* (*A*). *Then all solutions of* (1.1) *are oscillatory in case* (3.3) *holds for all* $\alpha \neq 0$. *In addition, if* (3.7) *holds, then* (3.3) *is also necessary.*

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Proof. Assume (3.3) holds for all $\alpha\neq 0$ and let $u(t)$ be a nonoscillatory solution of (1.1) with $u(t)>0$, $u(g(t))>0$ for $t\geq T$. As in Theorem 3.1, given $0<\alpha<1$ as in condition (A) there is a $T_\alpha\geq T$ such that

$$u''(t)+f(t, u(t), \alpha\mu(t)u(t)) \leq 0, \qquad t \geq T_\alpha, \tag{3.10}$$

Hence, we obtain a solution $y(t)$ of

$$y''(t)+f(t, y(t), \alpha\mu(t)y(t)) = 0 \tag{3.11}$$

with $0<u(T_\alpha)\leq y(t)\leq u(t)$, $t\geq T_\alpha$, by Theorem 7.4 of [12]. But now an application of Theorem 4 of [8] yields the desired contradiction.

Conversely, if (3.7) holds and (3.3) does not hold for some $\alpha\neq 0$, then by the monotonicity assumption (3.8) must hold for some $\alpha\neq 0$ which gives a nonoscillatory solution of (1.1) by Theorem 3.2.

As corollaries of these results, we obtain and extend the results of Gollwitzer [2] for the equation

$$y''(t)+p(t)(y(g(t)))^\gamma = 0 \tag{3.12}$$

where $p(t)\geq 0$ on $[T, +\infty)$ and $\gamma>1$ is the quotient of odd integers.

COROLLARY 3.4. *All solutions of* (3.12) *are oscillatory provided*

$$\int^\infty t^{1-\gamma}p(t)(g(t))^\gamma\, dt = +\infty \tag{3.13}$$

The converse is true in case (3.7) *holds.*

REMARK 3.5. Theorem 3.1 shows that (1.5) is a necessary condition for all solutions of (3.12) to oscillate, in the case $\gamma>1$, with just the assumption (1.2) on $g(t)$. However, (1.5) is no longer sufficient as the following example demonstrates:

Let $p(t)=t^\alpha/4$, $g(t)=t^{1/2}$, where $\alpha=-3/2-\gamma/4$ with $1<\gamma<2$, and γ the quotient of odd integers. For this example, $y(t)=t^{1/2}$ is a nonoscillatory solution but $\int^\infty tp(t)\, dt=+\infty$.

We shall now prove an extension of a second theorem of Gollwitzer [2] for equation (3.12) in the case $0<\gamma<1$, which is, in turn, a generalization of a result of Belohorec [13] which states that if $g(t)=t$, $0<\gamma<1$, then all solutions of (3.12) are oscillatory if, and only if, $\int^\infty t^\gamma p(t)\, dt=+\infty$. Gollwitzer shows this theorem remains true if $t-M\leq g(t)\leq t$ for some constant $M>0$.

THEOREM 3.6. *Let* $0<\gamma<1$ *in equation* (3.12). *Then all solutions of* (3.12) *are oscillatory if, and only if,*

$$\int^\infty (g(t))^\gamma p(t)\, dt = +\infty \tag{3.14}$$

Proof. Let $u(t)$ be a nonoscillatory solution of (3.12) with $u(t)>0$, $u(g(t))>0$ on $[T, +\infty)$. Then for any $0<k<1$, arguing as in Theorem 3.1, we obtain a solution $y(t)$ of

$$y''(t)+kp(t)(\mu(t))^{\gamma}(y(t))^{\gamma} = 0, \tag{3.15}$$

with $u(T_k)\leq y(t)\leq u(t)$ on $t\geq T_k\geq T$. Then by Theorem 1 of [13] it follows that $\int^{\infty} (g(t))^{\gamma}p(t)\, dt<+\infty$.

Conversely, if $\int^{\infty} p(t)(g(t))^{\gamma}\, dt<+\infty$ we may use an argument similar to that of [13] (see also [2]) to show the existence of a nonoscillatory solution $y(t)$ with $\lim y(t)/t=\alpha>0$. We omit the details.

REMARK 3.7. Ladas [6] has recently shown that an analogue of (3.14) for the more general equation (1.1) is necessary for all solutions to be oscillatory. More precisely, it is shown that if all solutions of (1.1) are oscillatory, then

$$\int^{\infty} f(t, kt, kg(t))\, dt = +\infty \quad \text{for all} \quad k > 0.$$

Obviously, this condition is not sufficient as the Euler equation demonstrates (i.e., Equation (2.6) with $g(t)=t$, $p(t)=t^2/4$).

We now give a sufficient condition for oscillation of all solutions of (1.1) based on a comparison theorem for nonlinear differential equations (see [12]).

THEOREM 3.8. *Let the partial derivative functions f_u, f_v be continuous and nonnegative on* $[0, +\infty)\times R^2$ *and assume f_u and f_v are nondecreasing in u and v for $u, v>0$. Then all solutions of* (1.1) *are oscillatory in case the linear equation*

$$z''+[f_u(t, \alpha, k\alpha\mu(t))+k\mu(t)f_v(t, \alpha, k\alpha\mu(t))]z = 0 \tag{3.16}$$

is oscillatory for all $\alpha\neq 0$ and some $0<k<1$.

Proof. If $u(t)$ is a nonoscillatory solution of (1.1) with $u(t)>0$, $u(g(t))>0$ on $[T, +\infty)$, then for any $0<k<1$. Lemma 2.1 shows that

$$u''(t)+f(t, u(t), k\mu(t)u(t)) \leq 0, \qquad t \geq T_k.$$

Then applying Theorem 7.7 and Theorem 7.8 of [12] we conclude that the equation

$$y''(t)+f(t, y(t), k\mu(t)y(t)) = 0 \tag{3.17}$$

has a solution $y_n(t)\in C^2[T_k, T_{k+n}]$ for each $n\geq 1$ satisfying $u(T_k)\leq y_n(t)\leq u(t)$ on $[T_k, T_{k+n}]$ and such that the variational equation of (3.17) with respect to $y_n(t)$,

$$z''+[f_u(t, y_n(t), k\mu(t)y_n(t))+k\mu(t)f_v(t, y_n(t), k\mu(t)y_n(t))]z = 0. \tag{3.18}$$

is disconjugate on (T_k, T_{k+n}). Hence, by the monotonicity assumptions and the Sturm comparison theorem, it follows that (3.16) with $\alpha=u(T_k)$ is disconjugate on (T_k, T_{k+n}) for all $n\geq 1$. This contradiction proves the theorem.

As a simple example which extends Corollary 3.4 and is the analogue of a result of Jones [14], we have

COROLLARY 3.9. *Let $\gamma_i>1$, $1\leq i\leq n$, be the quotient of odd integers, let $g_i(t)$ satisfy* (1.2) *and assume $p_i(t)$ are continuous and nonnegative for $t\geq T$, $1\leq i\leq n$. Define $\mu_i(t)\equiv g_i(t)/t$, $1\leq i\leq n$. Then all solutions of*

$$y''+\sum_{i=1}^{n} p_i(t)(y(g_i(t)))^{\gamma_i}=0$$

are oscillatory provided

$$\sum_{i=1}^{n}\int^{\infty} t(\mu_i(t))^{\gamma_i}p_i(t)=+\infty$$

The converse is true in case $\mu_i(t)$ satisfies (3.7) *for $1\leq i\leq n$.*

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