On Parseval Wavelet Frames via Multiresolution Analyses in H_G^2

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Abstract. We give a characterization of all Parseval wavelet frames arising from a given frame multiresolution analysis. As a consequence, we obtain a description of all Parseval wavelet frames associated with a frame multiresolution analysis. These results are based on a version of Oblique Extension Principle with the assumption that the origin is a point of approximate continuity of the Fourier transform of the involved refinable functions. Our results are written for reducing subspaces.

1 Introduction, Notation and Basic Definitions

We are interested in the study of methods for constructing tight wavelet frames in reducing subspaces of L^2. In this paper, we emphasis on tight wavelet frames can be constructed via multiresolution analyses and extension principles. Mallat [36] and Meyer [37] introduced the definition of multiresolution analysis (MRA) as a general method for constructing wavelets. In order to construct wavelet frames, the requirements on the definition of MRA were weakened. In this sense, the notion of a frame multiresolution analysis (FMRA) was formulated by Benedetto and Li [5] as a natural extension of MRA. Furthermore, a generalized multiresolution analysis (GMRA) was first introduced by Baggett, Medina and Merril [3] and Papadakis [39] independently; see also the paper by de Boor, DeVore and A. Ron [10]. Bownik and Rzeszotnik [9] show construction procedures for tight framelets and wavelets from a GMRA. The notions of MRA and FMRA in a reducing subspace of $L^2(\mathbb{R})$ are introduced by Li and Lian [31]. On $\mathbb{R}^n$ see [27]. Relationships of FMRA and MRA in reducing subspaces $L^2(\mathbb{R}^n)$ are studied by Li and Zhang [33]. Li and Zhou [34,35], develop GMRA-based construction procedures of frame wavelet systems in reducing subspaces of $L^2(\mathbb{R}^n)$. Jia and Li [25] present a refinable function-based construction of affine frames systems.

Gripenberg [19] and Hernández and Weiss [24] proved a characterization of all orthonormal wavelets associated to a multiresolution analysis in terms of its wavelet dimension function. That any orthonormal wavelet is associated with a generalized multiresolution analysis was proved by Papadakis [40]. In the paper by Kim, Kim and Lim [29] (see also Kim, Kim, Lee and Lim [28]), characterizations of the Riesz

wavelets which are associated with a multiresolution analysis were proved. A generalization of these results is given by Bownik and Garrigós [8]. Note that characterizations of biorthogonal wavelets from biorthogonal multiresolution analysis are proved in [29] and [8] (see also Calogero and Garrigós [12]). Zalik [48] introduced the notion of Riesz wavelet obtained by a multiresolution analysis. Moreover, he gave necessary and sufficient conditions on a given Riesz wavelet to be obtained by a multiresolution analysis. Bownik [7] studied both the notion of Riesz basis associated with a generalized multiresolution analysis and Riesz basis obtained by a generalized multiresolution analysis and proved that these two notions are equivalent. Characterizations of Parseval wavelet frames associated with generalized multiresolution analysis are proved by Baggett, Medina, and Merrill [3] and by Bakić [4].

A slightly different point of view for constructing wavelet frames was first proposed in the Unitary Extension Principle (UEP) by Ron and Shen [43] (see also [42]). The UEP leads to explicit constructions of tight wavelet frames generated by a refinable function. A more flexible way for constructing wavelet frames is the so-called Oblique Extension Principle (OEP). The OEP was introduced by Daubechies, Han, Ron, and Shen [17]. These extension principles have been developed by Benedetto and Treiber [6], Petukhov [41], Chui, He and Stöckler [14], Chui, He, Stöckler, and Sun [15], Bownik and Rzeszotnik [9], Han [21, 22], Li and Zhou [35], Stavropoulos [47], Atreas, Melas, and Stavropoulos [1, 2], and Li and Zhang [32]. Observe that in these papers, they proved that the obtained sufficient conditions are also necessary.

Extensive studies on multiresolution analysis and extension principles are enclosed, for instance, in [13] and [30].

We give a solution to the problem of characterizing all Parseval wavelet frames arising from a fixed frame multiresolution analysis in reducing subspaces of $L^2(\mathbb{R}^n)$. As a consequence, we obtain a new description of all Parseval wavelet frames associated with a frame multiresolution analysis. The proofs of these results are based on the characterization of the scaling functions in [27] and on a version of Oblique Extension Principle with no regularity at the origin on the modulus of the Fourier transform of the involved refinable functions.

Versions of the OEP without any extra condition on refinable functions have already been proved in the literature. The first version of the OEP was proved by Han [21, 22], where the context was in distribution spaces. The assumption at the origin of the Fourier transform of involved refinable functions is a limit in the sense of distributions. In the paper by Li and Zhang [32, Theorem 1.2], an OEP for frames in reducing subspaces of $L^2(\mathbb{R}^d)$ is proved. The authors use the condition $\lim_{j\to-\infty} \hat{\psi}_0(A^{*j}\cdot) = 1$ for almost every point of Ω, an A^*-invariant set of $\mathbb{R}^d$. Perhaps the main advance of our paper is the use of the classical notion of approximate continuity. This is why we emphasize this notion. In particular, this concept helps us to learn a bit of the behaviour on a neighborhood of the origin of the Fourier transform of refinable functions associated with Parseval wavelet frames. To illustrate our results we show some examples of Parseval wavelet frames where the origin of the Fourier transform of is not a point of continuity. At the end of this manuscript we will see that the conditions used by Han, by Li and Zhang and the conditions used here are of different nature.

Before formulating our results let us introduce some notation and definitions.

The sets of strictly positive integers, integers, real and complex numbers will be denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ respectively. We let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, $n \geq 1$, and with some abuse of the notation we consider also that $\mathbb{T}^n$ is the unit cube $[0,1)^n$.

We let $B_r(\mathbf{y})$ denote $\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{y}| < r\}$, and will write B_r if $\mathbf{y}$ is the origin. If $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, A^* will mean the adjoint of A. With some abuse of the notation, we write A to also denote the corresponding matrix with respect to the canonical basis. Moreover, $d_A = |\det A|$. For a Lebesgue measurable set $E \subset \mathbb{R}^n$, $E^c = \mathbb{R}^n \smallsetminus E$ and the Lebesgue measure of E in $\mathbb{R}^n$ will be denoted by $|E|_n$. If $\mathbf{x} \in \mathbb{R}^n$ then $\mathbf{x} + E = \{\mathbf{x} + \mathbf{y} : \text{for } \mathbf{y} \in E\}$. We will let $A(E)$ denote $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = A(\mathbf{t}) \text{ for } \mathbf{t} \in E\}$. The volume of E changes under A according to $|AE|_n = d_A|E|_n$. The characteristic function of a set $E \subset \mathbb{R}^n$ will be denoted by χ_E, *i.e.*, $\chi_E(\mathbf{x}) = 1$ if $\mathbf{x} \in E$ and $\chi_E(\mathbf{x}) = 0$ otherwise.

We write $L^p(\mathbb{R}^n)$, $n \geq 1$, $1 \leq p \leq \infty$ to mean the usual Lebesgue space. We write $f \in L_{\text{loc}}^p(\mathbb{R}^n)$ to mean the linear space of all measurable functions such that $f\chi_K \in L^p(\mathbb{R}^n)$ for every $K \subset \mathbb{R}^n$ compact sets.

If we take $f \in L^p(\mathbb{T}^n)$, we will understand that f is defined on the whole space $\mathbb{R}^n$ as a $\mathbb{Z}^n$-periodic function.

A key tool in the study of wavelet frames is the Fourier transform. Here, our convention is that if $f \in L^1(\mathbb{R}^n)$, then

$$\widehat{f}(\mathbf{x}) \coloneqq \int_{\mathbb{R}^n} f(\mathbf{t}) e^{-2\pi i \mathbf{t} \cdot \mathbf{x}} d\mathbf{t},$$

where $\mathbf{t} \cdot \mathbf{x}$ denotes the usual inner product of vectors $\mathbf{t}$ and $\mathbf{x}$ in $\mathbb{R}^n$. The definition of Fourier transform is extended as usual to functions in $L^2(\mathbb{R}^n)$.

A linear map A is called expansive if all (complex) eigenvalues of A have absolute value greater than 1. If A is invertible, we consider the operator D_A on $L^2(\mathbb{R}_n)$ defined by $D_A f(\mathbf{t}) = d_A^{1/2} f(A\mathbf{t})$. The translation of a function $f \in L^2(\mathbb{R}^n)$ by $\mathbf{b} \in \mathbb{R}^n$ will be denoted by $\tau_{\mathbf{b}} f(\mathbf{t}) = f(\mathbf{t} - \mathbf{b})$. For a subspace S of $L^2(\mathbb{R}^d)$,

$$D_A S = \{D_A^j f : f \in S\} \quad \text{and} \quad \tau_{\mathbf{b}} S = \{\tau_{\mathbf{b}} f : f \in S\}.$$

We write D_A^{-1} to mean the operator $D_{A^{-1}}$, D_A^0 is the identity map, and D_A^ℓ, $\ell \in \mathbb{N}$, is the ℓ-th composition of the operator D_A with itself.

A measurable set $E \subset \mathbb{R}^n$ such that $AE = E$ is called an A invariant set or an A-set. A shift-invariant subspace $S \subset L^2(\mathbb{R}^n)$ is called A-reducing if $D_A S = S$. If $G \subset \mathbb{R}^n$ is an A^*-set, we denote by H_G^2 the closed linear subspace of $L^2(\mathbb{R}^n)$ defined by

$$H_G^2 = \{f \in L^2(\mathbb{R}^n) : \widehat{f} = \widehat{f}\chi_G\}.$$

That S is an A-reducing subspace of $L^2(\mathbb{R}^n)$ if and only if S is H_G^2 for an A^*-set $G \subset \mathbb{R}^n$ was proved by Dai, Diao, Gu and Han [16].

For a given $\phi \in L^2(\mathbb{R}^n)$, set

$$[\phi, \phi](\mathbf{t}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} |\phi(\mathbf{t} + \mathbf{k})|^2$$

and denote

$$\mathcal{N}_\phi = \{\mathbf{t} \in \mathbb{R}^n \ : \ [\phi, \phi](\mathbf{t}) = 0\}.$$

For a measurable function $f : \mathbb{R}^n \to \mathbb{C}$ the support of f is defined to be $\text{supp}(f) = \{\mathbf{t} \in \mathbb{R}^n; f(\mathbf{t}) \neq 0\}$.

The sets are defined modulo a null measurable set and we will understand some equations as almost everywhere on $\mathbb{R}^n$ or $\mathbb{T}^n$. Moreover, in order to shorten the notation, we will consider $0/0 = 0$ or $0(1/0) = 0$ in some expressions where such an indeterminacy appears.

If $A : \mathbb{R}^n \to \mathbb{R}^n$ is an expansive linear invertible map such that $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$, then the quotient groups $\mathbb{Z}^n / A(\mathbb{Z}^n)$ and $A^{-1}(\mathbb{Z}^n)/\mathbb{Z}^n$ are well defined. We will denote by $\Omega_A \subset \mathbb{Z}^n$ and $\Gamma_A \subset A^{-1}(\mathbb{Z}^n)$ a full collection of representatives of the cosets of $\mathbb{Z}^n / A(\mathbb{Z}^n)$ and $A^{-1}(\mathbb{Z}^n)/\mathbb{Z}^n$ respectively. Recall that there are exactly d_A cosets of $\mathbb{Z}^n / A(\mathbb{Z}^n)$ (see [20]). Thus there are exactly d_A cosets of $A^{-1}(\mathbb{Z}^n)/\mathbb{Z}^n$.

The theory of frames was introduced by Duffin and Schaeffer [18]. A sequence $\{\phi_n\}_{n=1}^\infty$ of elements in a separable Hilbert space $\mathbb{H}$ is a *frame* for $\mathbb{H}$ if there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|h\|^2 \le \sum_{n=1}^\infty |\langle h, \phi_n \rangle|^2 \le C_2 \|h\|^2, \quad \forall h \in \mathbb{H},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{H}$. The constants C_1 and C_2 are called *frame bounds*. The definition implies that a frame is a complete sequence of elements of $\mathbb{H}$. A frame $\{\phi_n\}_{n=1}^\infty$ is *tight* if we may choose $C_1 = C_2$, and if in fact $C_1 = C_2 = 1$, we will call it a *Parseval frame*. A sequence $\{h_n\}_{n=1}^\infty$ of elements in a Hilbert space $\mathbb{H}$ is a *frame sequence* if it is a frame for $\overline{\text{span}}\{h_n\}_{n=1}^\infty$.

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an expansive linear map such that $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$ and $G \subset \mathbb{R}^n$ be an A^*-set. Let $\Psi = \{\psi_1, \dots, \psi_N\} \subset L^2(\mathbb{R}^n)$ be a set of functions, we call

(1.1) $$X_\Psi := \{D_A^j \tau_{\mathbf{k}} \psi_\ell : j \in \mathbb{Z}, \ \mathbf{k} \in \mathbb{Z}^d, \ 1 \le \ell \le N\}$$

the *affine system* generated by Ψ. The set of functions Ψ is called a *wavelet frame* associated with A for H_G^2, if the affine system (1.1) is a frame for H_G^2. In this case, the affine system (1.1) is usually called an *affine frame*. When the context is clear we do not write "associated with A for H_G^2". If this affine system is a tight frame for H_G^2 then Ψ is called a *tight wavelet frame* or *tight framelet* for H_G^2. In particular, a tight wavelet frame with frame bounds equal to 1 is usually called a *Parseval wavelet frame* or a *Parseval framelet* for H_G^2. The functions ψ_ℓ, $\ell = 1, \dots, N$ are called the *generators* of the wavelet frame.

Given a linear invertible map A and $G \subset \mathbb{R}^n$ as above, by a frame multiresolution analysis associated with the dilation A in H_G^2 (H_G^2-FMRA) we mean a sequence of closed subspaces V_j, $j \in \mathbb{Z}$, of the Hilbert space H_G^2 that satisfies the following conditions:

(i) $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$;
(ii) $f \in V_j$ if and only if $D_A f \in V_{j+1}$ for every $j \in \mathbb{Z}$;
(iii) $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = H_G^2$;
(iv) there exists a function $\phi \in V_0$, called a *scaling function*, such that $V_0 = \overline{\text{span}} \{\tau_{\mathbf{k}} \phi \ : \ \mathbf{k} \in \mathbb{Z}^n\}$.

When the system $\{\tau_{\mathbf{k}}\phi : \mathbf{k} \in \mathbb{Z}^n\}$ is an orthonormal basis for V_0, then the H_G^2-FMRA is called an orthonormal multiresolution analysis, or simply multiresolution analysis, in H_G^2 (H_G^2-MRA).

One of the possible ways for constructing an H_G^2-FMRA is to start with a scaling function $\phi \in H_G^2$. A function $\phi \in H_G^2$ *generates an H_G^2-FMRA* if $V_0 = \overline{\text{span}}\{\tau_{\mathbf{k}}\phi : \mathbf{k} \in \mathbb{Z}^n\}$ and the subspaces

$$V_j = \overline{\text{span}}\{D_A^j \tau_{\mathbf{k}}\phi; \mathbf{k} \in \mathbb{Z}^n\}, \quad j \in \mathbb{Z}$$

of the Hilbert space H_G^2 satisfy the conditions (i) and (iii).

If $\phi \in H_G^2$ generates an H_G^2-FMRA, then $\phi \in V_0 \subset V_1$. Thus,

(1.2)
$$\phi = \sum_{\mathbf{k}\in\mathbb{Z}^n} a_{\mathbf{k}} D_A \tau_{\mathbf{k}}\phi, \quad a_{\mathbf{k}} \in \mathbb{C},$$

where the convergence is in $L^2(\mathbb{R}^n)$. Taking the Fourier transform, we can write

$$\widehat{\phi}(A^*\mathbf{t}) = H(\mathbf{t})\widehat{\phi}(\mathbf{t}) \quad \text{a.e. on } \mathbb{R}^n$$

where H is a $\mathbb{Z}^n$-periodic measurable function. A function that satisfies an equality such as (1.2) is called *refinable*.

Given a multiresolution analysis, the following describes a standard procedure for constructing wavelet frames. If $\{V_j\}_{j\in\mathbb{Z}} \subset H_G^2$ is an H_G^2-FMRA, we denote by W_j the orthogonal complement of V_j in V_{j+1}. Thus, by condition (i) we have $V_{j+1} = W_j \oplus V_j$. Moreover, condition (iii) implies that $V_{j+1} = \oplus_{k<j} W_k$ and $H_G^2 = \oplus_{j\in\mathbb{Z}} W_j$. Observe that if $\{\tau_{\mathbf{k}}\psi_\ell : \mathbf{k} \in \mathbb{Z}^d, 1 \le \ell \le N\}$ is a Parseval frame for W_0, then the system $\{D_A^j \tau_{\mathbf{k}}\psi_\ell : j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, 1 \le \ell \le N\}$ is a Parseval frame for H_G^2.

Definition 1.1   Let $\{V_j\}_{j\in\mathbb{Z}} \subset H_G^2$ be an H_G^2-FMRA. A Parseval wavelet frame $\Psi = \{\psi_1, \ldots, \Psi_N\}$ for H_G^2 is said to be associated with $\{V_j\}_{j\in\mathbb{Z}}$ if $\{\tau_{\mathbf{k}}\psi_\ell : \mathbf{k} \in \mathbb{Z}^d, 1 \le \ell \le N\}$ is a Parseval frame for $W_0 = V_1 \ominus V_0$. We say that a Parseval wavelet frame is an H_G^2-FMRA wavelet frame if Ψ is Parseval wavelet frame associated with some H_G^2-FMRA.

A slightly more flexible type of Parseval wavelet frame is the following.

Definition 1.2   Let $\{V_j\}_{j\in\mathbb{Z}} \subset H_G^2$ be an H_G^2-FMRA. We say that $\Psi = \{\psi_1, \ldots, \Psi_N\} \subset H_G^2$ is a Parseval wavelet frame arising from $\{V_j\}_{j\in\mathbb{Z}}$ if $\Psi \subset V_1$ and the associated affine system (1.1) is a Parseval wavelet frame for H_G^2. Sometimes it is said that $\Psi = \{\psi_1, \ldots, \Psi_N\} \subset H_G^2$ is an H_G^2-FMRA based wavelet frame if Ψ is a Parseval wavelet frame arising from some H_G^2-FMRA.

The following definitions were introduced in [27].

Definition 1.3   Let $A: \mathbb{R}^n \to \mathbb{R}^n$ be an expansive linear map and let $G \subset \mathbb{R}^n$ be an A^*-set. It is said that $\mathbf{x} \in \mathbb{R}^n$ is a point of (A, G)-density for a measurable set $E \subset \mathbb{R}^n$, $|E|_n > 0$, if for any $r > 0$,

$$\lim_{j\to\infty} \frac{|E \cap [(G \cap A^{-j}B_r) + \mathbf{x}]|_n}{|G \cap A^{-j}B_r|_n} = 1.$$

Definition 1.4 Let $A: \mathbb{R}^n \to \mathbb{R}^n$ be an expansive linear map and let $G \subset \mathbb{R}^n$ be an A^*-set. Let $f: \mathbb{R}^n \longrightarrow \mathbb{C}$ be a measurable function. It is said that $\mathbf{x} \in \mathbb{R}^n$ is a point of (A, G)-approximate continuity of the function f if there exists $E \subset \mathbb{R}^n$, $|E|_n > 0$, such that $\mathbf{x}$ is a point of (A, G)-density for the set E and

$$\lim_{\substack{\mathbf{y} \to \mathbf{x} \\ \mathbf{y} \in E}} f(\mathbf{y}) = f(\mathbf{x}).$$

Observe that if $G = \mathbb{R}^n$ and $A = aI$, where $a > 1$ and I is the identity map on $\mathbb{R}^n$, then the definition of a point of (A, G)-approximate continuity coincides with the well-known definition of *approximate continuity* (*cf.* [11, 38]).

This paper is structured as follows. In Section 2, we present the main results of this paper with their proofs. In Section 3, we show some examples of Parseval wavelet frames. Section 4 is devoted to comparing conditions on the Fourier transform of involved refinable functions in different versions of Extension Principles.

2 Main Result

We present the main results of this paper. In what follows, we fix $A: \mathbb{R}^n \to \mathbb{R}^n$ an expansive linear map such that $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$. Let us fix $\Gamma_{A^*} = \{\mathbf{p}_k\}_{k=0}^{d_A - 1}$, where $\mathbf{p}_0 = \mathbf{0}$, a full collection of representatives of the cosets of $(A^*)^{-1}\mathbb{Z}^n / \mathbb{Z}^n$. Moreover, we fix an A^*-set $G \subset \mathbb{R}^n$ and the subspace H_G^2. In order to shorten the notation, if we write a wavelet frame we mean a wavelet frame for H_G^2 associated with the dilation A.

The following result characterizes all Parseval wavelet frames for some reducing subspace that can be constructed via the Oblique Extension Principle.

Theorem 2.1 *Let $\phi \in H_G^2$ such that*

$$\widehat{\phi}(A^* \mathbf{t}) = H_0(\mathbf{t})\widehat{\phi}(\mathbf{t}), \quad a.e.,$$

where $H_0 \in L^\infty(\mathbb{T}^n)$. Let $H_1, \dots, H_N \in L^\infty(\mathbb{T}^n)$ and define $\psi_1, \dots, \psi_N \in L^2(\mathbb{R}^n)$ by

$$\widehat{\psi_\ell}(A^* \mathbf{t}) = H_\ell(\mathbf{t})\widehat{\phi}(\mathbf{t}) \quad a.e., \quad \ell = 1, \dots, N.$$

Then the following are equivalent.

(i) *The set of functions $\{\psi_\ell : \ell = 1, \dots, N\}$ is a Parseval wavelet frame associated with A for H_G^2.*

(ii) *There exists S, a non-negative $\mathbb{Z}^n$-periodic measurable function such that $\sqrt{S}|\widehat{\phi}| \in H_G^2$ and also*

    (a) *the origin is a point of (A^*, G)-approximate continuity for $S|\widehat{\phi}|^2$, provided $S(\mathbf{0})|\widehat{\phi}(\mathbf{0})|^2 = 1$;*

    (b)

$$(2.1) \qquad S(A^* \mathbf{t})|H_0(\mathbf{t})|^2 + \sum_{\ell=1}^{N} |H_\ell(\mathbf{t})|^2 = S(\mathbf{t}), \quad a.e. \quad \mathbf{t} \in \mathbb{R}^n \setminus \mathcal{N}_{\widehat{\phi}};$$

    (c) *the equality*

$$(2.2) \qquad S(A^*\mathbf{t})H_0(\mathbf{t})\overline{H_0(\mathbf{t}+\mathbf{p}_k)} + \sum_{\ell=1}^{N} H_\ell(\mathbf{t})\overline{H_\ell(\mathbf{t}+\mathbf{p}_k)} = 0$$

holds for a.e. $\mathbf{t} \in \mathbb{R}^n \setminus \mathcal{N}_{\widehat{\phi}}$ and for any $\mathbf{p}_k$, $k = 1, \dots, d_A - 1$, such that $\mathbf{t} + \mathbf{p}_k \in \mathbb{R}^n \setminus \mathcal{N}_{\widehat{\phi}}$.

To prove (ii) implies (i) in Theorem 2.1 we need the following result.

Lemma 2.2 Let $\phi \in H_G^2$ such that $|\widehat{\phi}(\mathbf{t})| \le 1$ a.e. and the origin is a point of (A^*, G)-approximate continuity of $|\widehat{\phi}|$, provided $|\widehat{\phi}(\mathbf{0})| = 1$. Let $g \in L^2(\mathbb{R}^n)$ such that $\widehat{g}$ is continuous and compactly supported. Consider $f \in L^2(\mathbb{R}^n)$ defined by $\widehat{f} = \widehat{g}\chi_G$. Then for any $\varepsilon > 0$ there exists $J \in \mathbb{N}$ such that

$$(2.3) \qquad (1 - \varepsilon)\|f\|_2^2 \le \sum_{\mathbf{k} \in \mathbb{Z}^n} |\langle f, D_A^j \tau_{\mathbf{k}} \phi \rangle|^2 \le \|f\|_2^2, \quad \forall j \ge J.$$

Proof According to the hypotheses, $\operatorname{supp}(\widehat{f}) \subset B_R$ for a fixed $R > 0$ and there exists $K > 0$ such that $|\widehat{f}(\mathbf{t})| \le K$ for every $\mathbf{t} \in \mathbb{R}^n$. By Parseval's formula,

$$(2.4) \qquad \sum_{\mathbf{k} \in \mathbb{Z}^n} |\langle f, D_A^j \tau_{\mathbf{k}} \phi \rangle|^2 = \sum_{\mathbf{k} \in \mathbb{Z}^n} |\langle \widehat{f}, \widehat{D_A^j \tau_{\mathbf{k}} \phi} \rangle|^2 = \sum_{\mathbf{k} \in \mathbb{Z}^n} |\langle D_{A^*}^j \widehat{f}, \widehat{\tau_{\mathbf{k}} \phi} \rangle|^2$$

$$= \sum_{\mathbf{k} \in \mathbb{Z}^d} \left| \int_{(A^*)^{-j} B_R} D_{A^*}^j \widehat{f}(\mathbf{t}) \overline{\widehat{\phi}(\mathbf{t})} e^{2\pi \mathbf{k} \cdot \mathbf{t}} \, d\mathbf{t} \right|^2.$$

Since A^* is expansive, there exists $j_0 \in \mathbb{N}$ such that if $j \ge j_0$, then $(A^*)^{-j} B_R \subset [-1/2, 1/2]^d$. For those j, the sum over $\mathbf{k} \in \mathbb{Z}^n$ in (2.4) may be interpreted as the sum of the squares of the moduli of the $-\mathbf{k}$-th Fourier coefficients of the function $D_{A^{*j}} \widehat{f}(\mathbf{t}) \overline{\widehat{\phi}(\mathbf{t})}$. Thus

$$(2.5) \qquad \sum_{\mathbf{k} \in \mathbb{Z}^n} |\langle f, D_A^j \tau_{\mathbf{k}} \phi \rangle|^2 = \int_{(A^*)^{-j} B_R} |D_{A^{*j}} \widehat{f}(\mathbf{t})|^2 |\widehat{\phi}(\mathbf{t})|^2 \, d\mathbf{t} \quad \forall j \ge j_0.$$

Since $|\widehat{\phi}(\mathbf{t})| \le 1$ a.e., the right inequality of (2.3) follows. Now we prove the left inequality of (2.3).

Let $0 < \varepsilon < 1$ and take the set $\Lambda_\varepsilon = \{\mathbf{t} \in G : |\widehat{\phi}(\mathbf{t})| \le 1 - \frac{\varepsilon}{2}\}$. Since $|\widehat{\phi}(\mathbf{0})| = 1$, $|\widehat{\phi}(\mathbf{t})| \le 1$ a.e. and the origin is a point of (A^*, G)-approximate continuity of $|\widehat{\phi}|$,

$$\lim_{j \to \infty} \frac{\left| \left(G \cap (A^*)^{-j} B_R \right) \cap \Lambda_\varepsilon^c \right|_n}{|G \cap (A^*)^{-j} B_R|_n} = 1.$$

This implies that there exists $J \ge j_0$ such that if $j \ge J$, we have

$$\left| (A^*)^{-j} (G \cap B_R \cap A^{*j} \Lambda_\varepsilon) \right|_n = \left| \left(G \cap (A^*)^{-j} B_R \right) \cap \Lambda_\varepsilon \right|_n < \frac{\varepsilon}{2K^2 d_A^j} \|f\|_2^2.$$

Thus, if $j \ge J$

$$(2.6) \qquad |G \cap B_R \cap A^{*j} \Lambda_\varepsilon|_n < \frac{\varepsilon}{2K^2} \|f\|_2^2.$$

According to (2.5), if $j \geq J$ we obtain

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} |\langle f, D_A^j \tau_{\mathbf{k}} \phi \rangle|^2 \geq \left(1 - \frac{\varepsilon}{2}\right) \int_{G \cap (A^*)^{-j} B_R \cap \Lambda_\varepsilon^c} |D_{A^*} \widehat{f}(\mathbf{t})|^2 \, d\mathbf{t}$$

$$\geq \left(1 - \frac{\varepsilon}{2}\right) \|f\|_2^2 - \int_{G \cap B_R \cap A^{*j} \Lambda_\varepsilon} |\widehat{f}(\mathbf{t})|^2 \, d\mathbf{t}.$$

Furthermore, by the inequality (2.6), if $j \geq J$ we have

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} |\langle f, D_A^j \tau_{\mathbf{k}} \phi \rangle|^2 \geq \left(1 - \frac{\varepsilon}{2}\right) \|f\|_2^2 - K^2 |G \cap B_R \cap A^{*j} \Lambda_\varepsilon|_n$$

$$\geq \left(1 - \frac{\varepsilon}{2}\right) \|f\|_2^2 - K^2 \frac{\varepsilon}{2K^2} \|f\|_2^2 = (1 - \varepsilon) \|f\|_2^2.$$

This finishes the proof.   ∎

Proof of Theorem 2.1   We prove (i) implies (ii). Without loss of generality, we can assume that $H_\ell(\mathbf{t}) = 0$, a.e. $\mathbf{t} \in \mathcal{N}_{\widehat{\phi}}$, $\ell = 0, 1, \ldots, N$. The function

$$(2.7) \qquad \Theta(\mathbf{t}) = \sum_{m=0}^{\infty} \sum_{\ell=1}^{N} |H_\ell(A^{*m}\mathbf{t})|^2 \prod_{k=0}^{m-1} |H_0(A^{*k}\mathbf{t})|^2,$$

with the convention $\prod_{k=0}^{-1} |Q_0(A^{*k}\mathbf{t})|^2 = 1$, is usually called the *fundamental function* associated with $H_0, \ldots, H_N$. Let us mention that we have assumed $H_\ell(\mathbf{t}) = 0$ a.e. $\mathbf{t} \in \mathcal{N}_{\widehat{\phi}}$ to avoid problems with the measurability of Θ. Note that fundamental functions were introduced in [43].

　　We will see that the conditions in (ii) hold when we consider Θ instead of S. According to Lemma 2.3 and Lemma 2.10 in [32], we know that

$$1 = \lim_{J \to -\infty} \Theta(A^{*J}\mathbf{t}) |\widehat{\phi}(A^{*J}\mathbf{t})|^2 \quad \text{a.e. } \mathbf{t} \in G.$$

Thus, that the origin is a point of (A^*, G)-approximate continuity of $\Theta|\widehat{\phi}|^2$, if we set $\Theta(\mathbf{0})|\widehat{\phi}(\mathbf{0})|^2 = 1$, follows by [46, Lemma 2.5].

　　The remainder is proved in [32] (*e.g.*, see also [2]).

　　We prove (ii) implies (i). First, we prove this implication in the particular case when $S(\mathbf{t}) = 1$. This case is usually called the Unitary Extension Principle.

　　Our starting point is Lemma 2.2. Afterwards, the proof may be finished following the strategy by Benedetto and Treiber [6] (see also [32], [30]).

　　When S is a general function described in (ii), consider $\varphi \in H_G^2$ defined by $\widehat{\varphi}(\mathbf{t}) = \sqrt{S(\mathbf{t})}\widehat{\phi}(\mathbf{t})$. Observe that

$$\widehat{\varphi}(A^*\mathbf{t}) = \sqrt{S(A^*\mathbf{t})}\widehat{\phi}(A^*\mathbf{t}) = \sqrt{S(A^*\mathbf{t})}H_0(\mathbf{t})\widehat{\phi}(\mathbf{t}) = Q_0(\mathbf{t})\widehat{\varphi}(\mathbf{t}) \quad \text{a.e.,}$$

and

$$\widehat{\psi_\ell}(A^*\mathbf{t}) = \sqrt{\frac{1}{S(\mathbf{t})}}H_\ell(\mathbf{t})\sqrt{S(\mathbf{t})}\widehat{\phi}(\mathbf{t}) = Q_\ell(\mathbf{t})\widehat{\varphi}(\mathbf{t}), \quad \text{a.e. } \ell = 1, \ldots, N,$$

where $Q_0, Q_1, \ldots, Q_N$ are $\mathbb{Z}^n$-periodic measurable functions defined by

$$Q_0(\mathbf{t}) = \sqrt{\frac{S(A^*\mathbf{t})}{S(\mathbf{t})}} H_0(\mathbf{t}), \quad Q_\ell(\mathbf{t}) = \sqrt{\frac{1}{S(\mathbf{t})}} H_\ell(\mathbf{t}), \quad \ell = 1, \dots, N.$$

According to our hypotheses, the origin is a point of (A^*, G)-approximate continuity for $\widehat{\varphi}$, provided that $\widehat{\varphi}(\mathbf{0}) = 1$. Analogously to the proof of Proposition 1.11 in [17], we get that φ, ψ_ℓ, Q_0 and Q_ℓ, $\ell = 1, \dots, N$, satisfy the Unitary Extension Principles. Then the statement follows. ∎

The following result gives a characterization of all Parseval wavelet frames for an H_G^2 arising from a fixed H_G^2-FMRA.

Theorem 2.3 *Let $\{V_j\}_{j\in\mathbb{Z}} \subset H_G^2$ be an H_G^2-FMRA with a scaling function ϕ. Let $\varphi \in H_G^2$ defined by $\widehat{\varphi}(\mathbf{t}) = \widehat{\phi}(\mathbf{t})/[\widehat{\phi}, \widehat{\phi}]^{1/2}(\mathbf{t})$. Let $\Psi = \{\psi_1, \dots, \psi_N\} \subset H_G^2$. The following are equivalent.*

(i) *The set of functions Ψ is a Parseval wavelet frame for H_G^2 arising from $\{V_j\}_{j\in\mathbb{Z}}$.*

(ii) *There exist $H_0, H_1, \dots, H_N \in L^\infty(\mathbb{T}^n)$ such that*

    (a)

$$\widehat{\varphi}(A^*\mathbf{t}) = H_0(\mathbf{t})\widehat{\varphi}(\mathbf{t}), \ a.e., \ \widehat{\psi_\ell}(A^*\mathbf{t}) = H_\ell(\mathbf{t})\widehat{\varphi}(\mathbf{t}) \ a.e., \ \ell = 1, \dots, N,$$

    and

    (b) *there exists a non-negative $S \in L^\infty(\mathbb{T}^n)$ such that the origin is a point of (A^*, G)-approximate continuity for S if we set $S(\mathbf{0}) = 1$, and the equalities in (2.1) and (2.2) are satisfied.*

Proof We see that (i) implies (ii). There exist $H_0, H_1, \dots, H_N$ some $\mathbb{Z}^n$-periodic measurable functions such that

$$\widehat{\varphi}(A^*\mathbf{t}) = H_0(\mathbf{t})\widehat{\varphi}(\mathbf{t}) \quad a.e., \text{ and } \quad \widehat{\psi_\ell}(A^*\mathbf{t}) = H_\ell(\mathbf{t})\widehat{\varphi}(\mathbf{t}), \quad \ell = 1, \dots, N \quad a.e.,$$

because $\{\psi_1, \dots, \psi_N\} \subset V_1$. Since $\{\varphi(\cdot - \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^n\}$ is a Parseval frame for V_0, we have $\sum_{\mathbf{k}\in\mathbb{Z}^n} |\widehat{\varphi}(\mathbf{t}-\mathbf{k})|^2 = 1$ a.e. $\mathbf{t} \in \mathbb{T}^n \setminus \mathcal{N}_{\widehat{\varphi}}$. Thus H_0 can be taken in $L^\infty(\mathbb{T}^n)$. Indeed, it can be assumed that $H_\ell(\mathbf{t}) = 0$, $\ell = 0, 1, \dots, N$, a.e. on $\mathcal{N}_{\widehat{\varphi}}$.

Let Θ be the fundamental function associated with $H_0, H_1, \dots, H_N$ defined as in (2.7). Since Ψ is a Parseval frame for H_G^2, we have already said that Θ satisfies the condition (ii) in Theorem 2.1 if we consider Θ instead of S. It remains to see that $\Theta, H_1, \dots, H_N$ are in $L^\infty(\mathbb{T}^n)$ and the origin is a point of (A^*, G)-approximate continuity for Θ if we set $\widehat{\Theta}(\mathbf{0}) = 1$.

We know that $\Theta(\mathbf{t}) \sum_{\mathbf{k}\in\mathbb{Z}^n} |\widehat{\phi}(\mathbf{t}-\mathbf{k})|^2 \in L^\infty(\mathbb{T}^n)$ (*e.g.*, see (2.4) in [2] or proof of Lemma 2.12 in [32]). Furthermore, since $\sum_{\mathbf{k}\in\mathbb{Z}^n} |\widehat{\varphi}(\mathbf{t}-\mathbf{k})|^2 = 1$ a.e. $\mathbf{t} \in \mathbb{T}^n \setminus \mathcal{N}_{\widehat{\varphi}}$ and bearing in mind that $\Theta(\mathbf{t}) = 0$ a.e. on $\mathcal{N}_{\widehat{\varphi}}$, we conclude that Θ is in $L^\infty(\mathbb{T}^n)$. Hence $H_1, \dots, H_N$ can be taken in $L^\infty(\mathbb{T}^n)$.

By (ii) in Theorem 2.1 we know that the origin is a point of (A^*, G)-approximate continuity for $\Theta|\widehat{\varphi}|^2$ is we set $\Theta(\mathbf{0})|\widehat{\varphi}(\mathbf{0})|^2 = 1$. By Theorem 3.1 in [27], the origin is a point of (A^*, G)-approximate continuity for $\widehat{\varphi}$ is we set $\widehat{\varphi}(\mathbf{0}) = 1$. Hence, the origin is a point of (A^*, G)-approximate continuity for Θ is we set $\Theta(\mathbf{0}) = 1$ follows.

We check that (ii) implies (i). By (a) we know that $\Psi \subset V_1$. In addition, by Theorem 2.1, we conclude that Ψ is a Parseval wavelet frame for H_G^2. Hence the proof is finished. ∎

The following is a characterization of all Parseval wavelet frames associated with a fixed H_G^2-FMRA.

Corollary 2.4   *Let $\{V_j\}_{j \in \mathbb{Z}} \subset H_G^2$ be an H_G^2-FMRA with a scaling function ϕ. Let $\varphi \in L^2(\mathbb{R}^n)$ defined by $\widehat{\varphi}(\mathbf{t}) = \widehat{\phi}(\mathbf{t})/[\widehat{\phi}, \widehat{\phi}]^{1/2}(\mathbf{t})$. Let $\Psi = \{\psi_1, \ldots, \psi_N\} \subset H_G^2$. The following are equivalent.*

(α)   *The set of functions Ψ is a Parseval wavelet frame for H_G^2 associated with $\{V_j\}_{j \in \mathbb{Z}}$.*
(β)   *The condition* (ii) *in Theorem* 2.3 *holds and also*

$$(2.8) \qquad \sum_{k=0}^{d_A - 1} H_0(\mathbf{t} + \mathbf{p}_k) \overline{H_\ell(\mathbf{t} + \mathbf{p}_k)} = 0, \quad \text{for a.e. } \mathbf{t} \in \mathbb{R}^n \smallsetminus \mathcal{N}_{\widehat{\phi}}, \ \ell = \{1, 2, \ldots, N\}.$$

Proof   First, observe that condition (2.8) means that the ψ_i are orthogonal to the subspace V_0 (*e.g.*, see the proof of Theorem 2.4.3 in [30]). Therefore, by Theorem 2.3, (2.8), and the definition of W_j, $j \in \mathbb{Z}$, the statements follow. ∎

3   Examples

The main results in this note focus on Parseval wavelet frames with no regularity of the Fourier transform at the origin. Here, we give some examples to illustrate Theorems 2.1 and 2.3.

The following example exhibits orthonormal wavelets whose Fourier transforms are not continuous at the origin.

Example 3.1   Let $A \colon \mathbb{R}^n \to \mathbb{R}^n$ be an expansive linear map such that $A\mathbb{Z}^n \subset \mathbb{Z}^n$ and $G = \mathbb{R}^n$. Our starting point is $\phi \in L^2(\mathbb{R}^n)$, the scaling function of a multiresolution analysis associated with A that is constructed in [44, Theorem 2.2]. This function ϕ is defined by $\widehat{\phi} = \chi_E$, where $E \subset \mathbb{R}^n$ is a bounded measurable set satisfying

   (i)   the origin is a point of A^*-density for E and $\mathbf{0} \in E$;
   (ii) $\sum_{\mathbf{k} \in \mathbb{Z}^n} \chi_E(\mathbf{t}) = 1$;
   (iii) $(A^*)^{-1} E \subset E$;
   (iv)   for any open neighborhood of the origin, U, there exists an open set included in $E^c \cap U$.

By condition (ii), we have that $\{\phi(\cdot - \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^n\}$ is an orthonormal system. Acording to (ii) and (iii),

$$H_0(\mathbf{t}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \frac{\chi_E \big(A^*(\mathbf{t} + \mathbf{k}) \big)}{\chi_E(\mathbf{t} + \mathbf{k})}$$

is a well-defined $\mathbb{Z}^n$-periodic function that satisfies the following refinable equation

$$\widehat{\phi}(A^* \mathbf{t}) = H_0(\mathbf{t}) \widehat{\phi}(\mathbf{t}).$$

If $\{\mathbf{p}_k\}_{k=0}^{d_A - 1}$, where $\mathbf{p}_0 = \mathbf{0}$, is a full collection of representatives of the cosets of $(A^*)^{-1} \mathbb{Z}^n / \mathbb{Z}^n$, define

$$\widehat{\psi_\ell}(A^* \mathbf{t}) = H_0(\mathbf{t} + \mathbf{p}_\ell) \widehat{\phi}(\mathbf{t}), \quad \ell \in \{1, \ldots, d_A - 1\}.$$

It is not hard to prove that $\Psi = \{\psi_1, \ldots, \psi_{d_A-1}\} \subset L^2(\mathbb{R}^n)$ is a set of orthonormal wavelets in $L^2(\mathbb{R}^n)$ associated with A. In addition, according to conditions (iii) and (iv), some of the $\widehat{\psi_\ell}$ are not continuous at the origin.

For any H_G^2 where G is open, we construct Parseval wavelet frames where the Fourier transform of some generators is not continuous at the origin.

Example 3.2 Let $A: \mathbb{R}^n \to \mathbb{R}^n$ be an expansive linear map such that $A\mathbb{Z}^n \subset \mathbb{Z}^n$ and $G \subset$ be an open A^*-set.

Since A^* is expansive, there exists $R > 0$ such that $\bigcup_{j=0}^{\infty}(A^*)^{-j}B_R \subset [-\frac{1}{2}, \frac{1}{2})^n$. Let $F := G \cap \bigcup_{j=0}^{\infty}(A^*)^{-j}B_R$ and observe that the set F can be written as $F = G \cap \bigcup_{j=0}^{j_0}(A^*)^{-j}B_R$, where $j_0 \in \mathbb{N}$ is such that if $j \geq j_0$ then $(A^*)^{-j}B_R \subset B_R$. This j_0 exists because A^* is an expansive linear map. Further, the set $F \smallsetminus (A^*)^{-1}(\overline{F})$ is an open measurable set, because the set F is an open measurable set and A^* is a continuous map. Moreover, since A^* is expansive, it is clear that $|F \smallsetminus (A^*)^{-1}(\overline{F})|_n > 0$.

Now let $F_l = (A^*)^{-l}F \smallsetminus (A^*)^{-l-1}F$, $l \in \{0, 1, 2, \ldots\}$. Observe that the measurable sets F_l are disjoint. Moreover, $F \smallsetminus \{\mathbf{0}\} = \bigcup_{l=0}^{\infty}F_l$. If $\mathbf{x} \in F \smallsetminus \{\mathbf{0}\}$, then $\mathbf{x} \in \left(G \cap (A^*)^{-j}B_R\right)$ for some $j \in \{0, 1, 2, \ldots\}$. Let j_1 be the biggest number in $\{0, 1, 2, \ldots\}$ such that $\mathbf{x} \in \left(G \cap (A^*)^{-j_1}B_R\right)$. Thus $\mathbf{x} \in F_{j_1} = (A^*)^{-j_1}F \smallsetminus (A^*)^{-j_1-1}F$.

Furthermore, observe that there exists $\mathbf{y_0} \in \mathbb{R}^n$ and $r > 0$ such that $B_r + \mathbf{y_0} \subset F_0$. Let

$$E = \bigcup_{j=0}^{\infty} F_j \smallsetminus \left[(A^*)^{-j}B_{2^{-j}r} + \mathbf{y_0}\right].$$

It is easy to see that the set E satisfies $|E|_n > 0$, $E \subset [-\frac{1}{2}, \frac{1}{2})^n$, $(A^*)^{-1}E \subset E$, and for any open neighborhood of the origin, U, there exists an open set included in $E^c \cap U$. In addition, that the origin is a point of (A^*, G)-density for E can be proved as in Step 2 of the proof of Theorem 2.2 in [44].

From this set E, we continue our construction as in Example 1 to obtain $\Psi = \{\psi_1, \ldots, \psi_{d_A-1}\} \subset H_G^2$, a set of Parseval wavelet frames for H_G^2, where the restriction on the set G of some $\widehat{\psi_\ell}$ are not continuous at the origin.

The purpose of the following example is to show visually how the previous examples work.

Example 3.3 Let M be the linear map in $\mathbb{R}^2$ such that $M^*(x, y) = (2x + 2y, 2y)$. It is easy to see that

$$\{\mathbf{p_0} = (0, 0), \quad \mathbf{p_1} = (\tfrac{1}{2}, 0), \quad \mathbf{p_2} = (0, \tfrac{1}{2}), \quad \mathbf{p_3} = (\tfrac{1}{2}, \tfrac{1}{2})\}$$

is a full collection of representatives of the cosets of $(M^*)^{-1}\mathbb{Z}^2/\mathbb{Z}^2$.

Take the set $E = [-\frac{1}{2}, \frac{1}{2}]^2 \smallsetminus \{(x, y) \in (-\frac{1}{2}, \frac{1}{2})^2 : x, y > 0\}$. The function $\phi \in L^{(\mathbb{R}^2)}$ defined by $\widehat{\phi} = \chi_E$ is a scaling function of a frame multiresolution analysis associated with M.
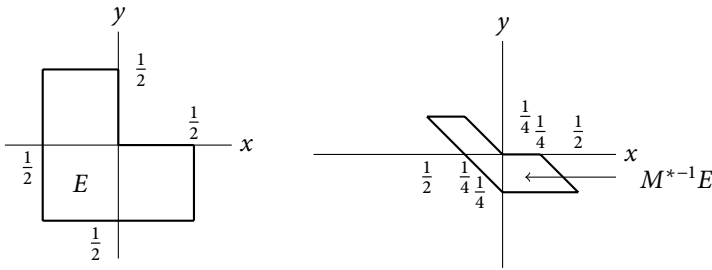
Define H_0 as the $\mathbb{Z}^2$-periodization of $\chi_{M^{*-1}E}$. Then $\{\psi_1, \psi_2, \psi_3\}$ defined by

$$\widehat{\psi_\ell}(M^*\mathbf{t}) = H_0(\mathbf{t} + \mathbf{p}_\ell)(\mathbf{t})\widehat{\phi}(\mathbf{t})$$

is a Parseval wavelet frame for $L^2(\mathbb{R}^2)$ associated with the dilation M.

Observe that in Theorem 2.1 the condition that the origin must be a point of approximate continuity of the Fourier transform of the involved refinable function when the function takes the value 1 at the origin, is not assumed. Then Theorem 2.1 provides a more flexible way for constructing wavelets frames than the different versions of the Oblique Extension Principle that appear in the literature. To show this fact, we give the following example.

Example 3.4   With the same notation as in Example 1, let $C \in (0, \infty) \smallsetminus \{1\}$. We take the function $\varphi := C\phi$. It is clear that the origin is a point of A^*-approximate continuity of $\widehat{\varphi}$ with $\widehat{\varphi}(\mathbf{0}) = C$ and

$$\widehat{\varphi}(A^*\mathbf{t}) = C\widehat{\phi}(A^*\mathbf{t}) = CH_0(\mathbf{t})\widehat{\phi}(\mathbf{t}) = H_0(\mathbf{t})\widehat{\varphi}(\mathbf{t}).$$

Moreover, for $\ell \in \{1, \ldots, d_A - 1\}$ we can write

$$\widehat{\psi_\ell}(A^*\mathbf{t}) = \frac{1}{C}H_0(\mathbf{t} + \mathbf{p}_\ell)\widehat{\varphi}(\mathbf{t}) = Q_\ell(\mathbf{t})\widehat{\varphi}(\mathbf{t}),$$

where $Q_\ell(\mathbf{t}) = \frac{1}{C}H_0(\mathbf{t} + \mathbf{p}_\ell)$. Therefore, the hypotheses and condition (ii) in Theorem 2.1 hold for φ, ψ_ℓ, the filters H_0, Q_ℓ, and $S(\mathbf{t}) = \frac{1}{c^2}$.

4 On Conditions at the Origin of the Fourier Transform of the Refinable Function

In this last section, we will make some comments on the conditions at the origin of the Fourier Transform of an involved refinable function in Extension Principles assumed by Han [21,22], by Li and Zhang [32] and the condition used in this note. In particular, we will see that those conditions are not equivalent.

The linear space of all compactly supported $C^\infty(\mathbb{R}^n)$ (test) functions with the usual topology will be denoted by $\mathcal{D}(\mathbb{R}^n)$. For $g \in \mathcal{D}(\mathbb{R}^n)$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, with some abuse in the notation, we will write

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(\mathbf{t})\overline{g(\mathbf{t})}\, d\mathbf{t}.$$

According to our context, the condition used by Han may be written as follows. Given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the following identity holds in the sense of distributions:

$$\tag{4.1} \lim_{j \to \infty} |f(A^{-j}\mathbf{t})| = 1.$$

More precisely,

$$\lim_{j \to \infty} \langle |f(A^{-j}\mathbf{t})|, g \rangle = \langle 1, g \rangle, \quad \forall g \in \mathcal{D}(\mathbb{R}^n).$$

Let f be the function in $L^1_{\text{loc}}(\mathbb{R})$ defined by

$$f(x) = \chi_{[-1,1]}(x) + \sum_{\ell=0}^{\infty} 2^{\ell+2} \chi_{(2^{-\ell-1}, 2^{-\ell-1}+2^{-2\ell-2})}(x).$$

We will see that the origin is a point of approximate continuity of f but (4.1) is not satisfied with $A = 2$.

Denote $F = [-1,1] \smallsetminus \bigcup_{\ell=0}^{\infty}(2^{-\ell-1}, 2^{-\ell-1}+2^{-2\ell-2})$. Since

$$\lim_{j \to \infty} \frac{|2^{-j}[-1,1] \cap F|}{|2^{-j}[-1,1]|}$$

$$= 1 - \lim_{j \to \infty} \frac{\left|2^{-j}[-1,1] \cap \left(\bigcup_{\ell=0}^{\infty}(2^{-\ell-1}, 2^{-\ell-1}+2^{-2\ell-2})\right)\right|}{|2^{-j}[-1,1]|}$$

$$= 1 - \lim_{j \to \infty} \frac{\left|\bigcup_{\ell=j}^{\infty}(2^{-\ell-1}, 2^{-\ell-1}+2^{-2\ell-2})\right|}{|2^{-j}[-1,1]|} = 1,$$

the origin is a point of density for the set F. It follows rapidly that the origin is a point of approximate continuity of f.

Now we see that the function f does not satisfy the condition (4.1) with $A = 2$. Take $g \in \mathcal{D}(\mathbb{R})$ such that g is nonnegative, with value 1 on the interval $[-1,1]$, supported on $[-2,2]$, increasing on $[-2,-1]$, and decreasing on $[1,2]$. We have the inequalities

$$\lim_{j \to \infty} \langle |f(2^{-j}t)|, g \rangle \geq \lim_{j \to \infty} \int_{-1}^{1} f(2^{-j}t)\, dt = \lim_{j \to \infty} 2^j \int_{-2^{-j}}^{2^{-j}} f(y)\, dy$$

$$= 2 + \lim_{j \to \infty} 2^j \int_{-2^{-j}}^{2^{-j}} \sum_{\ell=0}^{\infty} 2^{\ell+2} \chi_{(2^{-\ell-1}, 2^{-\ell-1}+2^{-2\ell-2})}(y)\, dy$$

$$= 2 + \sum_{k=0}^{\infty} 2^{-k} = 4 > \langle 1, g \rangle.$$

This implies that (4.1) does not hold.

By a second look at this example, we can conclude that the origin is a point of approximate continuity of f but the origin is not a point of Lebesgue of f. In addition, the origin is a point of M^*-approximate continuity of the function χ_E in Example 3 and the origin is not a point of Lebesgue of χ_E.

The condition used by Li and Zhang [32] on a neighborhood of the origin of the Fourier transform of a refinable function can be written as

$$\tag{4.2} \lim_{j \to \infty} |f(A^{-j}\mathbf{t})| = 1 \quad \text{a.e. on } G \subset \mathbb{R}^n, \text{ some } A\text{-set}.$$

We have already used that the condition (4.2) implies that the origin is a point of (A, G)-approximate continuity of f, provided that $f(\mathbf{0}) = 1$, see [46, Lemma 2.5].

The converse implication may be false, as is proved in [45, Proposition 2], where is shown an example of a function for which the origin is a point of approximate continuity but the condition (4.2) does not hold when A is the dyadic dilation and $G = \mathbb{R}^n$.

One more difference is the following. It is well known that there exist functions $f \colon \mathbb{R} \to [0, \infty)$ such that $\lim_{j \to \infty} f(4^{-j}t) = 1$ for all $t \in \mathbb{R}$ but $\lim_{j \to \infty} f(2^{-j}t)$ does not exist. In contrast to this, we have the following.

Proposition 4.1   *Let $m \in \{2, 3, \ldots\}$, let A be an expansive linear map on $\mathbb{R}^n$ and let $G \subset \mathbb{R}^n$ be an A-set. Let $f \colon \mathbb{R}^n \to \mathbb{R}^n$ be a measurable function such that the point $\mathbf{x}$ is a point of (A^m, G)-approximate continuity of f. Then $\mathbf{x}$ is a point of (A, G)-approximate continuity of f.*

Proof   Without loss in generality we assume that $\mathbf{x} = \mathbf{0}$. Since the origin is a point of (A^m, G)-approximate continuity of f, there exists an $E \subset \mathbb{R}^n$ such that the origin is a point of (A^m, G)-density for E.

Let $r > 0$. We consider $j \in \mathbb{N}$ of the form $j = mp + k$ with $p \in \{0, 1, \ldots\}$ and some fixed $k \in \{0, \ldots, m-1\}$. By the continuity of the linear map A, there exists $0 < r_k < R_k$ such that $B_{r_k} \subset A^{-k} B_r \subset B_{R_k}$. Then we have

$$
\frac{\left|E \cap \left[\left(G \cap A^{-(mp+k)} B_r\right)\right]\right|_n}{\left|\left(G \cap A^{-(mp+k)} B_r\right)\right|_n}
$$

$$
= 1 - \frac{\left|E^c \cap \left[\left(G \cap A^{-(mp+k)} B_r\right)\right]\right|_n}{\left|\left(G \cap A^{-(mp+k)} B_r\right)\right|_n}
$$

$$
\geq 1 - \frac{\left|A^{mp} E^c \cap \left[\left(G \cap B_{R_k}\right)\right]\right|_n}{\left|\left(G \cap B_{R_k}\right)\right|_n} \frac{\left|\left(G \cap B_{R_k}\right)\right|_n}{\left|\left(G \cap B_{r_k}\right)\right|_n} \to 1,
$$

as $p \to \infty$, because the origin is a point of (A^m, G)-density for E. Thus the origin is a point of (A, G)-density for E and the statement follows.   ∎

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