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NOTES ON OPTIONS, HEDGING, PRUDENTIAL RESERVES AND FAIR VALUES

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ABSTRACT

In this paper we present many investigations into the results of simulating the process of hedging a vanilla option at discrete times. We consider mainly a 'maxi' option (paying Max(A, B)), though calls, puts and 'minis' are also considered. We show the sensitivity of the variability of the hedging error to the actual investment strategy adopted, and to the many ways in which the simulated real world can diverge from the assumed option pricing model. We show how prudential reserves can be calculated, using conditional tail expectations, and how net premiums or fair values (which we present as the same) can be calculated, allowing for the necessary prudential reserves. We use two bond models, the very simple Black-Scholes one and a less unrealistic one. We also use the Wilkie model as an even more realistic real-world model, allowing for many complications in it to make it more realistic. We make observations on the important difference between real-world models and option pricing models, and emphasise the latter as the way of getting hedging quantities, and not just option prices.

KEYWORDS

Options; Hedging; Contingency Reserves; Quantile Reserves; Conditional Tail Expectations; Prudential Reserves; Shareholder Value; Fair Value; Investment Strategies; Hedging Errors; Transaction Costs; Practicability of Hedging; Fat-tailed Innovations; Stochastic Hypermodels; Stochastic Bridges; Brownian Bridges; Ornstein-Uhlenbeck Bridges

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1. Introduction, Summary and Conclusions

1.1 *Origins*

1.1.1 This paper has three sources. In Wilkie, Waters & Yang (2003) (referred to as WWY) (discussed at the Faculty in January 2003) we stated (¶C.12.3) that the results that had been found were "in contrast with results that we have found elsewhere for other types of option''. We present here the quite extensive results to which we referred, which relate to 'vanilla' options, i.e. ordinary calls and puts (and also 'maxi' and 'mini' options).

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1.1.2 Secondly, in the discussion at the Institute of Actuaries in November 2003 on Dullaway & Needleman (2004) one of us commented that the United Kingdom regulators appeared to be going in the wrong direction, in encouraging complicated option pricing methods with simple scenario tests. Our ideas on this subject, outlined in WWY, need amplification. In essence, we favour simple option pricing models which allow the hedging quantities to be calculated, and complicated real-world models in which empirical hedging, as we describe here, is investigated.

1.1.3 Thirdly, at the discussion at the Faculty in November 2003 (Faculty of Actuaries, 2004) on 'Asset models in life assurance; views from the Stochastic Accreditation Working Party', John Jenkins asked a question about how management actions (other than exact hedging) could be allowed for. In Section 10 we show how to provide an answer to his question.

1.2 *Outline: The Basic Model*

1.2.1 Section 2 describes the methodology we use. We define the options that we consider, 'vanilla' calls and puts, and also maxis and minis (which we defined in WWY); we consider only European options. We introduce the stochastic models, simple Brownian motion for a share (total return) index, and two models for the bond price. One, as in Black-Scholes, assumes constant interest rates; the other, as described in WWY, assumes a Vasicek-style model for the interest rate on a zero coupon bond maturing at the same time as the option. These models lead to similar formulae for the option price and the hedging quantities, the amounts to be invested in the share and the zero coupon bond in order to hedge the option.

1.2.2 We then explain the process of discrete hedging, describing various options about how, in practice, to invest when (inevitably) the proceeds after some discrete time do not match exactly what the theory requires. We show how we calculate the deficit (or surplus) at expiry, and discuss the options for discounting this to get a present value at the start of the option. We do not discount at some abstract and artificial rate, but we assume that the necessary contingency reserves to meet the deficit are invested in some practical way in one of our available assets. We calculate quantile reserves at various probability levels, such as 97.5%, 99%, etc., and we also calculate conditional tail expectations (CTEs) at the same levels. For the reasons described in WWY (Section 9), we prefer, and use, CTEs throughout.

1.2.3 We assume that the regulator follows our approach and lays down some level of practical prudential reserve in terms of a quantile reserve or CTE at a defined probability level. Alternatively, the management of the institution writing the options decides that some security level higher than the supervisor requires is desirable, and we are trying to meet that. We often assume that the option writer is a life office, with options embedded in its policies; but our methodology would apply to any institution writing options.

1.2.4 We then consider how the policy is financed. The first step is to calculate the pure option price, and assume that that has been invested in one of the ways we have considered. We also assume that the full prudential reserve is to be set up. Since the 'shareholders' will get the benefit of any surplus after the option has been paid off, it is to be expected that they will provide the bulk of the extra amount to set up the prudential reserve. However, as described in WWY (Section 8.2), we assume that the shareholders require an extra 'risk premium' on their investment. Therefore, we calculate the surplus or deficit *vis-à-vis* the prudential reserve, discounting surpluses at a small positive extra rate of interest, and (the quite rare) deficits at a small negative extra rate, to get the value of the policy to the shareholder. We assume that the policyholder pays the rest of what is required.

1.2.5 We then assert that the policyholder's 'net premium', calculated in this way (net because expenses and commission are excluded), is the same as the 'fair value' at which one life office would transfer the liability to another. This is the first important point in the paper.

1.2.6 In Section 3 we describe practical hedging with bond model A (Black-Scholes), assuming first that the real-world model is the same as that used to calculate the option price and hedging quantities. We show in detail how our calculations are carried out, and we demonstrate that, if hedging is reasonably frequent (we use hedging twice a month as our standard), our assumed investments do, in fact, replicate closely the required option pay-off, but do not replicate it exactly. We find that investing to match the required hedging quantity in the share, and investing the contingency reserve in the bond, gives generally good results; but this is not the only good strategy. In WWY we found that investing both in proportion to the required hedging quantities was best. We also calculate the CTEs, i.e. the prudential reserves, and the net premiums, i.e. the fair values.

1.3 *Outline: Variations on the Basic Model*

1.3.1 In Sections 4 to 7 we assume a great many variations on this basic model. First, we investigate different frequencies for hedging, from yearly down to 1,024 steps per month. The more frequent the hedging the smaller the hedging error. We also investigate modelling the share price movements, by simulating at annual intervals and then inserting intermediate steps by Brownian bridges. The results are quite similar.

1.3.2 In Section 5 we vary the parameters. First, we see the effect of the exercise price. In our basic model the option was nearly 'at the money'. Different exercise prices affect the initial option price greatly. They also affect the hedging error a bit. When the exercise price is increased, the standard deviation of the hedging error also increases somewhat. Next, we vary the mean return on shares; this parameter enters the real-world model, though it does not affect the option price. We see that it has a quite small effect on the hedging error, as we expected. Varying the mean return on cash, and therefore also the initial return, has a big effect; if we set the interest rate wrongly the option is mispriced; but current interest rates are always known, so there is no reason to misestimate this factor. When we vary the standard deviation on shares, the option is also mispriced, and we have the expected results: if the standard deviation is higher than we have allowed for we make losses, if smaller we make profits; but the variability of the hedging error increases in both directions.

1.3.3 So far we have ignored transaction costs. We investigate the effect of these in Section 6. We allow for transaction costs at two levels, high and low, and see what effect this has on the preferred hedging frequency. As the frequency increases, so do the transaction costs, though the hedging error decreases. There is an optimum frequency. With the high level of costs, the optimum is hedging once or twice a month. With the lower level, the optimum is around eight to 16 times per month. Note that we consider here only one option; with a portfolio of options, some of which might offset others, the net hedging required might be much smaller, and the transaction costs correspondingly reduced.

1.3.4 In Section 7 we vary the structure of the real-world model. Up to now we have assumed normally distributed innovations, corresponding with the Brownian motion assumed in the option pricing. In reality, almost all investment variables show high kurtosis, or 'fat tails'. We now allow the real-world model to have fat-tailed innovations, simulated as the difference between two independent log-normally distributed variables. The hedging error increases as the kurtosis increases. We next allow for uncertainty in the parameters of the real-world model, by selecting the parameters (in this case the mean and standard deviation of the return on shares) for each simulation from a multivariate normal distribution. We describe this as a 'hypermodel'. In general, the variability of the hedging error is increased. We 'hyperise' the parameters for the fat-tailed innovations too. This again increases the variability of the hedging error. If we include all the complications, the required prudential reserve is greatly increased, and the fair value/fair price ratio is also increased. Including transaction costs as well increases them all yet more.

1.4 *Outline: Other Models*

1.4.1 In Section 8 we move on to bond model B, in which the interest rate on the zero coupon bond is stochastic, rather than constant, and has a negative instantaneous correlation with share prices (so bond prices and share prices have positive correlation). The option price under this model, with the correlation coefficient we assume (-0.3), is very slightly less than the option price under the Black-Scholes model (because of this correlation), and the hedging quantities are very similar. We use bond model B as the real-world model, and carry out investigations similar to those that we carried out for the Black-Scholes model. Different hedging strategies show the same type of result as for the Black-Scholes model, so that investing correctly in the share quantity, letting the balance go into the bond, and discounting deficits at the bond yield is satisfactory. The numerical results are quite similar to those for bond model A.

1.4.2 We vary the parameters of the bond model. The mean interest rate and the autoregressive parameter of the Ornstein-Uhlenbeck process for the interest rate have rather little effect on the hedging error. The initial rate of interest, the standard deviation and the correlation coefficient can have a considerable effect. We investigate the effect of fat-tailed distributions for the innovations of the bond process. These, too, make a difference. We investigate, using a hypermodel for the parameters of the bond process; this makes rather little difference to the variability of the hedging error.

1.4.3 On the whole, the bond model has less effect on the hedging error than the share model has. This is not too surprising. Even when the interest rate fluctuates, the initial and final bond prices are always the same, both from one simulation to another, and between the two bond models.

1.4.4 In Section 9 we move on to represent the real world by the Wilkie model, with stochastic bridges (Brownian and Ornstein-Uhlenbeck) to simulate share prices and interest rates at intermediate points through the year (as in WWY). We use bond model B for the hedging strategy. The simplest form of the Wilkie model has fixed parameters and normally distributed innovations, both in the annual model and in the bridging model. We allow fat-tailed innovations in the bridging model, and hyperise each stage: the bridging parameters, the bridging innovations, and the annual parameters. We omit fat-tailed innovations (fixed or hyperised) for the annual model. As we put in more complications and more uncertainty, the standard deviation of the hedging error and the CTEs all increase.

1.4.5 In WWY we speculated (¶10.3.10) that the annual part of the Wilkie model might have rather little effect on the hedging error, whereas the bridging model would have a big effect. The latter is true, but we find that the annual model is not unimportant, when hedging is discrete. We try varying the parameters of the annual model, first doubling, then halving all the standard deviations in it, then hyperising all its parameters. The effects are not always what one might expect, but they are noticeable.

1.5 *Outline: Approximate Hedging*

In Section 10 we investigate the suggestion made by John Jenkins that an approximate hedging strategy might be of interest. We let the hedging proportions depend linearly between zero and unity on the 'moneyness' (the ratio of the current share price to the current value of the exercise price) or on the logarithm of the moneyness. We find that, if we assume that we know the real-world model exactly, as in the Black-Scholes model, failure to hedge exactly gives large hedging errors, so requires a large 'mismatching reserve'. However, if the real-world model is less close to the option model, as in our version of the Wilkie model with hyperised parameters and fat-tailed innovations, then the mismatching reserve, though not insignificant, is less than in the first case.

1.6 *Models*

1.6.1 We use several models in our investigations. We make a large and, we hope, clear distinction between real-world models and option pricing models. The real-world models are used in the simulations, and are intended to represent, to a greater or lesser degree, the behaviour of the real investment markets. We assume investment only in assets that behave according to the real-world model under consideration. The option pricing models are used only to calculate the initial option price and the hedging quantities at each rehedging point. We never use them for simulation (though, of course, they could be so used).

1.6.2 In some cases the real-world model is the equivalent of the option pricing model, as in what we describe as our basic model. To simulate hedging in the real-world equivalent of a chosen option pricing model, and to confirm that the resulting hedging errors are small, gives a useful check on our mathematical derivation of the hedging quantities (and on our programming). It is a standard against which we can compare more realistic real-world models.

1.6.3 The process according to which investment variables actually behave in the real world, if indeed there is one, is unknown to any of us. Any proposed real-world model is, therefore, only an approximation to reality. It is difficult ever to confirm that a proposed real-world model is 'correct', though it is not difficult to show that some models that might be proposed do not fit the evidence. The real world has so many obvious uncertainties and complications that we believe a model which reflects many of these features is better than one which does not. Therefore, we prefer to use a model like the Wilkie model, with fat-tailed innovations, and with hyperised parameters, rather than to use a model which does not have these features. We use the Wilkie model because it is available, published and well-known in actuarial circles; but, so far, it lacks a fully stochastic yield curve model, which would have been useful in our investigations. Nevertheless, the two-parameter yield structure in it (consols yield and base rate) seems much more realistic than any of the one-factor models that are sometimes used.

1.6.4 The requirements for an option pricing model are quite different from those of a real-world model. It needs to provide an initial option price and, most importantly, the hedging quantities at each step. It is, therefore, desirable for it to be mathematically tractable, so that the required quantities can be calculated easily at each step of each (real-world) simulation. This usually means that the option model should be based on Brownian motions, and that the relevant benefits, often transformed in terms of some numeraire, are lognormally distributed.

1.6.5 Some option benefits, however, do not lend themselves to such modelling. Examples would be a unit-linked policy where the final unit value is guaranteed to be not less than the average unit price each month for the last 36 months of the policy, or (as is currently on the market) a bond that provides 70% of the increase in the FTSE 100 index over the next five years, but where the price at maturity is taken as the daily average over the last 12 months of the policy. One approach is to simulate the option proceeds in a risk neutral or equivalent martingale model. This can provide an estimate of the option value, but not the hedging quantities immediately; these could often be calculated by repeated simulations and finite difference methods; but all this, requiring option simulations at each step within the real-world simulations, would take time. If sufficient computer power is readily available, perhaps on a large network of PCs, this may not matter. Our approach, instead, would be to investigate whether the benefits, which, in each of our examples, resemble an arithmetic average option, could be related to the price and hedging quantities of a geometric average option, which is much more tractable. However, we have not investigated this.

1.6.6 There are some who suggest using deflator models. While the deflators for both our bond models can be computed, that for bond model B is cumbersome. There seems little advantage in using deflators within simulations, necessarily based on the real-world equivalent of the option model (or some other model probabilistically equivalent to the option model), rather than on the desired realistic real-world model, in order to calculate option prices that can be calculated analytically far faster; and again, the hedging quantities do not appear from a deflator model immediately, though repeated simulations and finite difference methods might provide an answer.

1.7 *Progress Through the Contract*

1.7.1 All the calculations in this paper are carried out as if at the start of a ten-year contract. We do not consider what happens as we progress through the term of the contract. As events unfold, the probability of the assets being sufficient to meet the required payoff changes. What do we then do? One approach is the 'band' method, suggested by the Maturity Guarantees Working Party (Ford *et al.*, 1980), another is 'marking to market'; yet another would be to ignore the unfolding events, on the grounds that the initial probability remains unchanged and was chosen initially with suitable strength. One long-established actuarial principle is that a contract should be financed initially in such a way that there is little chance of further capital being needed, and the idea of marking to market is not compatible with this principle. However, the regulators are likely to wish to see the office demonstrate sufficient strength at each relevant time, whether annually or more frequently.

1.7.2 An approach, which we have not yet implemented, is to assume that the management wishes to set up an initial reserve sufficient to meet, not only the final proceeds with a suitably high probability, but also the supervisor's requirements throughout the duration of the contract. To be specific, imagine that the regulator requires a 95% level of solvency at each annual investigation, which must be calculated as the 95% quantile reserve using 1,000 simulations. Imagine, then, that the management wishes initially a 99% CTE, calculated with 10,000 simulations, to ensure that no further capital will be required during the term of the contract. The bases used by regulator and management could differ; the numbers of simulations and the probability levels could differ; each could require either quantiles or CTEs.

1.7.3 We start with some initial assumed reserve, perhaps the 99% CTE from 10,000 simulations carried out without considering intermediate points. We assume that this amount is invested in some chosen way. We then start a new set of 10,000 simulations (or the same set again), calculated using the management's basis. At each annual point within each simulation we carry out 1,000 simulations, calculated on the supervisor's basis, and see whether the proceeds of our investment are sufficient to satisfy the supervisor's 95% quantile reserve. If they are not we record the deficit, and discount at, say, the bond yield. The maximum discounted deficit over the number of intervening annual points gives the extra initial reserve required within that simulation to meet the supervisor's requirements. We then choose the 99% CTE to satisfy the management's requirements.

1.7.4 Such simulations within simulations would take much more computer time than the 10,000 simulations we have carried out, which are all quite quick (less than one minute with bond models A or B and with hedging twice a month, but about seven minutes with the Wilkie model). However, if we multiply these times by 8,000 or 9,000 (1,000 simulations at each of eight or nine intermediate annual points, perhaps including the first), the computer time required becomes significant. On the other hand, within our program, we calculate results for all four options and each possible investment strategy, so, if the calculations were reduced to only what is required for one combination, they could be speeded up.

1.8 *With-Profits Business*

1.8.1 All our examples assume the equivalent of a unit-linked contract, where the benefits are defined in terms of the value of some share index or portfolio. Many real life insurance contracts are with-profits. We now consider a mutual office; a '90/10' office can be treated as 90% mutual, 10% financed by shareholders. We would deal with with-profits contracts in the first place by considering the guaranteed sums assured and bonus, along with any options inherent in these, such as future bonus guarantees; we would replace our 'share index' by whatever asset portfolio we wished to target. The initial prudential reserve, which in our model is provided substantially by shareholders, needs to be included in the initial with-profits premium as a 'bonus loading'. This bonus loading remains unallocated unless circumstances have so improved that part of it can be released and declared as a guaranteed bonus. At maturity, the unused part of the bonus loading can be returned to the policyholder as terminal bonus.

1.8.2 It may be helpful to be specific. In our examples we assume a single premium contract with an investment premium of £100, which is assumed to be invested in 'units' of some share index. We assume an exercise price of £165 (Table 3.2.1), which corresponds to a guarantee of about 5% (continuous) interest on the £100 for our ten-year term contract. The extra premium for the option in our bond model B is £24.17 (Table 8.1.2). The 99% CTE reserve in our most realistic model (Wilkie hypermodel, variation (6), Table 9.3.2) is £30.96. We assume that this is made up by contributions of £25.46 from the shareholder, and £5.49 from the policyholder (Table 9.3.4). However, in a mutual office the policyholder has to contribute all of this, so his initial premium (ignoring expenses) is £155.13. The 'sum assured' is the guaranteed £165. The initial investment in shares is £62.01 (Table 8.1.2), all the rest being invested in the matching bond. A reversionary bonus only arises if the shares do much better than the 5% bond rate. However, at maturity the CTE reserve would have grown by $\exp(0.05 \times 10) = 1.65$ to about £51, and would be available either to meet the hedging error or as terminal bonus. In very many cases out of the 10,000 the terminal bonus would be substantial, because the average hedging error is close to zero. These seem not unreasonable numbers, and they can be derived easily from our standard calculations.

1.8.3 However, any 'asset share', derived from the accumulation of the whole initial premium, does not form part of these calculations. If it were to become a requirement that the whole (net) premium be treated as if invested in 'shares' and the resulting asset share be then guaranteed, then only unit-linked business with no minimum money guarantee is possible. The existence of guarantees requires both hedging of the investments and (usually) significant contingency reserves. In a with-profits office, these reserves are provided by the bonus loading, and can be returned as terminal bonus to the extent that they are not required.

1.9 *Final Observations*

1.9.1 We have no concluding section to the paper. Instead, we have put our concluding remarks into this introductory section, in the hope that those who find the extensive details that follow rather tedious may nevertheless read this section. We have not covered everything that we have thought of; no authors do. Many of our results are of the sort that others have discovered before, perhaps some time ago, or even that postgraduate students of options may have covered as exercises; but we have not seen such a comprehensive investigation published before, and certainly such an investigation has not been put before the actuarial profession.

1.9.2 Others have certainly studied some aspects of what we have done. Boyle & Emmanuel (1980) show that, in the Black-Scholes model, the one-step hedging error, suitably scaled, is distributed as chi-squared with one degree of freedom. The sum of the errors, though they have different scaling factors, is distributed in a way similar to chi-squared with many degrees of freedom, or roughly normally.

1.9.3 Wilmott (1998) shows that, in the Black-Scholes model, one can improve the hedging error when hedging discretely by adjusting the option price and the hedging quantities. We have not experimented with this, but, even if this improved the position in the basic situation, when the real world is the equivalent of the option model, when we move on to try out a real world with its realistic complications, the improvement might perhaps be found to be comparatively small.

1.9.4 Willmott (1998) also shows how the option price can be adjusted for transaction costs, again in the Black-Scholes model, and if transaction costs are strictly proportionate to the size of the deal with the same rates for all cases. This deserves further investigation. However, we allow for transaction costs varying for buying and selling, long and short positions, and our method could allow for transactions costs being non-linear, for example a constant if any transaction takes place, plus a proportionate charge.

1.9.5 Other authors have dealt with some aspects of discrete hedging. So far as we are aware, none has considered all the aspects that we have considered here.

2. Basic Principles

2.1 *The Options*

2.1.1 We consider only 'vanilla' options, the familiar call and put options, on shares for cash, and also what in WWY were defined as 'maxi' and 'mini' options. Many life insurance embedded options are of the maxi type.

2.1.2 If the (European) option expires at time T, the exercise price is K, and shares at time T have value S(T), then the payoffs of these four options are:

Maxi	$\mathrm{Max}(S(T), K)$
Mini	$\mathrm{Min}(S(T), K)$
Call	$\mathrm{Max}(0, S(T) - K)$
Put	$\mathrm{Max}(0, K - S(T))$.

2.1.3 The payoffs are such that we readily see the 'put-call parity' equation:

$$\text{Cash} + \text{Call} = \text{Share} + \text{Put}$$

but also:

$$\text{Cash} + \text{Call} = \text{Share} + \text{Put} = \text{Maxi}$$

and

$$\text{Cash} + \text{Share} = \text{Maxi} + \text{Mini}$$

or

$$\text{Mini} = \text{Share} - \text{Call} = \text{Cash} - \text{Put}$$

which is the payoff of a covered writer of a call or a put.

2.1.4 For pricing the options (which will be seen to be of secondary importance) and for calculating the hedging quantities (which is very important), we use first the usual Black-Scholes methodology, in which there are two tradeable assets: a unit fund, which we refer to as a 'share' without specifying how it is invested; and a bond which provides some default free return up to time T. The price of the former at time t is $S(t)$ and of the latter is $B(t)$.

2.1.5 We assume that the share price $S(t)$ is driven by the stochastic differential equation:

$$dS(t) = \mu_S().S(t).dt + \sigma_S.S(t).dW_1$$

where W_1 is a Wiener process, σ_S is a constant, and $\mu_S()$ is some function of t and $S(t)$. Usually we assume that $\mu_S() = \mu_S$, so that the logarithm of the share price performs a random walk with constant drift.

2.1.6 We investigate two different assumptions about the 'bond' price process.

2.1.7 First, in model A, we make the usual 'Black-Scholes' assumption that the cash yield (often called the 'risk free rate') is constant, so that the price of the 'bond' $B(t)$ changes deterministically in accordance with:

$$dB(t) = r.B(t).dt$$

where r is constant. We then have:

$$B(t) = \exp(-r.(T - t)) = B(0).\exp(r.t), \text{ since } B(0) = \exp(-r.T).$$

2.1.8 Secondly, in model B we make the same assumptions as WWY, that $B(t)$ is a zero coupon bond (zcb) maturing at T, whose price is driven by a single bond interest rate $R(t)$ (applicable to maturity at time T), which has the stochastic differential equation (very like the Vasicek model for short rates):

$$dR(t) = \mu_R().dt + \sigma_R.dW_2$$

where σ_R is a constant, $\mu_R()$ is some function of t and $R(t)$, and dW_1 and dW_2 have instantaneous correlation coefficient ρ. Then the price of the bond, $B(t)$, is given by:

$$B(t) = \exp(-(T - t).R(t)).$$

If $\mu_R() = 0$, $\sigma_R = 0$ and $R(0) = r$, this collapses to model A.

2.1.9 We then put the value of an investment in the bond which will provide K at time T as $K(t) = K.B(t)$, whence $K(T) = K$, since $B(T) = 1$.

2.1.10 In each case the option value and the hedging quantities at time t ($< T$) are given by the usual Black-Scholes formulae (see e.g. Baxter & Rennie, 1996), which we have slightly transformed. They are:

	Share quantity $H_S(t)$	Cash quantity $H_B(t)$
Maxi	$S(t).N(d_1)$	$K(t).N(d_2)$
Mini	$S(t).(1 - N(d_1)) = S(t).N(-d_1)$	$K(t).(1 - N(d_2)) = K(t).N(-d_2)$
Call	$S(t).N(d_1)$	$K(t).(N(d_2) - 1) = -K(t).N(-d_2)$
Put	$S(t).(N(d_1) - 1) = -S(t).N(-d_1)$	$K(t).N(d_2)$

where:

$$d_1 = \log\{S(t)/(K(t)\}/\Sigma + \Sigma/2$$

$$d_2 = -\log\{S(t)/(K(t)\}/\Sigma + \Sigma/2$$

but Σ is different for the two different bond models:

Model A $\quad \Sigma^2 = (T - t).\sigma_S^2$

Model B $\quad \Sigma^2 = (T - t).\sigma_S^2 + (T - t)^2.\rho.\sigma_R.\sigma_S + (T - t)^3.\sigma_R^2/3.$

The rationale for this result is explained in Appendix A.

2.1.11 The option price $H(t)$ is the sum of the hedging quantities:

$$H(t) = H_S(t) + H_B(t).$$

2.2 *Hedging*

2.2.1 In theory, if hedging were to be carried out continuously and costlessly, and if the model and the values of the parameters in the real world were the same as assumed in the option pricing model, then the proceeds of the hedge portfolio would exactly provide the payoff for the option. In practice, perfect hedging such as this is impossible. Therefore, as in WWY, we investigate the results of 'empirical hedging' by simulating the prices (of

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the share and the bond) at discrete intervals, then calculate the results of different hedging strategies, thus obtaining an estimate of the distribution of the hedging error. In practice we carry out $N = 10,000$ simulations.

2.2.2 We place ourselves in the position of the writer of one of the options. We make a preliminary estimate of the price (or premium) that we consider should be paid for the options by using the option formulae shown in ¶2.1.10. We assume that we have received this premium ($H(0) = V(0)$), and we invest it at time 0 in the share and cash in the quantities prescribed in ¶2.1.10. We denote the amounts invested in the share and the bond as $V_S(0)$ ($= H_S(0)$) and $V_B(0)$ ($= H_B(0)$) respectively. We then step forward a chosen time step h. At time $h-$, i.e. just before any rearrangement of the portfolio, the market values of our investments are:

$$V_S(h-) = V_S(0)/S(0).S(h)$$

and

$$V_B(h-) = V_B(0)/B(0).B(h).$$

2.2.3 The sum of these, $V(h-) = V_S(h-) + V_B(h-)$, will, in general, not equal the desired option price $H(h)$. We have to decide on a strategy to cover the deficit or surplus. We make our investment self-financing. That is, we do not assume that any extra capital comes in from or goes out to any outside fund. We ignore transaction costs at this stage. We have three obvious strategies:

(i) invest the correct amount in the share, and the balance in the bond;
(ii) invest the correct amount in the bond, and the balance in the share; and
(iii) invest the correct proportions in the share and the bond.

There are other possible investment strategies, but we do not investigate these at this stage. We also assume that we rearrange the portfolio at each step h. An alternative, discussed by Boyle & Hardy (1997), is to rearrange the portfolio only when the discrepancy between what we hold and what we would like to hold exceeds a certain size.

2.2.4 Thus, under the different strategies, at time $h+$, just after we have rearranged the portfolio, we hold:

(i) $V_S(h+) = H_S(h)$ and $V_B(h+) = V(h-) - V_S(h+)$;
(ii) $V_B(h+) = H_B(h)$ and $V_S(h+) = V(h-) - V_B(h+)$; and
(iii) $V_S(h+) = V(h-).H_S(h)/H(h)$ and $V_B(h+) = V(h-).H_B(h)/H(h)$.

At each step h:

$$V(h+) = V_S(h+) + V_B(h+) = V(h-)$$

so we make the strategy self-financing. Note that for call and put options one or other of $H_S(h)$ and $H_B(h)$ is negative, and in all cases either $V_S(h+)$ or $V_B(h+)$ or both may have a sign different from $H_S(h)$ or $H_B(h)$.

2.2.5 We continue in this way through successive time steps, until we reach the expiry date T. The proceeds are then $V(T) = V(T-)$, and we compare this with the payoff of the relevant option X(T). The deficit is $D(T) = X(T) - V(T)$. There may be a surplus, in which case D(T) is negative.

2.2.6 We then rank the values of the deficit from the N simulations in increasing order, so that:

$$D_1(T) \le D_2(T) \le \ldots \le D_n(T) \le \ldots \le D_N(T).$$

2.2.7 We can calculate the mean, variance, and higher moments of D(T). We would expect V(T) to be close to X(T), but to err on either side of it, so, although X(T) contains a large proportion of cases with the same value, either zero (for calls and puts) or K (for maxis and minis), the distribution of D(T) may be roughly symmetrical, so variances and higher moments are meaningful.

2.3 *Prudential Reserves*

2.3.1 We could, at this point, calculate for the deficit D(T) at time T, for any security level α (usually expressed as a percentage), the quantile reserve Q_α and the conditional tail expectation (CTE) T_α; but we need to consider first how the deficit might be financed. We have assumed that the initial option premium is invested in one of the ways described in ¶2.2.3; in simulation n it gives proceeds of $V_n(T)$ and leaves a deficit of $D_n(T)$. We assume that this deficit is funded by an extra initial amount of capital. This, in turn, can be invested in any one of several ways. We investigate three obvious ways, similar to the three ways of investing the option premium:

(a) invest the extra capital in the share;
(b) invest the extra capital in the bond; and
(c) invest the extra capital in the same proportions as the option has been invested; this really includes three subordinate ways, corresponding with the three ways described in ¶2.2.3.

2.3.2 Thus, if we assume that the extra capital is invested in one of these ways, the extra initial capital required, which we denote $VD_n(0)$, is:

(a) $VD_n(0) = D_n(T).S(0)/S_n(T)$;
(b) $VD_n(0) = D_n(T).B(0)/B_n(T) = D_n(T).B(0)$, since $B_n(T) = 1$ for all n; and
(c) $VD_n(0) = D_n(T).V(0)/V_n(T)$, for whatever investment strategy was used to derive V(T).

2.3.3 We now calculate CTEs and quantiles of VD(0), the present value of the deficit at time zero, for security level, α. We sort the values of VD(0)

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into increasing sequence. Then, putting $m = N.(1 - \alpha/100)$, we define $TD(0)_\alpha$ as the average of the m largest values of $VD_n(0)$ and $QD(0)_\alpha$ as the mth largest value. For example, if $\alpha = 99\%$ and $N = 10,000$, then $m = 100$, $QD(0)_{99} = VD_{9901}(0)$ and $TD(0)_{99} = \Sigma_{j=9901,10000} VD_j(0)/100$. It is these values, and other statistics of VD(0), that appear in many of the tables in the rest of the paper.

2.4 *Fair Values*

2.4.1 We now follow the rationale described in WWY, Section 8.2. It is easier to explain first in terms of quantile reserves. Assume that the total initial capital on a quantile basis, $H(0) + QD(0)_\alpha$, is either required by the regulator (whatever regulatory authority or supervisor is relevant to the institution we are considering), or is chosen by the management as being desirable and in excess of the minimum required. The initial $H(0) = V(0)$, invested in one of the ways we have specified, will provide $V_n(T)$ in simulation n at time T. The initial $QD(0)_\alpha$ will provide what we denote $QD_n(T)_\alpha$, whose value will depend on the investment assumptions:

(a) $QD_n(T)_\alpha = QD(0)_\alpha.S_n(T)/S(0)$;
(b) $QD_n(T)_\alpha = QD(0)_\alpha.B_n(T)/B(0) = QD(0)_\alpha/B(0)$; and
(c) $QD_n(T)_\alpha = QD(0)_\alpha.V_n(T)/V(0)$, for the strategy used to derive $V_n(T)$.

2.4.2 The total available at time T in simulation n is therefore $V_n(T) + QD_n(T)_\alpha$. In $\alpha\%$ of cases this will be more than sufficient to provide the payoff under the option. This surplus will fall back to the 'shareholders', or to whoever has provided the required initial capital. In the remaining $(100 - \alpha)\%$ of cases there will be a deficit, and we assume that the shareholders have to meet this deficit. We are imagining, for example, one portfolio of a life office or a bank. It is possible, in extreme circumstances, that the whole institution becomes insolvent, and limited liability would mean that shareholders do not have to provide for the deficit; but we assume that, in general, an institution carries the losses on any particular line of business, although it may not like doing so.

2.4.3 The same principles apply if the CTE $TD(0)_\alpha$ is the basis of the prudential reserve. However, the amount at time T will normally be sufficient to meet many more than $\alpha\%$ of cases. It would be possible to find what percentile value, say α^*, is such that $TD(0)_\alpha = QD(0)_{\alpha^*}$. We can then say that the CTE reserve will meet $\alpha^*\%$ of cases.

2.4.4 We now consider the value of this 'investment' to the shareholders. It has been invested in our chosen way, so provides that sort of return (as if in shares, the bond, etc.), but the outcome is risky. We assume that shareholders dislike extra risk, even if it is 'diversifiable risk'. We use an implicit utility function to value the shareholders' interest. An easy one to use is a 'kinked linear' utility function. In WWY, Section 8.2, we suggested that positive returns for the shareholders might be valued by discounting at a rate of interest of $j\%$ in addition to the return on the investment, and that negative returns might be valued by discounting at a rate of interest $-k\%$ in addition to the return on the investment. Thus, the value to the shareholders is estimated by:

Shareholder value

$$= [\Sigma_{+}\{TD(0)_{\alpha} - VD_{n}(0)\}.(1+j)^{-T} + \Sigma_{-}\{TD(0)_{\alpha} - VD_{n}(0)\}.(1-k)^{-T}]/N$$

where Σ_{+} is the sum of positive values of $TD(0)_{\alpha} - VD_{n}(0)$, and Σ_{-} is the sum of negative values. This 'shareholder value' is comparable with the 'certainty equivalent' of utility theory.

2.4.5 The effect of this basis is to value positive returns at less than 'face value' and negative returns at more than 'face value'. However, the effect of compounding is that the risk aversion, the difference or ratio between $(1+j)^{-T}$ and $(1-k)^{-T}$, increases with T. This may not be realistic. While we can imagine that shareholders may be less willing to undertake longer-term risks than shorter-term ones, the strength of the risk aversion may not increase as our model would indicate. Instead, shareholders may wish to consider a pair of functions to replace $(1+j)^{-T}$ and $(1-k)^{-T}$, which reduce with T, but not so strongly as do these compound interest functions.

2.4.6 In WWY we suggested, and gave examples for, j and k equal to 1% and 2%. We do not know what a suitable level would be. We believe that it is up to the market to set the level. If α is high, then j is much more important than k. A high value for j implies that shareholders expect a high 'risk premium' on their investment. As we shall see, this means that premiums are correspondingly high. In an efficient market, purchasers of options, who may be prospective life office policyholders, will choose the lowest premiums (assuming the same level of security of the company from whom they are purchasing). In order to get business, shareholders have to accept a low enough value for j, as low as the strongest competitor in the market, but the value of j in the market also needs to be high enough to attract enough capital to undertake this investment. On the other hand, if it is too high, then prices for options become too high, and purchasers are not available. So, the equilibrium value of j needs to be such as to attract a supply of sufficient capital to meet the demand for sales of the contracts.

2.4.7 What we have just described is what ought to happen in a competitive market, but the market for insurance appears to be very 'inefficient', in that offices can sell apparently identical products at surprisingly different prices. Since many insurance options are embedded in more complex products, it may be difficult to disentangle the prices being charged for them. For statutory purposes, it may become necessary for the regulator, perhaps in consultation with life offices, to lay down a satisfactory basis for j and k in addition to specifying the necessary value of α.

3. Practical Hedging: Bond Model A

3.1 *First Assumptions*

3.1.1 In order to investigate hedging at discrete intervals, we need a model for how investments are assumed to behave in the real world, a model for option pricing and hedging, and a set of rules that tells us how to invest, as we have discussed in Section 2.2. The real-world model and option pricing model do not need to be the same, but for the moment we assume that they are.

3.1.2 We assume, first, that the real-world model is the same as the option pricing model, that is a random walk for shares in discrete time and a fixed bond interest rate. We define the major time unit as a year, the minor time unit as one twelfth of a year (a 'month'), and the hedging time step as some fraction of a month, such as 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, . . . down to 1/1,024. We do not need to use binary fractions of a month, but it is convenient to do so. We describe the small unit of time over which we simulate as h of a year ($h = 1, \frac{1}{2}, \ldots, 1/12, 1/24, \ldots, 1/12{,}288$).

3.1.3 The model for shares is that they perform a logarithmic random walk, with steps of h with bias m_h and standard deviation s_h. Thus:

$$\ln \mathrm{S}(t+h) = \ln \mathrm{S}(t) + m_h + s_h.z(t+h)$$

where each $z(t+h)$ is independent and unit normally distributed (N(0, 1)).

3.1.4 We can calculate the $h$ly parameters from specified yearly ones as:

$$m_h = h.m$$

$$s_h = \sqrt{h}.s$$

where m is the annual mean rate of (logarithmic) growth and s is the annual standard deviation (or volatility). We put $m = 0.07$ and $s = 0.2$ in our first examples.

3.1.5 We first assume bond model A, and refer often to 'cash'. In this case cash grows with certainty at a uniform 'force' of r per annum, or rh per period. Thus $\mathrm{K}(t) = \mathrm{K}(\mathrm{T}).\exp(-ru)$, where $u = \mathrm{T} - t$, and:

$$\mathrm{K}(t+h) = \mathrm{K}(t).\exp(rh).$$

We put $r = 0.05$ in our first examples. We introduce bond model B in Section 8.

3.1.6 We assume that the option pricing model uses the same parameters as the real-world model, but we shall give them different names (with Greek letters), because later they will be different. The option pricing risk free rate is δ, where at present $\delta = r$ (= 0.05) and the option volatility is σ, where at present $\sigma = s$ (= 0.2).

3.1.7 In our first example we choose an overall time period, $T = 10$ years, a time step, one half of a month ($h = 1/24$), a share price at outset, $S(t = 0) = 100$, and an exercise price, $K(T = 10) = 165$. We choose this because $K(0) = 165 \times \exp(-10 \times 0.05) = 100.08$, so the option is very near to being 'at the (discounted) money'. We shall investigate 'out of the money' and 'in the money' options later, where, in each case, we compare S(0) and K(0), and not (as is colloquial in the traded options market) comparing $S(t)$ or S(0) and $K = K(T)$.

3.1.8 We simulate share prices at discrete intervals of h, using the formula in ¶3.1.3. An alternative would be to simulate yearly in the first place, and to fill in the gaps by 'Brownian bridges'. This is discussed in Section 4.2.

3.1.9 We consider all four types of option simultaneously (maxi, mini, call, and put), and for each we calculate the theoretical option price at time 0, using the relevant formula as shown in ¶2.1.10. We assume that the option is being granted by a 'life office' (us) to a policyholder, and that, in the first instance, the theoretical option price is paid at time $t = 0$ by the policyholder to the life office. 'We' then invest the option price in the share and in the bond, in accordance with the quantities given by the formulae shown in ¶2.1.10. We denote these as $V_S(0)$ and $V_B(0)$ respectively, with $V_S(0) + V_B(0) = V(0)$. We could put subscripts, say g, on these to show that they belong to option type g, $g = 1$ to 4, but there is no confusion without these subscripts.

3.1.10 One period later, at time $t + h$, the share price has changed randomly to $S(t + h)$ in accordance with the formula in ¶3.1.3, and cash has grown (deterministically) by $\exp(rh)$ in accordance with the formula in ¶3.1.5. We then rearrange the portfolio in accordance with the method described in Section 2.2. The new option prices depend on the new value of the share price $S(h)$, and the term to run is reduced to $T - h$. However, the parameters δ and σ remain fixed.

3.1.11 We described three investment strategies in ¶2.2.3. We also introduce strategy (iv) in which we invest the correct amounts in shares and bonds at $t = 0$, and then hold them up to the exercise date T, without any dynamic hedging. It is not suggested that this is a good strategy; it is selected purely in order to demonstrate how badly such a strategy performs in comparison with the dynamic hedging strategies.

3.1.12 We carry on with this simulation of a practical investment strategy until time T. At that point, we have to consider the outcome of the option. The various possible payoffs, which we denote X(T), are:

	If $S(T) < K(T)$	If $S(T) = K(T)$	If $S(T) > K(T)$
Maxi	K(T)	$S(T) = K(T)$	S(T)
Mini	S(T)	$S(T) = K(T)$	K(T)
Call	0	$0 = S(T) - K(T)$	$S(T) - K(T)$
Put	$K(T) - S(T)$	$0 = K(T) - S(T)$	0.

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3.1.13 If our hedging had been correct theoretically, we should have found that V(T−) exactly equalled the required payoff, but, because of our discrete hedging, we find that it is not exactly the right amount. We hope that, on average, we are correct, and that the error is not too great. We are concerned about having too little, rather than too much, so we define the deficit (at expiry) as D(T) = X(T) − V(T−), the required payoff minus the amount available. We consider statistics of D.

3.2 *First Examples*

3.2.1 We start with the parameters shown in Table 3.2.1, all fixed for the period of the simulation.

3.2.2 We now show in detail how the calculations work, using the first period of the first simulation as an example. The option prices and hedging quantities at time $t = 0$ are as in Table 3.2.2.

3.2.3 Note that, at $t = 0$, the options are very close to being 'at the discounted money'; the share price S(0) = 100.00 is very close to the present value of the exercise price K(0) = 100.08. Therefore, the values of the call and the put are very close, and the amounts to be invested in shares and cash for the maxi option are very close. All the other numbers can be derived from these.

Table 3.2.1. Parameters for first examples

Number of simulations	10,000
Years to exercise date	10
Number of periods per year	24
Share price at $t = 0$	100
Exercise price	165
Real-world parameters:	
Mean share return, m	0.07
Standard deviation of share return, s	0.2
Fixed return on 'cash', r	0.05
Option pricing model:	
Bond model	A
Standard deviation of share return, σ	0.2
Fixed return on 'cash', δ	0.05

Table 3.2.2. Hedging quantities and option values for first examples

	Share quantity	Cash quantity	Option value
Maxi	62.36	62.50	124.87
Mini	37.64	37.57	75.21
Call	62.36	−37.57	24.79
Put	−37.64	62.50	24.87

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Table 3.2.3. Values of investment after one period, first example

	Share value	Cash value	Total investment
Maxi	66.98	62.63	129.61
Mini	40.42	37.65	78.08
Call	66.98	−37.65	29.32
Put	−40.42	62.63	22.21

3.2.4 We use, as an example, the results from our first simulation. During the first period of this simulation (half a month or 1/24 of a year) the share price is assumed to increase, quite considerably, to $107.40 = S(1/24)$. The discounted value of the exercise price increases only slightly, to 100.29. Our investments for the four types of option now have the values shown in Table 3.2.3.

3.2.5 We now calculate the theoretical option prices and hedging quantities, on the basis of the new facts, S(1/24), K(1/24) and $T - t = 9 + 23/24$, and we get the values shown in Table 3.2.4.

3.2.6 Comparing the new option value with our actual investment, we see that we are short by 0.10 for the maxi, call and put, and have a surplus of 0.10 for the mini. We now consider what to do under three of our four investment strategies; under strategy (iv) we do nothing at intervening dates, and look at the results of the initial investment only when we reach the exercise date. Consider the maxi: we have 129.61 to invest (and no transaction costs to consider at this stage); under strategy (i) we set our share investment at the theoretically correct quantity 71.34, and invest the balance, 58.27, in cash; under strategy (ii) we set our cash investment at the theoretically correct quantity 58.37, and invest the balance, 71.24, in shares; under strategy (iii) we use the theoretical proportions, 55.00% in shares and 45.00% in cash, and we invest 71.29 in shares and 58.32 in cash.

3.2.7 Table 3.2.5 shows the results for all four types of option. Note that the values have been rounded correctly, so they may appear to sum incorrectly. (This comment applies *passim*.)

3.2.8 The calculations can be continued, period by period, up to the exercise date. They become rather extensive. At time $t = 0$ the investments for all three strategies (and also for strategy (iv)) were the same, so the results at the end of the first period (before rearranging the investments) were also

Table 3.2.4. Theoretical hedging quantities and option value after one period

	Share quantity	Cash quantity	Option value
Maxi	71.34	58.37	129.71
Mini	36.06	41.92	77.98
Call	71.34	−41.92	29.42
Put	−36.06	58.37	22.31

Table 3.2.5. Amounts invested under various strategies after one period, after rearrangement

	Total investment	Strategy (i) Shares	Strategy (i) Cash	Strategy (ii) Shares	Strategy (ii) Cash	Strategy (iii) Shares	Strategy (iii) Cash
Maxi	129.61	71.34	58.27	71.25	58.37	71.29	58.32
Mini	78.08	36.06	42.02	36.15	41.92	36.10	41.97
Call	29.32	71.34	−42.02	71.25	−41.92	71.11	−41.78
Put	22.21	−36.06	58.27	−36.15	58.37	−35.90	58.11

Table 3.2.6. Payoffs and investment results at end of ten years under various strategies

	Required payoff	Strategy (i) Assets	Strategy (i) Deficit	Strategy (ii) Assets	Strategy (ii) Deficit	Strategy (iii) Assets	Strategy (iii) Deficit	Strategy (iv) Assets	Strategy (iv) Deficit
Maxi	167.48	170.62	−3.14	171.28	−3.80	171.12	−3.64	207.49	−40.02
Mini	165.00	161.85	3.14	161.20	3.80	161.68	3.32	124.98	40.02
Call	2.48	5.62	−3.14	6.28	−3.80	5.44	−2.97	42.49	−40.02
Put	0.00	3.14	−3.14	3.80	−3.80	1.40	−1.40	40.02	−40.02

the same. This is not the case for investments during the second and subsequent periods.

3.2.9 At the exercise date, T = 10 years, after 240 periods, the share price in this first simulation is 167.48, a little above the exercise price. The payoff of the maxi option is therefore 167.48, of the mini 165, of the call 2.48 and of the put nil. The results of our investment strategies are shown in Table 3.2.6.

3.2.10 In this simulation, we have a surplus (shown as a negative deficit) for the maxi, call and put options, and a deficit for the mini, whatever the strategy. The surplus is modest (but not zero) for strategies (i), (ii) and (iii), and large for strategy (iv). This is not the case for all simulations. The results for strategies (i), (ii) and (iv) are the same for the maxi, call and put, and numerically the same, but with the sign reversed, for the mini. This is the case for all simulations. For strategy (iii) the results diverge for the different option types. One might argue from this one simulation that the options were overpriced. If 100 had been charged for either the maxi or the mini, and invested wholly in shares, the right amount would have been obtained for the maxi, and a little too much for the mini, but this is because the outcomes are so close. Different simulations produce very different results.

3.3 *First Results*

3.3.1 We now note some statistics of the results for this first example. First, in Table 3.3.1 we show statistics for the final share price (FSP),

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Table 3.3.1. Statistics for final share price (FSP) and deficit for different options, different investment strategies, bond model A, first assumptions

	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
FSP	245.7	174.3	21.0	2,002.9	768.4	924.7	1,155.4
Maxi:							
Strategy i	0.04	2.32	−11.48	12.68	5.35	6.42	7.75
Strategy ii	0.05	3.02	−14.79	30.04	7.49	9.22	11.58
Strategy iii	0.06	2.77	−13.63	26.22	6.75	8.22	10.21
Strategy iv	9.31	55.33	−40.94	650.79	186.15	244.98	331.83
Mini:							
Strategy i	−0.04	2.32	−12.68	11.48	5.09	5.99	7.07
Strategy ii	−0.05	3.02	−30.04	14.79	6.32	7.33	8.58
Strategy iii	−0.03	2.48	−13.20	12.13	5.51	6.49	7.64
Strategy iv	−9.31	55.33	−650.79	40.94	39.23	40.07	40.57
Call:							
Strategy i	0.04	2.32	−11.48	12.68	5.35	6.42	7.75
Strategy ii	0.05	3.02	−14.79	30.04	7.49	9.22	11.58
Strategy iii	−0.04	47.08	−2,312.62	2,584.61	37.11	58.82	110.88
Strategy iv	9.31	55.33	−40.94	650.79	186.15	244.98	331.83
Put:							
Strategy i	0.04	2.32	−11.48	12.68	5.35	6.42	7.75
Strategy ii	0.05	3.02	−14.79	30.04	7.49	9.22	11.58
Strategy iii	0.06	29.74	−1,173.77	1,454.51	29.44	50.59	102.87
Strategy iv	9.31	55.33	−40.94	650.79	186.15	244.98	331.83

followed by statistics for the deficit D(T), for four types of option for four investment strategies. As a reminder, we repeat the investment strategies:

(i) invest correct amount in shares, the balance in cash;
(ii) invest correct amount in cash, the balance in shares;
(iii) invest correct proportions in shares and cash; and
(iv) invest correct amounts in shares and cash at time zero, and then hold these amounts without dynamic hedging.

3.3.2 Note that in 6,234 of the 10,000 simulations the FSP = S(T) exceeded the exercise price K(T), and in 3,766 of them it fell short of the exercise price. In the former case call options were 'in the money' at expiry, in the latter case put options were.

3.3.3 The 95% CTE is the average of the 500 largest amounts, the 97.5% CTE is the average of the largest 250 amounts and the 99% CTE is the average of the 100 largest amounts out of the 10,000 simulations.

3.3.4 The simulated share price has a wide range, and the simulated values are close to being lognormally distributed (as the theoretical share price is), with observed mean and standard deviation close to the theoretically correct ones, which can be calculated as:

$$\text{Mean} = \text{S}(0).\exp(\text{T}(m + \tfrac{1}{2}s^2)) = 100\exp\{10 \times (0.07 + 0.02^2/2\} = 246.0$$

and

$$\begin{aligned}\text{Standard deviation} &= \text{Mean}.\surd(\exp(\text{T}s^2) - 1)\\ &= 246.0 \times \surd(\exp(10 \times 0.2^2) - 1) = 172.5\end{aligned}$$

so the simulated results, 245.7 and 174.3, are reasonably close to the correct values.

3.3.5 For strategies (i), (ii) and (iv), the results for maxi options, calls and puts are identical; for minis the mean and the range are the same with an opposite sign; the standard deviation is the same; and the CTE calculations are different, being based on the opposite tail of the distribution. For strategy (iii), the results for the different option types are different.

3.3.6 Strategy (iv) clearly fails to replicate the required payoff by a long way. This demonstrates what should be obvious, that it is the dynamic hedging process that comes close to replicating the option payoff, not a static 'buy and hold' policy. We omit strategy (iv) from further consideration.

3.3.7 For maxi and mini options, strategies (i), (ii) and (iii) all get reasonably close to the required payoff on average. Nevertheless, the range, the standard deviation and the CTEs are large enough to be uncomfortable. Dynamic hedging 24 times a year fails to replicate the payoff by quite a significant amount in a proportion of cases.

3.3.8 For call and put options, strategies (i) and (ii) have the same results as maxi options do, and the same comments apply. Strategy (iii), however, produces some extreme errors. This needs explaining.

3.3.9 We can investigate in detail one simulation in which a very large deficit of 1,671.55 for a call option under strategy (iii) has arisen. We discover that the problems begin as we approach the exercise date. At period 234, with six periods (half-months) still to go, the share price is 82.36, so very well below the exercise price of 165. The value of the call option, which is very far out of the money, is nearly zero, 0.72×10^{-11}. This should be hedged by two almost equal amounts, 0.5196×10^{-9} of share, and -0.5123×10^{-9} of cash. The accumulated investment is the modest sum of −6.26, which does not look too bad, at this point, but the proportional investment strategy (iii) requires investing 71.54 times this (7,154%) in shares and −70.54 times this in cash. So, we invest −448.00 in shares (i.e. go short) and 441.74 in cash. This immediately feels rash!

3.3.10 During the next period, the 235th, the share price rises to 88.01, still very far below the exercise price, but our investment has indeed been rather bad, being now worth −36.05. The theoretical option price has risen slightly, to 0.99×10^{-11}, but the theoretical proportions have widened to 77.68 times in shares and −76.68 times in cash. The program continues to follow the required policy (which, to a human observer, would seem an extraordinary one), and invests −2,800.72 in shares and 2,764.66 in cash.

3.3.11 During period 236 the share price rises further, to 94.21, and our investment is again bad, giving us a value of −227.48. The same strategy is continued, and over successive periods, as the share price happens to rise, our investment gets progressively worse, being worth −721.00 and −1,664.60 at the end of the next two periods. At this point we have −80,319.43 in shares and 78,654.74 in cash. At this point the computer calculates the option price as zero, even with double precision floating point calculations, and ends up putting nil in shares and the balance in cash (because the program calculates the cash proportion as one minus the share proportion), so the losses in the final two periods do not get any worse.

3.3.12 The conclusion that one draws is that the proportional strategy (iii), while not unreasonable for maxi and mini options, is quite unsuitable for call and put options, because of this potential problem as one approaches the exercise date with a well out-of-the-money option. Of course, if share prices had fallen, rather than risen, we could have made huge profits instead, but our objective is to match the option payoff, not to take gambles on share prices.

3.3.13 Strategies (i), (ii) and (iii) give reasonable results (i.e. without such extremes) for maxi and mini options. Strategy (i), however, gives the narrowest range, the smallest standard deviation, and the smallest CTE figures. Strategy (iii) is a little better than strategy (ii) in these respects. It is, perhaps, reasonable to suspect that matching exactly the more volatile investment, i.e. shares, will produce the closest match, but the differences are perhaps less than one might have imagined. For calls and puts, strategy (iii) is obviously unsuitable, and again strategy (i) is rather better then strategy (ii).

3.3.14 For guaranteed annuity options, a 'quanto' maxi type of option, WWY found that proportionate hedging was best, and referred (¶C.12.3) to the result we have shown here. It is interesting that different options give different relative results. One cannot automatically carry the results from one investigation across to another.

3.4 *Financing the Deficit*

3.4.1 So far we have considered the results at the exercise date. If we wish to cover ourselves against being unable to meet the liability for the payoff, we need to set up initial 'contingency reserves', i.e. an extra amount at time zero that will see us through all but the most extreme of outcomes, but how do we invest this contingency reserve? Three possibilities (and there could be more) are to put it all in shares, all in cash, or all in the same proportions as the option premium is being invested. Thus, for each strategy we should consider the present value of the deficit (PVD), D(T), when discounted either:

(a) as a share;
(b) as cash; or
(c) as if it were part of the option premium.

3.4.2 If the deficit is discounted as cash, with a fixed discount rate, clearly the ranking of outcomes is unchanged, but this does not follow for the other two discounting possibilities. A large deficit might be best matched by investment in shares or in the option portfolio. We need to investigate. However, one might suspect that investing in the option portfolio for calls or puts might be unsatisfactory, because the objective, when the option is out of the money, is to reach a zero result or close to it, and dividing by a near zero number (as is necessary) is likely to prove problematic; it does.

3.4.3 To be precise in the calculations: for a particular simulation n, the deficit which we have found for a specific type of option and investment strategy is $D_n(T)$. At expiry the share price is $S_n(T)$, and the exercise price is $K(T) = K$ (for all n). The result of our investment of the original option price is $V_n(T)$. At time zero, the share price is $S(0) = 100$; the value of the exercise price is $K(0)$; and the option price is $V(0) = H(0)$. The time zero value of the deficit, $PVD_n(0)$, is given by:

(a) $PVD_n(0) = D_n(T).S(0)/S_n(T)$ if we assume investment in shares;
(b) $PVD_n(0) = D_n(T).K(0)/K(T)$ if we assume investment in cash; and
(c) $PVD_n(0) = D_n(T).V(0)/V_n(T)$ if we assume investment in the same way as the option premium has been invested for that particular strategy.

3.4.4 Table 3.4.1 shows results for PVD(0) for all four types of option, for investment strategies (i) to (iii), for all three discounting methods. For calls and puts, the results for combinations i(*a*), i(*b*), ii(*a*) and ii(*b*) are just the same as for maxis, so are not shown. For calls and puts, the results for combinations i(*c*) and ii(c) require so many divisions by numbers close to zero that they cannot usefully be calculated.

3.4.5 It is clear that we can dismiss as unsuitable, because the extremes are too high, the following: strategy (iii) for calls and puts; and strategies with a (*c*) suffix for calls and puts. This leaves nine possibilities for maxis and minis, and four possibilities for calls and puts, all of which give rather similar results. For maxis strategies i(*b*), ii(*a*) and iii(*a*) give the best results (lowest standard deviations, lowest maxima, lowest CTEs), with ii(*a*) best on all counts; for minis strategies i(*c*) and iii(*c*) are best; for both calls and puts strategies i(*b*) and ii(*a*) are best. However, the answers are so similar that a different set of 10,000 simulations might well show different rankings.

3.4.6 One can confirm the results by redoing the same sequence of simulations, adding the possibility of an initial contingency reserve, equal, for example, to the highest value of PVD shown above for the particular combination of strategies; that is: investing the option premium according to strategy (i), (ii) or (iii), and investing the additional contingency reserve

Table 3.4.1. Statistics for present value of deficit (PVD), for different options, different investment strategies, different discounting methods, first assumptions

	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Maxi:							
Strategy i(*a*)	0.00	1.49	−7.00	9.13	3.69	4.52	5.55
Strategy i(*b*)	0.02	1.41	−6.96	7.69	3.25	3.89	4.70
Strategy i(*c*)	0.01	1.50	−7.83	9.91	3.67	4.55	5.71
Strategy ii(*a*)	0.00	1.38	−7.06	7.04	3.18	3.84	4.65
Strategy ii(*b*)	0.03	1.83	−8.97	18.22	4.54	5.59	7.03
Strategy ii(*c*)	0.00	1.57	−7.92	9.61	3.79	4.63	5.71
Strategy iii(*a*)	0.00	1.43	−6.74	8.62	3.40	4.12	4.97
Strategy iii(*b*)	0.04	1.68	−8.27	15.90	4.09	4.98	6.19
Strategy iii(*c*)	0.00	1.50	−7.87	9.10	3.56	4.36	5.38
Mini:							
Strategy i(*a*)	0.00	1.49	−9.13	7.00	3.43	4.06	4.81
Strategy i(*b*)	−0.02	1.41	−7.69	6.96	3.09	3.63	4.29
Strategy i(*c*)	−0.01	1.25	−6.66	5.64	2.77	3.24	3.81
Strategy ii(*a*)	0.00	1.38	−7.04	7.06	3.11	3.67	4.32
Strategy ii(*b*)	−0.03	1.83	−18.22	8.97	3.84	4.45	5.20
Strategy ii(*c*)	−0.01	1.44	−13.40	7.06	3.04	3.51	4.11
Strategy iii(*a*)	0.00	1.40	−7.78	7.37	3.28	3.91	4.61
Strategy iii(*b*)	−0.02	1.50	−8.01	7.36	3.34	3.94	4.63
Strategy iii(*c*)	0.00	1.23	−5.57	5.99	2.74	3.22	3.77
Call:							
Strategy iii(*a*)	−0.02	39.15	−1,846.11	2,163.03	22.28	38.73	82.84
Strategy iii(*b*)	−0.02	28.56	−1,402.68	1,567.64	22.51	35.68	67.25
Strategy iii(*c*)	22.19	2,824.10	−12,216.68	279,873.61	683.05	1,360.91	3,385.54
Put:							
Strategy iii(*a*)	0.06	15.00	−598.22	741.98	17.79	28.56	53.45
Strategy iii(*b*)	0.04	18.04	−711.93	882.20	17.86	30.68	62.39
Strategy iii(*c*)	−17.01	120.49	−11,192.02	1,038.11	24.35	44.86	96.98

according to strategy (*a*), (*b*) or (*c*), as appropriate. This confirms that the maximum deficit at expiry is then, in each case, zero, but with a different sequence of simulations it might not be.

3.4.7 On balance, strategy i(*b*), i.e. investing the option premium so that shares are matched exactly, and investing the contingency reserve in cash, is an intuitively sensible strategy and is one of the best for all option types, so further investigations will concentrate on this. Strategy ii(*a*) would have been a little better, on the basis of these results, but seems less intuitively sensible.

3.5 *Graphical Presentation*

3.5.1 It is interesting to see the results presented graphically. We concentrate first on the maxi option, with strategy i(*b*), and with steps of

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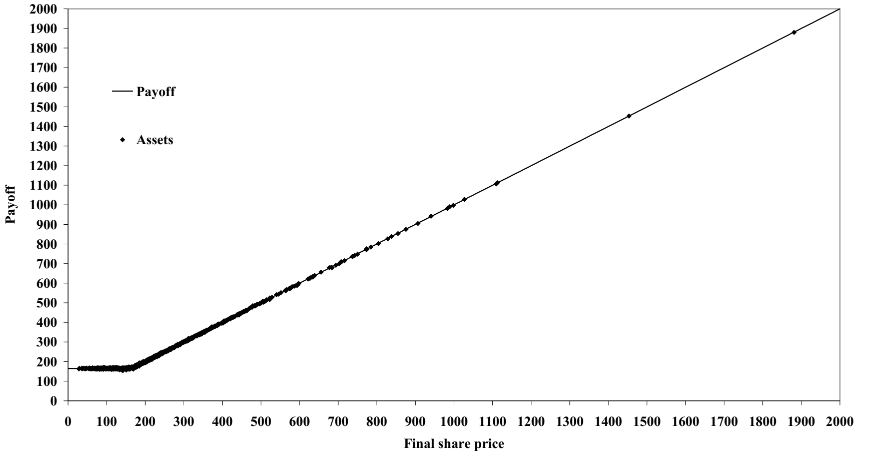

Figure 3.5.1. Maxi option, strategy (i), 1/24 year; payoff and final assets v. final share price

1/24 of a year. We use only the first 1,000 of the 10,000 simulated results. In Figure 3.5.1 we show the payoff and the final assets both plotted against the final share price, using the range from zero to 2,000, which covers all 1,000 results. It can be seen how closely the assets match the required payoff, even when the final share price is very far from the exercise price.

3.5.2 In Figure 3.5.2 we reduce the range to look at values of the final share price only from 100 to 230, a range equally spaced about the exercise price. The scatter looks much larger than in Figure 3.5.1, but remember that the bottom of the graph is not at zero. However, for a call option, the payoff in the level part to the left would be zero, so the errors are relatively large compared with the desired payoff.

3.5.3 In Figures 3.5.3 and 3.5.4 we show the present value of the deficit, discounted at the cash rate, plotted against the final share price, first showing the full range of values of the final share price, then showing the narrower range from zero to 500. One can see how the error, whether a surplus or a deficit, is larger when the final share price is close to the exercise price, and is smaller when it is further away, especially when it is very large. This last result is not unreasonable. As the share price gets very much 'into the money' for the call, the correct strategy is to invest a very high proportion in shares, and the amount in cash does not change very much. However, one should not be misled into thinking that any option in this position is correctly hedged by investing all the assets in shares at all times.

3.5.4 The fact that the error is larger when the final share price is near

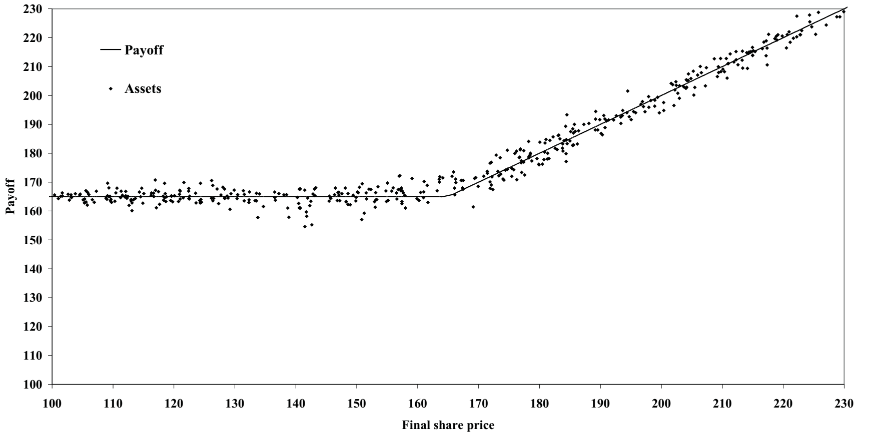

Figure 3.5.2. Maxi option, strategy (i), 1/24 year, payoff and final assets v. final share price, restricted range

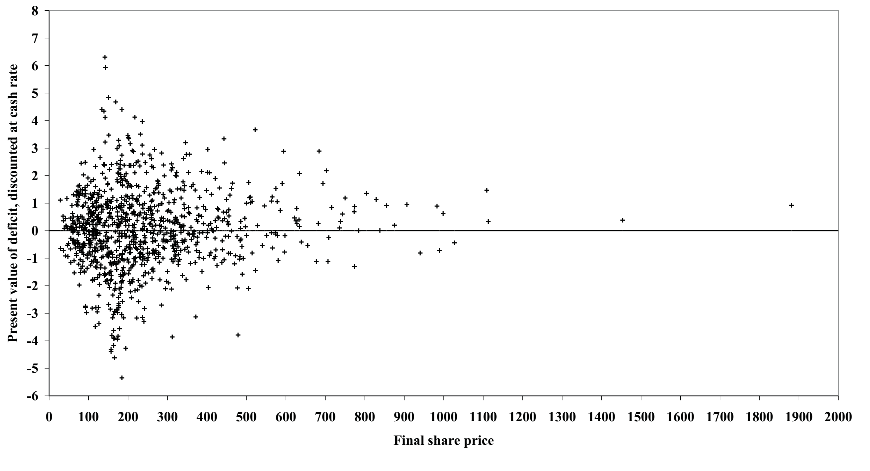

Figure 3.5.3. Maxi option, strategy i(b), 1/24 year, present value of deficit v. final share price

the exercise price suggests the following investigation. The 1,000 simulations are divided into ten subsets (deciles), based on the values of the final share price, the 100 smallest, then numbers 101 to 200, etc. Then we calculate the statistics of the present value of the deficit for each subset. These are shown in Table 3.5.1, for the maxi option, for strategy i(b). Note that the results for

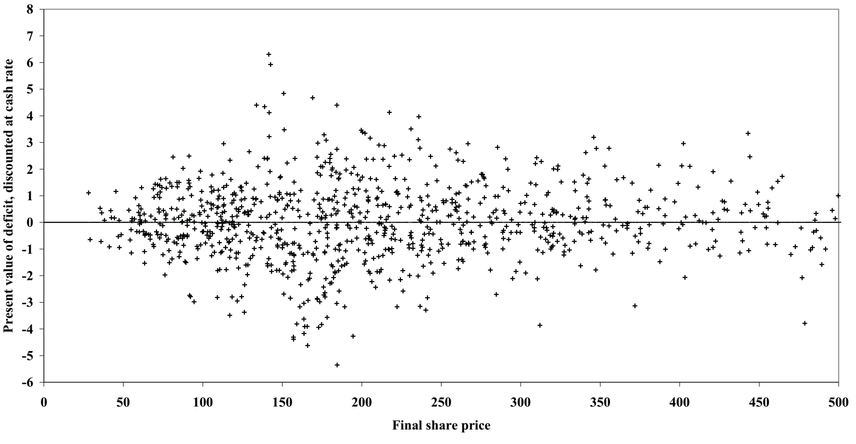

Figure 3.5.4. Maxi option, strategy i(*b*), 1/24 year, present value of deficit v. final share price, restricted range

Table 3.5.1. Statistics for separate deciles of 1,000 simulations

	Lowest FSP	Highest FSP	Mean	Standard deviation	Lowest	Highest
Maxi option:						
1-100	28.2	86.7	0.08	0.87	−1.97	2.45
101-200	86.8	116.3	0.03	1.16	−2.98	2.96
201-300	116.6	146.5	0.18	1.75	−3.49	6.31
301-400	146.8	175.4	−0.60	2.05	−4.62	4.84
401-500	175.4	200.4	0.04	1.82	−5.35	4.40
501-600	201.6	235.1	0.07	1.50	−3.17	4.13
601-700	235.4	277.1	0.18	1.37	−3.29	3.97
701-800	277.5	337.0	0.03	1.15	−3.86	2.82
801-900	337.6	447.4	0.30	1.24	−3.13	3.34
901-1,000	449.0	1,881.3	0.26	1.14	−3.79	3.66
Overall	28.2	1,881.3	0.06	1.44	−5.35	6.31

call and put options are the same as those for the maxi, and the results for the minis are the reverse of those for the maxi.

3.5.5 It is clear that the standard deviation rises to a maximum when the FSP is in the range 146.8 to 175.4, declining irregularly in either direction outside this subset. The highest and lowest values of the present value of the deficit are also at their peak in the central subsets, but not in the same one as the maximum standard deviation is found. The values move closer to zero as we go towards the extremities, but not uniformly.

3.6 *Premiums and Fair Values*

3.6.1 We can now consider how the principles described in Section 2 can be applied to pricing these options and calculating 'fair values', which we argue are the same as the 'net premium', i.e. the premium that should be charged excluding expenses, but including a 'margin for adverse deviation'. We first need to choose a prudential reserve, as described in Section 2.3, such as a regulator or prudent management might require. We use the 99% and 97.5% CTEs for our examples. For the 99% CTE we shall generally choose values of $j = k = 2\%$, and for the 97.5% CTE we choose $j = k = 1\%$. These are not equivalent, but are rather more extreme cases. If we chose (97.5%, 2%, 2%) or (99%, 1%, 1%) we would get intermediate answers, sometimes similar to one another. With extreme CTEs, the results are not very sensitive to the value of k.

3.6.2 We consider the maxi option in detail, with strategy i(b). The 'pure' option premium would be 124.87, as shown in Table 3.2.2. If we invest this according to strategy (i), i.e. putting the correct amount into shares and the balance into the bond at each step, we have a distribution of deficits/surpluses. If we finance the deficit by investing extra capital in the bond, i.e. using strategy i(b), we need 3.89 more initially to set up a 97.5% CTE and 4.70 more initially to set up a 99% CTE. We then consider the distribution of profit or loss at expiry, relative to the proceeds of the initial investment and the additional capital. We discount surpluses at the bond rate, and then multiply by $(1 + j)^{-T}$. We discount deficits at the bond rate, and then multiply by $(1 - k)^{-T}$. Since $T = 10$, these factors, for $j = k = 1\%$, are $1.01^{-10} = 0.9053$ and $0.99^{-10} = 1.1057$; for $j = k = 2\%$, the values are 0.8203 and 1.12239. The net value for the combination (97.5%, 1%, 1%) is 3.50, and for (99%, 2%, 2%) is 3.83. If the shareholders are willing to finance the capital requirement to this extent, this leaves 0.39 or 0.86, respectively, to be financed by the policyholder. We summarise this, and show other combinations, in Table 3.6.1. Note that the numbers are rounded and may not appear to sum

Table 3.6.1. Financing the initial capital, maxi option, strategy i(b), different combinations of (α, j, k)

Pure option price	α	CTE	j	k	Shareholder contribution	Policyholder contribution	Total premium
124.87	97.5%	3.89	1%	1%	3.50	0.39	125.25
124.87	97.5%	3.89	1%	2%	3.50	0.39	125.25
124.87	97.5%	3.89	2%	1%	3.17	0.72	125.59
124.87	97.5%	3.89	2%	2%	3.17	0.72	125.59
124.87	99%	4.70	1%	1%	4.23	0.47	125.33
124.87	99%	4.70	1%	2%	4.23	0.47	125.33
124.87	99%	4.70	2%	1%	3.83	0.86	125.73
124.87	99%	4.70	2%	2%	3.83	0.86	125.73

Table 3.6.2. Financing the initial capital, maxi option, different strategies, using (97.5%, 1%, 1%) and (99%, 2%, 2%)

Pure option price: 124.87	97.5% CTE	S'hdr	P'hdr	Total prem	99% CTE	S'hdr	P'hdr	Total premium
Strategy i(*a*)	4.52	4.08	0.43	125.30	5.54	4.55	1.00	125.87
Strategy i(*b*)	3.89	3.50	0.39	125.25	4.70	3.83	0.86	125.73
Strategy i(*c*)	4.55	4.11	0.44	125.30	5.71	4.68	1.03	125.90
Strategy ii(*a*)	3.84	3.47	0.36	125.23	4.65	3.81	0.84	125.70
Strategy ii(*b*)	5.59	5.03	0.56	125.42	7.03	5.73	1.29	126.16
Strategy ii(*c*)	4.63	4.19	0.44	125.31	5.71	4.68	1.03	125.89
Strategy iii(*a*)	4.12	3.73	0.39	125.26	4.97	4.07	0.90	125.76
Strategy iii(*b*)	4.98	4.48	0.51	125.37	6.19	5.04	1.14	126.01
Strategy iii(*c*)	4.36	3.94	0.42	125.28	5.38	4.41	0.97	125.84

Table 3.6.3. Financing the initial capital, various options, strategy i(*b*), using (97.5%, 1%, 1%) and (99%, 2%, 2%)

	Option price	97.5% CTE	S'hdr	P'hdr	Total prem	99% CTE	S'hdr	P'hdr	Total premium
Maxi	124.87	3.89	3.50	0.39	125.25	4.70	3.83	0.86	125.73
Mini	75.21	3.63	3.31	0.32	75.55	4.29	3.54	0.75	75.96
Call	24.79	3.89	3.50	0.39	25.18	4.70	3.83	0.86	25.65
Put	24.87	3.89	3.50	0.39	25.26	4.70	3.83	0.86	25.73

correctly. We see that, to the two decimal places shown, the value of k does not affect the premiums.

3.6.3 Similar calculations are shown in Table 3.6.2 for the maxi option, for different strategies. From this we see that strategy ii(*a*) is slightly better than strategy i(*b*) at the (97.5%, 1%, 1%) level, in that it gives a lower extra premium for the policyholder (0.36 instead of 0.39), and also a lower capital contribution from the shareholder (3.47 instead of 3.50). At the (99%, 2%, 2%) level the position is similar, with very slight improvements for both parties.

3.6.4 In Table 3.6.3 we show similar calculations, using strategy i(*b*), for all four types of option. The shareholder contributions and policyholder loadings for calls and puts are, with this strategy, the same as for the maxi, but as a percentage of the option premium they are larger. Nevertheless, they are quite a small extra for the policyholder to pay.

4. Variations in Assumptions: Frequency of Hedging

4.1 *Frequency of Hedging*

4.1.1 So far we have used a hedging frequency of twice per month, or 24 times per year. However, if the hedging were to be carried out theoretically

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Table 4.1. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(*b*), different frequencies, first assumptions

	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Maxi, call and put:							
Yearly	0.50	7.06	−20.94	44.59	17.99	21.89	27.07
Six-monthly	0.31	4.93	−17.43	29.18	12.11	14.43	17.45
Quarterly	0.16	3.42	−13.16	22.57	8.15	9.75	11.84
Two-monthly	0.11	2.82	−12.39	14.26	6.68	7.91	9.45
Monthly	0.06	1.99	−9.32	11.92	4.61	5.49	6.58
1/2 months	0.02	1.41	−6.96	7.69	3.25	3.89	4.70
1/4 months	0.01	1.01	−5.23	5.00	2.33	2.78	3.33
1/8 months	0.01	0.71	−3.52	4.10	1.61	1.90	2.30
1/16 months	0.00	0.51	−2.92	3.22	1.12	1.34	1.63
1/32 months	0.00	0.35	−1.82	2.48	0.78	0.94	1.15
1/64 months	0.00	0.25	−1.35	1.24	0.54	0.65	0.79
1/128 months	0.00	0.18	−1.08	1.34	0.40	0.48	0.60
1/256 months	0.00	0.13	−0.63	0.89	0.28	0.34	0.42
1/512 months	0.00	0.09	−0.42	0.50	0.20	0.24	0.29
1/1,024 months	0.00	0.06	−0.42	0.37	0.14	0.16	0.21

continuously, the match at expiry should be perfect. It is of interest to see what happens if we alter the hedging frequency. We continue with bond model A, the Black-Scholes model. Table 4.1 shows results for maxi, call and put options for yearly, six-monthly, quarterly, two-monthly, monthly, and binary fractions of a month, $\frac{1}{2}$, $\frac{1}{4}$, etc., down to 1/1,024, which is equivalent to hedging about every 40 minutes day and night. The figures are all calculated on strategy i(*b*), and show the discounted values of the deficit, discounted using the cash rate.

4.1.2 Note that, for this strategy i(*b*), the results for maxis, calls and puts are identical. Note also that the results for minis are the negative of those for maxis, so the standard deviation is the same, the mean, lowest and highest have the opposite signs, and the CTE values, being based on the opposite tail, are different; they are not shown.

4.1.3 The convergence is not particularly rapid. In order to reduce the variation by half, one must divide the step size by four, but it is clear that, as the step size is reduced, the error is converging to zero, which confirms that our programming seems to be correct, and demonstrates that the theoretical Black-Scholes result is right, at least for this idealised case. The level of error that is acceptable is a matter of judgement, and we could not decide, in any case, on these results, because we have not taken transaction costs into account. Nor have we yet considered the possibility that the real-world model might differ from the option pricing model which we are using. This is still to be considered.

4.1.4 Note that the results for steps of one half of a month are the same as those shown in Table 3.4.1.

4.1.5 Graphs of the results are shown for the two extremes: hedging at yearly intervals in Figures 4.1.1 and 4.1.2, and at intervals of 1/1,024 of a month in Figures 4.1.3 and 4.1.4. Observe how very closely the final assets match the payoff in the latter case, even in the middle region (Figure 4.1.3), where the errors were largest with hedging at 1/24 of a year. Observe, also, how the present value of the deficit has the same feature as before of being more scattered

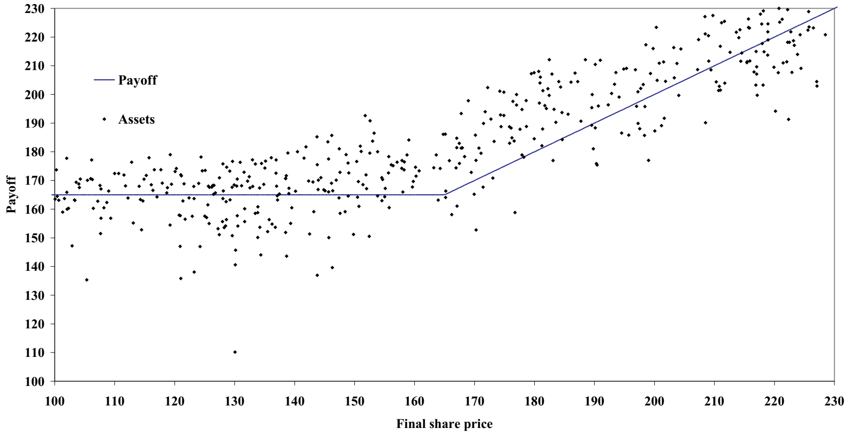

Figure 4.1.1. Maxi option, strategy (i), yearly, payoff and final assets v. final share price, restricted range

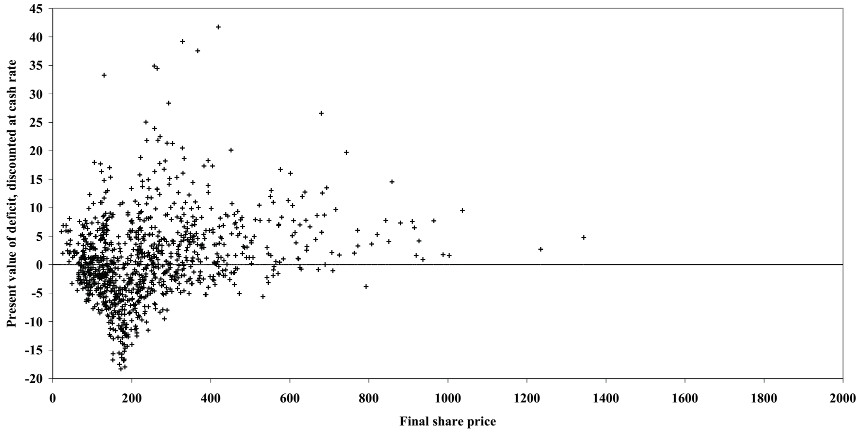

Figure 4.1.2. Maxi option, strategy i(*b*), yearly, present value of deficit v. final share price

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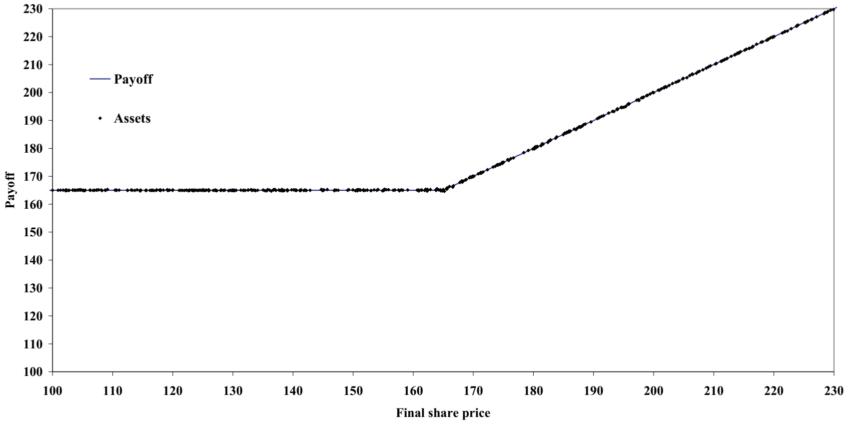

Figure 4.1.3. Maxi option, strategy (i), 1/1,024 month, payoff and final assets v. final share price, restricted range

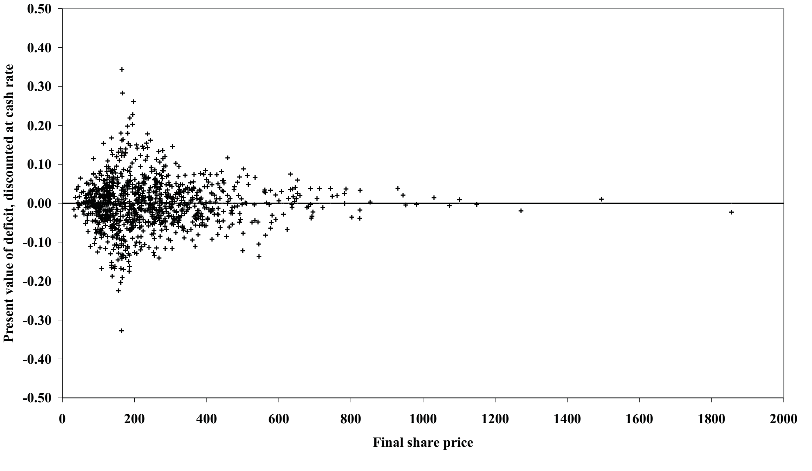

Figure 4.1.4. Maxi option, strategy i(*b*), 1/1,024 month, present value of deficit v. final share price

in the middle regions of final share price. However, observe the extremely different scales of Figure 4.1.4 as compared with those of Figure 4.1.2.

4.2 *Simulation using Brownian Bridges*

4.2.1 When we simulated over a fixed number of years with different

Table 4.2. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(*b*), different frequencies, first assumptions, simulated with Brownian bridges between annual steps

	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Yearly	0.50	7.06	−20.94	44.59	17.99	21.89	27.07
Six-monthly	0.24	4.84	−16.61	38.41	11.69	13.97	16.76
Quarterly	0.12	3.46	−16.73	17.83	8.04	9.62	11.71
Two-monthly	0.10	2.81	−13.26	16.19	6.55	7.88	9.49
Monthly	0.04	1.97	−8.39	10.06	4.46	5.34	6.55
1/2 months	0.01	1.42	−6.97	9.08	3.24	3.90	4.80
1/4 months	0.01	0.99	−5.21	6.99	2.23	2.66	3.20
1/8 months	0.00	0.69	−3.10	4.33	1.60	1.93	2.41
1/16 months	0.00	0.51	−3.37	3.27	1.15	1.39	1.70
1/32 months	0.00	0.35	−1.78	2.35	0.82	0.99	1.21
1/64 months	0.00	0.25	−1.36	1.37	0.57	0.68	0.82
1/128 months	0.00	0.18	−0.93	1.00	0.40	0.48	0.58

step sizes, we got different results for the final share price. This is because of the way in which the sequence of random numbers is used. It is possible that this may have produced more variation in the hedging results than would have been the case if the set of simulations had ended up at the same place each year with each step size. One way round this is to simulate the whole series using annual intervals in the first place, and then to interpolate, stochastically, using Brownian bridges. These are described fully in Appendix D of WWY.

4.2.2 Table 4.2 shows results similar to those in Table 4.1, using this method. Note that the results for yearly are the same here as in Table 4.1.

4.2.3 While these results are numerically different from those in Table 4.1, they are not qualitatively different. It is not obvious that either method is superior. This means that we can reasonably safely use the bridging method to interpolate when we turn to a model (such as the 'Wilkie' model) that is defined in annual steps and does not easily lend itself to a model with higher frequencies for all of its features.

4.2.4 It is also clear that the preliminary calculations with steps of one half of a month were qualitatively the same as those from higher frequencies. Since simulation with steps of 1/1,024 of a month takes about 512 times as long to calculate as simulation with steps of half of a month (and over 12,000 times as long as yearly), we restrict ourselves hereafter to steps of half of a month. We also restrict ourselves to quoting results for the maxi option, which covers both calls and puts too. Results for the mini option seem to add little more information.

5. Variations in Assumptions: Different Parameters

5.1 *Different Exercise Prices*

5.1.1 We now start to vary the conditions in different ways. We can try the effect of having different exercise prices, keeping all the other parameters the same. We use the base exercise price of 165 and add exercise prices of 105, 135, 195 and 225. The option prices are now different, and are shown in Table 5.1.1. Not surprisingly, the option prices for maxis, minis, and puts increase as the exercise price increases, whereas the prices of calls decrease.

5.1.2 Statistics of the present value of the deficit using strategy i(*b*), hedging twice a month, are shown in Table 5.1.2. We see that the variability of the present value of the deficit increases as the exercise price increases, i.e. as the call option gets more out of the money, or the put option is more in the money.

5.1.3 In Table 5.1.3 we show the statistics for pricing, for each of these exercise prices, for the maxi option. We can observe that the policyholder loading increases, both absolutely and as a percentage of the maxi option price, as the exercise price increases, i.e. as the call option is more out of the money or the put option is more in the money.

Table 5.1.1. Hedging quantities for maxi, and option values, different exercise prices

		Hedging quantities		Option values			
Exercise price K	K(0)	Share quantity	Cash quantity	Maxi	Mini	Call	Put
105	63.69	84.84	33.01	106.85	56.83	43.17	6.85
135	81.88	73.64	40.95	114.59	67.30	32.70	14.59
165	100.08	62.36	62.50	124.87	75.21	24.79	24.87
195	118.27	52.03	85.11	137.14	81.14	18.86	37.14
225	136.47	43.04	107.87	150.91	85.56	14.44	50.91

Table 5.1.2. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(*b*), first assumptions, varying exercise prices

Exercise price K	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
105	0.00	0.74	−4.34	5.05	1.71	2.09	2.59
135	0.02	1.12	−6.15	7.40	2.55	3.08	3.76
165	0.02	1.41	−6.96	7.69	3.25	3.89	4.70
195	0.03	1.64	−8.00	10.86	3.84	4.59	5.60
225	0.05	1.82	−9.28	11.48	4.36	5.23	6.38

Table 5.1.3. Financing the initial capital, maxi option, strategy i(b), using (97.5%, 1%, 1%) and (99%, 2%, 2%), various exercise prices

Exercise price K	Option price	97.5% CTE	S'hdr	P'hdr	Total premium	99% CTE	S'hdr	P'hdr	Total premium
105	106.85	2.09	1.89	0.20	107.05	2.59	2.12	0.47	107.32
135	114.59	3.08	2.77	0.31	114.89	3.76	3.07	0.69	115.28
165	124.87	3.89	3.50	0.39	125.25	4.70	3.83	0.86	125.73
195	137.14	4.59	4.13	0.46	137.60	5.60	4.57	1.03	138.16
225	150.91	5.23	4.69	0.54	151.44	6.38	5.19	1.18	152.09

5.2 *Different Mean Returns on Shares*

5.2.1 So far, we have assumed that the parameters of the real-world process were known, and those of the option price calculations were the same as the real-world ones. We now consider what happens if the real-world parameters are different from what is assumed in the option pricing calculations (but are still fixed). We start by varying the mean rate of return on shares, which, in the base calculations, was 0.07. We now use additional values of 0.03, 0.05, 0.09 and 0.11.

5.2.2 The mean rate of return on shares does not affect the option pricing calculations, so the initial option prices are unchanged. Table 5.2 shows the present value of the deficit. The means are still close to zero. We would expect this, since the theoretical option price does not depend on the mean return on shares, but the actual outcome does depend on the return on shares, and we see some variation, though this is irregular. On the whole, the further apart the real-world mean return and the interest rate assumed in the option pricing formula, the lower the standard deviation and the lower the CTEs, but the variation is not large. This may occur because, when the real-world mean return is either very high or very low, the option is likely to be very much either in or out of the money, and we have seen already that, in these cases, the hedging error is a little smaller.

Table 5.2. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(b), first assumptions, varying mean return on shares

Real-world mean return on shares, m	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
0.03	0.02	1.38	−7.83	7.27	3.30	3.96	4.80
0.05	0.00	1.41	−7.62	7.16	3.22	3.85	4.66
0.07	0.02	1.41	−6.96	7.69	3.25	3.89	4.70
0.09	0.05	1.38	−7.40	9.22	3.16	3.79	4.61
0.11	0.06	1.29	−6.09	9.05	2.93	3.53	4.41

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5.3 *Different Mean Returns on Cash*

5.3.1 We now consider what happens if the real-world return on cash is different from what is assumed in the option pricing calculations. In the base calculations we assumed 0.05. We now use additional values of 0.01, 0.03, 0.07 and 0.09.

5.3.2 The option pricing basis is unchanged, so the option prices are the same as before. Table 5.3.1 shows the present value of the deficit. Not surprisingly, if we earn more on cash than we expected when the pricing was done, we end up with bigger final assets and surplus, and vice versa if we earn less than expected. The mean values of the PVDs move in the expected direction, but the standard deviations and the range increase if we get it wrong, in either direction.

5.3.3 Instead of varying the real-world return on cash, we can vary the option pricing return on cash, keeping the real-world rate fixed at 0.05. As before, we use values of 0.01, 0.03, 0.07 and 0.09. Now the option prices are changed; they are shown in Table 5.3.2. As expected, the option price increases as the assumed risk free rate reduces.

5.3.4 The statistics for the PVDs are shown in Table 5.3.3. Qualitatively, the same effects are seen. If we price assuming too high a return on cash, we

Table 5.3.1. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(b), first assumptions, varying real world return on cash

Real-world return on cash r	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
0.01	24.42	12.23	2.23	51.49	45.66	46.49	47.28
0.03	11.10	5.51	0.95	26.10	20.83	21.32	21.86
0.05	0.02	1.41	−6.96	7.69	3.25	3.89	4.70
0.07	−9.19	4.38	−19.02	0.72	−1.98	−1.51	−1.03
0.09	−16.88	7.46	−31.82	−0.51	−4.56	−3.79	−3.01

Table 5.3.2. Hedging quantities for maxi, and option values, different returns on cash

	Hedging quantities			Option values			
Option cash return δ	K(0)	Share quantity	Cash quantity	Maxi	Mini	Call	Put
0.01	149.30	37.54	123.76	161.30	88.00	12.00	61.30
0.03	122.24	49.95	90.07	140.02	82.21	17.79	40.02
0.05	100.08	62.36	62.50	124.87	75.21	24.79	24.87
0.07	81.94	73.61	41.01	114.61	67.32	32.68	14.61
0.09	67.08	82.83	25.25	108.08	59.01	40.99	8.08

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Table 5.3.3. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(*b*), first assumptions, varying option return on cash

Option return on cash	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
0.01	−31.19	10.66	−48.65	−3.72	−11.47	−9.93	−8.29
0.03	−12.59	5.30	−24.36	0.34	−3.34	−2.67	−1.98
0.05	0.02	1.41	−6.96	7.69	3.25	3.89	4.70
0.07	7.97	4.52	−0.14	20.71	16.46	16.95	17.44
0.09	12.51	8.06	0.61	32.02	28.27	29.16	30.02

may almost guarantee that we have a deficit, and if we price using too low a return, we can almost guarantee a surplus.

5.3.5 There is, however, little excuse for getting the cash return wrong. Sufficiently many safe bonds (such as government stock) are usually available, with the right range of maturities, and, nowadays, zero coupon bonds are available, in the form of strips, in several countries. It would only be in exceptional circumstances that the present value of a zcb was unavailable.

5.3.6 Our model, however, has so far assumed that the return on cash is wholly predictable, or that the zcb always has the same yield. We consider what happens when the yield on the bond is stochastic in Section 8.

5.4 *Different Standard Deviations on Shares*

5.4.1 We now consider what happens if the real-world standard deviation on shares is different from what is assumed in the option pricing calculations. In the base calculations we assumed 0.2. We now use additional values of 0.1, 0.15, 0.25 and 0.3.

5.4.2 The option pricing basis is unchanged, so the option prices are the same as before. Table 5.4.1 shows the present value of the deficit. Not

Table 5.4.1. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(*b*), first assumptions, varying real-world share standard deviation

Real-world share SD s	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
0.1	−12.14	3.14	−20.17	−3.02	−6.10	−5.50	−4.76
0.15	−6.16	2.24	−14.24	−0.27	−2.18	−1.84	−1.45
0.2	0.02	1.41	−6.96	7.69	3.25	3.89	4.70
0.25	6.24	3.14	0.14	19.71	13.90	15.15	16.60
0.3	12.48	6.04	0.68	36.44	26.25	28.17	30.42

surprisingly, if the real-world standard deviation is smaller than that used for option pricing, a surplus is likely to emerge, and vice versa if the real-world volatility is larger. However, it is worth noting that the standard deviation of the deficit and the range of results increase as the difference between the actual SD and the assumed SD increases.

5.4.3 Instead of varying the real-world standard deviation on shares, we can vary the option pricing standard deviation on shares, keeping the real-world rate fixed at 0.2. As before, we use values of 0.1, 0.15, 0.25 and 0.3. Now the option prices are changed; they are shown in Table 5.4.2. As expected, the option price increases as the assumed standard deviation on shares increases.

5.4.4 The statistics for the present value of the deficit are shown in Table 5.4.3. Qualitatively, the same effects are seen. If we price assuming too low a standard deviation, we may almost guarantee that we have a deficit, and if we price using too high a standard deviation, we can almost guarantee a surplus. Again, the standard deviation and the range increase as the discrepancy between actual and assumed standard deviations on shares increases.

Table 5.4.2. Hedging quantities for maxi, and option values, different standard deviations for shares

	Hedging quantities		Option values			
Option share SD σ_S	Share quantity	Cash quantity	Maxi	Mini	Call	Put
0.1	56.19	56.42	112.61	87.47	12.53	12.61
0.15	59.31	59.48	118.79	81.28	18.72	18.79
0.2	62.36	62.50	124.87	75.21	24.79	24.87
0.25	65.33	65.46	130.79	69.29	30.71	30.79
0.3	68.21	68.32	136.53	63.55	36.45	36.53

Table 5.4.3. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(*b*), first assumptions, varying option share standard deviation

Option share SD σ_S	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
0.1	12.46	6.46	0.78	36.75	26.80	28.60	30.71
0.15	6.24	3.03	0.39	20.39	13.40	14.50	15.71
0.2	0.02	1.41	−6.96	7.69	3.25	3.89	4.70
0.25	−6.15	2.48	−15.82	0.18	−1.84	−1.47	−1.05
0.3	−12.21	3.92	−24.40	−1.59	−4.76	−4.02	−3.28

6. Allowing for Transaction Costs

6.1 *Transaction Costs*

6.1.1 Although the theoretical option pricing model assumes that one can hedge both continuously and costlessly, in practice hedging has to be done discretely, and also transaction costs are incurred. Allowing for transaction costs can be complicated. Although most costs can be taken as proportional to the value traded (as we assume throughout), shares and 'cash' (in practice perhaps a zero coupon bond) carry different rates from one another, there are different rates for buying and selling, and there probably would be different rates for long and short positions. Further, the costs may be, in effect, a constant if any trading is done, plus a proportion of the value; we do not consider this possibility.

6.1.2 While it is easy to understand the pricing of the purchase and sale of shares and bonds, the costs of going short are not so obvious. We assume that one 'borrows' shares from a willing counterparty, and sells them, incurring selling costs, to raise cash. When the time comes to repay the shares, one buys them in the market, incurring buying costs, and then returns the shares to the lender. One would expect the counterparty to exact some charges too, at least at the borrowing stage. Bonds are conceptually easier, since one can normally borrow cash from a bank; though the 'charge' is more likely to be in the form of an increased rate of interest than specific commission on borrowing and repaying (though that might happen too).

6.1.3 An alternative for shares is to go short on a traded index future, which should be much cheaper; but, so far as we understand, this cannot be done for the longer-term contracts that we are discussing, so a short position would need to be rolled over, at a cost, at regular intervals. We do not consider that possibility here.

6.1.4 We use two scales of charges, A and B. The rates for scale A are plausible for the purchase and sale of real shares (in the U.K., allowing for stamp duty of $\frac{1}{2}\%$ of purchases, and a small bid/offer spread). The rates for going short on shares allow for the costs of sale and repurchase, together with an extra 0.5% each way. The rates for bonds are all one-tenth of the rates for shares. The rates for scale B are arbitrarily taken as one-tenth of the rates for scale A, so are quite low. The rates assumed are as shown in Table 6.1.1.

6.1.5 Besides allowing for the costs of altering the portfolio during the hedging process, it is desirable (or at least consistent) to allow for the initial purchase and the final sale of the hedging assets. Thus, when an initial purchase of shares is made, the quantity bought allows for the fact that costs will be payable on the sale, and then the buying costs are added. This initial charge can be calculated at the outset, and is the same for all simulations. The values for the basic model are shown in Table 6.1.2.

6.1.6 The costs are a small percentage of the option price, much lower

Table 6.1.1. Rates of transaction costs assumed

	Scale A	Scale B
Buying shares	1%	0.1%
Selling shares	0.5%	0.05%
Borrowing shares	1%	0.1%
Repaying shares	1.5%	0.15%
Buying bonds	0.1%	0.01%
Selling bonds	0.05%	0.005%
Borrowing bonds	0.1%	0.01%
Repaying bonds	0.15%	0.015%

Table 6.1.2. Initial costs on purchase, scales A and B

	Option price	Initial charge A	Percentage A	Initial charge B	Percentage B
Maxi	124.87	1.03	0.82	0.10	0.08
Mini	75.21	0.62	0.82	0.06	0.08
Call	24.79	1.05	4.24	0.10	0.40
Put	24.87	1.02	4.10	0.10	0.40

than the usual bid/offer spread for unit trusts or unit-linked life policies, but these also have to bear initial commission to agents and the initial management expenses, neither of which are allowed for here. The charges, however, for calls and puts can be higher than for maxis (and in percentage terms they are much larger), because two quite large opposite positions need to be set up.

6.1.7 When the quantity of assets is increased or reduced during the hedging process, enough is held to allow for the selling (or repayment) costs at the close. Then, at settlement the shares and bonds are assumed to be sold (or repaid), and the relevant charges are calculated too. One can calculate the present value at the outset of all charges, discounted at the bond rate. This is similar to the initial charge plus the increase in the present value of the deficit at settlement.

6.1.8 It is reasonable to assume that the effect of transaction costs will be greater the more frequent the hedging. We therefore investigate the costs for various frequencies of hedging. The results for calls and puts are no longer the same as for maxis, because the portfolios to be held carry different costs; they are, however, similar. There is no need to go much beyond some optimum position, because one must expect the transaction costs to become excessive as the frequency reduces to very short intervals.

6.1.9 The statistics for the present value of the deficit, discounted at the bond rate, for transaction costs scale A, are shown in Table 6.1.3.

6.1.10 We see that the mean PVD increases with the frequency of hedging. The standard deviation falls to a minimum with hedging either

Table 6.1.3. Statistics for present value of deficit, for all option types, investment strategy i(*b*), first assumptions, with transaction costs scale A, various frequencies of hedging

	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Maxi:							
Yearly	1.69	7.29	−20.43	47.70	19.86	23.88	29.09
Six-monthly	1.99	5.15	−16.65	32.57	14.54	16.97	20.08
Quarterly	2.45	3.68	−10.72	26.05	11.54	13.29	15.51
Two-monthly	2.89	3.13	−9.20	19.13	10.76	12.19	13.96
Monthly	3.93	2.56	−4.89	18.18	10.37	11.53	12.94
1/2 monthly	5.42	2.49	−0.13	17.12	11.43	12.40	13.45
1/4 monthly	7.61	3.02	0.71	18.28	14.18	15.02	16.76
1/8 monthly	10.64	3.98	1.22	24.04	18.62	19.46	20.39
1/16 monthly	15.03	5.53	1.79	31.05	25.55	26.43	27.45
1/32 monthly	21.08	7.75	2.17	40.67	35.90	37.02	38.11
Mini:							
Yearly	0.42	7.03	−43.09	21.48	14.14	15.89	17.81
Six-monthly	1.07	4.91	−27.11	18.98	11.39	12.88	14.49
Quarterly	1.83	3.45	−18.52	15.40	9.47	10.60	11.90
Two-monthly	2.37	2.91	−10.00	15.41	9.04	10.06	11.32
Monthly	3.51	2.33	−5.71	13.76	9.11	9.94	10.84
1/2 monthly	5.07	2.34	−0.99	14.70	10.37	11.07	11.83
1/4 monthly	7.29	2.93	0.43	16.23	13.27	13.89	14.56
1/8 monthly	10.33	3.92	1.61	20.72	17.90	18.54	19.21
1/16 monthly	14.73	5.51	2.18	28.91	25.16	25.88	26.67
1/32 monthly	20.76	7.76	2.74	39.68	35.63	36.65	37.62
Call:							
Yearly	1.72	7.31	−20.45	47.80	19.94	23.97	29.22
Six-monthly	2.01	5.17	−16.69	33.04	14.65	17.07	20.17
Quarterly	2.52	3.70	−10.72	26.66	11.69	13.45	15.68
Two-monthly	2.98	3.16	−9.32	19.60	10.95	12.41	14.21
Monthly	4.06	2.61	−4.74	18.75	10.65	11.83	13.29
1/2 monthly	5.63	2.56	0.00	17.27	11.79	12.81	13.87
1/4 monthly	7.91	3.14	0.74	18.47	14.70	15.56	16.63
1/8 monthly	11.09	4.14	1.29	24.29	19.38	20.24	21.21
1/16 monthly	15.66	5.77	1.90	31.96	26.66	27.58	28.65
1/32 monthly	21.95	8.08	2.24	43.63	37.43	38.60	39.77
Put:							
Yearly	1.46	7.09	−20.55	46.20	19.23	23.29	28.60
Six-monthly	1.81	5.01	−15.72	32.79	14.23	16.66	19.85
Quarterly	2.41	3.63	−10.64	27.64	11.52	13.28	15.56
Two-monthly	2.94	3.15	−9.07	20.49	11.07	12.54	14.34
Monthly	4.21	2.73	−4.61	19.40	11.10	12.36	13.85
1/2 monthly	6.04	2.84	0.16	19.13	12.81	13.90	15.14
1/4 monthly	8.73	3.63	1.01	21.34	16.48	17.41	18.50
1/8 monthly	12.42	4.83	1.92	28.02	22.14	23.11	24.16
1/16 monthly	17.76	6.75	2.69	37.51	30.73	31.80	32.91
1/32 monthly	25.12	9.49	3.39	51.40	43.48	44.87	46.22

Table 6.1.4. Statistics for present value of deficit, for all option types, investment strategy i(*b*), first assumptions, with transaction costs scale B, various frequencies of hedging

	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Maxi:							
Yearly	0.62	7.08	−20.87	44.90	18.17	22.09	27.27
Six-monthly	0.47	4.95	−17.30	29.51	12.35	14.68	17.71
Quarterly	0.40	3.44	−12.91	22.92	8.48	10.01	12.20
Two-monthly	0.39	2.84	−12.06	14.69	7.08	8.33	9.89
Monthly	0.44	2.01	−8.87	12.55	5.15	6.07	7.19
1/2 monthly	0.56	1.43	−6.28	8.63	4.01	4.70	5.53
1/4 monthly	0.77	1.06	−4.38	6.26	3.41	3.93	4.54
1/8 monthly	1.08	0.82	−1.98	6.10	3.15	3.54	3.99
1/16 monthly	1.50	0.75	−0.59	5.55	3.33	3.63	4.03
1/32 monthly	2.10	0.85	0.13	6.32	4.00	4.27	4.59
1/64 monthly	2.97	1.13	0.36	6.61	5.23	5.46	5.75
1/128 monthly	4.19	1.59	0.59	8.92	7.25	7.52	7.82
Mini:							
Yearly	−0.41	7.05	−44.44	20.98	13.32	15.10	17.06
Six-monthly	−0.17	4.92	−28.94	17.58	9.99	11.49	13.14
Quarterly	0.04	3.42	−22.17	13.38	7.25	8.35	9.63
Two-monthly	0.14	2.81	−13.83	12.69	6.16	7.12	8.37
Monthly	0.30	1.98	−11.30	9.73	4.73	5.45	6.32
1/2 monthly	0.49	1.41	−6.76	7.60	3.76	4.33	5.00
1/4 monthly	0.72	1.03	−3.77	6.10	3.22	3.69	4.28
1/8 monthly	1.02	0.80	−2.24	5.06	3.01	3.38	3.84
1/16 monthly	1.48	0.74	−1.03	5.42	3.29	3.57	3.94
1/32 monthly	2.08	0.85	0.21	5.19	3.95	4.19	4.44
1/64 monthly	2.94	1.12	0.39	6.34	5.17	5.39	5.63
1/128 monthly	4.17	1.58	0.51	8.27	7.16	7.40	7.64
Call:							
Yearly	0.62	7.08	−20.89	44.91	18.18	22.09	27.28
Six-monthly	0.48	4.95	−17.29	29.56	12.36	14.69	17.72
Quarterly	0.40	3.44	−12.91	22.98	8.50	10.11	12.22
Two-monthly	0.40	2.84	−12.04	14.74	7.10	8.35	9.91
Monthly	0.46	2.01	−8.86	12.60	5.17	6.10	7.22
1/2 monthly	0.58	1.43	−6.25	8.65	4.04	4.73	5.57
1/4 monthly	0.81	1.06	−4.33	6.28	3.46	3.98	4.59
1/8 monthly	1.12	0.83	−1.90	6.13	3.22	3.61	4.07
1/16 monthly	1.56	0.77	−0.52	5.61	3.44	3.73	4.15
1/32 monthly	2.20	0.88	0.13	6.50	4.16	4.43	4.76
1/64 monthly	3.11	1.18	0.38	6.84	5.45	5.69	5.98
1/128 monthly	4.38	1.66	0.62	9.34	7.57	7.85	8.16

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Table 6.1.4 (continued).

	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Put:							
Yearly	0.60	7.06	−20.90	44.75	18.11	22.03	27.22
Six-monthly	0.46	4.93	−17.23	29.54	12.31	14.65	17.69
Quarterly	0.39	3.43	−13.90	23.08	8.47	10.09	12.21
Two-monthly	0.40	2.83	−12.05	14.83	7.10	8.36	9.92
Monthly	0.48	2.01	−8.84	12.68	5.21	6.14	7.27
1/2 monthly	0.63	1.44	−6.21	8.85	4.13	4.83	5.68
1/4 monthly	0.90	1.08	−4.16	6.58	3.62	4.15	4.76
1/8 monthly	1.27	0.87	−1.62	6.35	3.47	3.88	4.34
1/16 monthly	1.79	0.85	−0.14	6.03	3.83	4.14	4.58
1/32 monthly	2.53	1.02	0.25	7.28	4.75	5.05	5.41
1/64 monthly	3.60	1.38	0.50	7.90	6.32	6.59	6.92
1/128 monthly	5.10	1.95	0.73	10.78	8.85	9.16	9.51

monthly or twice a month, and then rises. The CTEs generally reach a minimum with monthly hedging, and then rise. This suggests that, with transaction costs on scale A, hedging more frequently than monthly may not be worth the expense.

6.1.11 The results with the lower transaction costs of scale B are shown in Table 6.1.4.

6.1.12 We can see that the advantages of more frequent hedging continue to a higher frequency than for scale A. Nevertheless, all the means increase beyond monthly hedging, and all the standard deviations increase beyond hedging 16 times a month. The highest values and the CTEs reach their minimal values at frequencies of either eight or 16 times per month. Thus, even these very low transaction costs bite if hedging is frequent enough.

6.1.13 An alternative to hedging fully at regular time intervals is to use a 'distance-based' strategy, and hedge when the discrepancy between the actual holdings and the desired holdings exceeds some threshold (as suggested by Boyle & Hardy, 1997). This threshold might increase as one approaches expiry close to being at the money, when the maximum changes in position are to be expected. This approach has not been investigated.

6.1.14 A further, and very important, consideration is that we are considering here only one option contract, rather than a portfolio of contracts. A portfolio may happen to contain offsetting contracts, so that the transactions required to maintain the aggregate hedge position might be less, possibly much less, than the sum of the individual transactions. This would reduce the transaction costs, and therefore also the contingency reserves required.

Table 6.2.1. Financing the initial capital, maxi option, strategy i(b), using (97.5%, 1%, 1%) and (99%, 2%, 2%), hedging twice monthly, various transaction costs

Transaction costs	97.5% CTE	S'hdr	P'hdr	Total premium	99% CTE	S'hdr	P'hdr	Total premium
None	3.89	3.50	0.39	125.25	4.70	3.83	0.86	125.73
Scale A	12.40	6.32	6.09	130.95	13.45	6.58	6.87	131.73
Scale B	4.70	3.74	0.96	125.82	5.53	4.07	1.46	126.32

6.2 *Pricing*

In Table 6.2.1 we show the shareholder and policyholder contributions when transaction costs are allowed for. One can see that the policyholder contribution increases very greatly with the transactions costs, quite reasonably, because a certain level of transaction costs is almost certain to be incurred. Nevertheless, the uncertainty about transaction costs increases the shareholder contribution considerably also.

7. Varying the Real-World Model

7.1 *Fat-Tailed Distribution of Innovations*

7.1.1 So far we have assumed that the real-world model within which we are simulating is the Black-Scholes world, in which the values of the parameters are known, and the share price is driven by a Wiener process, which therefore has normally distributed increments. We now investigate what happens if we change these assumptions. First, we consider simulating with a different distribution for the innovations of the model for shares. It is well known that the distribution of share price changes is fatter tailed than the normal distribution, so we move in that direction.

7.1.2 One way to simulate a fat-tailed distribution is to simulate: $Z = X_1 - X_2$, where both X_1 and X_2 are independently lognormally distributed. X is lognormally distributed if $Y = \ln X$ is normally distributed. The lognormal distribution is usually parameterised by the mean and variance of the underlying normal μ and σ^2. We denote the parameters of X_1 and X_2 by subscripts. For some purposes it is convenient to reparameterise the lognormal with parameters λ and δ, putting:

$$\lambda = \exp(\mu)$$

$$\delta = \exp(\sigma^2/2).$$

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7.1.3 The moments about zero of the lognormal are then given by:

$$E[X^r] = \lambda^r \delta^{r \times r}$$

and we can calculate other features of the distribution, skewness, kurtosis, cumulants, etc., from these. We see that λ is purely a scale parameter and δ (or σ) controls the shape of the distribution.

7.1.4 If X_1 and X_2 have the same parameters, so that $\lambda_1 = \lambda_2$ and $\delta_1 = \delta_2$, then Z is symmetric, and has zero mean. We can arrange that Z has unit variance by putting:

$$\lambda_1^2 = \lambda_2^2 = 1/\{2(\delta_1^4 - \delta_1^2)\}$$

thus defining the equivalent of a unit normal (which has zero mean and unit variance). If X_1 and X_2 have different values of δ, i.e. $\delta_1 \neq \delta_2$, we can still choose λ_1 and λ_2 (or μ_1 and μ_2) so that Z has zero mean and unit variance. If $\delta_1 > \delta_2$, then Z is positively skewed, and if $\delta_1 < \delta_2$, then Z is negatively skewed. If δ_1 and δ_2 are reversed, this is equivalent to changing the sign of Z. Z is fatter tailed (higher kurtosis) than the normal distribution, and it can be made more or less fat tailed and either symmetric or skewed, within limits, by a suitable choice of δ_1 and δ_2 (or σ_1 and σ_2).

7.1.5 There are many other possible fat-tailed distributions, but normal random variates are easy to simulate, so the lognormal difference is a convenient one to use.

7.1.6 In order to choose parameters for the distribution, it is useful to investigate the actual distribution of the changes in the logarithms of share prices in the U.K. This is discussed in Appendix B. The statistics of these over the whole period and selected sub-periods are shown in Table 7.1.1. The last three periods shown are the 120-month periods, which show the smallest and the largest values of the skewness and kurtosis, the last of which has both the smallest (largest negative) skewness, −2.42, and the largest kurtosis, 15.31.

Table 7.1.1. Statistics of U.K. total return index for shares for various periods

Period	Number of differences	Mean	Standard deviation	Skewness	Kurtosis
Dec 1923 to Jun 2004	966	0.0086	0.0493	−0.08	11.59
Dec 1923 to Mar 1964	483	0.0071	0.0398	−0.74	6.66
Mar 1964 to Jun 2004	483	0.0102	0.0572	0.10	11.19
Dec 1946 to Jun 2004	690	0.0096	0.0523	0.06	11.49
Nov 1974 to Nov 1984	120	0.0228	0.0691	1.62	12.39
Jan 1949 to Jan 1959	120	0.0090	0.0386	−0.50	2.86
May 1940 to May 1950	120	0.0083	0.0384	−2.42	15.31

Table 7.1.2. Symmetric difference of lognormals, standardised: kurtosis

	$\mu_1 = \mu_2$	Half month	Month	Year	Ten years
normal		3	3	3	3
$\sigma_1 = \sigma_2 = 0.02$	35.3447	3.00	3.00	3.00	3.00
$\sigma_1 = \sigma_2 = 0.12$	5.8292	3.12	3.06	3.00	3.00
$\sigma_1 = \sigma_2 = 0.22$	3.0994	3.42	3.21	3.02	3.00
$\sigma_1 = \sigma_2 = 0.32$	2.0459	3.95	3.48	3.04	3.00
$\sigma_1 = \sigma_2 = 0.42$	1.4740	4.84	3.92	3.08	3.01
$\sigma_1 = \sigma_2 = 0.52$	1.1085	6.30	4.65	3.14	3.01
$\sigma_1 = \sigma_2 = 0.62$	0.8522	8.73	5.87	3.24	3.02
$\sigma_1 = \sigma_2 = 0.72$	0.6620	12.94	7.97	3.41	3.04
$\sigma_1 = \sigma_2 = 0.82$	0.5159	20.64	11.82	3.73	3.07
$\sigma_1 = \sigma_2 = 0.92$	0.4014	35.59	19.29	4.36	3.14

7.1.7 From these values, it looks reasonable to target a kurtosis of about 12 for monthly differences. The overall skewness is small, though large falls, such as occurred in June 1940 (−0.23), March 1974 (−0.23) and October 1987 (−0.31), produce negative skewness from time to time, whereas large rises, such as occurred in January 1975 (0.43) and February 1975 (0.22), produce positive skewness. If our distribution is chosen to be symmetric, a kurtosis of 12 for monthly steps is equivalent to a kurtosis of 21 for half-monthly steps. This can be provided approximately by choosing $\sigma_1 = \sigma_2 = 0.82$.

7.1.8 Table 7.1.2 shows the kurtosis for intervals of one half of a month, one month, one year, and ten years for a symmetric distribution with various values of σ_1 ($= \sigma_2$), from 0.02 to 0.92. Note that the kurtosis of a normal distribution is three, and as σ_1 ($= \sigma_2$) tends to zero the distribution tends to normality; but if we do set $\sigma_1 = \sigma_2 = 0$, the two lognormals collapse to point distributions at one, and the difference is always zero, so a small positive value is the smallest one practicable. In all cases the long-run experience becomes asymptotically normal, so the distribution of the final share price might be expected to be similar in all cases.

7.1.9 As a specimen, we show in Table 7.1.3 the skewness and kurtosis for a few asymmetric distributions for the same intervals. Again, the long-run experience tends towards normality.

Table 7.1.3. Asymmetric difference of lognormals, standardised: skewness and kurtosis

		Half month		Month		Year		Ten years	
σ_1	σ_2	Skew.	Kurt.	Skew.	Kurt.	Skew.	Kurt.	Skew.	Kurt.
0.82	0.62	1.69	20.15	1.19	11.57	0.34	3.71	0.11	3.07
0.82	0.72	0.92	18.50	0.66	10.75	0.19	3.65	0.06	3.06
0.82	0.92	−1.16	31.21	−0.82	17.10	−0.24	4.18	−0.08	3.12

7.1.10 Table 7.1.4 shows the results for the distributions specified above. First, the final share price is shown, which, indeed, is very similar to the normal for all the symmetric distributions. However, the asymmetric distributions are rather more skew than we might have expected.

7.1.11 Table 7.1.4 then shows the PVD for maxi, call and put options. The statistics of the PVD show quite a small mean in all cases, which indicates that the option pricing formula and hedging method are doing part of their job. However, the standard deviations for all the fatter-tailed distributions are larger than for the normal, and the highest values and the CTEs are significantly higher. Skewness makes it quite a lot worse.

7.1.12 Thus, the existence of fatter-tailed distributions for the innovations justifies a higher contingency reserve than if the distributions were known to be normally distributed.

7.2 *Uncertainty in the Parameters of the Model: a Hypermodel*

7.2.1 So far, we have assumed that the parameters of the real-world model were known. In practice, we can only estimate what appear to be reasonable values to be used. A way of dealing with this uncertainty is to use a 'hypermodel', by which we mean a model in which, for each simulation, the values of the parameters to be used are drawn from some multivariate distribution of the parameters. For the time being we remain with our assumption of a 'Black-Scholes world'. In this world the only parameters are the fixed interest rate on the bond and the mean and standard deviation of the share price increment. We assume that the initial interest rate on the bond is known. As noted in ¶5.3.5, there is little reason to get this wrong, though, of course, the assumption in the Black-Scholes model that this rate never changes is unrealistic.

7.2.2 We therefore consider uncertainty in the parameters m and s, the mean and standard deviation of the annual log return on shares. We have used $m = 0.07$ and $s = 0.2$ in our standard examples so far. We consider, again, the data-set discussed in Appendix B. Over the whole period of 966 months the monthly mean is 0.0086, equivalent to an annual mean of 12 times this or 0.1036, higher than the 0.07 we have assumed (but we have already seen that the value of m makes little difference to the hedging error). The monthly standard deviation is 0.0493, equivalent to an annual value of $\sqrt{12}$ times this, or 0.1707, rather lower than the 0.2 we have assumed.

7.2.3 If the data series were homogeneous and were normally distributed, then the standard error of our estimate of the annual mean would be 0.0055, quite a small value, but we know that the distribution is quite fat tailed, and it is likely that it is non-homogenous. One argument could be that share prices, which depend ultimately on company earnings and dividends, are influenced by inflation, and inflation has run at very different levels during the period of investigation. Another approach is to look at the range of results of the 847 (overlapping) periods of 120 months, as in Appendix B.

Table 7.1.4. Final share price and present value of deficit for various real-world share models

σ_1, σ_2	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Final share price							
Basic	245.7	174.3	21.0	2,002.9	768.4	924.7	1,155.4
Symmetric:							
0.02	245.6	174.6	17.1	2,804.0	767.5	914.1	1,145.3
0.12	245.6	174.9	17.5	2,786.0	769.6	915.8	1,146.9
0.22	245.6	175.1	17.8	2,806.7	772.0	917.7	1,146.6
0.32	245.6	175.3	18.2	2,868.1	773.7	919.9	1,147.1
0.42	245.7	175.4	18.8	2,899.1	775.5	922.2	1,146.9
0.52	245.7	175.4	19.2	2,895.2	776.6	925.0	1,146.4
0.62	245.7	175.5	18.6	2,855.5	777.6	927.9	1,146.2
0.72	245.8	175.7	18.2	2,776.4	779.2	930.9	1,150,6
0.82	245.8	176.1	17.9	2,659.0	782.0	935.2	1,159.2
0.92	245.9	176.8	17.9	2,506.2	786.5	941.1	1,174.8
Asymmetric:							
0.82, 0.62	247.1	184.7	23.1	2,596.3	819.6	991.7	1,247.8
0.82, 0.72	246.5	180.7	20.1	2,648.2	802.0	965.6	1,207.1
0.92, 0.82	246.7	182.3	19.8	2,559.7	809.0	977.8	1,230.7
0.62, 0.82	244.5	167.4	15.5	2,713.8	740.3	871.9	1,058.8
0.72, 0.82	245.1	171.3	16.5	2,724.9	759.3	901.4	1,102.9
0.82, 0.92	245.0	170.9	16.4	2,583.1	759.1	900.5	1,106.3
PVD: Maxi, call and put:							
Basic	0.02	1.41	−6.96	7.69	3.25	3.89	4.70
Symmetric:							
0.02	0.01	1.42	−6.86	12.41	3.23	3.89	4.83
0.12	0.02	1.45	−6.31	11.79	3.32	3.98	4.93
0.22	0.01	1.55	−6.83	10.98	3.62	4.37	5.43
0.32	0.01	1.72	−8.14	13.96	4.07	4.93	6.13
0.42	0.01	1.95	−9.55	18.80	4.69	5.71	7.08
0.52	0.00	2.26	−10.13	22.78	5.56	6.85	8.61
0.62	−0.02	2.69	−11.45	27.49	6.84	8.56	11.12
0.72	−0.03	3.29	−11.70	54.85	8.69	11.04	14.55
0.82	−0.05	4.14	−13.03	96.53	11.23	14.67	20.17
0.92	−0.06	5.35	−14.61	161.64	14.71	19.75	28.28
Asymmetric:							
0.82, 0.62	0.28	4.49	−11.00	108.55	12.82	17.18	24.17
0.82, 0.72	0.13	4.20	−12.15	91.09	11.68	15.42	21.39
0.92, 0.82	0.15	5.51	−13.02	158.24	15.33	20.88	30.46
0.62, 0.82	−0.32	3.40	−11.95	42.56	9.13	11.82	15.74
0.72, 0.82	−0.20	3.55	−12.67	62.18	9.44	12.19	16.32
0.82, 0.92	−0.25	4.39	−14.06	102.76	12.02	15.78	21.66

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The annualised observed means vary from −0.0167 to 0.2778. If the distribution were normal and i.i.d. (independent and identically distributed), then the standard error of a sample of 120 would be about 0.0156; the observed range is many times this. We have chosen to model the value of m as being distributed normally, with mean 0.07 (to be consistent with our calculations so far) and standard deviation 0.04.

7.2.4 The variance of a sample from a unit normal distribution is approximately distributed as χ_n^2/n, with n, the degrees of freedom, equal to the number of observations. A χ_n^2 distribution has mean n and variance $2n$ (Kendall & Stuart, 1977, p398). If there are 966 observations, χ_n^2/n is distributed almost approximately normally, with mean one and standard error about 0.046. The observed annualised variance of 0.0291, with a standard error of 0.0013, is equivalent to a standard deviation of about 0.17 with a standard error of roughly 0.004. The variance of our 847 periods of 120 months ranges from 0.0107 to 0.0779, with standard deviations ranging from 0.1035 to 0.2791. We have chosen to model the value of s^2 as being distributed normally, with mean 0.04 (again to be consistent with our calculations so far) and standard deviation 0.012, equivalent to a standard deviation with mean 0.2 and standard deviation roughly 0.03.

7.2.5 We assume first that the values of m and s^2 are normally distributed and are independent. If a population is distributed normally, then estimates of the means and standard deviations of samples from it are indeed independent, but they can easily be modelled as correlated, and we assume both positive and negative correlations, as examples. It seems, perhaps, more likely that positive correlation would exist. If the mean return on shares were relatively high, then it seems plausible that the variability of that return might also be high.

7.2.6 The results for some specimen values are shown in Table 7.2.1. We start with the basic model, with no uncertainty about the mean ($m = 0.07$)

Table 7.2.1. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(b), first assumptions, varying real-world share parameters (m and s)

Hyperised	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Neither	0.02	1.41	−6.96	7.69	3.25	3.89	4.70
m (0.04)	0.04	1.36	−6.23	7.65	3.15	3.83	4.80
s (0.012)	−0.21	4.30	−20.23	20.38	9.01	10.57	12.55
m (0.04) s (0.012)	−0.11	4.10	−21.05	24.03	9.03	10.73	12.79
ditto $r = 0.5$	−0.13	4.07	−20.60	19.06	8.75	10.38	12.37
ditto $r = -0.5$	−0.06	3.93	−15.96	24.11	8.97	10.72	13.16
m (0.02) s (0.006)	−0.01	2.41	−12.16	13.44	5.52	6.59	7.95
m (0.08) s (0.024)	−0.19	6.82	−24.78	31.34	15.14	18.00	21.50

and variance ($s^2 = 0.04$) of the share price return. We keep these values as the means of our parameters. Then we allow the value of m to be normally distributed with standard deviation 0.04, keeping s^2 fixed. Then we allow the value of s^2 to be normally distributed with variance 0.012 (and limit it to be non-negative; this limit would very rarely apply), keeping m fixed. Then we allow both to vary, with these standard deviations; then we include a correlation between the values of m and s^2 of +0.5, then −0.5. Then we halve both standard deviations, then double them, with zero correlation.

7.2.7 It can be seen that varying the value of m has a very small effect, and such effect as there is is beneficial, reducing the standard deviation. Varying the value of s^2 has a big effect, which is very slightly reduced if m also varies. The correlation also has only a small effect, which can be slightly beneficial or slightly harmful. Larger values for the standard deviations of the parameters have a bigger effect, and smaller values a lesser effect, but not exactly proportional.

7.3 *Uncertainty in the Parameters of the Innovations*

We also do not know the true values of the parameters of the fat-tailed distribution which we have used for simulating the innovations, even if the distribution we have used, the difference between two lognormals, can be taken as a true representation of real world innovations. We can 'hyperise' these parameters too. We start with a symmetric model where $\sigma_1 = \sigma_2 = 0.82$, and both values are fixed. Then we allow the value of σ_1 to be normally distributed, with mean 0.82 and standard deviation 0.1; then we vary σ_2 in the same way; then both of them, independently. Then we include a positive correlation coefficient of 0.5; then a negative one of −0.5. The results are shown in Table 7.3.1.

7.4 *Allowing for Every Complication*

7.4.1 Finally in these experiments, we can put all our complications together, and allow for a fat-tailed distribution for the innovations, parameter

Table 7.3.1. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(b), first assumptions, varying innovation parameters (σ_1 and σ_2)

Hyperised fat tails	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
$\sigma_1 = \sigma_2 = 0.82$	−0.05	4.14	−13.03	96.53	11.23	14.67	20.17
$\sigma_1 \sim N(0.82, 0.1)$	−0.03	4.49	−12.12	84.98	12.48	16.79	24.15
$\sigma_2 \sim N(0.82, 0.1)$	−0.05	4.34	−13.00	103.47	11.95	15.69	22.00
Both vary	−0.03	4.72	−13.59	91.72	13.21	17.87	26.17
Correlation +0.5	−0.04	4.53	−12.94	90.74	12.54	16.82	24.33
Correlation −0.5	−0.03	4.88	−13.10	91.73	13.78	18.77	27.71

uncertainty for the mean parameters, parameter uncertainty for the innovation parameters, and also transaction costs, on both scales A and B, considered in Section 6. To be precise, we allow for the mean rate of return on shares, m, to be distributed normally N $(0.07, 0.04^2)$, for the variance of the return on shares to be distributed N $(0.04, 0.012^2)$, independently from the mean; and for the two parameters of the innovation, σ_1 and σ_2, to be distributed independently N $(0.82, 0.1^2)$. We then allow for no transaction costs, costs on scale A and costs on scale B. The results are shown in Table 7.4.1. We show the results with transaction costs only for the maxi option; the others are reasonably similar.

7.4.2 One can see that the extra complications for which we have allowed increase both the mean and the standard deviation of the share price after ten years. The extremes are now very much further out. The extra complications have little effect on the mean PVD, but increase the standard deviation and all the quantiles considerably. Transaction costs put up almost

Table 7.4.1. Statistics for the final share price and the present value of deficit, for maxi, call and put options, investment strategy i(b), first assumptions, with all extra complications

Transaction costs	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Final share price:							
Basic	245.6	174.3	21.0	2,002.9	768.4	924.7	1,155.4
With all extras	268.4	240.1	9.6	3,518.5	1,029.2	1,282.5	1,646.6
Maxi, call, put:							
Basic	0.02	1.41	−6.96	7.69	3.25	3.89	4.70
With all extras and no costs	−0.15	5.95	−20.68	105.88	15.87	20.96	29.77
Maxi option:							
Scale A	3.85	6.52	−19.52	112.31	21.81	27.07	35.98
Scale B	0.25	5.99	−20.56	106.52	16.43	21.55	30.37

Table 7.4.2. Financing the initial capital, maxi option, strategy i(b), using (97.5%, 1%, 1%) and (99%, 2%, 2%), hedging twice monthly, various transaction costs

Pure option price: 124.87	97.5% CTE	S'hdr	P'hdr	Total premium	99% CTE	S'hdr	P'hdr	Total premium
Basic	3.89	3.50	0.39	125.25	4.70	3.83	0.86	125.73
With all complications:								
None	20.96	11.54	1.11	125.97	29.77	24.52	5.25	130.11
Scale A	27.07	21.01	6.06	130.93	35.98	26.34	9.64	134.50
Scale B	21.55	19.26	2.29	127.15	30.27	24.69	5.68	130.55

all the values, but by perhaps rather less with the complications than they did without them (see Tables 6.1.3 and 6.1.4).

7.4.3 The effect of all these complications on the premiums and fair values can now be shown, in Table 7.4.2. One can see that the initial capital required from the shareholders, if based on a CTE at 97.5% or 99%, is far from trivial, and that the extra charge to the policyholder is also very significant. Guarantees do not come cheap.

8. Bond Model B

8.1 *Assumptions*

8.1.1 We now try bond model B, which we described in ¶2.1.8, with $B(t)$ as a zcb maturing at T, whose price is driven by the bond interest rate $R(t)$, which has the stochastic differential equation:

$$dR(t) = \mu_R().dt + \sigma_R.dW_2$$

where σ_R is a constant, and $\mu_R()$ is some function of t and $R(t)$. The price of the bond $B(t)$ is given by:

$$B(t) = \exp(-(T - t).R(t)).$$

8.1.2 For calculating option prices, we do not need to define the form of $\mu_R()$, but, in order to simulate the real-world model we do. As in WWY, we put:

$$\mu_R() = \alpha_R(\mu_R - R(t))$$

with α_R and μ_R constants. This gives us the differential equation for $R(t)$ as:

$$dR(t) = \alpha_R(\mu_R - R(t)).dt + \sigma_R.dW_2$$

which is an Ornstein-Uhlenbeck process. The discrete real-world equivalent is:

$$R(t + h) = m_R + a_{R,h}.(R(t) - m_R) + s_{R,h}.z_R(t + h)$$

where $m_R = \mu_R$, $a_{R,h} = \exp(-\alpha_R h)$, $s_{R,h} = \sigma_R\sqrt{\{(1 - a_{R,h}^2)/(2\alpha_R)\}}$ and z_R is a unit normal random variable, correlated with $z_S(t + h)$ with correlation coefficient ρ. This is a first order autoregressive, or AR(1), time series model for $R(t)$. Again we use Roman letters m, a, s to denote the real-world values, because they may well be different from the option pricing parameters. We also need a value for $R(0)$.

8.1.3 The bond price is calculated from the yield as:

$$B(t) = \exp(-(T - t).R(t)).$$

8.1.4 The hedging quantities and option value are calculated using the same formulae as previously, but, as noted in ¶2.1.10, instead of $\Sigma^2 = (T - t).\sigma_S^2$, we use:

$$\Sigma^2 = (T - t).\sigma_S^2 + (T - t)^2.\rho.\sigma_R\sigma_S + (T - t)^3.\sigma_R^2/3.$$

8.1.5 In our first examples with this model we make the real-world model and the option model correspond, and we use the same parameters as in Table 3.2.1, except as shown in Table 8.1.1. These are the same as we used in WWY (but there m_R was denoted θ_R). Note that R(0) is not equal to m_R, but is taken as the same in both real-world and option pricing models.

8.1.6 The option prices and hedging quantities at time $t = 0$ are as in Table 8.1.2.

8.1.7 All these values are very close to the values shown for bond model A in Table 3.2.2, and, indeed, for the maxi, call and put are slightly smaller. This is because the assumed negative correlation between changes in share prices and changes in the bond yield offsets the additional uncertainty introduced by the stochastic bond yield. If we had chosen a correlation coefficient of −0.0208333, the values would have been identical. This critical

Table 8.1.1. Additional parameters for bond model B

Real-world parameters:	
Mean rate of interest m_R	0.065
Autoregressive parameter a_R	0.125
Standard deviation of interest rate s_R	0.0125
Initial interest rate R(0)	0.05
Option pricing model:	
Bond model	B
Standard deviation of interest rate σ_R	0.0099
Correlation coefficient ρ	−0.3
Initial interest rate R(0)	0.05

Table 8.1.2. Hedging quantities and option values, bond model B

	Share quantity	Cash quantity	Option value
Maxi	62.01	62.16	124.17
Mini	37.99	37.92	75.91
Call	62.01	−37.92	24.09
Put	−37.99	62.16	24.17

value for the correlation coefficient can readily be calculated as:

$$\rho = -(\mathrm{T} - t).\sigma_{\mathrm{R}}/(3\sigma_{\mathrm{S}}).$$

Had we used zero correlation, the option values would have been about 1.5 larger for maxi, call and put, and 1.5 smaller for the mini. The range of values for the maxi option is from 117.83, with $\rho = -1$, to 132.53, with $\rho = +1$. All these numbers alter as the term to go decreases, and, of course, with the values of the other parameters.

8.2 *Results*

8.2.1 In Table 8.2.1 we show results corresponding with those for bond model A, shown in Table 3.3.1. We omit strategy (iv). In this bond model the bond prices at times zero and T are the same as before, and the prices at the intervening dates are irrelevant for this strategy; so the results are identical. The distribution of the final share price is the same as before, so we omit that too.

8.2.2 As before, the results under strategies (i) and (ii) are the same for maxi, call and put options, and for the mini are the negative of these; and the results for strategy (iii) for call and put options are sometimes quite extreme (so are excluded from the remarks which follow). In all other cases, the means, as before, are close to zero. However, the standard deviations, the extremes and the CTEs are almost all rather *lower* than before. (This result is not true, however, for all possible combinations of parameter values for

Table 8.2.1. Statistics for deficit for different options, different investment strategies, bond model B

	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Maxi:							
Strategy i	0.05	2.25	−11.90	12.01	5.26	6.31	7.58
Strategy ii	0.05	2.91	−12.86	19.87	7.30	8.89	11.06
Strategy iii	0.06	2.68	−11.45	17.84	6.59	7.97	9.85
Mini:							
Strategy i	−0.05	2.25	−12.01	11.90	4.90	5.80	6.96
Strategy ii	−0.05	2.91	−19.87	12.86	6.11	7.19	8.49
Strategy iii	−0.03	2.38	−13.53	12.66	5.26	6.26	7.45
Call:							
Strategy i	0.05	2.25	−11.90	12.01	5.26	6.31	7.58
Strategy ii	0.05	2.91	−12.86	19.87	7.30	8.89	11.06
Strategy iii	−0.45	31.65	−2,136.98	597.71	28.43	41.67	68.11
Put:							
Strategy i	0.05	2.25	−11.90	12.01	5.26	6.31	7.58
Strategy ii	0.05	2.91	−12.86	19.87	7.30	8.89	11.06
Strategy iii	0.20	31.11	−1,082.97	1,518.67	31.15	53.18	108.42

bond model B.) Strategy (i), investing the right amount in the share and the balance in the bond, is best for all options, and distinctly better than strategies (ii) or (iii), except for strategy (ii) for the mini.

8.2.3 In Table 8.2.2 we show the results corresponding to those in Table 3.4.1, giving the PVD for different options, with different discounting methods. For calls and puts the results for the four useable strategies ((i) and (ii) combined with (a) and (b)) are the same as for maxis.

8.2.4 These results are very similar to those shown in Table 3.4.1 for bond model A, with the standard deviations, extremes and CTEs being often a little lower, but sometimes a little higher than previously, but again, these conclusions depend on the particular parameter values being used.

8.3 *Varying the Parameters: Mean Interest Rate*

8.3.1 We restrict our further investigations to those relating to the parameters of bond model B (of which there are several). First, we consider changing the real-world mean interest rate m_R. For the basic model we used $m_R = 0.065$. We now use $m_R = 0.035$, 0.05 and 0.08. The results for strategy i(b) are shown in Table 8.3.1.

Table 8.2.2. Statistics for present value of deficit (PVD), for different options, different investment strategies, different discounting methods, bond model B

	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Maxi, call and put:							
Strategy i(*a*)	0.02	1.48	−7.14	17.05	3.76	4.63	5.76
Strategy i(*b*)	0.03	1.37	−7.22	7.29	3.19	3.82	4.60
Strategy ii(*a*)	0.01	1.33	−6.80	7.19	3.12	3.74	4.52
Strategy ii(*b*)	0.03	1.76	−7.80	12.05	4.43	5.39	6.71
Maxi:							
Strategy i(*c*)	0.01	1.46	−8.40	9.28	3.64	4.48	5.59
Strategy ii(*c*)	0.00	1.51	−7.99	9.76	3.67	4.50	5.56
Strategy iii(*a*)	0.02	1.41	−6.48	13.52	3.44	4.16	5.11
Strategy iii(*b*)	0.04	1.62	−6.94	10.82	4.00	4.83	5.97
Strategy iii(*c*)	0.01	1.45	−7.56	8.94	3.49	4.27	5.23
Mini:							
Strategy i(*a*)	−0.02	1.48	−17.05	7.14	3.30	3.91	4.69
Strategy i(*b*)	−0.03	1.37	−7.29	7.22	2.97	3.52	4.22
Strategy i(*c*)	−0.02	1.24	−12.30	5.99	2.68	3.17	3.76
Strategy ii(*a*)	−0.01	1.33	−7.19	6.80	2.96	3.53	4.20
Strategy ii(*b*)	−0.03	1.76	−12.05	7.80	3.71	4.36	5.15
Strategy ii(*c*)	−0.02	1.40	−9.02	6.13	2.96	3.47	4.09
Strategy iii(*a*)	−0.01	1.36	−7.73	7.60	3.12	3.74	4.47
Strategy iii(*b*)	−0.02	1.45	−8.21	7.68	3.19	3.80	4.52
Strategy iii(*c*)	−0.01	1.20	−5.75	6.31	2.63	3.12	3.70

Table 8.3.1. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(b), bond model B, varying mean interest rate

Real-world mean interest rate m_R	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
0.035	0.03	1.37	−7.25	7.35	3.20	3.83	4.61
0.05	0.03	1.37	−7.24	7.32	3.20	3.83	4.60
0.065	0.03	1.37	−7.22	7.29	3.19	3.82	4.60
0.08	0.03	1.36	−7.18	7.23	3.19	3.82	4.59

8.3.2 We see that the mean interest rate makes almost no difference to the hedging results. This is what we should have expected. However, changing the initial interest rate would make a big difference, as it does for bond model A.

8.4 *Varying the Parameters: Autoregressive Parameter*

8.4.1 We now vary the autoregressive parameter a_R. For the basic model we used $a_R = 0.125$, a moderately slow regression to the mean. We now try $a_R = 0.05$ (slower regression) and 0.25, 0.5, 1, 2 and 4 (faster regression). We also try one special case, in which we put $a_R = 0$, so that there is no regression to the mean (the value of which is therefore irrelevant), and interest rates follow a random walk with no drift. In each of these cases we keep the instantaneous standard deviation, s_R, the same, but the annual and half-monthly standard deviations depend on the value of a_R, as shown in Table 8.4.1.

8.4.2 It is interesting that the half-monthly parameters reduce quite slowly, but the annual ones reduce much more quickly, so that, for $a_R = 4$, the yearly autoregressive parameter is quite small and the yearly standard deviation is about one third of the continuous value.

Table 8.4.1. Equivalent autoregessive parameters and standard deviations at different frequencies

Real-world autoregressive parameter a_R	Half-monthly autoregressive parameter	Yearly autoregressive parameter	Continuous standard deviation s_R	Half-monthly standard deviation	Yearly standard deviation
0.0	1.0000	1.0000	0.0125	0.0026	0.0125
0.05	0.9979	0.9512	0.0125	0.0025	0.0122
0.125	0.9948	0.8825	0.0125	0.0025	0.0118
0.25	0.9896	0.7788	0.0125	0.0025	0.0111
0.5	0.9794	0.6065	0.0125	0.0025	0.0099
1	0.9592	0.3679	0.0125	0.0025	0.0082
2	0.9200	0.1353	0.0125	0.0024	0.0062
4	0.8465	0.0183	0.0125	0.0024	0.0044

Table 8.4.2. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(*b*), bond model B, varying autoregressive parameter

Real-world autoregressive parameter a_R	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
0.0	0.03	1.37	−7.17	7.29	3.20	3.84	4.61
0.05	0.03	1.37	−7.19	7.23	3.19	3.83	4.61
0.125	0.03	1.37	−7.22	7.29	3.19	3.82	4.60
0.25	0.03	1.36	−7.25	7.35	3.19	3.82	4.59
0.5	0.03	1.36	−7.29	7.42	3.19	3.82	4.60
1	0.05	1.36	−7.32	7.47	3.21	3.84	4.62
2	0.08	1.36	−7.34	7.49	3.24	3.87	4.66
4	0.13	1.36	−7.28	7.48	3.31	3.95	4.74

8.4.3 The results are shown in Table 8.4.2, again for strategy i(*b*). We can see that the value of the autoregressive parameter makes very little difference, except that the CTE values start to rise as the value of a_R becomes large.

8.5 *Varying the Parameters: Standard Deviation of Interest Rates*

We now vary the standard deviation of the rate of interest s_R. For the basic model we used $s_R = 0.0125$. We now use 0.005, 0.025 and 0.05. The results are shown in Table 8.5.1. It is not surprising that, if the value of s_R (which is used for the real-world model) is much larger than the value of σ_R (which is used for the option pricing model), then our initial option value is too low, and we have much larger deficits. However, if the value of s_R is reduced, then our position is not improved, and, indeed, becomes slightly worse. This can be explained by considering the value of Σ, which is calculated from

$$\Sigma^2 = (\mathrm{T} - t).\sigma_S^2 + (\mathrm{T} - t)^2.\rho.\sigma_R.\sigma_S + (\mathrm{T} - t)^3.\sigma_R^2/3.$$

Keeping the values of $(\mathrm{T} - t)$, σ_S and ρ fixed, then, as σ_R varies, Σ reaches a minimum at $\sigma_R = -1.5\rho.\sigma_S/(\mathrm{T} - t)$. We are using $\mathrm{T} - t = 10$, $\sigma_S = 0.2$ and $\rho = -0.3$. With these values, Σ reaches a minimum at $\sigma_R = 0.009$. The values of the options are also at a minimum when Σ is minimised. In effect, if we use too small a value for σ_R the beneficial effect of a negative value for ρ ceases to operate, and the option price rises again. Thus, if the real-world value of s_R is less than 0.009 (other things remaining unchanged), then we have charged too little for the option by setting $\sigma_R = 0.0125$, and the deficit increases.

Table 8.5.1. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(*b*), bond model B, varying standard deviation of interest rates

Real-world interest rate standard deviation s_R	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
0.005	0.07	1.37	−6.68	7.72	3.32	3.99	4.77
0.0125	0.03	1.37	−7.22	7.29	3.19	3.82	4.60
0.025	2.54	2.56	−4.89	42.16	10.08	12.15	15.26
0.05	15.66	4.16	1.52	32.66	24.23	25.43	26.94

8.6 *Varying the Parameters: Correlation Coefficient*

We now vary the correlation coefficient between the innovations for shares and interest rates in the real-world model *r*. For the basic model we used $r = -0.3$. We now use −1, 0, 0.3 and +1. The results are shown in Table 8.6.1. We see that, if the (negative) correlation is greater than we have assumed, we, on average, make profits, and if it is smaller, or zero, or becomes positive, we make losses. However, it is economically plausible that there is a negative correlation between interest rates and share price changes, at least in the shorter term, and the evidence is also in that direction, as shown in Appendix B.

8.7 *Varying the Model: Fat-Tailed Innovations for Interest Rates*

8.7.1 Just as we did for the share innovations, we can allow fat-tailed innovations for the bond model. From Appendix B, we see that the kurtosis in the bond model has a mean value of 13.41, but ranges for the various 120-month periods from 3.0 (practically normal) to the very large 64.18. The mean value is close enough to 12 for us to use the same model (the difference of two independent lognormals), and the same range of parameters as for

Table 8.6.1. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(*b*), bond model B, varying correlation coefficient

Real-world interest correlation coefficient ρ	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
−1.0	−5.96	1.43	−12.87	1.58	−3.21	−2.81	−2.25
−0.3	0.03	1.37	−7.22	7.29	3.19	3.82	4.60
0.0	2.37	1.60	−5.08	10.66	6.22	6.89	7.72
0.3	4.59	1.97	−2.91	13.94	9.24	9.99	10.88
1.0	9.43	3.17	1.12	21.24	16.27	17.18	18.29

Table 8.7.1. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(*b*), bond model B with fat-tailed innovations; share model normal

σ_1, σ_2	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Basic	0.03	1.37	−7.22	7.29	3.19	3.82	4.60
Symmetric:							
0.02	0.04	1.37	−7.01	7.23	3.19	3.82	4.66
0.12	0.04	1.37	−6.94	7.22	3.19	3.82	4.67
0.22	0.04	1.37	−6.85	7.20	3.20	3.83	4.69
0.32	0.04	1.37	−6.74	7.17	3.21	3.84	4.71
0.42	0.04	1.38	−6.61	7.12	3.22	3.86	4.73
0.52	0.04	1.39	−6.57	7.07	3.24	3.88	4.76
0.62	0.04	1.40	−6.64	7.00	3.28	3.93	4.81
0.72	0.04	1.42	−6.72	7.21	3.35	4.02	4.93
0.82	0.03	1.46	−6.79	9.13	3.47	4.20	5.20
0.92	0.03	1.53	−6.86	15.39	3.69	4.53	5.75
Asymmetric:							
0.82, 0.62	0.06	1.48	−6.75	12.50	3.58	4.34	5.42
0.82, 0.72	0.05	1.46	−6.77	10.80	3.50	4.24	5.27
0.92, 0.82	0.02	1.48	−6.82	7.90	3.52	4.29	5.30
0.62, 0.82	0.02	1.44	−6.71	7.63	3.37	4.06	5.00
0.72, 0.82	0.02	1.44	−6.75	7.22	3.38	4.07	4.98
0.82, 0.92	0.05	1.53	−6.84	17.96	3.75	4.59	5.89

shares. We first introduce fat-tailed innovations for bonds, leaving the share model in its basic form. The results are shown in Table 8.7.1.

8.7.2 We see that fat-tailed innovations in the bond model make rather little difference to the results, much less than fat-tailed innovations in the share model. The standard deviation increases when the distribution is asymmetric, but even then not by much.

8.7.3 Part of this may be caused by the way in which correlation is effected. When there are no fat tails we use the following procedure. The correlation between the share and bond innovations is ρ; we calculate the complement of ρ, ρ_c, where $\rho^2 + \rho_c^2 = 1$; then, for each step t, we simulate a unit normal innovation for the share model $Z_S(t)$, then simulate another independent unit normal $Z_2(t)$, and calculate the unit normal innovation for the bond model $Z_B(t)$ as $\rho.Z_S(t) + \rho_c.Z_2(t)$. This works correctly when both are normally distributed. However, when the bond model is fat tailed, we generate $Z_2(t)$ as an independent unit fat-tailed variate, and add it in as before. Thus, the bond innovation consists of part of a unit normal and part of a fat-tailed innovation. A better way to deal with the simulation might be through the use of a copula instead of a single correlation coefficient, but we have not explored this.

Table 8.7.2. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(b), bond model B with fat-tailed innovations; share model also with fat-tailed innovations $\sigma_{1S} = \sigma_{2S} = 0.82$

σ_{1B}, σ_{2B}	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Basic (both normal)	0.03	1.37	−7.22	7.29	3.19	3.82	4.60
Bond normal $\sigma_{1S} = \sigma_{2S} = 0.82$	−0.02	3.87	−11.84	97.19	9.93	13.04	18.09
Symmetric:							
0.02	−0.03	3.69	−12.59	96.17	10.02	13.23	18.23
0.12	−0.03	3.69	−12.55	96.17	10.02	13.21	18.20
0.22	−0.03	3.69	−12.51	96.16	10.01	13.19	18.17
0.32	−0.03	3.69	−12.48	96.15	10.01	13.17	18.14
0.42	−0.03	3.69	−12.44	96.13	10.00	13.16	18.11
0.52	−0.03	3.69	−12.41	96.10	10.00	13.15	18.09
0.62	−0.03	3.69	−12.37	96.06	10.01	13.14	18.07
0.72	−0.03	3.70	−12.34	96.02	10.02	13.14	18.06
0.82	−0.03	3.72	−12.31	95.96	10.06	13.18	18.09
0.92	−0.03	3.75	−12.28	95.90	10.15	13.30	18.31
Asymmetric:							
0.82, 0.62	−0.01	3.73	−12.05	96.22	10.14	13.28	18.21
0.82, 0.72	−0.02	3.72	−12.18	96.09	10.09	13.23	18.14
0.92, 0.82	−0.04	3.73	−12.43	95.84	10.04	13.15	18.08
0.62, 0.82	−0.05	3.71	−12.59	95.78	10.00	13.12	18.08
0.72, 0.82	−0.04	3.71	−12.46	95.88	10.01	13.13	18.06
0.82, 0.92	−0.02	3.76	−12.14	96.02	10.20	13.37	18.42

8.7.4 We now make the share innovations fat tailed too. We use, for the share innovations, only the symmetric $\sigma_{1S} = \sigma_{2S} = 0.82$, in combination with all the σ_{1B} and σ_{2B} values for the bond model. The results are shown in Table 8.7.2.

8.7.5 We see now that the introduction of fat-tailed innovations for the share model makes a great deal of difference, larger with this bond model than with the Black-Scholes model. This may be because part of the share innovation is carried into the bond innovation because of the way we have implemented the correlation. However, thereafter fat-tailed innovations for the bond model make almost no difference, indeed slightly reduce the standard deviation and the maximum value, though generally not the CTEs.

8.8 *Varying the Model: a Hypermodel for the Real-World Parameters*

8.8.1 We can allow the parameters of the bond model to be chosen randomly for each simulation, from a prescribed distribution, on the same lines as for the share model discussed in Section 7.2. We can, of course, 'hyperise' all the parameters. We use the parameters shown in Table 8.8.1, with the correlation matrix shown in Table 8.8.2; but we introduce them by

Table 8.8.1. Means and standard deviations for parameters for hypermodel

	Mean	S Dev
Bond mean	0.065	0.03
Bond alpha	0.65	0.4
Bond variance	0.0125^2	0.0005
Correlation	−0.3	0.125
Share mean	0.07	0.04
Share variance	0.2^2	0.012

Table 8.8.2. Correlation coefficients for parameters for hypermodel

	Bond mean	Bond alpha	Bond variance	Correlation	Share mean	Share variance
Bond mean	1.0					
Bond alpha	0.3	1.0				
Bond variance	0.9	0.55	1.0			
Correlation	−0.2	−0.05	−0.2	1.0		
Share mean	0.65	0.4	0.6	−0.25	1.0	
Share variance	0.75	0.15	0.75	−0.25	0.2	1.0

stages. The only change of the means is to use 0.065 instead of 0.125 for the bond alpha. The standard deviations and correlation coefficients are taken from Appendix B, heavily rounded.

8.8.2 We introduce the hyperparameters in the following sequence:
(i) bond mean and alpha only;
(ii) bond mean and alpha and bond sigma;
(iii) bond mean and alpha and correlation coefficient;
(iv) all four bond parameters;
(v) all bond and share parameters, independent; and
(vi) all bond and share parameters, correlated.

The results are shown in Table 8.8.3.

8.8.3 Varying the bond model parameters makes rather little difference to the results. Varying the share model parameters makes much more difference, as we observed in Section 7.2.

8.8.4 One can see, from the investigations described in Appendix B, that the distributions of some of the parameters in the sample periods considered there are very fat tailed. This would justify picking the parameters in a hypermodel from a fat-tailed distribution, rather than a normal one. We have not indulged in this complication (yet).

8.9 *Varying the Model: a Hypermodel for the Innovation Parameters*

8.9.1 We now introduce variation in the innovation parameters for the bond model, as we did for the share model in Section 7.3. We use the same

Table 8.8.3. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(b), bond model B, varying real-world parameters

Hyperised	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Basic	0.02	1.41	−6.96	7.69	3.25	3.89	4.70
(i)	0.07	1.36	−7.20	7.36	3.21	3.82	4.59
(ii)	0.11	1.37	−7.09	7.58	3.28	3.91	4.67
(iii)	0.06	1.68	−7.28	7.95	3.95	4.64	5.47
(iv)	0.09	1.68	−7.19	8.19	4.02	4.72	5.54
(v)	−0.04	3.93	−19.61	23.32	8.74	10.34	12.32
(vi)	−0.16	3.76	−20.28	23.50	8.40	10.11	12.42

variations, first keeping the share innovation parameters fixed, then allowing the values of σ_{1S} and σ_{2S} to be normally distributed $N(0.82, 0.1^2)$. The results are shown in Table 8.9.1.

8.9.2 We see from Table 8.9.1 that fat-tailedness of the bond innovations has very little effect on the results, whereas fat-tailedness of the share

Table 8.9.1. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(b), bond model B, varying innovation parameters (σ_{1B} and σ_{2B}), with various share models

Hyperised fat-tails	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Share model normal:							
$\sigma_{1B} = \sigma_{2B} = 0.82$	0.03	1.46	−6.79	9.13	3.47	4.20	5.20
$\sigma_{1B} \sim N(0.82, 0.1)$	0.04	1.47	−6.77	14.67	3.54	4.29	5.31
$\sigma_{2B} \sim N(0.82, 0.1)$	0.04	1.46	−6.86	8.97	3.48	4.21	5.19
Both vary	0.04	1.48	−6.78	15.78	3.55	4.31	5.37
Correlation +0.5	0.04	1.47	−6.74	14.56	3.53	4.28	5.31
Correlation −0.5	0.04	1.49	−6.82	16.69	3.58	4.36	5.43
$\sigma_{1S} = \sigma_{2S} = 0.82$:							
$\sigma_{1B} = \sigma_{2B} = 0.82$	−0.02	3.70	−12.38	95.69	10.00	13.08	17.98
$\sigma_{1B} \sim N(0.82, 0.1)$	−0.02	3.71	−12.58	95.69	10.03	13.13	18.03
$\sigma_{2B} \sim N(0.82, 0.1)$	−0.02	3.71	−12.43	95.73	10.00	13.09	18.02
Both vary	−0.02	3.72	−12.62	95.73	10.04	13.13	18.04
Correlation +0.5	−0.02	3.71	−12.51	95.73	10.03	13.11	18.01
Correlation −0.5	−0.02	3.72	−12.71	95.72	10.06	13.15	18.04
$\sigma_{1S}, \sigma_{2S} N(0.82, 0.1)$:							
$\sigma_{1B} = \sigma_{2B} = 0.82$	−0.06	3.96	−11.80	56.00	11.09	14.83	21.30
$\sigma_{1B} \sim N(0.82, 0.1)$	−0.05	3.97	−11.91	55.99	11.13	14.88	21.32
$\sigma_{2B} \sim N(0.82, 0.1)$	−0.06	3.96	−11.86	56.03	11.10	14.84	21.33
Both vary	−0.06	3.97	−11.81	56.03	11.13	14.87	21.33
Correlation +0.5	−0.06	3.97	−11.81	56.03	11.12	14.86	21.30
Correlation −0.5	−0.06	3.98	−11.85	56.02	11.15	14.89	21.33

innovations has a substantial effect, which is made a bit worse when the parameters controlling the innovations are themselves uncertain.

8.10 *Varying the Model: All the Complications*

8.10.1 We now pack in all the features that we have explored so far, for the bond model as well as for the share model, as we did for bond model A in Section 7.4. Thus, we allow for the parameters of the models for both assets to be hyperised, and for the innovations to be fat tailed, also with hyperised parameters. We also allow for transaction costs in three ways: none, scale A and scale B. To be precise, we use the parameters of the model parameters shown in Table 8.8.1, with the correlations shown in Table 8.8.2. We use the fat-tailed parameters with each one being distributed independently N(0.82, 0.1^2). The results are shown in Table 8.10.1.

8.10.2 It is clear that the complications of a realistic model add enormously to the range of possible outcomes, and that the required CTEs are very much bigger than if it is assumed that the real world behaves exactly as the option pricing model assumes. It is made worse by transaction costs. The effect of all this on 'fair values' is shown in Table 8.10.2.

Table 8.10.1. Statistics for the present value of deficit, for maxi, call and put options, investment strategy i(*b*), bond model B, with all extra complications

Transaction costs	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Maxi, call, put:							
Basic	0.03	1.37	−7.22	7.29	3.19	3.82	4.60
With all extras and no costs	−0.26	5.07	−20.71	99.83	12.86	16.69	22.94
Maxi option with costs:							
Scale A	3.71	5.64	−20.07	107.45	18.89	22.97	29.48
Scale B	0.13	5.11	−20.57	100.59	13.43	17.29	23.57

Table 8.10.2. Financing the initial capital, maxi option, strategy i(b), using (97.5%, 1%, 1%) and (99%, 2%, 2%), hedging twice monthly, bond model B, with all complications, various transaction costs

Pure option price: 124.17	97.5% CTE	S'hdr	P'hdr	Total premium	99% CTE	S'hdr	P'hdr	Total premium
Basic model	3.82	3.43	0.39	124.56	4.60	3.75	0.85	125.02
With all complications:								
None	16.69	15.34	1.35	125.52	22.94	19.03	3.92	128.09
Scale A	22.97	17.42	5.55	129.71	29.48	21.13	8.35	132.52
Scale B	17.29	15.52	1.77	125.94	23.57	19.21	4.36	128.52

9. Using the Wilkie Model

9.1 *Introduction*

9.1.1 An alternative real-world model, which attempts to model reality rather more fully than our simplified share and bond models, is the Wilkie model. We can use this just as well for our real-world model in the simulations, provided that, as in WWY, we use stochastic bridges to interpolate at intervals shorter than the annual intervals for which the Wilkie model is designed. We can also add some of our complications, including fat-tailed innovations and hyperised parameters, to the Wilkie model, either to the annual model, or to the bridging models, or to both. We explore certain options in this section.

9.1.2 The Wilkie model, as described by Wilkie (1995), is defined for annual time steps, and simulates, *inter alia*, values for a share total return index S(t), a long-term 'consols' redemption yield C(t) and a short-term 'cash' rate B(t). In the basic model the parameters are fixed and the innovations are normally distributed. We use the parameters as defined in Wilkie (1995), with the exception that we set the mean rate of inflation at 2.5%, which seems more realistic in current conditions, and we allow for dividend yields being now 'actual' rather than gross by setting YMU = 3.75% (instead of 4.0%). The parameters that are relevant for our simulations here are: QMU = 0.025 (not 0.047), QA = 0.58, QSD = 0.0425, YW = 1.8, YMU = 3.75%, YA = 0.55, YSD = 0.155, DD = 0.13, DW = 0.58, DMU = 0.016, DY = −0.175, DB = 0.57, DSD = 0.07, CD = 0.045, CMU = 3.05%, CA = 0.9, CY = 0.34, CSD = 0.185, BA = 0.74, BMU = −0.23, BSD = 0.18. We use initial conditions as at June 2004, but modified so that the initial ten-year zero coupon yield is 0.05 (see below). The values of the relevant parameters are: I(0) = 0.029885, Y(0) = 3.16% ('actual yield'), Y(−1) = 3.43%, DM(0) = 0.032765, DE(0) = 012767, C(0) = 5.123511% (instead of the actual 4.83%), CM(0)0.047431, B(0) = 4.5%.

9.1.3 In order to provide values of the relevant variables more frequently than yearly, we use stochastic bridges, as described in WWY. We use a Brownian bridge for the total return share index S(t), and Ornstein-Uhlenbeck bridges for the logarithm of the consols yield C(t) and the log spread log(B(t)/C(t)). We use the same parameters as in WWY, except for the mean log consols yield, viz:

for the share total return: $\sigma_y = 0.2$;
for the log consols model: $\mu_y = -3.0$, $\alpha_y = 0.94$ and $\sigma_y = 0.095$; and
for the log spread model: $\mu_y = -0.23$, $\alpha_y = 0.74$ and $\sigma_y = 0.18$.

In WWY we used −2.56 for the mean log consols yield, corresponding to a yield of 7.7305%, which seems high compared with our mean inflation rate of

2.5%; the value we use now, of -3.0, corresponds to a consols yield of 4.9787%. We now assume that the bridging innovations are correlated (in WWY we assumed that they were independent), with correlation coefficients:

for shares and consols: $\rho = -0.3$;
for shares and spread: $\rho = -0.3$; and
for consols and spread: $\rho = 0.0$.

9.1.4 In order to provide a zero-coupon yield for a bond that starts as a ten-year one and reduces with each time step simulated, we use a yield curve, similar to what is described in WWY, Appendix B, but constructed in greater detail. We use C(t) and B(t) at each time t. We assume that B(t) is a continuous rate; we assume that C(t) is a rate of interest convertible annually; we convert it to a rate convertible with the frequency of simulation, e.g. half-monthly. We assume that B(t) and the adjusted C(t) are the redemption yields on bonds standing at par, with interest payable at the frequency of simulation, and are for durations zero and infinity respectively. We fit a yield curve, using the parameter $\beta = 0.39$ (as described in WWY). We then derive from this a series of zero coupon rates, at steps of the frequency of simulation. This is the same method as we have used in Appendix B.

9.1.5 Using the data as at June 2004, when B(t) = 4.5% and C(t) = 4.83%, we get an initial zcb yield (continuous) of 0.047181. When we alter C(t) to 5.123511% we get an initial zcb yield of 0.05 almost exactly. This was done so as to provide comparability with our earlier calculations.

9.1.6 The basic annual model has fixed parameters and normally distributed innovations. The basic bridging model also has fixed parameters and normally distributed innovations. Either or both of the innovations can be made 'fat tailed'. Any of the resulting four sets of parameters (annual model, annual innovations, bridging model, bridging innovations) can be 'hyperised'. There are, in all, 36 possible combinations. We do not investigate all of these.

9.2 *Basic Results*

9.2.1 We start by keeping the annual model with fixed parameters and normal innovations. We vary the bridging model in six ways:

(1) fixed parameters and normal innovations ('fn');
(2) hyperised parameters and normal innovations ('hn');
(3) fixed parameters and fat-tailed innovations ('ff');
(4) hyperised parameters and fat-tailed innovations ('hf');
(5) fixed parameters and hyperised fat-tailed innovations ('fh'); and
(6) hyperised parameters and hyperised fat-tailed innovations ('hh').

Table 9.2.1. Means and standard deviations for hyperised Wilkie bridging parameters

	Mean	Standard deviation
Share variance	0.2^2	0.012
Log consols mean	−3.0	0.05
Log consols alpha	0.94	0.1
Log consols variance	0.0095^2	0.0005
Log spread mean	−0.23	0.05
Log spread alpha	0.74	0.05
Log spread variance	0.018^2	0.01
Correlation share/consols	−0.3	0.05
Correlation share/spread	−0.3	0.05
Correlation consols/spread	0.0	0.05

9.2.2 The means and standard deviations of the hyperised bridging parameters are as shown in Table 9.2.1. The means are the same as the fixed values when these are used. The standard deviations are chosen as plausible and reasonable values, but without detailed statistical justification. We assume independence of the parameter values.

9.2.3 The means and standard deviations of the hyperised innovation parameters are all 0.82 and 0.1 respectively. As with the bridging parameters, the means are the same as the fixed values when these are used. The standard deviations are the same as those chosen for the hyperised fat-tailed innovations in Section 8.7. Again we assume independence of the parameter values.

9.2.4 The option pricing formula used is exactly as in Section 8, i.e. bond model B. Since the initial zcb rate is exactly the same, at 0.05, the option price and the hedging quantities are also the same.

9.2.5 Results for the six variations described in ¶9.2.1 are shown in Table 9.2.2, followed by those values selected from Tables in Section 8 that most closely correspond.

9.2.6 We can see that the basic Wilkie model (1) shows a mean very close to zero. This is perhaps fortuitous; if it had not been so we would have modified the bridging parameters we were using so that it was so. The correct procedure would be to choose the bridging parameters to be as realistic as one could make them, and then adjust the option parameters so that the mean was close to zero; but we wished to retain comparability with our earlier results. The standard deviation is larger than for the bond model B basic, but not very much so (1.54 against 1.37). The CTEs are larger, but again not very much so.

9.2.7 When we introduce more variability, the results from the Wilkie model remain a bit higher than for bond model B, but the same pattern is retained. The hypermodel for the bridging parameters (2) makes a lot of difference in both cases. Fat tails for the innovations (3) increase the

Table 9.2.2. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(*b*), Wilkie model, with different variations

	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Wilkie variations							
(1) 'fn'	−0.01	1.54	−6.46	8.73	3.54	4.27	5.09
(2) 'hn'	−0.08	4.45	−18.56	22.66	9.95	11.66	13.78
(3) 'ff'	−0.14	4.22	−12.93	89.48	11.65	15.20	21.16
(4) 'hf'	−0.17	5.95	−18.78	123.21	15.64	19.61	25.74
(5) 'fh'	−0.18	4.54	−12.77	68.82	12.94	17.14	24.02
(6) 'hh'	−0.21	6.23	−18.81	96.92	16.95	21.65	29.11
Bond model B							
Basic	0.03	1.37	−7.24	7.32	3.20	3.83	4.60
8.8.3 (v) cf (2)	−0.04	3.93	−19.61	23.32	8.74	10.34	12.32
8.7.2 0.82 cf (3)	−0.03	3.72	−12.31	95.96	10.06	13.18	18.09
8.9.1 both cf (5)	−0.06	3.96	−11.86	56.03	11.10	14.84	21.33
8.10.1 no costs cf (6)	−0.26	5.07	−20.71	99.83	12.86	16.69	22.94

standard deviation a bit less than does (2), but increases the CTEs by much more. A hypermodel for the innovations makes things only a bit worse (c.f. (5) with (3) and (6) with (4)). Hyperising all round (6) is clearly the worst. All these statements are true for both models.

9.2.8 Table 9.2.3 shows the financing costs for these variations of the Wilkie model. In each of variations (2) to (5), the shareholders have to put up much bigger initial funding than in case (1), and the policyholder should also be asked to pay more.

9.3 *Varying the Wilkie Model*

9.3.1 We speculated, in WWY (¶10.3.10), that variation in the parameters of the annual Wilkie model might have little effect on the results from hedging. We tried two extreme variations to test this. First, we doubled the

Table 9.2.3. Financing the initial capital, maxi option, using (97.5%, 1%, 1%) and (99%, 2%, 2%), hedging twice monthly, Wilkie variations, no transaction costs

	97.5% CTE	S'hdr	P'hdr	Total premium	99% CTE	S'hdr	P'hdr	Total premium
Wilkie variations								
(1) 'fn'	4.27	3.87	0.39	124.56	5.09	4.18	0.90	125.07
(2) 'hn'	11.66	10.63	1.04	125.20	13.78	11.37	2.41	126.58
(3) 'ff'	15.20	13.88	1.32	125.49	21.16	17.46	3.70	127.87
(4) 'hf'	19.61	17.89	1.71	125.88	25.74	21.25	4.49	128.66
(5) 'fh'	17.14	15.67	1.47	125.64	24.02	19.84	4.18	128.35
(6) 'hh'	21.65	19.77	1.88	126.04	29.11	24.03	5.08	129.24

Table 9.3.1. Means and standard deviations for hyperised Wilkie bridging parameters

	Basic	Double	Half
QSD	0.0425	0.085	0.02125
YSD	0.155	0.310	0.0775
DSD	0.07	0.14	0.035
CSD	0.185	0.037	0.0925
BSD	0.18	0.36	0.09

size of all the standard deviations in the model, and then we halved them all. These represent very large changes. The resulting values are shown in Table 9.3.1.

9.3.2 The corresponding results are shown in Table 9.3.2 for variations (1) and (6). Doubling the standard deviations makes the CTEs in (1) bigger, though the standard deviation in (1) is reduced; in (6) both standard deviation and CTEs are reduced. Halving the standard deviations makes almost no difference to the CTEs in (1), though the standard deviation in (1) is increased; in (6) both standard deviations and CTEs are increased. This all seems perverse. One possible explanation, which we have not investigated further, is that when the standard deviations in the annual model are greatly increased, the share price spends much more time either well in the money or well out of it. We have already seen that there is a tendency for the hedging errors to be smaller in these circumstances. Table 9.3.2 also shows the results of using a hypermodel for the Wilkie parameters, which we discuss next.

9.3.3 We can also vary the annual model parameters by hyperising them. We have done this for the parameters of the annual Wilkie model; we leave the innovations normally distributed. The means are the same as the fixed parameters used in the unhyperised version. The standard deviations have

Table 9.3.2. Statistics for present value of deficit, for maxi, call and put options, investment strategy i(*b*), Wilkie model, varying the annual model

	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Wilkie variations							
(1) basic	−0.01	1.54	−6.46	8.73	3.54	4.27	5.09
(1) double sds	1.10	1.47	−5.20	8.58	4.75	5.44	6.27
(1) halve sds	−0.51	1.81	−7.94	8.43	3.53	4.22	5.07
(1) hypermodel	−0.04	1.52	−6.19	8.72	3.47	4.19	5.14
(6) basic	−0.21	6.23	−18.81	96.92	16.95	21.65	29.11
(6) double sds	1.06	5.04	−15.24	73.12	15.26	19.97	27.28
(6) halve sds	−0.87	7.24	−21.18	85.47	18.74	23.72	31.23
(6) hypermodel	−0.10	6.27	−21.95	95.50	17.48	22.73	30.96

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Table 9.3.3. Means and standard deviations for hyperised Wilkie model parameters

	Mean	Standard deviation
QSD^2		
QMU	0.025	0.015
QA	0.58	0.08
QSD^2	0.0425^2	0.00034
YW	1.8	0.6
YMU	3.75%	0.2%
YA	0.55	0.1
YSD^2	0.155^2	0.00465
DD	0.13	0.08
DW	0.58	0.2
DX	0.42	0.2
DMU	0.016	0.015
DY	−0.175	0.05
DB	0.57	0.15
DSD^2	0.07^2	0.0014
CD	0.045	0.01
CMU	3.05%	0.65%
CA	0.9	0.05
CY	0.034	0.15
CSD^2	0.0185^2	0.0074
BMU	−0.23	0.1
BA	0.74	0.1
BSD^2	0.18^2	0.0072

been chosen to reflect roughly the standard errors of the parameter estimates as shown in Wilkie (1995). It is assumed that the values of the parameters are distributed normally and independently. Closer investigation of the data would be necessary to establish plausible correlation coefficients for them. The values of the means and standard deviations used are shown in Table 9.3.3. Note that normally DX = 1 − DW, which maintains 'unit gain' of dividends on inflation, but we now allow both to vary independently.

9.3.4 The results for variations (1) and (6) are shown in Table 9.3.2. It can be seen that the results for (1) are very similar to those for the unhyperised model. For (6) the CTEs are increased a bit, but by no more than halving the standard deviations did. Our conclusion is that the parameters of the Wilkie model do have an effect on the hedging results, but, for realistic variations of them, it might be relatively small.

9.3.5 Table 9.3.4 shows the financing costs for these variations. One interesting feature is that in variation (1), when the standard deviations of the annual model are halved, for the basis (97.5%, 1%, 1%), the policyholder extra contribution is negative. This is because the average result of the hedging process is to produce a small profit (mean deficit −0.51, i.e. mean profit +0.51). The shareholders get the benefit of this, and are assumed to be

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Table 9.3.4. Financing the initial capital, maxi option, using (97.5%, 1%, 1%) and (99%, 2%, 2%), hedging twice monthly, Wilkie parameter variations, no transaction costs

	97.5% CTE	S'hdr	P'hdr	Total premium	99% CTE	S'hdr	P'hdr	Total premium
Wilkie variations								
(1) basic	4.27	3.87	0.39	124.56	5.09	4.18	0.90	125.07
(1) double sds	5.44	3.93	1.51	125.68	6.27	4.24	2.03	126.20
(1) halve sds	4.22	4.28	−0.06	124.11	5.07	4.58	0.50	124.66
(1) hypermodel	4.19	3.83	0.36	124.53	5.14	4.25	0.89	125.06
(6) basic	21.65	19.77	1.88	126.04	29.11	24.03	5.08	129.24
(6) double sds	19.97	17.11	2.86	127.03	27.28	21.50	5.78	129.95
(6) halve sds	23.72	22.24	1.48	125.64	31.23	26.32	4.92	129.08
(6) hypermodel	22.73	20.66	2.08	126.24	30.96	25.46	5.49	129.66

prepared to contribute a bit more towards the initial CTE reserves. Further, since the 97.5% CTE is slightly reduced as compared with the basic model, the policyholder is not asked to contribute as much. Some other variations show negative mean deficits (see Tables 9.2.2 and 9.3.2), and while these act towards reducing the policyholder's contribution, none goes so far as this example.

9.3.6 Having reached the end of the complications that we wish to introduce, it is desirable to check whether the strategy which we have chosen to adopt is still a good one. Table 9.3.5 shows the results for 'Wilkie hypermodel variation (6)' for each of the possible strategies, for a maxi option only. We see that any of several strategies may seem acceptable. Strategies i(*a*), i(*b*), ii(*a*), iii(*a*) and iii(*b*) all have relatively low standard deviations and low CTEs. Strategy iii(*a*), investing the premium according to the option proportions, and investing the contingency reserves in shares, is

Table 9.3.5. Statistics for present value of deficit (PVD), for maxi option, different investment strategies, different discounting methods, Wilkie hypermodel, variation (6)

	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Strategy i(*a*)	−0.12	6.23	−22.39	77.99	17.56	23.16	31.85
Strategy i(*b*)	−0.10	6.27	−21.95	95.50	17.48	22.73	30.96
Strategy i(*c*)	0.22	7.65	−22.57	173.19	22.42	30.79	44.95
Strategy ii(*a*)	−0.13	6.37	−21.07	120.81	17.26	22.55	31.38
Strategy ii(*b*)	−0.16	7.79	−26.97	141.92	21.52	28.19	38.88
Strategy ii(*c*)	0.24	9.27	−21.24	381.12	24.35	33.53	50.45
Strategy iii(*a*)	−0.27	5.75	−21.41	57.21	15.39	19.66	26.24
Strategy iii(*b*)	−0.30	6.60	−23.03	80.10	17.82	22.66	29.86
Strategy iii(*c*)	0.04	7.13	−21.53	165.98	20.13	26.90	38.28

best on all measures. However, as with all results based on simulations, one should also try this with different, non-overlapping simulation runs, to check whether the sampling errors of the simulation process have affected the results.

10. Approximate Hedging

10.1 We now turn to the question asked by John Jenkins at the discussion on 'Asset models in life assurance; views from the Stochastic Accreditation Working Party', at the Faculty on 17 November 2003 (Faculty of Actuaries, 2004). He asked:

> "I have a supplementary question to those of Professor Wilkie. I note his comments on the closed form solution for valuing options and guarantees under fair values. What I do not understand is how he would allow for management actions (for example changes in investment mix) to reduce the likelihood that the guarantee will actually apply."

10.2 A later communication from him ran:

> "The normal situation I see is that companies have say a portfolio of with-profits endowments, with a sum assured (SA) and reversionary bonus (RB) guarantee at maturity. The main thing the management can change is the EBR (i.e. the proportion invested in equities). In practice they often do not formally hedge, they just 'manage' the EBR to balance the guarantee risk with the desire to achieve maximum long term returns.
>
> The sort of management action which they may well be thinking of is something like:
>
> Set the EBR at 50% (say), but if for any policy the gap between the asset share and the discounted value of SA + RB falls below 20% (say) then reduce to EBR on a sliding scale down to 20%.
>
> Thus, the closer the guarantees get to biting, the more action a company will take to reduce the likelihood that they will actually bite.
>
> It is the above sort of management action which I was querying whether the closed form solution could allow for. With a stochastic projection model, clearly one can build such things in."

10.3 Interpreting this in the context of our maxi option, where we assume 100% investment in shares in respect of the liability, rather than a standard equity backing ratio of only 50%, would perhaps mean that the proportion of the available assets invested in shares would be set at 100%, provided the option was at least 20% 'in the money', i.e. $S(t) \geq 1.2K(t)$, and would reduce linearly from 100% at $S(t) = 1.2K(t)$ to 40% at, say, $S(t) = K(t)$, with 40% if $S(t) \leq K(t)$. However, if the share is sufficiently far 'out of the money', then the share proportion tends to zero, so a better, and symmetrical, rule might be to reduce linearly to 0% at, say, $S(t) = 0.8K(t)$ and below.

10.4 Before doing any calculations, however, it is best to investigate this suggestion mathematically. We know from Section 2.1 that, for a maxi option, the hedging quantities are given by:

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$$\text{Share quantity:} \quad H_S(t) = S(t).N(d_1)$$
$$\text{Cash/bond quantity:} \quad H_B(t) = K(t).N(d_2)$$

where:

$$d_1 = \log\{S(t)/K(t)\}/\Sigma + \Sigma/2$$
$$d_2 = -\log\{S(t)/K(t)\}/\Sigma + \Sigma/2$$

and Σ is different for the two different bond models:

Model A $\quad \Sigma^2 = (T - t).\sigma_S^2$

Model B $\quad \Sigma^2 = (T - t).\sigma_S^2 + (T - t)^2.\rho.\sigma_R\sigma_S + (T - t)^3.\sigma_R^2/3.$

10.5 We now define the 'moneyness' as $M(t) = S(t)/K(t)$ and the proportions to be invested in the share and the bond as $P_S(M, t)$ and $P_B(M, t)$. We then get:

$$\begin{aligned}\text{Share proportion} = P_S(M, t) &= H_S(t)/\{H_S(t) + H_B(t)\}\\ &= S(t).N(d_1)/\{S(t).N(d_1) + K(t).N(d_2)\}\\ &= M(t).N(d_1)/\{M(t).N(d_1) + N(d_2)\}\end{aligned}$$

$$\begin{aligned}\text{Bond proportion} = P_B(M, t) &= H_B(t)/\{H_S(t) + H_B(t)\}\\ &= N(d_2)/\{M(t).N(d_1) + N(d_2)\}\end{aligned}$$

and

$$d_1 = \log\{M(t)\}/\Sigma + \Sigma/2$$
$$d_2 = -\log\{M(t)\}/\Sigma + \Sigma/2.$$

Thus, the share and bond proportions depend only on the moneyness, the term to go $(T - t)$ and the fixed parameters included in Σ.

10.6 We can, however, make this symmetrical by using log moneyness, $m(t) = \log M(t)$, in which case we have:

$$P_S(m, t) = \exp(m(t)).N(d_1)/\{\exp(m(t)).N(d_1) + N(d_2)\}$$

$$P_B(m, t) = N(d_2)/\{\exp(m(t)).N(d_1) + N(d_2)\}$$

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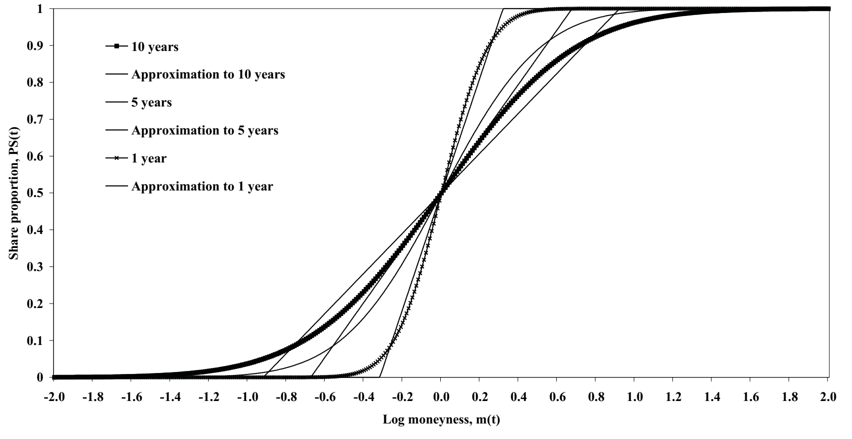

Figure 10.1. Proportion invested in shares, $P_S(m; t)$, and linear approximations, terms to go of ten years, five years, one year

with:

$$d_1 = m(t)/\Sigma + \Sigma/2$$
$$d_2 = -m(t)/\Sigma + \Sigma/2.$$

In this case, $P_S(m, t) = P_B(-m, t) = 1 - P_S(-m, t)$.

10.7 In Figure 10.1 (the S-shaped curves) we show $P_S(m, t)$, for the Black-Scholes model with $\sigma = 0.2$, graphed as a function of m, for $T - t = 10$ years (the most spread out), five years (intermediate) and one year (the steepest). We see how the graphs are symmetrical about $m = 0$, where $M = 1$, or $S(t) = K(t)$, and the option is exactly 'at the (discounted) money', and $P_S(t) = P_B(t) = 0.5$ for all t. A linear approximation to the S-shaped curves would be plausible, and a good fit for each curve is shown, based on minimising the maximum difference between the curve and the straight line. The slopes of the straight lines increase as the term to go reduces.

10.8 The three straight lines shown in Figure 10.1 can be denoted as $PA_S(m, t)$, and can be expressed as:

$$PA_S(m, t) = \max(0, \min(1, b(T - t).m))$$

where $b(T - t)$ has the values: $b(10) = 0.5428$, $b(5) = 0.74$, $b(1) = 1.57$. The lines reach zero close to $m = -0.92$, -0.68 and -0.32 respectively, and reach unity at the corresponding positive values. These correspond to ranges of the moneyness of 39.9% to 250.9%, 50.7% to 197.4% and 72.6% to 137.7% respectively, much shallower slopes than the linear 80% to 20% suggested in ¶10.3.

10.9 We can easily test out these hedging strategies within some real-world model. We still assume that the liability is defined in terms of a share index, not in relation to the actual portfolio of the office. We then take two extreme examples. First we assume that the real world corresponds exactly with the Black-Scholes model, as in the examples in Section 3. We consider only a maxi option, because these approximate strategies are designed to hedge only that type of option. The initial amount invested is taken to be equal to the Black-Scholes option price. Hedging is carried out twice per month, and there are no transaction costs. The deficit is assumed to be financed by investment in the bond (sub-strategy (*b*) from Section 3). The statistics of the present value of the deficit are shown in Table 10.1.

10.10 We see that the standard deviations of the PVD are very much larger than if hedging is carried out exactly. However, the strategies can turn out to be very profitable, since the lowest deficits in each case are large and negative. However, the worst results and the quantile reserves are much higher than in the basic case. In this case, not hedging properly is extremely risky, and requires large contingency reserves, in effect large 'mismatching reserves'. The best strategy, among those tried, is that based on log moneyness, scaled from −0.68 to +0.68, which corresponds to a log linear approximation to the correct hedging proportions at term five years (with five years to go), about the middle of the duration of the option.

10.11 Our second example uses the Wilkie model, in one of its more extreme varieties, with fat-tails in the bridging, and with hypermodels for the annual parameters, the bridging parameters, and the parameters that control the fat tails. We assume bond model B for the initial option price. This corresponds with the model whose results are shown in the last line of Table 9.3.2. The results are shown in Table 10.2.

10.12 In this case, all the standard deviations and all the CTEs are increased, as compared with variation (6). However, the increase to allow for mismatching is less than that in our first example, and in the last two rows the increase in the CTEs is not enormous. Perhaps we can explain this by saying that, if the real-world model were precisely known, as in our first example, not hedging in accordance with that model requires a large

Table 10.1. Statistics for present value of deficit, for maxi option, approximate investment strategies 1 to 4, Black-Scholes model

Strategy	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Basic	0.02	1.41	−6.96	7.69	3.25	3.89	4.70
S1 80/120	−2.68	20.06	−188.04	49.28	26.16	29.25	32.94
S2 ±0.32	−1.69	14.33	−141.60	38.08	17.36	19.31	21.76
S3 ±0.68	0.51	6.74	−32.32	24.62	12.83	14.21	16.02
S4 ±0.92	1.89	11.53	−18.64	80.39	25.73	28.87	33.59

Table 10.2. Statistics for present value of deficit, for maxi option, approximate investment strategies 1 to 4, Wilkie hypermodel, variation (6)

Strategy	Mean	Standard deviation	Lowest	Highest	95% CTE	97.5% CTE	99% CTE
Variation (6)	−0.10	6.27	−21.95	95.50	17.48	22.73	30.96
S1 80/120	2.72	16.83	−106.48	113.86	41.80	49.84	61.05
S2 ±0.32	2.06	12.60	−82.28	102.05	32.39	39.41	49.29
S3 ±0.68	−0.28	8.63	−22.54	77.17	20.60	25.17	32.08
S4 ±0.92	−1.20	10.98	−22.68	63.69	24.49	28.71	34.35

Table 10.3. Financing the initial capital, maxi option, using (97.5%, 1%, 1%) and (99%, 2%, 2%), approximate hedging twice monthly, no transaction costs

	97.5% CTE	S'hdr	P'hdr	Total premium	99% CTE	S'hdr	P'hdr	Total premium
Example 1 option price 124.87								
Basic	3.89	3.50	0.39	125.25	4.70	3.83	0.86	125.73
S1 80/120	29.25	28.89	0.35	125.22	32.94	29.21	3.73	128.59
S2 ±0.32	19.31	19.00	0.31	125.17	21.76	19.23	2.53	127.40
S3 ±0.68	14.21	12.39	1.81	126.68	16.02	12.72	3.30	128.17
S4 ±0.92	28.87	24.42	4.45	129.32	33.59	26.00	7.59	132.46
Example 2 option price 124.17								
Basic	22.73	20.66	2.08	126.24	30.96	25.46	5.49	129.66
S1 80/120	49.84	42.63	7.21	131.27	61.05	48.83	13.22	137.39
S2 ±0.32	39.41	33.80	5.61	129.78	49.29	38.73	10.56	134.73
S3 ±0.68	25.17	23.03	2.15	126.31	32.08	26.53	5.55	129.71
S4 ±0.92	28.71	27.07	1.64	125.81	34.35	29.16	5.19	129.36

mismatching reserve, but if, on the other hand, there is uncertainty about the real-world model and the values of its parameters, the contingency reserve required to allow for this model/parameter uncertainty is already large, and the mismatching reserve required for hedging badly is relatively less.

10.13 We conclude by showing the pricing corresponding to the examples in this section, in Table 10.3. In the first example, the CTEs are all greatly increased as compared with the basic model, and the contributions of both shareholder and policyholder are also increased. In the second example, this is true to a much greater extent for the first two approximate hedging strategies shown, but for S3 the policyholder contribution is only slightly increased as compared with the basic model, and for S4 it is reduced, more for (97.5%, 1%, 1%) than for (99%, 2%, 2%). Again, this can be explained because the mean deficit is sufficiently negative (−1.20) to justify the shareholders contributing more than simply a fraction of the increase in the required CTE.

References

Baxter, M. & Rennie, A. (1996). *Financial calculus.* Cambridge University Press.

Boyle, P.P. & Emmanuel, D. (1980). Discretely adjusted option hedges. *Journal of Financial Economics*, **8**, 259-282.

Boyle, P.P. & Hardy, M.R. (1997). Reserving for maturity guarantees: two approaches. *Insurance: Mathematics and Economics*, **21**, 113-127.

Dullaway, D. & Needleman, P.D. (2004). Realistic liabilities and risk capital margins for with-profits business. *British Actuarial Journal*, **10**, 185-316.

Faculty of Actuaries (2004). Asset models in life assurance; views from the Stochastic Accreditation Working Party, a discussion meeting. *British Actuarial Journal*, **10**, 153-165.

Ford, A., Benjamin, S., Gillespie, R.G., Hager, D.P., Loades, D.H., Rowe, B.N., Ryan, J.P., Smith, P. & Wilkie, A.D. (1980). Report of the Maturity Guarantees Working Party. *Journal of the Institute of Actuaries*, **107**, 103-212.

Kendall, M.G. & Stuart, A. (1977). *The advanced theory of statistics, volume 1, distribution theory.* 4th edition. Charles Griffin & Company, London.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, **5**, 177-188.

Wilkie, A.D. (1995). More on a stochastic investment model for actuarial use. *British Actuarial Journal*, **1**, 777-964.

Wilkie, A.D., Waters, H.R. & Yang, S. (2003). Reserving, pricing and hedging for policies with guaranteed annuity options. *British Actuarial Journal* , **9**, 263-425.

Willmot, P. (1998). *The theory and practice of financial engineering*. Wiley.

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APPENDIX A

DERIVATION OF OPTION FORMULAE

A.1 *Assumptions*

A.1.1 In this Appendix we summarise the derivation of options prices and hedging quantities using our bond model B. In the course of this we also derive the Black-Scholes option formula (bond model A). We follow closely Appendix C of WWY, and refer to the results therein.

A.1.2 We start at time $t = 0$, and consider European call, put, maxi and mini options expiring at time T. We assume two traded assets, a share and a zero coupon bond, a 'zcb' maturing at time T. The market price of a unit share at time t is $S(t)$, and at time T it has a value $S(T)$. The zcb pays one at time T, and prior to that has value $B(t, T)$ or just $B(t)$. $B(T) = 1$. The exercise price of all the options is K. The amount of the bond to match the exercise price is therefore also K, with value at time t of $K.B(t) = K(t)$.

A.1.3 We now consider the Brownian motions 'driving' the prices. We assume two separate Brownian motions, W_1, and W_2. W_1 and W_2 have instantaneous correlation ρ. The W_is are related to two independent Brownian motions, Z_1, and Z_2:

$$dW_1 = dZ_1$$
$$dW_2 = \rho.dZ_1 + \rho_c.dZ_2$$

where:

$$\rho^2 + \rho_c^2 = 1.$$

A.1.4 We assume that the share price, $S(t)$, is driven by the stochastic differential equation:

$$dS(t) = \mu_S().S(t).dt + \sigma_S.S(t).dW_1$$

where σ_S is a constant, and $\mu_S()$ is some function of t and $S(t)$, to be defined later. For shares in practice we choose $\mu_S() = \mu_S$.

A.1.5 In bond model A, the zcb price $B(t)$ is derived from the zcb interest rate $R(t)$, which is assumed to be constant, and equal to its initial value:

$$dR(t) = r = r_0.$$

A.1.6 In bond model B, we let the zcb price $B(t)$ be driven by the zcb interest rate $R(t)$, which has the stochastic differential equation:

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$$dR(t) = \mu_R().dt + \sigma_R.dW_2$$

where σ_R is a constant, and $\mu_R()$ is some function of t and $R(t)$, similar to $\mu_S()$. In practice, we shall choose $\mu_R() = \alpha_R(\theta_R - R(t))$, an Ornstein-Uhlenbeck process. This is similar to the Vasicek (1977) model, but this is usually applied to the short rate, rather than to a zcb rate for constant maturity date.

A.1.7 In both models, the zcb price B(t) is related to the zcb interest rate R(t), by:

$$B(t) = \exp(-(T - t).R(t))$$

but in model A it is deterministic, in model B stochastic.

A.2 *Stochastic Derivatives*

A.2.1 We put $C(t) = 1/B(t)$, and express it as a function of R(t):

$$C(t) = 1/B(t) = \exp((T - t).R(t)).$$

For bond model A, C(t) is deterministic, with:

$$C(t) = \exp((T - t).r)$$
$$dC(t) = -r\exp((T - t).r) = -r.dC(t).dt.$$

A.2.2 In bond model B, C(t) is stochastic, and we use Ito's formula (see WWY C.6) to get the differential equation of C(t). We have $C(t) = f(R(t))$, with derivatives:

$$f'() = \partial C/\partial R = (T - t).\exp((T - t).R(t)) = (T - t).C(t)$$

and

$$f''() = \partial^2 C/\partial R^2 = (T - t)^2.\exp((T - t).R(t)) = (T - t)^2.C(t).$$

Also:

$$\partial C/\partial t = -R(t).\exp((T - t).R(t)) = -R(t).C(t).$$

We also have:

$$dR(t) = \mu_R().dt + \sigma_R.dW_2.$$

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Hence:

$$\begin{aligned} dC(t) &= (T - t).C(t).dR + \tfrac{1}{2}\sigma_R^2.(T - t)^2.C(t).dt - R(t).C(t).dt \\ &= C(t).\{(T - t).\mu_R() + \tfrac{1}{2}\sigma_R^2.(T - t)^2 - R(t)\}.dt + (T - t).C(t).\sigma_R.dW_2. \end{aligned}$$

A.2.3 We now choose the zcb as numeraire. The value of the share relative to the zcb is $S(t)/B(t) = S(t).C(t)$, which we denote as $H(t)$. The value of the zcb relative to itself is unity.

A.2.4 We now get the stochastic differential equation for $H(t)$. We use the product rule (WWY, C.7.2):

$$d(XY) = X.dY + Y.dX + \rho_{XY}.\sigma_X.\sigma_Y.dt.$$

A.2.5 In bond model A we have:

$$dS(t) = \mu_S().S(t).dt + \sigma_S.S(t).dW_1$$

and

$$dC(t) = -r.C(t).dt$$

so:

$$\begin{aligned} dH(t) &= S(t).dC(t) + C(t).dS(t) \\ &= S(t).(-r.C(t).dt) + C(t).(\mu_S().S(t).dt + \sigma_S.S(t).dW_1) \\ &= H(t).((\mu_S() - r).dt + \sigma_S.dW_1) \\ &= H(t).((\mu_S() - r).dt + \sigma_S.dZ_1). \end{aligned}$$

A.2.5 In bond model B we have:

$$dS(t) = \mu_S().S(t).dt + \sigma_S.S(t).dW_1 = \mu_S().S(t).dt + \sigma_S.S(t).dZ_1$$

and

$$\begin{aligned} dC(t) &= C(t).\{(T - t).\mu_R() + \tfrac{1}{2}\sigma_R^2.(T - t)^2 - R(t)\}.dt + (T - t).C(t).\sigma_R.dW_2 \\ &= C(t).\{(T - t).\mu_R() + \tfrac{1}{2}\sigma_R^2.(T - t)^2 - R(t)\}.dt + (T - t).C(t).\sigma_R(\rho.dZ_1 + \rho_c.dZ_2) \end{aligned}$$

so:

$$\begin{aligned} dH(t) &= S(t).dC(t) + C(t).dS(t) + \{\sigma_S.S(t).(T - t).C(t).\rho.\sigma_R\}.dt \\ &= H(t)[(T - t).\mu_R() + \mu_S() + \tfrac{1}{2}\sigma_R^2(T - t)^2 - R(t) + (T - t).\rho.\sigma_S.\sigma_R].dt \\ &\quad + H(t).[\{\sigma_S + (T - t).\rho.\sigma_R\}.dZ_1 + (T - t).\rho_c.\sigma_R.dZ_2]. \end{aligned}$$

A.3 *Equivalent Martingales*

A.3.1 We now change the stochastic differential equations so that the process for each of the tradeables (relative to the new numeraire) is a martingale. We do this by adjusting the Zs to new values:

$$Z_i^*(t) = Z_i(t) + \int \gamma_i(t).dt$$

or:

$$dZ_i^* = dZ_i + \gamma_i(t)$$

in such a way that the tradeables are martingales. We denote adjusted values by an asterisk.

A.3.2 The zcb is easy in either model, since its value relative to the new numeraire is always one.

A.3.3 The value of the share, relative to the new numeraire, is $H(t)$, with stochastic differential equation in bond model A:

$$dH(t) = H(t).((\mu_S() - r).dt + \sigma_S.dZ_1).$$

We replace dZ_1 by $dZ_1^* - \gamma_1(t)dt$, with:

$$\gamma_1(t) = (\mu_S() - r)/\sigma_S$$

to give:

$$dH^*(t) = H^*(t).\sigma_S.dZ_1^*$$

so that $H^*(t)$ is a martingale.

A.3.4 In bond model B:

$$\begin{aligned} dH(t) = {} & H(t)[(T-t).\mu_R() + \mu_S() + \tfrac{1}{2}\sigma_R^2(T-t)^2 - R(t) + (T-t).\rho.\sigma_S.\sigma_R].dt \\ & + H(t).[\{\sigma_S + (T-t).\rho.\sigma_R\}.dZ_1 + (T-t).\rho_c.\sigma_R.dZ_2]. \end{aligned}$$

We define a new Brownian motion Z_3, with:

$$\sigma_3(t).dZ_3 = \{\sigma_S + (T-t).\rho.\sigma_R\}.dZ_1 + (T-t).\rho_c.\sigma_R.dZ_2$$

and

$$\begin{aligned} \sigma_3(t)^2 &= \{\sigma_S + (T-t).\rho.\sigma_R\}^2 + \{(T-t).\rho_c.\sigma_R\}^2 \\ &= \sigma_S^2 + 2(T-t).\rho.\sigma_S.\sigma_R + (T-t)^2.\sigma_R^2 \end{aligned}$$

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so that:

$$\begin{aligned} dH(t) = {} & H(t).[(T-t).\mu_R() + \mu_S() + \tfrac{1}{2}\sigma_R^2(T-t)^2 - R(t) + (T-t).\rho.\sigma_S.\sigma_R].dt \\ & + H(t).\sigma_3(t).dZ_3 \end{aligned}$$

and we replace dZ_3 by $dZ_3^* - \gamma_3(t)dt$ with:

$$\gamma_3(t) = [(T-t).\mu_R() + \mu_S() + \tfrac{1}{2}\sigma_R^2(T-t)^2 - R(t) + (T-t).\rho.\sigma_S.\sigma_R]/\sigma_3(t)$$

to give:

$$dH^*(t) = H^*(t).\sigma_3(t).dZ_3^*$$

so that $H^*(t)$ is also a martingale.

A.4 *The Option Payoff*

A.4.1 The value of any payoff X, at time T, which is a function of $H(t)$, can now be calculated as the expected value of X, expressed as a function of $H^*(t)$, i.e. under the equivalent martingale measure. For the various options we have an exercise price of K, with value at time t of $K(t)$. For the maxi option, we can express the payoff as:

$$X(T) = \max(S(T), K(T)).$$

We express this in terms of the numeraire, so that the payoff is:

$$\begin{aligned} X(T)/B(T) &= \max(S(T), K(T))/B(T) \\ &= \max(H(T), K(T)). \end{aligned}$$

Thus, the value of the maxi option in terms of the numeraire is:

$$E^*[X(T)/B(T)] = E[\max(H^*(T), K(T))].$$

Re-expressed in pounds at time zero, it is:

$$V(0) = B(0).E^*[X(T)/B(T)] = B(0).E[\max(H^*(T), K(T))].$$

We treat time zero as the starting point of the option, with it being exercised T time units later.

A.4.2 We now consider the distribution of $H^*(T)$. We have, in the two bond models:

$$dH^*(t) = H^*(t).\sigma_S.dZ_1^*$$

and

$$dH^*(t) = H^*(t).\sigma_3(t).dZ_3^*$$

with:

$$\sigma_3(t)^2 = \sigma_S^2 + 2(T-t).\rho.\sigma_S.\sigma_R + (T-t)^2.\sigma_R^2.$$

A.4.3 Let $L(t) = \log(H^*(t))$. In bond model A we get:

$$\begin{aligned} dL(t) &= \{1/H^*(t)\}.H^*(t).\sigma_S.dZ_1^* + \tfrac{1}{2}H^*(t)^2.\sigma_S^2.(-1/H^*(t)^2).dt \\ &= -\tfrac{1}{2}\sigma_S^2.dt + \sigma_S.dZ_1^*. \end{aligned}$$

We put:

$$\begin{aligned} L(T) &= L(0) + \int_0^T\{-\tfrac{1}{2}\sigma_S^2.dt + \sigma_S.dZ_1^*\} \\ &= L(0) - \int_0^T \tfrac{1}{2}\sigma_S^2.dt + \int_0^T \sigma_S.dZ_1^*. \end{aligned}$$

The first integral is deterministic, the second stochastic. The second integral is normally distributed with mean zero, and variance: $\int_0^T \sigma_S^2.dt$.
Thus, L(T) is normally distributed with mean:

$$E[L(T)] = L(0) - \int_0^T \tfrac{1}{2}\sigma_S^2.dt$$

and variance:

$$Var[L(T)] = \int_0^T \sigma_S^2.dt = T.\sigma_S^2.$$

Since $L(T) = \log H^*(T)$ is normally distributed, $H^*(T)$ is lognormally distributed.

A.4.4 In bond model B the same arguments lead us to:

$$\begin{aligned} dL(t) &= -\tfrac{1}{2}\sigma_3(t)^2.dt + \sigma_3(t).dZ_3^* \\ L(T) &= L(0) + \int_0^T\{-\tfrac{1}{2}\sigma_3(t)^2.dt + \sigma_3(t).dZ_3^*\} \\ &= L(0) - \int_0^T \tfrac{1}{2}\sigma_3(t)^2.dt + \int_0^T \sigma_3.dZ_3^*. \end{aligned}$$

And L(T) is normally distributed, with mean:

$$E[L(T)] = L(0) - \tfrac{1}{2}T.\sigma_S^2\sigma_S.\sigma_R + T^3.\sigma_R^2/3$$

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and variance:

$$\begin{aligned}\mathrm{Var}[\mathrm{L(T)}] &= \textstyle\int_0^{\mathrm{T}} \sigma_3(t)^2.\mathrm{d}t \\ &= [\mathrm{T}.\sigma_{\mathrm{S}}^2 + \mathrm{T}^2.\rho.\sigma_{\mathrm{S}}.\sigma_{\mathrm{R}} + \mathrm{T}^3.\sigma_{\mathrm{R}}^2/3].\end{aligned}$$

Again $\mathrm{H}^*(\mathrm{T})$ is lognormally distributed.

A.4.5 We put in bond model A:

$$\Sigma^2 = \mathrm{T}.\sigma_{\mathrm{S}}^2$$

and in bond model B:

$$\Sigma^2 = \mathrm{T}.\sigma_{\mathrm{S}}^2 + \mathrm{T}^2.\rho.\sigma_{\mathrm{S}}.\sigma_{\mathrm{R}} + \mathrm{T}^3.\sigma_{\mathrm{R}}^2/3$$

and L(T) has mean $\mathrm{L}(0) - \frac{1}{2}\Sigma^2$ and variance Σ^2. Both models are now expressed similarly.

A.4.6. To calculate the value of the maxi option, we note that, if X is normally distributed, with mean μ and variance σ^2, then:

$$\begin{aligned}\mathrm{E}[\mathrm{Y}^r; a, b] &= \textstyle\int_a^b y^r.f_{\mathrm{Y}}(y).\mathrm{d}y \\ &= \exp(r\mu + r^2\sigma^2/2).[\mathrm{N}\{(\log b - \mu - r\sigma^2)/\sigma\} - \mathrm{N}\{(\log a - \mu - r\sigma^2)/\sigma\}].\end{aligned}$$

Whence:

$$\mathrm{E}[\mathrm{Y}; a, \infty] = \exp(\mu + \tfrac{1}{2}\sigma^2).[1 - \mathrm{N}\{(\log a - \mu - \sigma^2)/\sigma\}]$$

and

$$\mathrm{E}[1; 0, b] = \mathrm{N}\{(\log b - \mu)/\sigma\} - 0.$$

A.4.7 We put: $\mu = \mathrm{L}(0) - \frac{1}{2}\Sigma^2$ and $\sigma = \Sigma$. $\mathrm{L}(0) = \log \mathrm{H}^*(0) = \log \mathrm{H}(0) = \log(\mathrm{S}(0)/\mathrm{B}(0))$. The value of the maxi option, expressed in units of the numeraire, is:

$$\mathrm{E}[\max(\mathrm{H}^*(\mathrm{T}), \mathrm{K})] = \textstyle\int_0^{\mathrm{K}} \mathrm{K}.f(\mathrm{H}^*).\mathrm{dH}^* + \int_{\mathrm{K}}^{\infty} \mathrm{H}^*(\mathrm{T}).f(\mathrm{H}^*).\mathrm{dH}^*$$

and expressed in units of currency is:

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$$\begin{aligned}
V(0) &= B(0).E[\max(H^*(T), K)] \\
&= B(0).[\textstyle\int_0^K K.f(H^*).dH^* + \int_K^\infty H^*(T).f(H^*).dH^*] \\
&= B(0).[K.N\{(\log K - \mu)/\sigma\} + \exp(\mu + \tfrac{1}{2}\sigma^2).[1 - N\{(\log K - \mu - \sigma^2)/\sigma\}] \\
&= S(0).[N\{(\log(S(0)/K(0)))/\Sigma + \tfrac{1}{2}\Sigma\}] \\
&\quad + K(0).[1 - N\{(\log(S(0)/K(0)))/\Sigma - \tfrac{1}{2}\Sigma\}.
\end{aligned}$$

A.4.8 The formula above applies equally at general time t if S(0) and K(0) are replaced by S(t) and K(t), T is replaced by $T - t$, and $1 - N(x)$ is replaced by $N(-x)$, giving:

$$V(t) = S(t).[N\{(\log(S(t)/K(t)))/\Sigma + \tfrac{1}{2}\Sigma\}] + K(t).[N\{-(\log(S(t)/K(t)))/\Sigma + \tfrac{1}{2}\Sigma\}]$$

or

$$V(t) = S(t).N(d_1) + K(t).N(d_2)$$

with:

$$\begin{aligned}
d_1 &= (\log(S(t)/K(t)))/\Sigma + \tfrac{1}{2}\Sigma \\
d_2 &= -(\log(S(t)/K(t)))/\Sigma + \tfrac{1}{2}\Sigma
\end{aligned}$$

where, in bond model A:

$$\Sigma^2 = (T - t).\sigma_S^2$$

and in bond model B:

$$\Sigma^2 = (T - t).\sigma_S^2 + (T - t)^2.\rho.\sigma_S.\sigma_R + (T - t)^3.\sigma_R^2/3.$$

A.5 *The Hedging Quantities*

A.5.1 We now find the hedging quantities. We define the amounts to be invested in the share and the bond to be $\varphi_S(t)$ and $\varphi_B(t)$ respectively, so that:

$$V(t) = \varphi_S(t) + \varphi_B(t)$$

and put:

$$\begin{aligned}
\varphi_H(t) &= \varphi_S(t)/B(t) \\
\varphi_A(t) &= \varphi_B(t)/B(t) \\
U(t) &= V(t)/B(t) = H(t).N(d_1) + K.N(d_2).
\end{aligned}$$

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A.5.2 Then, as in WWY C10, we calculate:

$$\begin{aligned} dU(t) &= d\varphi_H(t) + d\varphi_A(t) \\ &= \varphi_H(t)/H(t).dH(t) + \varphi_A(t)/A(t).dA(t) \end{aligned}$$

where A(t) is the value of the zcb in terms of the zcb, so $A(t) = 1$ and $dA(t) = 0$ for all t. We then find that

$$dU(t) = \partial U/\partial H.dH(t) + \text{non-stochastic terms}$$

so we need:

$$\varphi_H(t)/H(t) = \partial U/\partial H$$

and

$$\varphi_A(t) = U(t) - \varphi_H(t).$$

A.5.3
$$\begin{aligned} \partial U/\partial H &= \partial\{H(t).N(d_1) + K.N(d_2)\}/\partial H \\ &= N(d_1) \end{aligned}$$

so:

$$\begin{aligned} \varphi_H(t) &= H(t).\partial U/\partial H = H(t).N(d_1) \\ \varphi_A(t) &= U(t) - \varphi_H(t) \end{aligned}$$

whence:

$$\begin{aligned} \varphi_S(t) &= \varphi_H(t).B(t) \\ \varphi_B(t) &= \varphi_A(t).B(t) \end{aligned}$$

or, directly:

$$\begin{aligned} \varphi_S(t) &= S(t).N(d_1) \\ \varphi_B(t) &= V(t) - \varphi_S(t) = K(t).N(d_2). \end{aligned}$$

This applies to both bond models.

A.6 *Simulating the Real-World Equivalent Model*

A.6.1 In many of our investigations, we use as the real-world model the model that corresponds with the option pricing model, which we simulate over time steps of length h. Since $\mu_S()$ and $\mu_R()$ do not enter the option pricing formula, we can make them any functions that we like (within limits). We therefore choose $\mu_S() = \mu_S$, a constant, and $\mu_R() = \alpha_R(\mu_R - R(t))$.

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A.6.2 The differential equation for S(t) is:

$$dS(t) = \mu_S.S(t).dt + \sigma_S.S(t).dW_1$$

from which we can derive (using Ito again):

$$d \log S(t) = (\mu_S - \tfrac{1}{2}\sigma_S^2).dt + \sigma_S.dW_1$$

whence:

$$\log S(t+h) = \log S(t) + m_{S,h} + s_{S,h}.W_1(t+h)$$

where $m_{S,h} = (\mu_S - \frac{1}{2}\sigma_S^2)h$ and $s_{S,h} = \sigma_S\sqrt{h}$, and W_1 is a unit normal random variable. This is a random walk model with drift for $\log S(t)$.

A.6.3 The differential equation for R(t) is:

$$dR(t) = \alpha_R(\mu_R - R(t)).dt + \sigma_R.dW_2$$

which is an Ornstein-Uhlenbeck process, whence:

$$R(t+h) = m_R + a_{R,h}.(R(t) - m_R) + s_{R,h}.W_2(t+h)$$

with $m_R = \mu_R$, $\alpha_{R,h} = \exp(-\alpha_R h)$, $s_{R,h} = \sigma_R\sqrt{\{(1-\alpha_{R,h}^2)/(2\alpha_R)\}}$, and W_2 is a unit normal random variable. This is a first order autoregressive, or AR(1), time series model for R(t).

A.6.4 Note that W_1 and W_2 are related through:

$$W_1 = Z_1$$
$$W_2 = \rho.Z_1 + \rho_c.Z_2$$

where Z_1 and Z_2 are independent unit normal variates.

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APPENDIX B

DATA ANALYSIS

B.1 In this Appendix we describe the analyses we have carried out into actual data, in order to support some of the numerical assumptions which we have made in the paper.

B.2 We have available three series, each at monthly intervals, with end-of-month values for December 1923 to June 2004, inclusive, giving 967 values of each series. The series represent:

(i) A total return (rolled-up) share index on U.K. shares. In recent years this has been the total return index for the All-Share Index in the FTSE-Actuaries U.K. Share Indices. In earlier years it is an index that we have constructed from similar indices from the past. In early years this is gross of tax, but, in the most recent years, it includes dividends on an 'actual' basis. We denote the index in month t as $S(t)$.

(ii) An indicator of long-term interest rates which we describe as a 'Consols yield', and denote as $C(t)$. In recent years this has been the irredeemables yield index from the FTSE-Actuaries BGS Indices. In earlier years, it was the quoted yield on $2\frac{1}{2}\%$ Consols.

(iii) An indicator of short-term interest rates, which we denote as $B(t)$. This is the rate of bank rate, minimum lending rate or bank base rates, as determined by the Bank of England, taken as at the end of each month. This is not necessarily the best indicator of short-term interest rates, for which, nowadays, a suitable LIBOR rate might be better, but the series we use is available for the whole period which we have studied.

B.3 We start by calculating certain derived series from the source data. First, we take the logarithm of each of the values of the share total return index, and then take the differences. This gives us a series of monthly total returns, for $t =$ January 1924 to June 2004, of 966 values:

$$L(t) = \ln(S(t)) - \ln(S(t-1)).$$

B.4 Next, we transform the Consols yield (which is quoted as a percentage) to a monthly percentage yield. We assume that $C(t)$ represents a yield convertible half-yearly. This is correct for the recent values, taken from the FTSE-Actuaries BGS indices. We do not know how the earlier values were calculated. We calculate:

$$Cm(t) = 100 \times ((1 + C(t)/200)^{1/6} - 1).$$

Note that this gives a rate of interest per cent per month. We also calculate the equivalent monthly rate for $B(t)$. We assume that bank rate is convertible

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monthly. It is not clear that any prescribed frequency is intended, and custom no doubt varies from bank to bank and from time to time. This gives us:

$$\mathrm{Bm}(t) = \mathrm{B}(t)/12.$$

B.5 We now assume that Bm(t) and Cm(t) represent redemption yields on redeemable stocks, with interest payable monthly, and standing at par, with terms of zero and infinity respectively. Now we proceed as described in WWY Appendix B. We assume that a par yield curve can be interpolated between these values with an exponential curve, so that the par yield for a stock of term u, at time t, P(u, t), is given by:

$$\mathrm{P}(u, t) = \mathrm{Cm}(t) + (\mathrm{Bm}(t) - \mathrm{Cm}(t)).\exp(-\beta.u)$$

where β is taken, as in WWY, as 0.39 (annually, equal to 0.39/12 monthly) for all t.

B.6 Next, for each month t, we derive a series of zcb yields, with terms at monthly intervals, again following WWY. We denote the present value function of a zcb of term u at time t by V(u, t), and we calculate these:

$$\mathrm{V}(1, t) = 1/(1 + \mathrm{P}(1, t)/100)$$
$$\mathrm{V}(u, t) = (1 - \mathrm{P}(u, t)\Sigma_{s=1,u-1}\mathrm{V}(s, t))/(1 + \mathrm{P}(u, t)/100).$$

We then convert these to annual percentage zcb rates, assumed payable continuously, R(u, t), by:

$$\mathrm{R}(u, t) = -1200 \times \ln(\mathrm{V}(u, t))/u.$$

We now have zcb rates for each date from December 1923 to June 2004, for each term from zero months to 120 months.

B.7 Next, we construct series of zcb rates maturing at a fixed date. The first series is assumed to mature in December 1933, and starts with the ten-year (120-month) rate in December 1923, followed by the 119-month rate in January 1924, the 118-month rate in February 1924, and so on up to the zero-month rate in December 1933. Each of the 847 series has 121 values. The first matures in December 1933, the last in June 2004. We denote the values by R(u, T), where u is the outstanding term and runs down from 120 to zero, and T is the maturity date and runs from December 1933 to June 2004.

B.8 To go with the 847 series of zcb yields, we construct 847 matching series of 121 values of the share index for the same dates as the zcb yields. These overlap one another considerably. Each series of 121 values of the index allows 120 values of the log differences, and we identify them as

L(u, T), $u = 120, \ldots, 0$, T = December 1933 to June 2004, where L(u, T) = L(T − u).

B.9 We can now analyse the statistics of each of our 847 pairs of series. We start with the zcb yields, and record, for each series, the maximum and minimum values, the mean, variance, standard deviation, skewness and kurtosis. We also calculate the first autocorrelation coefficient, and the second and third partial autocorrelation coefficients. For series T, we denote the mean zcb yield as Rm(T) and the first autocorrelation coefficient as Ra(T). Using these, we calculate a further series of 120 'residuals', denoted Re(u, T):

$$\mathrm{Re}(u, \mathrm{T}) = (\mathrm{R}(u, \mathrm{T}) - \mathrm{Rm}(\mathrm{T})) - \mathrm{Ra}(\mathrm{T}).(\mathrm{R}(u+1, \mathrm{T}) - \mathrm{Rm}(\mathrm{T})).$$

Note that the previous value, for month T − u − 1, is for term $u + 1$.

B.10 We now calculate the same set of statistics for each of the 847 series of residuals. We also calculate the same set of statistics for the log share differences, and also the (simultaneous) correlation coefficient between log share differences and yield residuals, L(u, T) and Re(u, T).

B.11 There is a big variation in the statistics which we have calculated. For each of the statistics, the means, standard deviations, etc., we have 847 values, and we can calculate just the same statistics for each of the series statistics. The results are shown in Table B.1 for the zcb statistics, and in

Table B.1. Statistics for statistics of zcb and zcb residuals for each 120-month period

	Minimum	Maximum	Mean	Standard deviation	Skewness	Kurtosis
Zcb rates:						
Mean (%)	2.76	12.49	6.52	3.15	0.40	1.75
Autocorrelation coefficient	0.7281	0.9855	0.9460	0.0307	−1.87	8.36
Continuous autocorrelation factor	0.1754	3.8074	0.6726	0.4028	2.09	10.25
Zcb residuals:						
Variance	0.001586	0.495032	0.099646	0.126637	1.55	4.05
Standard deviation (%)	0.0398	0.7036	0.2614	0.1770	0.99	2.78
Skewness	−1.74	6.88	1.30	1.59	1.32	4.73
Kurtosis	3.00	64.18	13.41	11.99	1.96	7.19
Continuous S.D. % (annualised)	0.1298	3.0147	0.9353	0.7155	1.15	3.10
967 months:						
Base Rate %	2.00	17.00	6.0204	3.7282	0.92	3.04
Consols Yield %	2.53	17.20	6.5214	3.2365	0.86	2.84

Table B.2 for the share statistics. Table B.2 also includes the statistics for the whole sample of 966 log share returns.

B.12 Consider first the zcb yields. The means Rm(T) range from 2.76% (the series from January 1941 to January 1951) to 12.49% (the series from August 1972 to August 1982), with a mean value of 6.52% and a standard deviation of 3.15%. The skewness of the means is 0.40 and the kurtosis a very low 1.75 (for a normal distribution the skewness is zero and the kurtosis is three). The autocorrelation coefficients Ra(T) range from 0.7281 (the series from March 1942 to March 1952) to 0.9855 (the series from February 1994 to February 2004), with a mean value of 0.9460 and a standard deviation of 0.0307. The skewness of the autocorrelation coefficients is −1.87 and the kurtosis is 8.36; but these coefficients are not expected to be normally distributed.

B.13 The monthly autocorrelation coefficient corresponds with a continuous factor (per annum; this is what we require for the continuous model), Ralpha(T), calculated as:

$$\text{Ralpha(T)} = -12 \times \ln(\text{Ra(T)}).$$

These have also been calculated, and the results are also shown in Table B.1. They range from 0.1754 (corresponding with Ra(T) = 0.9855) to 3.8074 (corresponding with Ra(T) = 0.7281), with a mean value of 0.6726 and a standard deviation of 0.4028. Their skewness is 2.09 and kurtosis is 10.25.

B.14 Table B.1 also shows the statistics for the 967 values of the original base rate and the Consols yield. Not shown in the table are the monthly autocorrelation coefficients, 0.9911 and 0.09955, respectively, corresponding with continuous annualised values of 0.1079 and 0.0547. These are much lower than the mean value of 0.6726 quoted above. WWY used a value of 0.125 (as we have done in many of our calculations), which was derived from other, less detailed, investigations. As shown in Section 8.4, the value of the autoregressive parameter makes very little difference to the hedging results.

B.15 From the series of zcb residuals, we obtain the standard deviations; note that these are percentage values, since the yields are expressed as percentages. These range from 0.0398 to 0.7036 with a mean of 0.2614 and a standard deviation of 0.1770; their skewness is 0.99 and their kurtosis is 2.78. For the variances the corresponding figures are: range 0.0016 to 0.4950, mean 0.0996, standard deviation 0.1266, skewness 1.55, kurtosis 4.05. The monthly standard deviation, denoted Rs(T), corresponds with a continuous standard deviation, denoted Rsigma(T), calculated as:

$$\text{Rsigma(T)} = \text{Rs(T)}.\sqrt{}\{(2 \times \text{Ralpha(T)})/(1 - \text{Ra(T)}^2)\}.$$

The statistics of Rsigma(T) (still expressed as a percentage) are: range 0.1298 to 3.0147, mean 0.9353, standard deviation 0.7155, skewness 1.15, kurtosis 3.10.

B.16 We are also interested in the skewness and kurtosis of the residuals. The statistics for the skewness are: range −1.74 to 6.88, mean 1.30, standard deviation 1.59, skewness 1.32, kurtosis 4.73. The statistics for the kurtosis are: range 3.00 to 64.18, mean 13.41, standard deviation 11.99, skewness 1.96, kurtosis 7.19.

B.17 For shares, we are interested in the mean log difference over each ten-year period. The statistics of the mean are: range −0.0014 (June 1930 to June 1940) to 0.0231 (December 1974 to December 1984), mean 0.0091, standard deviation 0.0043, skewness 0.39, kurtosis 3.05. We are also interested in the standard deviations whose statistics are: range 0.0299 to 0.0806, mean 0.0486, standard deviation 0.1345, skewness 1.25, kurtosis 3.50. The numbers used in our modelling are the annual equivalents for the mean and standard deviation, calculated by multiplying by 12 and $\sqrt{12}$ respectively. The statistics for these are also shown in Table B.2.

B.18 A further analysis is of the residuals of the zcb rates, rearranged so that the 847 values of the residuals for a 119-month term are put into one series, then the 847 residuals of the 118-month term, and so on. This gives us 120 series, each with 847 values. Each series is the residual after fitting different means and autocorrelation factors, but the interesting features are the standard deviations of these series. They are plotted in Figure B.1. The standard deviations have a value of around 0.3 for most of the terms, but the value rises to over 0.5 as the term shortens below 24 months or so. This suggests that our bond model, with a constant value of σ_R, could, perhaps, be improved by allowing a deterministically varying value $\sigma_R(t)$.

B.19 Note that the values of the standard deviations shown in Figure B.1 are higher than the mean standard deviation shown in Table B.1 of about

Table B.2. Statistics for statistics of share log difference for each 120-month period, and for whole period of 966 months

	Minimum	Maximum	Mean	Standard deviation	Skewness	Kurtosis
Mean	−0.0014	0.0231	0.0091	0.0043	0.39	3.05
Variance	0.000893	0.006493	0.002541	0.001543	1.56	4.15
Standard deviation	0.0299	0.0806	0.0486	0.1345	1.25	3.50
Skewness	−2.42	1.62	−0.57	0.90	0.22	2.49
Kurtosis	2.86	15.31	6.55	3.34	0.52	1.67
Correlation with zcb residuals	−0.5708	−0.0358	−0.3075	0.1231	−0.22	2.38
Annualised mean	−0.0167	0.2778	0.1095	0.0516	0.39	3.05
Annualised stand. dev.	0.1035	0.2791	0.1683	0.0466	1.25	3.50
966 months:						
log difference	−0.3056	0.4300	0.0086	0.0493	−0.08	11.59

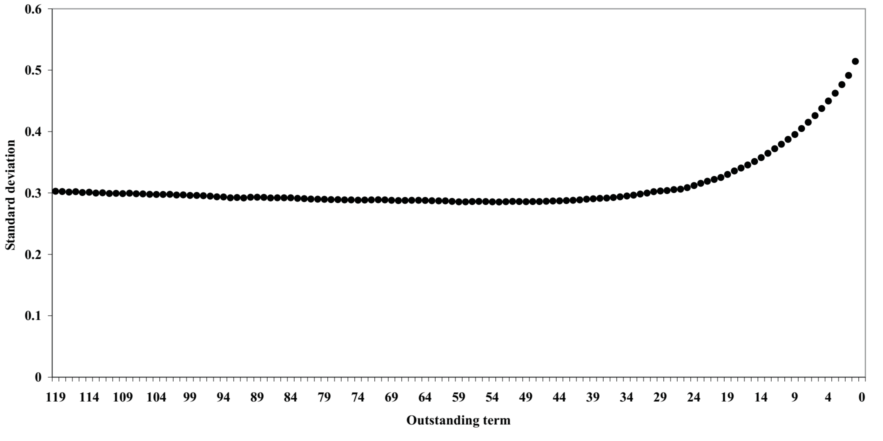

Figure B.1. Standard deviations of zcb residuals by outstanding term

0.26, but those show variations about the 120-month trends, whereas the figures under consideration now show variations about the whole series of 70 years of values. When the underlying means are clearly varying so much, measurement of the variations from those means is difficult. The approach in the Wilkie model of removing, first, the effect of inflation, and then treating the remaining 'real' yield as having a constant mean, is probably more realistic, but it would require a more complex option pricing bond model, and would require more complex hedging, including investment in the RPI or its equivalent (such as index-linked stock), to match the liability. We have not considered this in this paper.

B.20 We have also calculated the correlation coefficients between selected statistics, those that are needed for our real-world model. These are shown in Table B.3, along with the means and standard deviations of the values. We have included the initial bond yield, the ten-year rate at the start of each sample. All these are calculated from the 847 samples, which overlap greatly, so standard statistical tests do not necessarily apply.

B.21 Many of these correlation coefficients are large. Some can be explained. When the initial bond rate is high, the average bond rate for the next ten years is also high. This has occurred at times of high inflation, so the return on shares has also been high. When the initial bond rate, or the mean bond rate, is high, the standard deviation (sigma) of the bond rate is also high. Therefore, a log transform might be a better way of modelling the real world (as in the Wilkie model), but it would alter our option pricing model. This evidence suggests that fuller investigations than are appropriate here would be interesting.

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Table B.3. Correlation coefficients of selected statistics

	Bond mean %	Bond rate 0	Bond alpha	Bond sigma %	Share/bond correlation	Share mean	Share standard deviation
Mean	6.5185	6.4661	0.6726	0.9353	−0.3075	0.1095	0.1683
Standard deviation	3.1500	3.2923	0.4028	0.7155	0.1231	0.0516	0.0466
Correlation coefficient							
Bond mean	1.0	0.8859	0.2947	0.8962	−0.1901	0.6623	0.7593
Bond rate 0	0.8859	1.0	0.3197	0.7663	−0.1524	0.7172	0.5259
Bond alpha	0.2947	0.3197	1.0	0.5495	−0.0454	0.4153	0.1466
Bond sigma	0.8962	0.7663	0.5495	1.0	−0.2121	0.5827	0.7792
Share/bond correlation	−0.1901	−0.1524	−0.0454	−0.2121	1.0	−0.2687	−0.2367
Share mean	0.6623	0.7172	0.4153	0.5827	−0.2687	1.0	0.2166
Share standard deviation	0.7593	0.5259	0.1466	0.7792	−0.2367	0.2166	1.0