Proceedings of the Royal Society of Edinburgh, **146A**, 195–212, *2016*
DOI:10.1017/S0308210515000426

The regularity and stability of solutions to semilinear fourth-order elliptic problems with negative exponents

Baishun Lai
Institute of Contemporary Mathematics, Henan University,
Kaifeng 475004, People's Republic of China
(laibaishun@henu.edu.cn)

(MS received 29 April 2014; accepted 16 December 2014)

We examine the regularity of the extremal solution of the nonlinear eigenvalue problem

$$\Delta^2 u = \frac{\lambda}{(1-u)^p}$$

on a general bounded domain Ω in $\mathbb{R}^N$, with Navier boundary condition $u = \Delta u$ on $\partial\Omega$. Firstly, we prove the extremal solution is smooth for any $p > 1$ and $N \leqslant 4$, which improves the result of Guo and Wei (*Discrete Contin. Dynam. Syst.* A **34** (2014), 2561–2580). Secondly, if $p = 3$, $N = 3$, we prove that any radial weak solution of this nonlinear eigenvalue problem is smooth in the case $\Omega = \mathbb{B}$, which completes the result of Dávila *et al*. (*Math. Annalen* **348** (2009), 143–193). Finally, we also consider the stability of the entire solution of $\Delta^2 u = -1/u^p$ in $\mathbb{R}^N$ with $u > 0$.

Keywords: biharmonic equation; entire solution; regularity; stability

2010 *Mathematics subject classification:* Primary 35J91; 35B08; 35B53; 35B40

1. Introduction

In this paper, we consider the following fourth-order elliptic problems with negative exponents:

$$\left.\begin{aligned} \Delta^2 u &= \frac{\lambda}{(1-u)^p} && \text{in } \Omega, \\ 0 < u &\leqslant 1 && \text{in } \Omega, \\ u = \Delta u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \qquad (F)_\lambda$$

and

$$\Delta^2 u = -\frac{1}{u^p} \quad \text{in } \mathbb{R}^N \text{ with } u > 0, \qquad (E)$$

where $0 < \lambda$, $0 < p$ and Ω is smooth and bounded in $\mathbb{R}^N$. Recently, higher-order equations with a singular nonlinearity have attracted the interest of many researchers. In particular, the corresponding second-order problem for $(F)_\lambda$ has been intensively studied; this models a simple electrostatic microelectromechanical systems (MEMS) device (see, for example, [7, 19] and the references therein).

Recently, Lin and Yang [16] derived $(F)_\lambda$ with $p = 2$ in the study of charged plates in electrostatic actuators. Here $\lambda = aV^2$, where V is the voltage and a is a positive constant. The following energy functional is associated with $(F)_\lambda$:

$$E(u) = \int_\Omega \left\{ \frac{|\Delta u|^2}{2} - \frac{\lambda}{1-u} \right\} \mathrm{d}x,$$

where the first term is the bending energy and the second is the potential energy. Lin and Yang considered two kinds of boundary conditions: the Navier boundary condition,

$$u = \Delta u = 0 \quad \text{on } \partial\Omega,$$

and the Dirichlet boundary condition,

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

They found that there exists $0 < \lambda^* < \infty$, called the pull-in threshold, such that, for $\lambda \in (0, \lambda^*)$, $(F)_\lambda$ has a smooth solution u_λ. For $\lambda > \lambda^*$, $(F)_\lambda$ does not have any smooth solution. Physically, this is a natural relation because a higher supply voltage results in greater elastic deformation or deflection.

Equation (E) has its roots in Riemannian geometry for $N = 3$. Let us briefly describe the background of this equation. Let $g = (g_{ij})$ be the standard Euclidean metric on $\mathbb{R}^N$, $N \geqslant 3$, with $g_{ij} = \delta_{ij}$. Let $\bar{g} = u^{4/(N-4)}g$, $N \neq 4$, be a second metric derived from g by the positive conformal factor $u\colon \mathbb{R}^N \to \mathbb{R}$. Then u satisfies

$$\Delta^2 u = \frac{N-4}{2} Q_{\bar{g}} u^{(N+4)/(N-4)},$$

where $Q_{\bar{g}}$ is the scalar curvature of $\bar{g}$. If we assume that $Q_{\bar{g}} > 0$ is a constant, we can obtain (E) via scaling. The existence and properties of the solution have been considered by various authors (for details see [6, 10, 17] and the references therein).

The first aim of this paper is to consider the regularity of the extremal solution defined below, and the weak solutions of $(F)_\lambda$, and the second aim is to study the stability of the solution associated with (E).

1.1. Navier boundary-value problem

In this subsection, we state some results associated with $(F)_\lambda$. First, we list some of the known results one comes to expect when studying $(F)_\lambda$ (for more details see [2–4] and the references therein).

(i) There exists a extremal parameter $0 < \lambda^* < \infty$ such that for each $0 < \lambda < \lambda^*$ there exists a unique stable classical solution u_λ for $(F)_\lambda$. Moreover, u_λ is the minimal solution and $\lambda \to u_\lambda$ is increasing. By the minimal solution, we mean here that if v is another solution of $(F)_\lambda$, then $v \geqslant u_\lambda$ almost everywhere (a.e.) in Ω.

(ii) For $0 < \lambda < \lambda^*$ the minimal solution u_λ is semi-stable in the sense that

$$p\lambda \int_\Omega (1-u_\lambda)^{-p-1}\psi^2 \,\mathrm{d}x \leqslant \int_\Omega |\Delta\psi|^2 \,\mathrm{d}x \quad \forall \psi \in H^2(\Omega) \cap H_0^1(\Omega). \tag{1.1}$$

https://doi.org/10.1017/S0308210515000426 Published online by Cambridge University Press

(iii) No weak solution exists for $\lambda > \lambda^*$.

(iv) The monotone limit $u^* = \lim_{\lambda \to \lambda^*} u_\lambda$ belongs to $H^2(\Omega) \cap H_0^1(\Omega)$, called the extremal solution.

An interesting problem is the regularity of the extremal solution u^*. This is of interest since one can then apply the Crandall–Rabinowitz bifurcation theorem to start a second branch of solutions from (λ^*, u^*). If the domain Ω is the unit ball, then one can use the method of [3,5] to obtain optimal results for the radial extremal solution in the case when $p = 2$ (see, for example, [18]). For the general case, Cowan and Ghoussoub proved in [2, 4] that if

$$\frac{N}{4} < \frac{p}{p+1} + \frac{p-1}{p+1}\left(\sqrt{\frac{2p}{p+1}} + \sqrt{\frac{2p}{p-1} - \sqrt{\frac{2p}{p-1}}} - \frac{1}{2}\right), \quad 1 < p \neq 3,$$

then u^* is smooth. The restriction for $p \neq 3$ is because of the borderline Sobolev imbedding theorem (for details see [2, 4]). Very recently, Guo and Wei [12] proved that if the following hold, then the extremal solution u^* is smooth:

$$N = 3, \quad p \in (p^1(3), p^2(3)) \subset (1, +\infty);$$
$$N = 4, \quad p \in (p^1(4), p^2(4)) \subset (1, +\infty);$$
$$5 \leqslant N \leqslant 12, \quad 1 < p_*(N) < p;$$

the definitions of $p^1(3)$, $p^2(3)$, $p^1(4)$, $p^2(4)$, and $p_*(N)$ are found in [12].

Inspired by the arguments in [9, 13], we shall prove that the regularity of the extremal solution u^* is very well understood for $N \leqslant 4$, $p > 1$, i.e. u^* is smooth for $N \leqslant 4$, $p > 1$.

Another aim concerning $(F)_\lambda$ is the regularity of the weak solutions. By a weak solution of $(F)_\lambda$, we mean here that

$$\frac{1}{(1-u)^p}, \quad u_\lambda \in L^1(\Omega),$$

and u_λ satisfies

$$\int_\Omega u_\lambda \Delta^2 \phi \, \mathrm{d}x = \int_\Omega \frac{\lambda \phi}{(1-u_\lambda)^p} \, \mathrm{d}x \quad \forall \phi \in C_0^\infty(\Omega);$$

a weak solution u_λ to $(F)_\lambda$ is called singular if $\|u_\lambda\|_{L^\infty(\Omega)} = 1$ and regular if $\|u_\lambda\|_{L^\infty(\Omega)} < 1$. In addition, if $\Omega = \mathbb{B}$, the radial singular solutions can be only singular at the origin by the maximum principle.

To illustrate the ideas in this paper in detail, we now recall some corresponding results for $(F)_\lambda$.

- For $N \geqslant 4$ and $p > 1$, or $N = 3$ and $1 < p < 3$, $(F)_\lambda$ with $\Omega = \mathbb{B}$ admits a unique weak solution, $u_\lambda(r)$, which is singular and such that

  $$\lim_{r \to 0} r^{-4/(p+1)}(1 - u_\lambda(r))$$

  exists [6].

- For $N = 3$, $p > 3$ or $N = 2$, $p > 1$, any weak solution of $(F)_\lambda$ is regular [11,13].

A natural question is 'what about the critical case?'; namely, for $0 < p \leqslant 1$ or $N = 3$ and $p = 3$, is there regularity of the weak solution of $(F)_\lambda$? We consider a simple case, $\Omega = \mathbb{B}$, and prove that any radial weak solution of $(F)_\lambda$ is regular for $N = 3$, $p = 3$. In order to achieve this proof, we suppose by contradiction that $(F)_\lambda$ admits a weak solution $u_\lambda(r)$ with $u_\lambda(0) = 1$. Then, according to some ordinary differential equation techniques, we see that

$$\begin{aligned} \lim_{r\to 0} r^{-1}(-\log r)^{-1/4}(1-u_\lambda(r)) &= 2^{1/4} && \text{if } N = 3,\ p = 3; \\ \lim_{r\to 0} r^{-2}(-\log r)^{-1/2}(1-u_\lambda(r)) &= \sqrt{(N(N-2))^{-1}} && \text{if } N \geqslant 4,\ p = 1; \\ \lim_{r\to 0} r^{-2}(1-u_\lambda(r)) &= c_0 && \text{if } N \geqslant 5,\ 0 < p < 1. \end{aligned}$$

From this asymptotic, we immediately obtain a contraindication to $(1-u_\lambda)^{-3} \in L^1$ for $N = 3$. However, for other critical cases, it is unclear whether any weak solution is regular or singular. Note that the rate of vanishing of $1 - u_\lambda(r)$ for the critical case as $r \to 0$ is in striking contrast to the case when $N \geqslant 4$ and $p > 1$ or $N = 3$ and $1 < p < 3$.

Making use of the above discussion, we state our results associated with $(F)_\lambda$ as follows.

Theorem 1.1.

(i) *For any $p > 1$, the extremal solution u^* of $(F)_{\lambda^*}$ is regular for dimensions $N \leqslant 4$.*

(ii) *Let Ω be a unit ball centred at 0. Then any radial weak solution of $(F)_\lambda$ is regular for $p = 3$, $N = 3$.*

Remark 1.2. In our recent work [15], we proved that (E) has no positive entire solution for $0 < p \leqslant 1$. From the blow-up point of view, we conjecture that the extremal solution of $(F)_\lambda$ is smooth for $0 < p \leqslant 1$.

1.2. Stability of entire solutions

In this subsection, we state some stability results concerning (E). In this paper we only consider the classical solution of (E), where of course $u > 0$ in $\mathbb{R}^N$. We now start by explaining what we mean by stability.

Definition 1.3. A solution $u \in C^4(\mathbb{R}^N)$ to (E) is stable if

$$\int_{\mathbb{R}^N} |\Delta\varphi|^2 \,\mathrm{d}x - p\int_{\mathbb{R}^N} u^{-p-1}\varphi^2 \,\mathrm{d}x \geqslant 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

A solution $u \in C^4(\mathbb{R}^N)$ to (E) is stable outside the compact set K if

$$\int_{\mathbb{R}^N\setminus K} |\Delta\varphi|^2 \,\mathrm{d}x - p\int_{\mathbb{R}^N\setminus K} u^{-p-1}\varphi^2 \,\mathrm{d}x \geqslant 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N \setminus K).$$

First, we note that, according to Warnaut's result [20], (E) admits no stable solution for $N = 3, 4$.

https://doi.org/10.1017/S0308210515000426 Published online by Cambridge University Press

Proposition 1.4. *Equation (E) admits no stable solution for $N = 3, 4$.*

Now we discuss some stability results for the radial case, and we rewrite the radial version of (E) as follows:

$$\left.\begin{aligned} \Delta^2 u(r) = -u^{-p} \quad \text{for } r \in (0, R_{\max}(\beta)), \\ u(0) = 1, \qquad \Delta u(0) = \beta, \qquad u'(0) = (\Delta u)'(0) = 0, \end{aligned}\right\} \tag{1.2}$$

where $[0, R_{\max}(\beta))$ is the interval of existence of the solution. We say that a solution of (1.2) is entire (respectively, local) if $R_{\max}(\beta) = \infty$ (respectively, $R_{\max}(\beta) < \infty$).

We now list the properties one comes to expect when studying (1.2).

Proposition 1.5. *Assume $p > 1$. Then there exists a unique $\beta_0 > 0$ such that the following hold.*

(a) *If $\beta < \beta_0$, then $R_{\max}(\beta) < \infty$.*

(b) *If $\beta \geqslant \beta_0$, then $\lim_{r\to\infty} \Delta u_\beta(r) \geqslant 0$ and $R_{\max}(\beta) = \infty$.*

(c) *For $\beta = \beta_0$, we have the following:*

(1) *if $N = 3$, $p > 3$, then*

$$\lim_{r\to\infty} \frac{u_{\beta_0}}{r} = \alpha > 0 \quad (\textit{for some fixed } \alpha);$$

(2) *if $N = 3$, $p = 3$, then*

$$\lim_{r\to\infty} \frac{u_{\beta_0}}{r(\log r)^{1/4}} = 2^{1/4};$$

(3) *if $N \geqslant 4$, $p > 1$ or $N = 3$, $1 < p < 3$, then*

$$\lim_{r\to\infty} r^{-4/(p+1)} u_{\beta_0}(r) = \left(-Q_4\left(\frac{-4}{p+1}\right)\right)^{-1/(p+1)},$$

where Q_4 is defined in (1.3).

The proofs of (a), (b) and (3) can be found in [6]. The proofs of (1) and (2) were obtained in [10]. And the argument to prove (1) and (2) is the complex analysis of a dynamical system. In fact, we can simplify the proof of [10], according to the arguments that we use to prove lemma 2.5.

Although (E) admits no stable solution for $N = 3, 4$, it has a solution that is stable outside a compact set under a certain range of p. First, we need the following notation, which will be used throughout the paper.

Set

$$m = -\frac{4}{p+1}, \qquad Q_4(\beta) := \beta(\beta+2)(N-2-\beta)(N-4-\beta). \tag{1.3}$$

The polynomial $Q_4(\beta)$ is yielded by the following identity:

$$\Delta^2 |x|^{-\beta} = Q_4(\beta)|x|^{-\beta-4} \quad \forall x \in \mathbb{B} \setminus \{0\}.$$

https://doi.org/10.1017/S0308210515000426 Published online by Cambridge University Press

This polynomial is closely related to Rellich's inequality,

$$\int_{\mathbb{R}^N} (\Delta u)^2 \, dx \geqslant \frac{N^2(N-4)^2}{16} \int_{\mathbb{R}^N} |x|^{-4} u^2 \, dx,$$

which is valid for each $u \in H^2(\mathbb{R}^N)$ and each $N > 4$. Namely, the constant that appears on the right-hand side is merely the unique local maximum value of $Q_4(\beta)$. From [6], we see that there exist $p_c > 0$, $p_c^+ > 0$ such that if

$$p = \begin{cases} p_c \text{ or } p_c^+ & \text{for } N = 3, \\ p_c & \text{for } N \geqslant 4, \end{cases} \tag{1.4}$$

then $-pQ_4(m) = \frac{1}{16}(N^2(N-4)^2)$. And if

$$N = 3 \text{ and } p_c < p < p_c^+ \quad \text{or} \quad 4 \leqslant N \leqslant 12 \text{ and } p_c < p < +\infty,$$

then $-pQ_4(m) > \frac{1}{16}(N^2(N-4)^2)$. In addition, we always assume that β_0 is defined as proposition 1.5 and that $u_\beta(r)$ is a radial solution of (E).

Theorem 1.6. *Let $N = 3$, $p > 1$. Then we have the following.*

(i) *If $p_c^+ < p < 3$ or $1 < p < p_c$, then $u_{\beta_0}(r)$ is stable outside a compact set; if $p_c < p < p_c^+$, $u_{\beta_0}(r)$ is unstable outside every compact set; if $p \geqslant 3$, then $u_{\beta_0}(r)$ is stable outside a compact set.*

(ii) *If $\beta > \beta_0$, then $u_\beta(r)$ is stable outside a compact set.*

Theorem 1.7. *Let $N = 4$. Then u_{β_0} is unstable outside every compact set and u_β is stable outside a compact set for $\beta > \beta_0$.*

Question 1.8 (open problem). What is the stability behaviour outside compact sets for $\beta = \beta_0$, $N = 3$, $p = p_c$, $p = p_c^+$?

The case $N \geqslant 5$ is strikingly different from the cases $N = 3, 4$. For this we have the following.

Theorem 1.9. *Let $N \geqslant 5$. The following statements hold.*

(i) *If $5 \leqslant N \leqslant 12$ and $1 \leqslant p \leqslant p_c$ or $N \geqslant 13$, $p > 1$, then u_β is stable for every $\beta \geqslant \beta_0$.*

(ii) *For $5 \leqslant N \leqslant 12$ and $p < p_c$, u_{β_0} is unstable outside every compact set; if $\beta \in (\beta_0, \beta_1)$, then u_β is unstable; u_β is stable outside a compact set, and is stable for $\beta \in [\beta_1, +\infty)$.*

2. Proof of theorem 1.1

2.1. Proof of theorem 1.1(i)

We start by proving the following lemma.

Lemma 2.1. *Let u_λ be the minimal solution of $(F)_\lambda$. Then there exists a constant C independent of λ such that*

$$\int_\Omega |\Delta u_\lambda|^2 \,\mathrm{d}x + \int_\Omega (1-u_\lambda)^{-p-1}\,\mathrm{d}x \leqslant C. \tag{2.1}$$

Proof. Testing $(F)_\lambda$ on $u_\lambda \in H^2(\Omega) \cap H_0^1(\Omega)$, we see that

$$\lambda \int_\Omega u_\lambda (1-u_\lambda)^{-p}\,\mathrm{d}x = \int_\Omega (\Delta u_\lambda)^2\,\mathrm{d}x \geqslant p\lambda \int_\Omega u_\lambda^2 (1-u_\lambda)^{-p-1}\,\mathrm{d}x.$$

In particular, for $\delta > 0$ small we have

$$\begin{aligned} \int_{u_\lambda \geqslant \delta} (1-u_\lambda)^{-p-1}\,\mathrm{d}x &\leqslant \frac{1}{\delta^2} \int_{u_\lambda \geqslant \delta} u_\lambda^2 (1-u_\lambda)^{-p-1}\,\mathrm{d}x \\ &\leqslant \frac{1}{\delta^2} \int_\Omega (1-u_\lambda)^{-p}\,\mathrm{d}x \\ &\leqslant \delta^{-p-1} \int_\Omega (1-u_\lambda)^{-p-1}\,\mathrm{d}x + C_\delta \end{aligned}$$

by means of Young's inequality with δ. Since, for δ small,

$$\int_{u_\lambda \leqslant \delta} (1-u_\lambda)^{-p-1}\,\mathrm{d}x \leqslant C$$

for some $C > 0$, we get that

$$\int_\Omega (1-u_\lambda)^{-p-1}\,\mathrm{d}x \leqslant C$$

for some $C > 0$ and for every $\lambda \in (0, \lambda^*)$. Since

$$\int_\Omega |\Delta u_\lambda|^2\,\mathrm{d}x = \lambda \int_\Omega u_\lambda (1-u_\lambda)^{-p}\,\mathrm{d}x \leqslant \int_\Omega (1-u_\lambda)^{-p-1} \leqslant C,$$

we obtain

$$\int_\Omega |\Delta u_\lambda|^2\,\mathrm{d}x + \int_\Omega (1-u_\lambda)^{-p-1}\,\mathrm{d}x \leqslant C,$$

where C is an absolute constant. □

Proof of theorem 1.1(i). As already observed, estimate (2.1) implies $(1-u^*)^{-p} \in L^{(p+1)/p}(\Omega)$. We need to show that

$$u^*(x_0) = \sup_{x \in \Omega} u^*(x) < 1 \quad \text{for some } x_0 \in \Omega$$

to get the regularity of u^*. In fact, on the contrary, suppose that $u^*(x_0) = 1$. The standard elliptic regularity theory shows that $u^* \in W^{4,(p+1)/p}(\Omega)$. The Sobolev imbedding theorem, i.e. $W^{4,(p+1)/p}(\Omega) \hookrightarrow C^m(\bar{\Omega})$ $(0 < m \leqslant 4 + pN/(p+1),$ $1 \leqslant N \leqslant 4)$, yields that u^* is a Lipschitz function in Ω for $1 \leqslant N \leqslant 3$.

Now suppose $u^*(x_0) = 1$ and $1 \leqslant N \leqslant 2$. Since

$$\frac{1}{1-u^*} \geqslant \frac{C}{|x - x_0|} \quad \text{in } \mathbb{B}_R(x_0) \subset \Omega$$

for some $C > 0$, $R > 0$, we see that

$$+\infty = C^{p+1} \int_{\mathbb{B}_R(x_0)} |x - x_0|^{-p-1} \, \mathrm{d}x \leqslant \int_{\Omega} (1 - u^*)^{-p-1} \, \mathrm{d}x < +\infty.$$

A contradiction arises, and hence u^* is regular for $1 \leqslant N \leqslant 2$.

For $N = 3$, by the Sobolev imbedding theorem, we have $u^* \in C^{(p+4)/(p+1)}(\bar{\Omega})$. If $(p+4)/(p+1) \geqslant 2$, then $u^*(x_0) = 1$, $\nabla u^*(x_0) = 0$ and

$$|\nabla u^*(\varepsilon) - \nabla u^*(x_0)| \leqslant M|\varepsilon - x_0| \leqslant M|x - x_0|,$$

where $0 < |\varepsilon - x_0| < |x - x_0|$. Thus,

$$|u^*(x) - u^*(x_0)| \leqslant |\nabla u^*(\varepsilon)| \, |x - x_0| \leqslant M|x - x_0|^2.$$

This inequality shows that

$$+\infty = M^{-p-1} \int_{\mathbb{B}_R(x_0)} |x - x_0|^{-2p-2} \, \mathrm{d}x \leqslant \int_{\Omega} (1 - u^*)^{-p-1} \, \mathrm{d}x < +\infty.$$

A contradiction arises and hence u^* is regular for $N = 3$; if $(p+4)/(p+1) < 2$, then

$$|\nabla u^*(\varepsilon) - \nabla u^*(x_0)| \leqslant M|\varepsilon - x_0|^{(p+4)/(p+1)-1} \leqslant M|x - x_0|^{3/(p+1)}$$

where $0 < |\varepsilon - x_0| < |x - x_0|$. Thus,

$$|u^*(x) - u^*(x_0)| \leqslant |\nabla u^*(\varepsilon)| \, |x - x_0| \leqslant M|x - x_0|^{(p+4)/(p+1)},$$

and a contradiction is obtained as above.

For $N = 4$, by the Sobolev imbedding theorem, we have $u^* \in C^{4/(p+1)}(\bar{\Omega})$. If $1 < 4/(p+1) < 2$, then $u^*(x_0) = 1$, $\nabla u^*(x_0) = 0$ and

$$|\nabla u^*(\varepsilon) - \nabla u^*(x_0)| \leqslant M|\varepsilon - x_0|^{4/(p+1)-1} \leqslant M|x - x_0|^{4/(p+1)-1},$$

where $0 < |\varepsilon - x_0| < |x - x_0|$. Thus,

$$|u^*(x) - u^*(x_0)| \leqslant |\nabla u(\varepsilon)||x - x_0| \leqslant M|x - x_0|^{4/(p+1)}.$$

If $4/(p+1) \leqslant 1$, then u^* is a Hölder continuity,

$$1 - u^*(x) \leqslant M|x - x_0|^{4/(p+1)}$$

and we obtain a contradiction, as above. Combining the above elements of discussion, we complete the proof of theorem 1.1(i). □

2.2. Proof of theorem 1.1(ii)

Let $u_\lambda(r)$, $r = |x|$, be a radial singular solution of $(F)_\lambda$. Then, according to the maximum principle, $u_\lambda(r)$ may be singular only at the origin, and hence $u_\lambda \in C^4(\mathbb{B} \setminus \{0\})$. In what follows, we always denote u_λ by u for simplicity. We now rewrite the radial version of $(F)_\lambda$ as follows:

$$\frac{\mathrm{d}^4 u}{\mathrm{d}r^4} + \frac{2(N-1)}{r}\frac{\mathrm{d}^3 u}{\mathrm{d}r^3} + \frac{(N-1)(N-3)}{r^2}\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} - \frac{(N-1)(N-3)}{r^3}\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\lambda}{(1-u)^p}, \quad 0 < r < 1. \tag{2.2}$$

https://doi.org/10.1017/S0308210515000426 Published online by Cambridge University Press

In order to make the ordinary differential equation techniques accessible to our present work, we first set

$$U(x) := 1 - u(r/\sqrt[4]{\lambda}), \quad v(t) := \mathrm{e}^{mt}U(\mathrm{e}^t), \quad t \in (-\infty, \tfrac{1}{4}\log\lambda). \tag{2.3}$$

Then $v(t)$ solves the autonomous equation

$$v^{\mathrm{iv}} + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t) + K_0 v(t) = -\frac{1}{v^p(t)}, \quad t \in (-\infty, \tfrac{1}{4}\log\lambda), \tag{2.4}$$

where K_0 is defined in (1.3) and K_1, K_2, K_3 are fixed constants dependent only on N and p.

Obviously, (2.4) admits a constant solution only if $K_0 < 0$; this is given by

$$v \equiv (-K_0)^{-1/(p+1)} = (-Q_4(m))^{-1/(p+1)}.$$

But condition $Q_4(m) < 0$ holds if and only if

$$N = 3, \quad 1 < p < 3,$$
$$N \geqslant 4, \quad 1 < p.$$

From this, we know that (2.4) does not have a constant solution if $0 \leqslant p \leqslant 1$ or if $N = 3$ and $p \geqslant 3$. In addition, (2.4) may be rewritten as

$$Lv := (\partial_t - \nu_1)(\partial_t - \nu_2)(\partial_t - \nu_3)(\partial_t - \nu_4)v(t) = -\frac{1}{v^p}, \quad t \in (-\infty, \tfrac{1}{4}\log\lambda), \tag{2.5}$$

where

$$\nu_1 = m - N + 4, \qquad \nu_2 = m - N + 2, \qquad \nu_3 = m + 2, \qquad \nu_4 = m.$$

For $K_0 \geqslant 0$, we have the following.

LEMMA 2.2. *Let $0 \leqslant p < 1$, or $N = 3$ and $p \in [3, +\infty)$, or $N = 2$ and $p > 1$, and let v be a solution of (2.4) corresponding to a solution of $(F)_\lambda$. Then v is unbounded in $(-\infty, t_0)$.*

Proof. Suppose by contradiction that v is bounded. Indeed, since $0 < p \leqslant 1$, or $N = 3$ and $p \in [3, +\infty)$, or $N = 2$ and $p > 1$, we have $K_0 \geqslant 0$, and there exists $\varepsilon > 0$ such that

$$-K_0 v(t) - v^{-p}(t) \leqslant -v^{-p}(t) < -\varepsilon \quad \forall t \in (-\infty, t_0). \tag{2.6}$$

After integration in (2.4), by (2.6) we obtain

$$v'''(t) + K_3 v''(t) + K_2 v'(t) > -\varepsilon(t - t_0) + O(1) \quad \text{as } t \to -\infty.$$

Two further integrations yield

$$v'(t) > -\tfrac{1}{6}\varepsilon(t - t_0)^3 + O(t^2) \quad \text{as } t \to -\infty.$$

This contradicts the fact that $v(t) > 0$ for any $t \in (-\infty, t_0)$. □

Inspired by [8], we give the following lemma.

https://doi.org/10.1017/S0308210515000426 Published online by Cambridge University Press

Lemma 2.3. *Let v be a solution of (2.4). Then,*

$$\lim_{t\to-\infty}\inf v(t) > 0.$$

Proof. Suppose by contradiction that there exist $t_k \to -\infty$ such that

$$v'(t_k) \geqslant 0 \quad \text{and} \quad \lim_{t_k\to-\infty} v(t_k) = 0. \tag{2.7}$$

Define

$$\lambda_k = v^{-p-1}(t_k),$$

so that

$$\lim_{k\to+\infty} \lambda_k = +\infty.$$

Since (2.4) is an autonomous equation, the translated function

$$\tilde{v}_k(t) = v(t + t_k - \tfrac{1}{4}\log\lambda_k), \quad t \in (-\infty, \tfrac{1}{4}\log\lambda - t_k + \tfrac{1}{4}\log\lambda_k),$$

also solves (2.4). In particular, the function

$$\tilde{U}_k(r) = r^{-m}\tilde{v}_k(\log r)$$

is a radial solution of

$$\Delta^2 u = -u^{-p}, \quad x \in \mathbb{B}_{\tilde{a}}(0)\ (\tilde{a} = \lambda^{1/4}\lambda_k^{1/4}\mathrm{e}^{-t_k}) \tag{2.8}$$

and satisfies the conditions

$$\tilde{U}_k(\sqrt[4]{\lambda_k}) = \lambda_k^{1/(p+1)}\tilde{v}_k(\tfrac{1}{4}\log\lambda_k) = \lambda_k^{1/(p+1)}v(t_k) = 1$$

and, by (2.7),

$$\begin{aligned}\tilde{U}_k'(\sqrt[4]{\lambda_k}) &= -m\lambda_k^{(-m-1)/4}\tilde{v}_k(\tfrac{1}{4}\log\lambda_k) + \lambda_k^{(-m-1)/4}\tilde{v}_k'(\tfrac{1}{4}\log\lambda_k)\\ &> -m\lambda_k^{(-m-1)/4}v(t_k)\\ &> 0.\end{aligned}$$

Next, we define the radial function

$$u_k(r) = 1 - \tilde{U}_k(\sqrt[4]{\lambda_k}r) = 1 - \lambda_k^{-1/(p-1)}r^{-4/(p-1)}\tilde{v}_k(t + \tfrac{1}{4}\log\lambda_k).$$

So that we have

$$\begin{gathered}\Delta^2 u_k(r) = \lambda_k(1-u_k)^{-p}, \quad u_k > 0 \quad \text{in } \mathbb{B},\\ u_k = 0 \quad \text{on } \partial\mathbb{B}, \qquad \frac{\partial u_k}{\partial n} \leqslant mv(t_k) < 0 \quad \text{on } \partial\mathbb{B}.\end{gathered}$$

This boundary-value problem is solved in a weak sense, since $\tilde{U}(r)$ is a weak solution of (2.8). This also shows that u_k is a supersolution of the following equation with parameter $\lambda = \lambda_k$:

$$\begin{aligned}\Delta^2 u(r) &= \lambda(1-u)^{-p} \quad &&\text{in } \mathbb{B},\\ u &= \frac{\partial u}{\partial n} = 0 &&\text{on } \partial\mathbb{B}.\end{aligned}$$

https://doi.org/10.1017/S0308210515000426 Published online by Cambridge University Press

Then we obtain a solution of the above equation for any k by the method of sub- and supersolutions. From [3], we see that there exists $\lambda^* > 0$ such that the above equation admits no solution for $\lambda > \lambda^*$. And then we obtain a contradiction from the fact that $\lambda_k \to +\infty$. □

LEMMA 2.4. *If a smooth function h satisfies*

$$(\partial_t - m)h \geqslant C_0 \mathrm{e}^{nt} \quad \text{as } t \to +\infty,$$

where $m < n$, $C_0 > 0$, then there exists a constant $C_2 > 0$ such that

$$h \geqslant C_2 \mathrm{e}^{nt} \quad \forall t \gg 1.$$

The proof of this lemma is trivial, so we omit it here.

The following lemma plays a key role in the proof of theorem 1.1(ii).

LEMMA 2.5. *Let u be a radial singular solution of $(F)_\lambda$.*

(i) *If $N = 3$, $p = 3$, then*

$$\lim_{r\to 0}(1 - u(r)) \cdot r^{-1}(-\log r)^{-1/4} = 2^{1/4}.$$

(ii) *If $N \geqslant 4$, $p = 1$, then u_s is such that*

$$\lim_{r\to 0}(1 - u(r)) \cdot r^{-2}(-\log r)^{-1/2} = \sqrt{d_1},$$

where $d_1 = 1/(N(N-2))$.

(iii) *If $N \geqslant 5$, $0 < p < 1$, then there exists a constant c_0 such that*

$$\lim_{r\to 0}(1 - u(r)) \cdot r^{-2} = c_0.$$

Proof. First we give the proof of (i) (the argument for (ii) is the same as for (i), so we omit it here). Let $v(t)$ be defined as in (2.3). We first claim $v(t) \to +\infty$ as $t \to -\infty$. Combining this with lemma 2.2, we only need to prove $v'(t) < 0$ for large $|t|$. Indeed, let $s = -t$, $v'(t) = w(s)$. Then $w(s)$ satisfies

$$(\partial_s - 2)(\partial_s - 1)(\partial_s + 1)w(s) = v^{-3} > 0.$$

Multiplying by e^{-2s}, we get $(\mathrm{e}^{-2s}(\partial_s - 1)(\partial_s + 1)w(s))' > 0$. We first claim

$$\lim_{s\to+\infty} \mathrm{e}^{-2s}(\partial_s - 1)(\partial_s + 1)w(s) \leqslant 0.$$

Indeed, if

$$\lim_{s\to+\infty} \mathrm{e}^{-2s}(\partial_s - 1)(\partial_s + 1)w(s) = C_0,$$

then for large s we deduce $(\partial_s - 1)(\partial_s + 1)w(s) > C\mathrm{e}^{2s}$. From lemma 2.4, we have $(\partial_s + 1)w(s) > C\mathrm{e}^{2s}$. Again using lemma 2.4, one can see that

$$v'(t) = w(s) > C\mathrm{e}^{-2t} \quad \text{for large } |t|,$$

where the constant $C > 0$ is different from line to line. This contradicts the fact that $v(t) > 0$ for any $t \in (-\infty, t_0)$. So our claim is proved, and then $(\mathrm{e}^{-s}(\partial_s + 1)w(s))' < 0$. If $\lim_{s\to+\infty} \mathrm{e}^{-s}(\partial_s + 1)w(s) > 0$, we have a contradiction by the above argument, so $\lim_{s\to+\infty} \mathrm{e}^{-s}(\partial_s + 1)w(s) := C_1 \leqslant 0$. If $C_1 < 0$, then, since $v(t)$ is unbounded, there exists $s_0 > 0$ such that $w(s_0) < 0$. We immediately obtain that $v'(t) = w(s) < 0$ for $s > s_0$. If $C_1 = 0$, then $(\mathrm{e}^s w(s))' > 0$. Since there is a sequence $s_k \to +\infty$ such that $w(s_k) < 0$, we have $\lim_{s\to+\infty} \mathrm{e}^s w(s) \leqslant 0$, and again $w(s) < 0$.

Now, by the variation of parameters formula, $v(t)$ can be represented by

$$v(t) = \sum_{i=1}^{4} C_i \mathrm{e}^{v_i t} + 2^{-1} \int_t^{t_0} v^{-3}(\tau)\,\mathrm{d}\tau - \sum_{i=2}^{3} d_i \int_{-\infty}^{t} \mathrm{e}^{v_i(t-\tau)} v^{-3}(\tau)\,\mathrm{d}\tau - d_4 \int_t^{t_0} \mathrm{e}^{(t-\tau)} v^{-3}(\tau)\,\mathrm{d}\tau,$$

where $v_1 = 0$, $v_2 = -2$, $v_3 = -1$, $v_4 = 1$ and $t_0 < 0$ is fixed. Since $v(t) = o(\mathrm{e}^t)$ and

$$-\sum_{i=2}^{3} d_i \int_{-\infty}^{t} \mathrm{e}^{v_i(t-\tau)} v^{-3}(\tau)\,\mathrm{d}\tau - d_4 \int_t^{t_0} \mathrm{e}^{(t-\tau)} v^{-3}(\tau)\,\mathrm{d}\tau \to 0 \quad \text{as } t \to -\infty,$$

we have $C_2 = C_3 = 0$ and we can rewrite $v(t)$ as

$$v(t) = C + 2^{-1} \int_t^{t_0} v^{-3}(\tau)\,\mathrm{d}\tau + r(t), \tag{2.9}$$

where

$$r(t) = -\sum_{i=2}^{3} d_i \int_{-\infty}^{t} \mathrm{e}^{v_i(t-\tau)} v^{-3}(\tau)\,\mathrm{d}\tau - d_4 \int_t^{t_0} \mathrm{e}^{(t-\tau)} v^{-3}(\tau)\,\mathrm{d}\tau + C_4 \mathrm{e}^t = o(1) \quad \text{as } t \to -\infty.$$

Moreover, we have

$$r'(t) = o(1) \quad \text{as } t \to -\infty.$$

From (2.9), we have

$$v'v^3 = 2^{-1} + r'(t)v^3(t) \quad \text{and} \quad v'(t) \to 0 \quad \text{as } t \to -\infty. \tag{2.10}$$

By l'Hôpital's rule and the fact that $v'(t) \to 0$ as $t \to -\infty$, we have $r'(t)v^3(t) \to 0$ as $t \to -\infty$. Integrating the first equality of (2.10), we have

$$\tfrac{1}{4} v^4(t) = 2^{-1}(t_0 - t) + o(t_0 - t) \quad \text{for } t \ll -1.$$

From this, we immediately have

$$\lim_{t\to-\infty} \frac{v(t)}{\sqrt[4]{-t}} = 2^{1/4}, \quad \text{i.e. } \lim_{r\to 0} (1 - u(r)) \cdot r^{-1}(-\log r)^{-1/4} = 2^{1/4}.$$

Now we prove (iii). Since

$$v(t) = \sum_{i=1}^{4} C_i \mathrm{e}^{v_i t} + \sum_{i=1}^{4} d_i \int_{-\infty}^{t} \mathrm{e}^{v_i(t-s)} v^{-1}(s)\,\mathrm{d}s,$$

where

$$v_1 = -\frac{4}{p+1} - N + 4, \quad v_2 = -\frac{4}{p+1} - N + 2, \quad v_3 = -\frac{4}{p+1} + 2, \quad v_4 = -\frac{4}{p+1},$$

and since $v(t) = o(\mathrm{e}^{-4t/(p+1)})$ as $t \to -\infty$, we have $C_1 = C_2 = C_4 = 0$. Now we claim $C_3 \neq 0$. Indeed, by contradiction, if $C_3 = 0$, then

$$v(t) = \sum_{i=1}^{4} d_i \int_{-\infty}^{t} \mathrm{e}^{v_i(t-s)} v^{-1}(s)\,\mathrm{d}s.$$

But from lemma 2.3 and $v_i < 0$, $i = 1, 2, 3, 4$, we have

$$\int_{-\infty}^{t} \mathrm{e}^{v_i(t-s)} v^{-1}(s)\,\mathrm{d}s = O(1),$$

which contradicts $v(t)$ being unbounded. So, we have

$$\lim_{r\to 0} r^2(1 - u(r)) = C,$$

where C denotes a constant depending on u and λ. □

Proof of theorem 1.1(ii). From lemma 2.5, we immediately obtain the proof of theorem 1.1(ii). Indeed, if we suppose by contradiction that there is a weak solution $u(r)$ such that $\|u(r)\|_{L^\infty(\mathbb{B})} = 1$, we then have

$$\lim_{r\to 0} (1 - u(r)) \cdot r^{-1}(-\log r)^{-1/4} = 2^{1/4},$$

which contradicts that $(1 - u(r))^{-3} \in L^1(\mathbb{B})$. □

3. Stability of the entire solution

In this section, we study the stability of the entire solution of (E). For completeness, we give the proof of proposition 1.4, which appears in [20].

Proof of proposition 1.4. We argue by contradiction, assuming that (E) admits a stable solution for $N = 3, 4$. Then we have

$$\int_{\mathbb{R}^N} |\Delta\varphi|^2\,\mathrm{d}x \geqslant p \int_{\mathbb{R}^N} u^{-p-1}\varphi^2\,\mathrm{d}x \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

First, we consider the case $N = 3$ and take $\eta \in C_0^\infty(\mathbb{R}^3)$ such that

$$\eta = 1 \text{ on } \mathbb{B}_1, \quad \eta = 0 \text{ on } \mathbb{R}^N \setminus B_2 \quad \text{and} \quad 0 \leqslant \eta \leqslant 1.$$

We set $\eta_R(r) = \eta(r/R)$. Then

$$p \int_{\mathbb{B}_R} u^{-p-1}\,\mathrm{d}x \leqslant \int_{\mathbb{R}^3} |\Delta\eta_R(|x|)|^2\,\mathrm{d}x \leqslant CR^{-4} \int_0^{2R} r^2\,\mathrm{d}r \to 0 \quad \text{as } R \to \infty,$$

which is a contradiction to $u > 0$. Now we consider the case when $N = 4$. For this we set

$$\varphi_R(r) = \begin{cases} 1 & \text{on } [0, R], \\ \dfrac{1}{2\ln R}\left(-\dfrac{R^2}{r^2} + 1 + 4\ln R - 2\ln r\right) & \text{on } [R, R^2], \\ \dfrac{R^2(R^2-1)}{2r^2 \ln R} & \text{on } [R^2, +\infty). \end{cases}$$

By a simple calculation, we have $\varphi_R(r) \in C^1[0, +\infty)$, $\Delta\varphi_R(r) = 0$ on $[R^2, +\infty)$ and $\varphi_R(r) \in \dot{H}^2(\mathbb{R}^4)$, where $\dot{H}^2(\mathbb{R}^4)$ is the closure of $C_0^\infty(\mathbb{R}^4)$ for the semi-norm $\|\Delta \cdot\|_{L^2}$. Moreover, we have

$$\begin{aligned} \int_{\mathbb{R}^4} |\Delta\varphi_R(|x|)|^2 \,\mathrm{d}x &= \frac{C}{2\ln R}\int_R^{R^2} \left|\Delta\left(-\frac{R^2}{r^2} + 1 + 4\ln R - 2\ln r\right)\right|^2 r^3 \,\mathrm{d}r \\ &= \frac{C}{\ln R} \to 0 \quad \text{as } R \to \infty. \end{aligned}$$

Then we obtain the contradiction, as for the case when $N = 3$. □

In what follows we shall consider the radial case, and we now give some Hardy–Rellich inequalities in exterior domains, which play a key role in the research of the stability of radial solutions.

Lemma 3.1. *Let $\mathbb{B}_R$ a the ball of radius $R > 0$, centred at the origin in $\mathbb{R}^N$. The following Hardy-type inequalities with optimal constants hold true:*

$$2\int_{\mathbb{R}^2\setminus\mathbb{B}_R} |\Delta\varphi|^2 \,\mathrm{d}x \geqslant \int_{\mathbb{R}^2\setminus\mathbb{B}_R} \frac{\varphi^2}{r^4\log^2 r}\,\mathrm{d}x \qquad \forall\varphi \in C_0^\infty(\mathbb{R}^2 \setminus \bar{\mathbb{B}}_R), \tag{3.1}$$

$$\frac{16}{9}\int_{\mathbb{R}^3\setminus\mathbb{B}_R} |\Delta\varphi|^2 \,\mathrm{d}x \geqslant \int_{\mathbb{R}^3\setminus\mathbb{B}_R} \frac{\varphi^2}{r^4}\,\mathrm{d}x \qquad \forall\varphi \in C_0^\infty(\mathbb{R}^3 \setminus \bar{\mathbb{B}}_R), \tag{3.2}$$

$$\int_{\mathbb{R}^4\setminus\mathbb{B}_R} |\Delta\varphi|^2 \,\mathrm{d}x \geqslant \int_{\mathbb{R}^4\setminus\mathbb{B}_R} \frac{\varphi^2}{r^4\log^2 r}\,\mathrm{d}x \qquad \forall\varphi \in C_0^\infty(\mathbb{R}^4 \setminus \bar{\mathbb{B}}_R), \tag{3.3}$$

$$\int_{\mathbb{R}^N\setminus\mathbb{B}_R} |\Delta\varphi|^2 \,\mathrm{d}x \geqslant \frac{N^2(N-4)^2}{16}\int_{\mathbb{R}^N\setminus\mathbb{B}_R} \frac{\varphi^2}{r^4}\,\mathrm{d}x \quad \forall\varphi \in C_0^\infty(\mathbb{R}^N \setminus \bar{\mathbb{B}}_R),\ N \geqslant 5. \tag{3.4}$$

For the proof, see [1, corollaries 4.3 and 4.4].

Proof of theorem 1.6.
(i) First, we consider the case $p_c^+ < p < 3$, $1 < p < p_c$. Indeed, for this case we have that

$$\lim_{r\to\infty} r^{-4/(p+1)} u_{\beta_0}(r) = (-Q_4(m))^{-1/(p+1)} \quad \text{and} \quad 0 < -pQ_4(m) < \tfrac{9}{16}.$$

https://doi.org/10.1017/S0308210515000426 Published online by Cambridge University Press

And using the Hardy-type inequality (3.2), we see there exist $R > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^3\setminus\mathbb{B}_R} |\Delta\varphi|^2\,\mathrm{d}x - p\int_{\mathbb{R}^3\setminus\mathbb{B}_R} u^{-p-1}\varphi^2\,\mathrm{d}x &\geqslant \int_{\mathbb{R}^3\setminus\mathbb{B}_R} |\Delta\varphi|^2\,\mathrm{d}x + (-pQ_4(m) - \epsilon_R)\int_{\mathbb{R}^3\setminus\mathbb{B}_R} \frac{\varphi^2}{r^4}\,\mathrm{d}x \\ &> \int_{\mathbb{R}^3\setminus\mathbb{B}_R} |\Delta\varphi|^2\,\mathrm{d}x - \frac{9}{16}\int_{\mathbb{R}^3\setminus\mathbb{B}_R} \frac{\varphi^2}{r^4}\,\mathrm{d}x \\ &> 0\end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^3 \setminus \bar{\mathbb{B}}_R)$, where $\epsilon_R \to 0$ as $R \to \infty$. Then u_{β_0} is stable outside a compact set.

Second, we consider the case $p_c < p < p_c^+$. Indeed, by contradiction, we assume that u_{β_0} is stable outside a compact set K. We can always choose R so large that $K \subset \mathbb{B}_R(0)$ and then

$$\int_{\mathbb{R}^3\setminus\mathbb{B}_R} |\Delta\varphi|^2\,\mathrm{d}x - p\int_{\mathbb{R}^3\setminus\mathbb{B}_R} u_{\beta_0}^{-p-1}\varphi^2\,\mathrm{d}x > 0 \tag{3.5}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^3 \setminus \mathbb{B}_R)$. Now let $\epsilon_R \to 0^+$ as $R \to \infty$. Since

$$-pQ_4(m) > \tfrac{9}{16} \quad \text{for } p_c < p < p_c^+,$$

we can choose R so large that $-pQ_4(m) - \epsilon_R > \frac{9}{16}$. Combining proposition 1.5(3) with (3.5), we have

$$\begin{aligned}\int_{\mathbb{R}^3\setminus\mathbb{B}_R} |\Delta\varphi|^2\,\mathrm{d}x - (\tfrac{9}{16} + \epsilon_R)\int_{\mathbb{R}^3\setminus\mathbb{B}_R} \frac{\varphi^2}{r^4}\,\mathrm{d}x &\geqslant \int_{\mathbb{R}^3\setminus\mathbb{B}_R} |\Delta\varphi|^2\,\mathrm{d}x + (-pQ_4(m) - \epsilon_R)\int_{\mathbb{R}^3\setminus\mathbb{B}_R} \frac{\varphi^2}{r^4}\,\mathrm{d}x \\ &\geqslant \int_{\mathbb{R}^3\setminus\mathbb{B}_R} |\Delta\varphi|^2\,\mathrm{d}x + p\int_{\mathbb{R}^3\setminus\mathbb{B}_R} u_{\beta_0}^{-p-1}\varphi^2\,\mathrm{d}x \\ &> 0,\end{aligned}$$

which contradicts the optimality of the Hardy–Rellich inequality (3.2). And so $u_{\beta_0}(r)$ is unstable outside every compact set for this case.

Finally, we consider the case $p \geqslant 3$. Since in this situation the solution u_{β_0} satisfies

$$\lim_{r\to\infty} \frac{u_{\beta_0}}{r(\log r)^{1/4}} = 2^{1/4}, \qquad p = 3,$$
$$\lim_{r\to\infty} \frac{u_{\beta_0}}{r} = \alpha > 0, \quad p > 3,$$

by combining the Hardy–Rellich inequality (3.2) with the argument above, we obtain that u_{β_0} is stable outside a compact set.

(ii) Since

$$\lim_{r\to\infty} \Delta u_\beta(r) \to \gamma > 0 \quad \text{for } \beta > \beta_0,$$

https://doi.org/10.1017/S0308210515000426 Published online by Cambridge University Press

we immediately have

$$\lim_{r\to\infty}\frac{u_\beta(r)}{r^2}=\gamma_1>0 \quad \text{for } \beta>\beta_0.$$

Combining this with the Hardy–Rellich inequality, (3.2), we find the solutions are stable outside a compact set, and this completes the proof of this theorem. □

Proof of theorem 1.7. We prove by contradiction that $u_{\beta_0}(r)$ is stable outside a compact set K. Choosing R so large that $K\subset\mathbb{B}_R(0)$ and $\log^2 R\cdot(pQ_4(m)+\epsilon_R)>1$, and using proposition 1.5(3), we have, for all $\varphi\in C_0^\infty(\mathbb{R}^3\setminus\mathbb{B}_R)$,

$$\begin{aligned}0&<\int_{\mathbb{R}^4\setminus\mathbb{B}_R}|\Delta\varphi|^2\,\mathrm{d}x-p\int_{\mathbb{R}^4\setminus\mathbb{B}_R}u^{-p-1}\varphi^2\,\mathrm{d}x\\&=\int_{\mathbb{R}^4\setminus\mathbb{B}_R}|\Delta\varphi|^2\,\mathrm{d}x+(-pQ_4(m)-\epsilon_R)\int_{\mathbb{R}^4\setminus\mathbb{B}_R}\frac{\varphi^2}{|x|^4}\,\mathrm{d}x\\&<\int_{\mathbb{R}^4\setminus\mathbb{B}_R}|\Delta\varphi|^2\,\mathrm{d}x-\log^2 R\cdot(-pQ_4(m)+\epsilon_R)\int_{\mathbb{R}^4\setminus\mathbb{B}_R}\frac{\varphi^2}{|x|^4\log^2|x|}\,\mathrm{d}x,\end{aligned}$$

which is a contradiction of the optimality of the Hardy–Rellich inequality (3.3).

For the case when $\beta>\beta_0$, the proof is the same as for theorem 1.6 and we omit it here. □

Now, we give the following lemma, which plays a key role in the proof of theorem 1.9.

Lemma 3.2. *If $N\geqslant 13$, or $4<N\leqslant 12$ and $1<p\leqslant p_c$, then*

$$u_\beta(r)>(-Q_4(m))^{-1/(p+1)}r^{4/(p+1)} \quad \textit{for all } r>0,\ \beta\geqslant\beta_0.$$

For the proof of this lemma, see [14].

Proof of theorem 1.9.

(i) Since p, N satisfy

$$N\geqslant 13 \quad\text{or}\quad 5\leqslant N\leqslant 12 \text{ and } 1<p\leqslant p_c$$

from lemma 3.2, we have

$$u_\beta(r)>(-Q_4(m))^{-1/(p+1)}r^{4/(p+1)} \qquad \text{for } r>0,\ \beta\geqslant\beta_0 \text{ and } -pQ_4(m)>\tfrac{1}{16}(N^2(N-4)^2).$$

Combining this with the following Hardy-type inequalities:

$$\int_{\mathbb{R}^N}|\Delta\varphi|^2\,\mathrm{d}x\geqslant\frac{N^2(N-4)^2}{16}\int_{\mathbb{R}^N}\frac{\varphi^2}{r^4}\,\mathrm{d}x \quad \forall\varphi\in C_0^\infty(\mathbb{R}^N),\ N\geqslant 5, \tag{3.6}$$

we see for $\beta\geqslant\beta_0$ that

$$\int_{\mathbb{R}^N}|\Delta\varphi|^2\,\mathrm{d}x-p\int_{\mathbb{R}^N}u_\beta^{-p-1}\varphi^2\,\mathrm{d}x\geqslant\int_{\mathbb{R}^N}|\Delta\varphi|^2\,\mathrm{d}x+pQ_4(m)\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^4}\,\mathrm{d}x>0,$$

which implies u_β is stable and completes the proof of (i).

(ii) The case $\beta = \beta_0$ is similar to theorem 1.7, so we omit it here. Now we consider the case when $\beta > \beta_0$. Indeed, we first recall that

$$u_\beta \geqslant u_{\beta_0} + \frac{\beta - \beta_0}{2N} r^2 \quad \text{for all } r \geqslant 0.$$

Simply choose $\bar{\beta} > \beta_0$ and r_0 such that

$$pu_{\bar{\beta}}^{-p-1} \leqslant \begin{cases} \dfrac{N^2(N-4)^2}{16} \dfrac{1}{r^4} & \text{for } r \leqslant r_0, \\ p\left(\dfrac{\bar{\beta} - \beta_0}{2N}\right)^{-p-1} r^{-2(p+1)} \leqslant \dfrac{N^2(N-4)^2}{16} \dfrac{1}{r^4} & \text{for } r > r_0. \end{cases}$$

Combining the above inequality with the Hardy–Rellich inequality (3.6), we deduce that u_β is stable for $\beta \geqslant \bar{\beta}$. So, we may define

$$\Lambda = \{\beta > \beta_0 \mid u_\beta \text{ is stable}\} \quad \text{and} \quad \beta_1 = \inf\{\Lambda\}.$$

By standard ordinary differential equation theory, one may easily prove that $\beta_1 = \min\{\Lambda\}$. Since u_{β_0} is unstable, $\beta_1 > \beta_0$. Also, the solutions are ordered: if $\beta_2 > \beta_1$, then $u_{\beta_2} > u_{\beta_1}$. So, Λ is the interval $[\beta_1, +\infty)$. Obviously, u_β is stable outside a compact set for $\beta \in (\beta_0, \beta_1)$. □

Acknowledgements

The author thanks Professor Dong Ye of the East China Normal University for some valuable suggestions. This research is supported by the NSFC (Grant nos 11201119 and 11471099) and the International Cultivation of Henan Advanced Talents programme (Grant no. yqpy20140043).

References

1 P. Caldiroli and R. Musina. Rellich inequalities with weights. *Calc. Var. PDEs* **45** (2012), 147–164.

2 C. Cowan and N. Ghoussoub. Regularity of semi-stable solutions to fourth order nonlinear eigenvalue problems on general domains. *Calc. Var. PDEs* **49** (2014), 291–305.

3 C. Cowan, P. Esposito, N. Ghoussoub and A. Moradifam. The critical dimension for a fourth order elliptic problem with singular nonlinearity. *Arch. Ration. Mech. Analysis* **198** (2010), 763–787.

4 C. Cowan, P. Esposito and N. Ghoussoub. Regularity of extremal solutions in fourth order nonlinear eigenvalue problems on general domains. *Discrete Contin. Dynam. Syst.* **28** (2010), 1033–1050.

5 J. Dávila, L. Dupaigne, I. Guerra and M. Montenegro. Stable solutions for the bi-Laplacian with exponential nonlinearity. *SIAM J. Math. Analysis* **39** (2007), 565–592.

6 J. Dávila, I. Flores and I. Guerra. Multiplicity of solutions for a fourth order equation with power-type nonlinearity. *Math. Annalen* **348** (2009), 143–193.

7 P. Esposito, N. Ghoussoub and Y. Guo. *Mathematical analysis of partial differential equations modeling electrostatic MEMS*. Courant Lecture Notes in Mathematics, vol. 20 (Providence, RI: American Mathematical Society, 2010).

8 A. Ferrero and H.-C. Grunau. The Dirichlet problem for supercritical biharmonic equations with power-type nonlinearity. *J. Diff. Eqns* **234** (2007), 582–606.

9 A. Ferrero and G. Warnault. On solutions of second and fourth order elliptic equations with power-type nonlinearities. *Nonlin. Analysis* **70** (2009), 2889–2902.

10 I. Guerra. A note on nonlinear biharmonic equations with negative exponents. *J. Diff. Eqns* **253** (2012), 3147–3157.

11 Z. Guo and J. Wei. On a fourth order nonlinear elliptic equation with negative exponent. *SIAM J. Math. Analysis* **40** (2009), 2034–2054.

12 Z. Guo and J. Wei. Liouville type results and regularity of the extremal solutions of biharmonic equation with negative exponents. *Discrete Contin. Dynam. Syst.* A **34** (2014), 2561–2580.

13 Z. Guo, B. S. Lai and D. Ye. Revisiting the biharmonic equation modelling electrostatic actuation in lower dimensions. *Proc. Am. Math. Soc.* **142** (2014), 2027–2034.

14 B. S. Lai and Q. Luo. Biharmonic equations with singular nonlinearity. (Submitted.)

15 B. S. Lai and D. Ye. Remarks on entire solutions for two fourth-order elliptic problems. *Proc. Edinb. Math. Soc.* (2015), doi:10.1017/S0013091515000371.

16 F. Lin and Y. Yang. Nonlinear non-local elliptic equation modelling electrostatic actuation. *Proc. R. Soc. Lond.* A **463** (2007), 1323–1337.

17 P. J. Mckenna and W. Rechel. Radial solutions of singular nonlinear biharmonic equations and applications to conformal geometry. *Electron. J. Diff. Eqns* **37** (2003), 1–13.

18 A. Moradifam. On the critical dimension of a fourth order elliptic problem with negative exponent. *J. Diff. Eqns* **248** (2010), 594–616.

19 J. A. Pelesko and A. A. Bernstein. *Modeling MEMS and NEMS* (Boca Raton, FL: Chapman Hall/CRC Press, 2002).

20 G. Warnault. Liouville theorems for stable radial solutions for the biharmonic operator. *Asymp. Analysis* **69** (2010), 87–98.