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CARLESON INTERPOLATING SEQUENCES FOR BANACH SPACES OF ANALYTIC FUNCTIONS

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Abstract This paper presents an approach, based on interpolation theory of operators, to the study of interpolating sequences for interpolation Banach spaces between Hardy spaces. It is shown that the famous Carleson result for H^∞ can be lifted to a large class of abstract Hardy spaces. A description is provided of the range of the Carleson operator defined on interpolation spaces between the classical Hardy spaces in terms of uniformly separated sequences. A key role in this description is played by some general interpolation results proved in the paper. As by-products, novel results are obtained which extend the Shapiro–Shields result on the characterisation of interpolation sequences for the classical Hardy spaces H^p. Applications to Hardy–Lorentz, Hardy–Marcinkiewicz and Hardy–Orlicz spaces are presented.

Keywords and phrases: Carleson interpolating sequences, interpolation of operators, Hardy spaces

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1. Introduction

Let H^∞ denote the space of bounded analytic functions on the unit disc $\mathbb{D}$ of the complex plane, and assume that $\lambda = \{\lambda_j\}_{j=1}^\infty$ is a sequence of distinct points in $\mathbb{D}$. A linear mapping

T_λ given by

$$T_\lambda f = \{f(\lambda_j)\}_{j=1}^{\infty}, \quad f \in H^\infty,$$

is a contraction from H^∞ into ℓ^∞. If, in addition, T_λ is surjective, then the sequence λ is called a (*universal*) *interpolating sequence* (*for* H^∞). The term 'interpolating' is justified by the following observation: if T_λ is surjective, then by the open mapping theorem there is $\gamma > 0$ such that for a given sequence $w = \{w_j\}_{j=1}^{\infty} \in \ell^\infty$, there exists $f \in H^\infty$ with $f(\lambda_j) = w_j$, $j \in \mathbb{N}$ and $\|f\|_{H^\infty} \leqslant \gamma \|w\|_{\ell^\infty}$.

The characterisation of interpolating sequences for H^∞ was given by Carleson in 1958 in his seminal paper [9]. He proved that λ is an interpolating sequence for H^∞ if and only if λ is *uniformly separated* – that is,

$$\inf_{k\in\mathbb{N}} \prod_{\substack{j=1\\ j\neq k}}^{\infty} \left| \frac{\lambda_j - \lambda_k}{1 - \bar{\lambda}_k \lambda_j} \right| > 0.$$

Formulation of an analogous problem for Hardy spaces H^p, $p \in [1,\infty)$, requires weighting the situation. A sequence λ is called an *interpolating sequence for* H^p if an operator T_λ given by

$$T_\lambda f = \left\{(1 - |\lambda_j|^2)^{1/p} f(\lambda_j)\right\}_{j=1}^{\infty}, \quad f \in H^p,$$

is surjective from H^p onto ℓ^p. It was proved by Shapiro and Shields in [26] that λ is an interpolating sequence for H^p if and only if λ is uniformly separated. We refer the reader to Duren's book [11] for more background information on the subject. The study of interpolating sequences for various spaces (e.g., Bergman spaces) has also attracted much attention in recent years (see, e.g., [25] and the references therein). The classical theorems related to these topics for Hardy spaces H^p raise the question whether it is possible to extend these results for abstract Hardy spaces. We mention here the paper in which Hartmann [15] studied the associated notion of so-called free-interpolating sequences for Hardy–Orlicz spaces (see also [14]).

Interpolating sequences play a remarkable role in function theory on the disc and related operators – for example, within the theory of model spaces, multipliers and Toeplitz and Hankel operators [10]. Moreover, through their connection with Carleson measures, interpolating sequences are of importance far beyond the theory of analytic functions, such as in harmonic analysis [1, 12] and linear systems [13]. We point out that uniformly separated sequences have found an important application in the study of the existence of invariant subspaces for polynomially bounded operators on a complex Banach space with spectra containing the unit circle [3].

The aim of this paper is to analyse very general variants of the classical results mentioned and prove a kind of interpolation formula for interpolating sequences (notice here the double meaning of the term 'interpolation'). We cast the problem of interpolating sequences into the framework of Banach spaces of analytic functions on the unit disc generated by Banach lattices and study the surjectivity of a suitable operator T_λ for such abstract settings. To give a taste of the ideas that support our research, we sketch one of the main results of this paper (see Theorem 7 for a precise formulation, and

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compare with Theorem 11). Let F be an exact interpolation functor satisfying some Köthe duality condition and let $H(X)$ be the Hardy space generated by some X, which is an interpolation space between $L^{p_0}(\mathbb{T})$ and $L^{p_1}(\mathbb{T})$ given by $X = F(L^{p_0}(\mathbb{T}), L^{p_1}(\mathbb{T}))$, with $p_0, p_1 \in [1,\infty]$, $p_0 < p_1$. Then the sequence $\{\lambda_j\}$ is uniformly separated if and only if

$$\Big\{\{f(\lambda_j)\}_{j=1}^{\infty}; f \in H(X)\Big\} = E,$$

where E is an interpolation Banach sequence space between $\ell^{p_0}(\nu)$ and $\ell^{p_1}(\nu)$ given by $E = F(\ell^{p_0}(\nu), \ell^{p_1}(\nu))$ with measure ν on $2^{\mathbb{N}}$ defined by $\nu(\{j\}) = (1 - |\lambda_j|^2)$ for each $j \in \mathbb{N}$.

We point out that such an approach allows us to give a new proof of the Shapiro–Shields result. Moreover, we provide thorough studies of functors that can be used as in the foregoing. In particular, we prove that any K- or J-method of interpolation is a suitable functor that can be used in this interpolation formula. The key ingredient of our considerations that allows us to use advanced interpolation methods is that Hardy spaces behave well under interpolation – that, is for any interpolation functor F, we have $HF(L^{p_0,p_1}) = F(H^{p_0}, H^{p_1})$, $p_0, p_1 \in [1,\infty]$ (see [29]).

As an outcome, we apply the obtained results for the particular functors and we receive direct generalisations of the interpolating problem for Hardy–Lorentz, Hardy–Marcinkiewicz and Hardy–Orlicz spaces, which play an important role in the theory of rearrangement-invariant spaces. Note that we manage to derive the characterisation of interpolating sequences for abstract Hardy spaces from the Carleson theorem solely. We present a novel approach with some modern refinements based on interpolation methods in the theory of operators and Banach spaces.

2. Preliminaries

In this section we collect the required notation and prove some auxiliary results. If X and Y are topological linear spaces, then $X \hookrightarrow Y$ means that $X \subset Y$ and the inclusion map is continuous. In the case where X and Y are normed spaces, we write $X = Y$ whenever $X \hookrightarrow Y$ and $Y \hookrightarrow X$ – that is, $X = Y$ up to equivalence of norms. We also write $X \cong Y$ if $X = Y$ with *equality* of norms. Given two nonnegative functions f and g defined on the same set A, we write $f \sim g$ if there exist positive constants γ_1 and γ_2 such that $\gamma_1 g(x) \leqslant f(x) \leqslant \gamma_2 g(x)$ for all $x \in A$.

Throughout the paper, we consider only complete σ-finite measure spaces. For a given measure space (Ω, Σ, μ), we let $L^0(\Omega) := L^0(\Omega, \Sigma, \mu)$ denote the space of all real-valued measurable functions on Ω with the topology of convergence in measure on μ-finite sets. If $f, g \in L^0(\Omega)$, then $f \leqslant g$ means that $f(t) \leqslant g(t)$ for μ-almost all $t \in \Omega$. A Banach space $X \subset L^0(\mu)$ is said to be a Banach lattice on (Ω, Σ, μ) (on Ω for short) if for all $f \in L^0(\Omega)$ it holds that $|f| \leqslant |g|$ with $g \in X$ implies $f \in X$ and $\|f\|_X \leqslant \|g\|_X$. We work only with Banach lattices X such that $\operatorname{supp} X = \Omega$ (up to a set of measure 0) – that is, there exists $h \in X$ with $h > 0$ μ-almost everywhere on Ω (for more details, we refer to [4, Chapter 10]). The *weighted Banach space* $E(w)$, where w is a weight (that is a measurable positive

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function on Ω) consists of those $f \in L^0(\Omega)$ for which $fw \in E$ and is equipped with the norm $\|f\|_{E(w)} = \|fw\|_E$.

A lattice modelled on natural numbers is called a *sequence space*. The set of all sequences will be denoted by $\omega(\mathbb{N})$. As usual, for each $j \in \mathbb{N}$, e_j denotes the standard unit vector. For simplicity of the presentation we will avoid writing indices when defining sequences (unless it leads to ambiguity). The notation $\{a_j\}$ should be read as $\{a_j\}_{j=1}^{\infty}$. For simplicity of presentation, we will often consider the sequence space $\ell^p(\nu)$ as the $L_p(\nu)$-space, where for a given weight sequence $\{w_j\}$ modelled on the set $J = \mathbb{N}$ or $J = \mathbb{Z}$, the measure ν is given by $\nu(\{j\}) := w_j$ for each $j \in J$. Clearly, $\ell^p(\nu)$ coincides isometrically with the weighted space $\ell^p(\{w_j^{1/p}\})$.

An important class of Banach lattices is rearrangement-invariant spaces. Given $f \in L^0(\Omega)$, its *distribution function* is defined by $\mu_f(s) = \mu(\{t \in \Omega; |f(t)| > s\})$, $s \geqslant 0$. A Banach lattice X is called *rearrangement invariant* (an *r.i. space*, for short) if for any $f \in X$ and $g \in L^0(\Omega)$ such that $\mu_f = \mu_g$, we have $g \in X$ and $\|f\|_X = \|g\|_X$. It is well known that if X is an r.i. space for some finite measure space Ω, then $L^\infty(\Omega) \hookrightarrow X \hookrightarrow L^1(\Omega)$ (see [18]).

If X is an r.i. space, then for any measurable set A, the expression $\|\chi_A\|$ depends only on $\mu(A)$. Thus, for every $t \in \{\mu(A); A \in \Sigma\}$, we define a function ϕ_X by the formula $\phi_X(t) = \|\chi_A\|$, where A is any measurable set with $\mu(A) = t$. This function is called the *fundamental function* of X. If X is an r.i. space on a nonatomic measure space (Ω, Σ, μ), then ϕ_X is quasi-concave on $[0, \tau)$ with $\tau = \mu(\Omega)$ – that is, $\phi_X(0) = 0$ and both ϕ_X and $t \mapsto \frac{t}{\phi_X(t)}$ are positive and nondecreasing on $(0, \tau)$ (see [5]). Note also that ϕ_X is continuous at 0 if and only if $X \neq L^\infty$.

The Köthe dual space $X^\times$ of a normed space $X \subset L^0(\Omega)$ is defined as the space of all $f \in L^0(\Omega)$ such that $fg \in L^1(\Omega)$ for every $g \in X$. Note that $X^\times$ is a Banach lattice on (Ω, Σ, μ) when equipped with the norm $\|f\|_{X^\times} := \sup\{|\int_\Omega fg \mathrm{d}\mu| ; \|g\|_X \leqslant 1\}$. A Banach lattice X is said to be *maximal* if $X^{\times\times} \cong X$. Equivalently, X possesses the *Fatou property* – that is, for any sequence $\{f_n\}$ in X and $f \in L^0(\Omega)$ satisfying $0 \leqslant f_n \uparrow f$ μ-almost everywhere as $n \to \infty$ and $\sup\{\|f_n\|_X; n \in \mathbb{N}\} < \infty$, it follows that $f \in X$ and $\|f_n\|_X \to \|f\|_X$ as $n \to \infty$. We say that X has the *weak Fatou property* if for any sequence $\{f_n\}$ in X and $f \in X$ such that $0 \leqslant f_n \uparrow f$ μ-almost everywhere, it follows that $\|f_n\|_X \to \|f\|_X$ as $n \to \infty$.

A point $f \in X$ is said to have an *order-continuous norm* if for any sequence $\{f_n\} \subset X$ such that $0 \leqslant f_n \leqslant |f|$ and $f_n \to 0$ μ-almost everywhere on Ω, we have $\|f_n\|_X \to 0$. The symbol X_a will denote the subspace of all order-continuous elements of X. A Banach lattice X is called *order-continuous* if every element of X has an order-continuous norm – that is, $X = X_a$. If X is an order-continuous Banach lattice on (Ω, μ), then the dual space X^* can be identified with $X^\times$. We note that $X^\times$ always has the Fatou property.

In what follows we will consider *complex* Banach lattices. The term *complex space* refers to the *complexification* of a real Banach lattice space – that is, if X denotes the (real) Banach lattice, the complexification X^c of X is the Banach space of all complex-valued measurable functions f on Ω such that the element $|f|$ defined by $|f|(t) = |f(t)|$ for $t \in \Omega$ is in X and $\|f\|_{X^c} = \||f|\|_X$. For simplicity of presentation, we will often write *Banach* *lattice* or *r.i. space* instead of *complex Banach lattice* or *complex r.i. space*, and avoid using the symbol X^c.

2.1. Hardy spaces

In this paper we study Hardy spaces generated by Banach lattices. The prototype for them is the classical Hardy spaces H^p (see Duren's monograph [11]). For the convenience of the reader we start with the definitions of those standard objects.

The space of analytic functions on $\mathbb{D}$ will be denoted by $H(\mathbb{D})$. As usual, we let $\mathbb{T}$ denote the unit circle, $\mathbb{T}=\{z\in\mathbb{C};|z|=1\}$. Throughout the paper we let m be the probability Lebesgue measure on $\mathbb{T}$. Let $L^p(\mathbb{T})$, $p\in[1,\infty)$, and $L^\infty(\mathbb{T})$ denote the usual Lebesgue spaces (of equivalence classes) of measurable functions $f\colon\mathbb{T}\to\mathbb{C}$ normed by $\|f\|_{L^p}^p=\int_{\mathbb{T}}|f|^p\,\mathrm{d}m=\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(e^{i\theta}\right)\right|^p\mathrm{d}\theta$ and $\|f\|_{L^\infty}=\operatorname{ess\,sup}\left\{\left|f\left(e^{i\theta}\right)\right|;\theta\in[0,2\pi)\right\}$. The *Fourier coefficients* of a function $f\in L^1(\mathbb{T})$ are given by

$$\widehat{f}(n)=\frac{1}{2\pi}\int_0^{2\pi}f\left(e^{i\theta}\right)e^{-in\theta}\mathrm{d}\theta,\quad n\in\mathbb{Z}.$$

If a function $f\in H(\mathbb{D})$ is such that

$$\widetilde{f}\left(e^{i\theta}\right):=\lim_{r\to1^-}f\left(re^{i\theta}\right),\quad\theta\in[0,2\pi),$$

exists almost everywhere on $\mathbb{T}$, then $\widetilde{f}$ is called the radial limit function of f.

For $p\in[1,\infty)$, the *Hardy space* H^p consists of functions $f\in H(\mathbb{D})$ such that

$$\|f\|_{H^p}^p:=\sup_{r\in[0,1)}\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p\mathrm{d}\theta<\infty.$$

It is well known that for $f\in H^p$, the radial limit function $\widetilde{f}$ exists almost everywhere on $[0,2\pi)$. Moreover, if $f\in H^p$, $p\in[1,\infty)$, then $\widetilde{f}\in L^p(\mathbb{T})$ and $\|f\|_{H^p}=\|\widetilde{f}\|_{L^p(\mathbb{T})}$ (see [11, Theorem 2.2]).

Let $X\subset L^0(\mathbb{T})$ be a Banach lattice such that $X\subset L^1(\mathbb{T})$. A *Hardy space* $H(X)$ consists of all $f\in H^1$ such that $\widetilde{f}\in X$.

We will use the following lemma, which is surely well known to specialists:

Lemma 1. *Let X be a Banach lattice on $\mathbb{T}$ such that $X\subset L^1(\mathbb{T})$.*

(i) *The Hardy space $H(X)$ is a Banach space equipped with the norm*

$$\|f\|_{H(X)}:=\|\widetilde{f}\|_X,\quad f\in H(X).$$

(ii) *The map $f\mapsto\widetilde{f}$, $f\in H(X)$, is an isometric isomorphism from $H(X)$ onto $H[X]$, where $H[X]$ denotes the space of all $g\in X$ for which the negative Fourier coefficients vanish and which is equipped with the norm $\|g\|_{H[X]}:=\|g\|_X$.*

In the next sections, when considering Banach spaces of analytic functions, we will need to know that the limit function of a sequence of elements from the space, uniformly convergent on compact subsets, belongs to the space. It appears that this happens in a wide class of spaces, as Lemma 2 indicates. Since this result seems to be of independent

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interest, we decided to introduce the following definition: we say that a Banach space X of analytic functions on a domain $\Omega \subset \mathbb{C}$ has the *analytic* (*resp., weak*) Fatou property if for every bounded sequence $\{f_k\}$ in X that converges uniformly on compact subsets of Ω to a function f (resp., $f \in X$), we have $f \in X$ and $\|f\|_X \leqslant \liminf_{k\to\infty} \|f_k\|_X$ (resp., $\|f\|_X \leqslant \liminf_{k\to\infty} \|f_k\|_X$). It is well known that H^p has the analytic Fatou property for any $p \in [1,\infty]$. The following lemma shows that a wide class of Hardy spaces $H(X)$ also possesses this property:

Lemma 2. *Let X be a Banach lattice on $\mathbb{T}$ with the (resp., weak Fatou) property.*

(i) *If $X \hookrightarrow L^1(\mathbb{T})$ and $\operatorname{supp}(X^\times)_a = \mathbb{T}$, then the Hardy space $H(X)$ has the analytic (resp., weak) Fatou property.*

(ii) *If X is an r.i. space on $\mathbb{T}$, then the Hardy space $H(X)$ has the analytic (resp., weak) Fatou property.*

Proof. (i) From the well-known theorem due to Nakano [4], it follows that the set of order-continuous functionals on a Banach lattice with the weak Fatou property is a norming set – that is,

$$\|h\|_X = \sup\left\{\left|\int_{\mathbb{T}} gh\,\mathrm{d}m\right|;\ \|g\|_{X^\times} \leqslant 1\right\}, \quad h \in X.$$

The assumption $\operatorname{supp}(X^\times)_a = \mathbb{T}$ implies that for any $g \in X^\times$, there exists a sequence $\{g_n\} \subset (X^\times)_a$ such that $0 \leqslant g_n \uparrow |g|$ m-almost everywhere. Hence,

$$\|h\|_X = \sup\left\{\left|\int_{\mathbb{T}} gh\,\mathrm{d}m\right|;\ \|g\|_{(X^\times)_a} \leqslant 1\right\}, \quad h \in X.$$

Since $(X^\times)_a$ is order-continuous, the set of simple functions of $(X^\times)_a$ is dense in $(X^\times)_a$. Then, by the results of Luzin and Weierstrass, the set $\mathcal{P}$ of trigonometric polynomials on $\mathbb{T}$ is dense in $(X^\times)_a$. Thus,

$$\|h\|_X = \sup\left\{\left|\int_{\mathbb{T}} hp\,\mathrm{d}m\right|;\ \|p\|_{(X^\times)_a} \leqslant 1, p \in \mathcal{P}\right\}, \quad h \in X. \tag{1}$$

Take any bounded sequence $\{f_k\}$ in $H(X)$ such that $f_k \to f$ uniformly on compact subsets of $\mathbb{D}$. Note that from the Cauchy integral formula, it follows that

$$\lim_{k\to\infty} \widehat{f}_k(n) = \widehat{f}(n), \quad n \geqslant 0.$$

Since $H(X) \hookrightarrow H^1$, $\widehat{f}(n) = \widehat{\widetilde{f}}(n)$ for all $f \in H(X)$ and $n \in \mathbb{Z}$, with $\widehat{f}(n) = \widehat{\widetilde{f}}(n) = 0$ for $n < 0$, we get

$$\lim_{k\to\infty} \int_0^{2\pi} \widetilde{f}_k\left(e^{i\theta}\right) e^{-in\theta}\mathrm{d}\theta = \int_0^{2\pi} \widetilde{f}\left(e^{i\theta}\right) e^{-in\theta}\mathrm{d}\theta, \quad n \in \mathbb{Z}.$$

This implies that for any $p \in \mathcal{P}$, we have

$$\lim_{k\to\infty} \int_{\mathbb{T}} \widetilde{f}_k p\,\mathrm{d}m = \int_{\mathbb{T}} \widetilde{f} p\,\mathrm{d}m.$$

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Therefore, for every $p \in \mathcal{P}$ on $\mathbb{T}$,

$$\Big|\int_{\mathbb{T}} \widetilde{f} p \,\mathrm{d}m\Big| = \liminf_{k\to\infty}\Big|\int_{\mathbb{T}} \widetilde{f}_k p \,\mathrm{d}m\Big| \leqslant \liminf_{k\to\infty} \|\widetilde{f}_k\|_X \,\|p\|_{(X^\times)_a}$$

and so based on our hypothesis on X, we conclude from equation (1) the required statement.

(ii) We consider two cases: $\phi_{X^\times}(0+)=0$ and $\phi_{X^\times}(0+)>0$. Clearly, $\phi_{X^\times}(0+)=0$ implies $(X^\times)_a \neq \{0\}$. Since $(X^\times)_a$ is an r.i. space, $\operatorname{supp}(X^\times)_a = \mathbb{T}$ by $L^\infty(\mathbb{T}) \hookrightarrow (X^\times)_a$. Thus the required statement follows from i.

If $\gamma = \phi_{X^\times}(0+) > 0$, we have, by the well-known formula [5]

$$\phi_{X^\times}(t) = \frac{t}{\phi_X(t)}, \quad t \in (0,1],$$

that $\phi_X(t) \leqslant \gamma^{-1} t$ for all $t \in (0,1]$. This easily yields $L^1(\mathbb{T}) \hookrightarrow X$ and so $X = L^1(\mathbb{T})$. In consequence, $H(X) = H^1$, and the proof is completed. □

3. Interpolating sequences

In this section we introduce the notion of E-interpolating sequences and study them for a wide class of Banach spaces of analytic functions on the unit disc $\mathbb{D}$. This part generalises results for the classical Hardy spaces H^p from [11, Chapter 9], [22] and [28].

Let $\lambda = \{\lambda_j\}_{j=1}^\infty$ be a sequence of distinct points of the open unit disc $\mathbb{D}$. If, in addition, $\sum_{j=1}^\infty (1-|\lambda_j|) < \infty$, then λ will be called a *Blaschke sequence*. It is commonly known that the infinite zero set of any not identically zero function $f \in H^p$ is a Blaschke sequence. If E is a complex Banach sequence space, then λ is said to be an *E-interpolating sequence* for a Banach space $\mathcal{X} \subset H(\mathbb{D})$ if

$$E \subset \{\{f(\lambda_j)\};\, f \in \mathcal{X}\}.$$

This notion extends the well-known concept of interpolation sequences. Indeed, it is evident that the sequence $\{\lambda_j\}$ is called an interpolating sequence if it is an ℓ^∞-interpolating sequence for H^∞.

We will need some preliminary results. Let X be a Banach space, $\mathcal{Y}$ a topological linear space and $T\colon X \to \mathcal{Y}$ a linear and continuous map. We denote the *range* $\{Tx;\, x \in X\}$ of T by $R(T)$.

The following technical result is obvious, and so we omit the proof:

Proposition 1. *Let X be a Banach space, $\mathcal{Y}$ a topological linear space and $T\colon X \to \mathcal{Y}$ be a continuous and linear map. Then $R(T)$ is a Banach space equipped with the norm*

$$\|y\|_{R(T)} := \inf\{\|x\|_X;\, Tx = y\}, \quad y \in R(T).$$

In addition, $R(T) \hookrightarrow Y$, and moreover, if Y is a Banach space with $Y \subset R(T)$ and $Y \hookrightarrow \mathcal{Y}$, then there exists $\gamma > 0$ such that $\|y\|_{R(T)} \leqslant \gamma \|y\|_Y$ for all $y \in Y$.

Furthermore, we will use the following corollary:

Corollary 1. *Assume that a Banach space $\mathcal{X} \subset H(\mathbb{D})$ is such that for every $z \in \mathbb{D}$, the functional $\delta_z : \mathcal{X} \to \mathbb{C}$ given by $\delta_z f = f(z)$, $f \in \mathcal{X}$, is continuous. If $\lambda = \{\lambda_j\}$ is an E-interpolating sequence for $\mathcal{X}$, then there exists a constant $\gamma > 0$ such that for all $w = \{w_j\} \in E$,*

$$\inf\{\|f\|_{\mathcal{X}};\, f(\lambda_j) = w_j,\ j \in \mathbb{N}\} \leqslant \gamma \|w\|_E.$$

Proof. Let $T_\lambda : \mathcal{X} \to \omega(\mathbb{N})$ be the linear mapping defined by

$$T_\lambda f = \{f(\lambda_j)\}_{n=1}^{\infty}, \quad f \in \mathcal{X}.$$

Then it easily follows from the closed graph theorem that T_λ is continuous. Applying Proposition 1 and using the fact that $E \hookrightarrow \omega(\mathbb{N})$ give the required statement. □

Let X be a Banach space. For a closed subspace Y of X, the *annihilator* $Y^\perp$ of Y is defined as

$$Y^\perp = \{x^* \in X^*;\, x^*(y) = 0 \text{ for all } y \in Y\}.$$

Recall that if $Q : X \to X/Y$ is the quotient map, then $Q^* : (X/Y)^* \to X^*$ is an isometric isomorphism onto $Y^\perp$. From this we obtain the following well-known fact:

Lemma 3. *Let Y be a closed subspace of X, $x \in X$. Then*

$$\inf\{\|x - y\|_X;\, y \in Y\} = sup\left\{|y^*(x)|;\, y^* \in Y^\perp,\, \|y^*\|_{X^*} \leqslant 1\right\}.$$

This lemma easily yields the following statement:

Theorem 1. *Let X be an order-continuous Banach lattice on $\mathbb{T}$ such that $X \hookrightarrow L^1(\mathbb{T})$. For every $f \in X$, the following formula holds:*

$$\inf\{\|f - g\|_X;\, g \in H(X)\} = sup\left\{\left|\int_{\mathbb{T}} fg\,\mathrm{d}m\right|;\, g \in H_0\left[X^\times\right], \|g\|_{H[X^\times]} \leqslant 1\right\},$$

where $H_0[X^\times] := \left\{h \in H[X^\times];\, \widehat{h}(0) = 0\right\}$.

Our goal in this section is to characterise E-interpolating sequences in terms of an appropriate inclusion. As in the classical case, the main role is played by Blaschke products. Let $\{\lambda_j\}$ be a sequence in $\mathbb{D}$ such that $\sum_{j=1}^{\infty}(1 - |\lambda_j|) < \infty$. For each $n \in \mathbb{N}$, we define a finite Blaschke product

$$B_n(z) = \prod_{j=1}^{n} \frac{z - \lambda_j}{1 - \overline{\lambda}_j z}, \quad z \in \mathbb{D},$$

and for each $j \in \{1, \ldots, n\}$,

$$B_{n,j}(z) = \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{z - \lambda_k}{1 - \overline{\lambda}_k z}, \quad z \in \mathbb{D}.$$

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We also define

$$b_{n,j} := B_{n,j}(\lambda_j), \quad \rho_{n,j} := |b_{n,j}| \quad \text{and} \quad \rho_j := \lim_{n\to\infty} \rho_{n,j}.$$

In what follows, we will need a technical fact which states that if $L^\infty(\mathbb{T}) \hookrightarrow X \hookrightarrow L^1(\mathbb{T})$, then for any $z \in \mathbb{D}$, $\delta_z : H(X) \to \mathbb{C}$ given by $\delta_z f = f(z)$ is a bounded linear functional on $H(X)$. Indeed, when $X \hookrightarrow L^1(\mathbb{T})$, then $H(X) \hookrightarrow H^1$. Thus for any $f \in H(X)$, the radial limit function $\widetilde{f}$ of f exists and we have, by the well-known Poisson representation,

$$f(z) = \int_{\mathbb{T}} P_z \widetilde{f} \mathrm{d}m, \quad z \in \mathbb{D}.$$

Since $L^\infty(\mathbb{T}) \hookrightarrow X^\times$, it follows that

$$\left|f(z)\right| \leqslant \|P_z\|_{X^\times} \|\widetilde{f}\|_X \leqslant c\|P_z\|_{L^\infty(\mathbb{T})} \|f\|_{H(X)} \leqslant \frac{2c}{1-|z|} \|f\|_{H(X)}, \tag{2}$$

where c is the norm of the inclusion map $L^\infty(\mathbb{T}) \hookrightarrow X^\times$.

The following theorem will play a key role in our investigations:

Theorem 2. *Let $\lambda = \{\lambda_j\}$ be a Blaschke sequence. Consider the following statements for Banach lattices $E \subset \omega(\mathbb{N})$ and minimal X such that $L^\infty(\mathbb{T}) \hookrightarrow X \hookrightarrow L^1(\mathbb{T})$:*

(i) $E \subset \{\{f(\lambda_j)\}; f \in H(X)\}$.

(ii) *There exists a constant $\gamma > 0$ such that for all $w = \{w_j\} \in E$ one has*

$$\sup_{\|g\|_{H(X^\times)} \leqslant 1} \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} (1 - |\lambda_j|^2) g(\lambda_j) \right| \leqslant \gamma \|w\|_E, \quad n \in \mathbb{N}.$$

Then (i) implies (ii). If in addition $H(X)$ has the analytic Fatou property, then (i) and (ii) are equivalent.

Proof. (i)⇒(ii). For each $n \in \mathbb{N}$ and every $w = \{w_j\} \in E$, we define

$$m_{E,n}(w) := \inf_{f \in H(X)} \left\{ \|f\|_{H(X)}; f(\lambda_j) = w_j,\ 1 \leqslant j \leqslant n \right\}$$

and

$$f_n(z) := \sum_{j=1}^{n} w_j \frac{B_{n,j}(z)}{b_{n,j}} = \sum_{j=1}^{n} w_j \frac{B_{n,j}(z)}{B_{n,j}(\lambda_j)}, \quad z \in \mathbb{D}.$$

Clearly, $f_n \in H^\infty$ and $f_n(\lambda_j) = w_j$ for each $j \in \{1, 2, \dots, n\}$. Since $H^\infty \hookrightarrow H(X)$,

$$m_{E,n}(w) = \inf_{h \in H(X)} \|f_n - B_n h\|_{H(X)}. \tag{3}$$

Observe that since $\left|\widetilde{B}_n\left(e^{i\theta}\right)\right| = 1$,

$$m_{E,n}(w) = \inf_{h \in H(X)} \|\widetilde{f}_n - \widetilde{B}_n \widetilde{h}\|_X = \inf_{h \in H(X)} \|\frac{\widetilde{f}_n}{\widetilde{B}_n} - \widetilde{h}\|_X.$$

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Applying Theorem 1, we obtain

$$m_{E,n}(w) = \sup_{\|g\|_{H(X^\times)} \leqslant 1} \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\widetilde{f}_n\left(e^{i\theta}\right)}{\widetilde{B}_n\left(e^{i\theta}\right)} e^{i\theta} \widetilde{g}\left(e^{i\theta}\right) \mathrm{d}\theta \right|. \tag{4}$$

Fix $g \in H\left(X^\times\right)$ and notice that

$$\frac{\widetilde{f}_n\left(e^{i\theta}\right)}{\widetilde{B}_n\left(e^{i\theta}\right)} = \sum_{j=1}^{n} \frac{w_j}{B_j\left(\lambda_j\right)} \frac{1-\overline{\lambda}_j e^{i\theta}}{e^{i\theta}-\lambda_j}, \quad \theta \in [0,2\pi),$$

whence it follows that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\widetilde{f}_n\left(e^{i\theta}\right)}{\widetilde{B}_n\left(e^{i\theta}\right)} e^{i\theta} \widetilde{g}\left(e^{i\theta}\right) \mathrm{d}\theta = \sum_{j=1}^{n} \frac{w_j}{B_j\left(\lambda_j\right)} \frac{1}{2\pi} \int_0^{2\pi} \frac{1-\overline{\lambda}_j e^{i\theta}}{e^{i\theta}-\lambda_j} e^{i\theta} \widetilde{g}\left(e^{i\theta}\right) \mathrm{d}\theta.$$

Since $H\left(X^\times\right) \hookrightarrow H^1$, the function G given by $G(z) = \left(1-\overline{\lambda}_j z\right) g(z)$ for all $z \in \mathbb{D}$ belongs to H^1, and consequently, for each $j \in \{1,\ldots,n\}$,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \frac{1-\overline{\lambda}_j e^{i\theta}}{e^{i\theta}-\lambda_j} e^{i\theta} \widetilde{g}\left(e^{i\theta}\right) \mathrm{d}\theta &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\widetilde{G}\left(e^{i\theta}\right)}{1-e^{-i\theta}\lambda_j} \mathrm{d}\theta \\ &= G\left(\lambda_j\right) = \left(1-|\lambda_j|^2\right) g\left(\lambda_j\right). \end{aligned}$$

Combining this with equation (4), we obtain another, crucial, representation:

$$m_{E,n}(w) = \sup_{\|g\|_{H(X^\times)} \leqslant 1} \left| \sum_{j=1}^{n} \frac{w_j}{b_{n,j}} \left(1-|\lambda_j|^2\right) g\left(\lambda_j\right) \right|. \tag{5}$$

(ii)⇒(i). Assume that $E \subset \{\{f\left(\lambda_j\right)\}; f \in H(X)\}$. Since δ_z is a continuous linear functional on $H(X)$ for every $z \in \mathbb{D}$, it follows from Corollary 1 that there exists a constant $\gamma > 0$ such that

$$\inf\left\{\|f\|_{H(X)}; f\left(\lambda_j\right) = w_j,\ j \in \mathbb{N}\right\} \leqslant \gamma \|w\|_E, \quad w = \{w_j\} \in E.$$

It is clear that for each $n \in \mathbb{N}$, we have

$$m_{E,n}(w) \leqslant \inf\left\{\|f\|_{H(X)}; f\left(\lambda_j\right) = w_j,\ j \in \mathbb{N}\right\},$$

and hence

$$m_{E,n}(w) \leqslant \gamma \|w\|_E, \quad w \in E, \tag{6}$$

which is the required statement.

Assume now that there exists $\gamma > 0$ such that for all $w = \{w_j\} \in E$,

$$\sup_{n \in \mathbb{N}} \sup_{\|g\|_{H(X^\times)} \leqslant 1} \left| \sum_{j=1}^{n} \frac{w_j}{b_{nj}} \left(1-|\lambda_j|^2\right) g\left(\lambda_j\right) \right| \leqslant \gamma \|w\|_E.$$

This estimate combined with equations (3) and (5) yields, for every $w \in E$ and each $n \in \mathbb{N}$,

$$\inf_{h \in H(X)} \|f_n - B_n h\|_{H(X)} \leqslant \gamma \|w\|_E.$$

We can therefore find, for each n, a function $h_n \in H(X)$ such that $\|g_n\|_{H(X)} \leqslant M$, where $g_n = f_n - B_n h_n \in H(X)$ and $M := \gamma\|w\|_E + 1$. It follows from formula (2) that the sequence $\{g_n\}$ is locally uniformly bounded in $\mathbb{D}$. By Montel's theorem we can extract a subsequence $\{g_{k_n}\}_{n=1}^{\infty}$ of $\{g_n\}$ such that $g_{k_n} \to g \in H(\mathbb{D})$ uniformly on compact subsets of $\mathbb{D}$. Since X is order-continuous, it has the weak Fatou property, and so Lemma 2 applies. Thus we conclude that $g \in H(X)$. In particular, this yields, for each $j \in \mathbb{N}$,

$$w_j = \lim_{n \to \infty} g_{k_n}(\lambda_j) = g(\lambda_j).$$

This completes the proof. □

In what follows, we will use Theorem 2 in the proof of the main result of this section. One of the main tools we use is interpolation theory. Here we present some basic concepts; for a more detailed study, we refer the reader to [5, 6, 7, 18].

Let X_0 and X_1 be Banach spaces. The pair $\vec{X} = (X_0, X_1)$ is called a *Banach couple* if there exists a Hausdorff topological vector space $\mathcal{X}$ such that $X_k \hookrightarrow \mathcal{X}$, $k \in \{0,1\}$. A Banach space X is called an *intermediate space* with respect to $\vec{X}$ if $X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$.

If $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ are Banach couples and $T\colon X_0 + X_1 \to Y_0 + Y_1$ is a linear map such that $T|_{X_k}\colon X_k \to Y_k$ for $k \in \{0,1\}$, then we write $T\colon \vec{X} \to \vec{Y}$.

Banach spaces X and Y are said to be *interpolation spaces* with respect to $\vec{X}$ and $\vec{Y}$ if X and Y are intermediate with respect to $\vec{X}$ and $\vec{Y}$, respectively, and if T maps X into Y for every $T\colon \vec{X} \to \vec{Y}$. A mapping F from the category of all couples of Banach spaces into the category of all Banach spaces is said to be a bounded *interpolation functor* if there exists a constant $\gamma > 0$ such that for every Banach couple $\vec{X}$, $F(\vec{X})$ is a Banach space intermediate with respect to $\vec{X}$, and if, for any Banach couples $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$, the condition $T\colon \vec{X} \to \vec{Y}$ implies $T(F(\vec{X})) \subset F(\vec{Y})$ with

$$\|T\|_{X \to Y} \leqslant \gamma \max\left\{\|T|_{X_0}\|_{X_0 \to Y_0}, \|T|_{X_1}\|_{X_1 \to Y_1}\right\}.$$

If $\gamma = 1$, then F is said to be an *exact* interpolation functor. Later we will need to quantify an intermediate space with respect to a given Banach couple. For that we define the characteristic function of a functor.

Let F be an exact interpolation functor. The *characteristic function ψ_F of F* is defined by

$$\psi_F(s,t)\mathbb{R} \cong F(s\mathbb{R}, t\mathbb{R}), \quad s,t > 0,$$

where $\alpha\mathbb{R}$ with $\alpha > 0$ is equipped with the norm $\|x\| = \alpha|x|$ for all $x \in \mathbb{R}$. It is easy to see that ψ_F is homogeneous of degree 1 – that is, $\psi_F(\gamma s, \gamma t) = \gamma\psi_F(s,t)$ for all $\gamma, s, t > 0$ – and in addition, ψ_F is nondecreasing in each variable. For applications of characteristic functions in interpolation of operators on Banach spaces, we refer the reader to [23].

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We are ready to prove the main result of this section. Let us recall (see just after Theorem 1) that for a Blaschke sequence $\{\lambda_j\}$ and finite Blaschke product B_n,

$$\rho_j = \lim_{n\to\infty} \rho_{n,j},$$

where $\rho_{n,j} = |B_{n,j}(\lambda_j)|$ and $B_{n,j}(z) = \prod_{\substack{k=1\\k\neq j}}^{n} \frac{z-\lambda_k}{1-\overline{\lambda}_k z}$.

Theorem 3. *Let $E \subset \omega(\mathbb{N})$ be a Banach sequence lattice and let X be minimal and maximal Banach lattice such that $L^\infty(\mathbb{T}) \hookrightarrow X \hookrightarrow L^1(\mathbb{T})$. If $\lambda = \{\lambda_j\}$ is a Blaschke sequence, then $E \subset \{\{f(\lambda_j)\}; f \in H(X)\}$ if and only if there exists a constant $\gamma > 0$ such that for all $g \in H(X^\times)$,*

$$\left\|\left\{\rho_j^{-1}\left(1-|\lambda_j|^2\right) g(\lambda_j)\right\}\right\|_{E^\times} \leqslant \gamma \|g\|_{H(X^\times)}.$$

Proof. Assume that there exists a constant $\gamma > 0$ such that

$$\sup_{\|g\|_{H(X^\times)}\leqslant 1} \left\|\left\{\rho_j^{-1}\left(1-|\lambda_j|^2\right) g(\lambda_j)\right\}\right\|_{E^\times} \leqslant \gamma.$$

Since $\sum_{j=1}^{\infty}(1-|\lambda_j|) < \infty$, $\rho_{n,j} \geqslant \rho_j > 0$ for each positive integer $j \leqslant n$. This yields

$$\begin{aligned}
\sup_{\|g\|_{H(X^\times)}\leqslant 1} &\left|\sum_{j=1}^{n} \frac{w_j}{b_{n,j}}\left(1-|\lambda_j|^2\right) g(\lambda_j)\right| \\
&\leqslant \sup_{\|g\|_{H(X^\times)}\leqslant 1} \sum_{j=1}^{n} |w_j|\,\rho_{n,j}^{-1}\left(1-|\lambda_j|^2\right)|g(\lambda_j)| \\
&\leqslant \sup_{\|g\|_{H(X^\times)}\leqslant 1} \left\|\left\{\rho_j^{-1}\left(1-|\lambda_j|^2\right)|g(\lambda_j)|\right\}\right\|_{E^\times} \|w\|_E \\
&\leqslant \gamma \|w\|_E.
\end{aligned}$$

Thus Theorem 2 applies and the implication follows.

To prove the opposite implication, assume that $E \subset \{\{f(\lambda_j)\}; f \in H(X)\}$. From Theorem 2 it follows that there exists $\gamma > 0$ such that

$$\sup_{\|w\|_E\leqslant 1} \sup_{n\in\mathbb{N}} \sup_{\|g\|_{H(X^\times)}\leqslant 1} \left|\sum_{j=1}^{n} \frac{w_j}{b_{n,j}}\left(1-|\lambda_j|^2\right) g(\lambda_j)\right| \leqslant \gamma.$$

Clearly, for all $g \in H(X^\times)$, we have (by $|b_{n,j}| = \rho_{n,j}$)

$$\left\|\left\{\rho_{n,j}^{-1}\left(1-|\lambda_j|^2\right)|g(\lambda_j)|\right\}\right\|_{E^\times} \leqslant \gamma \|g\|_{H(X^\times)}.$$

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Now observe that $\rho_{n,j} \geqslant \rho_{n+1,j}$ for each $j,n \in \mathbb{N}$, and $\rho_{n,j} \to \rho_j$ for each $j \in \mathbb{N}$ as $n \to \infty$. This means that the sequence $\{x_n\}_{n=1}^{\infty}$ given by

$$x_n = \left\{\rho_{n,j}^{-1}\left(1-|\lambda_j|^2\right)|g(\lambda_j)|\right\}_{j=1}^{\infty}$$

satisfies $\|x_n\|_{E^\times} \leqslant \gamma\|g\|_{H(X^\times)}$ and $0 \leqslant x_n \uparrow x$ in $\omega(\mathbb{N})$, where

$$x := \left\{\rho_j^{-1}\left(1-|\lambda_j|^2\right)|g(\lambda_j)|\right\}_{j=1}^{\infty}.$$

Since $E^\times$ has the Fatou property, $x \in E^\times$ and $\|x\|_{E^\times} \leqslant \gamma\|g\|_{H(X^\times)}$ – that is,

$$\left\|\left\{\rho_j^{-1}\left(1-|\lambda_j|^2\right)g(\lambda_j)\right\}\right\|_{E^\times} \leqslant \gamma\|g\|_{H(X^\times)}.$$

This completes the proof. □

Theorem 4. *Let $\lambda = \{\lambda_j\}$ be a Blaschke sequence and let $E \subset \omega(\mathbb{N})$ be a Banach sequence lattice. Then*

$$E \subset \{\{f(\lambda_j)\}; f \in H^\infty\}$$

if and only if there exists a constant $\gamma > 0$ such that for all $f \in H^1$,

$$\left\|\left\{\rho_j^{-1}\left(1-|\lambda_j|^2\right)g(\lambda_j)\right\}\right\|_{E^\times} \leqslant \gamma\|g\|_{H^1}.$$

Proof. Following the notation from the proof of Theorem 2, with $E = \ell^\infty$ for $w = \{w_j\} \in \ell^\infty$, we define

$$m_{\ell^\infty,n}(w) := \inf\{\|f_n - B_n g\|_{H^\infty}; g \in H^\infty\}, \quad n \in \mathbb{N},$$

where $f_n(z) = \sum_{j=1}^{n} b_{n,j}^{-1} w_j B_{n,j}(z)$ for all $z \in \mathbb{D}$. Then by [16, pp. 197–198] we have

$$m_{\ell^\infty,n}(w) = \sup\left\{\left|\sum_{j=1}^{n}\frac{w_j}{b_{n,j}}f(\lambda_j)\left(1-|\lambda_j|^2\right)\right|; \|f\|_{H^1} \leqslant 1\right\}.$$

Since H^∞ has the analytic Fatou property, we can now complete the proof of the theorem by repeating the proof of Theorem 3. □

We conclude this section with two corollaries. The first one follows from Theorems 2 and 3.

Corollary 2. *Let $\lambda = \{\lambda_j\}$ be a Blaschke sequence and let $E \subset \omega(\mathbb{N})$ be a Banach sequence lattice. Then for any $p \in [1,\infty]$, we have*

$$E \subset \{\{f(\lambda_j)\}; f \in H^p\}$$

if and only if there exists $\gamma > 0$ such that for all $g \in H^q$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\left\|\left\{\rho_j^{-1}\left(1-|\lambda_j|^2\right)g(\lambda_j)\right\}\right\|_{E^\times} \leqslant \gamma\|g\|_{H^q}.$$

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Corollary 3. *The following statements about the Blaschke sequence $\lambda = \{\lambda_j\}$ are equivalent*:

(i) $\ell^\infty \subset \{\{f(\lambda_j)\}; f \in H^\infty\}$.

(ii) *There exists $\gamma > 0$ such that for all $g \in H^1$,*

$$\sum_{j=1}^{\infty} \rho_j^{-1}\left(1-|\lambda_j|^2\right)|g(\lambda_j)| \leqslant \gamma \|g\|_{H^1}.$$

(iii) *The sequence λ is uniformly separated.*

Proof. For the equivalence of (i) and (ii) we apply Corollary 2 for $p = \infty$ and $E = \ell^\infty$. Since (i) and (iii) are equivalent by the Carleson theorem, the proof is complete. □

Note here that we could equally use the outcome of [28] to get the equivalence of Corollary 3(ii) and (iii). Similarly, we could refer to [22, Corollary 4] for the equivalence of (i) and (ii).

4. The result of Shapiro and Shields revisited

In this section we give a new proof of the characterisation of interpolating sequences for the classical Hardy spaces H^p, $p \in [1,\infty)$. We note that a necessary and sufficient condition for the sequence to be interpolating for the Bergman space A^p, $p \in (0,\infty)$, was discovered by Schuster and Seip in [24]. They also noticed that their argument yields a constructive proof that the notation of uniformly separated sequence is a sufficient condition for interpolation in H^p with $p \in (1,\infty)$. A constructive proof of this fact with a different nature was given also by Amar in [2].

Our approach is heavily based on Banach space theory and interpolation of operators. The result of this part serves as a motivation for considering more general and abstract settings in the next sections.

To obtain a characterisation of interpolating sequences for H^p spaces, we will use the complex method of interpolation introduced by Calderón in [8]. For a complex Banach couple $\vec{X} = (X_0, X_1)$, we define $\mathcal{F}(\vec{X})$ to be the space of all continuous functions on the closed strip $\{z \in \mathbb{C}; \operatorname{Re}(z) \in [0,1]\}$ with values in $X_0 + X_1$ that are analytic on the interior, bounded and continuous into X_j on the complex line $j + i\mathbb{R}$, $j \in \{0,1\}$, equipped with the norm

$$\|f\|_{\mathcal{F}(\vec{X})} = \max\left\{\sup_{t\in\mathbb{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X_1}\right\}.$$

For a given $\theta \in (0,1)$, the *complex interpolation space* $[X_0, X_1]_\theta$ is defined as $\left\{f(\theta); f \in \mathcal{F}(\vec{X})\right\}$ and equipped with the quotient norm

$$\|x\|_{[X_0,X_1]_\theta} = \inf\left\{\|f\|_{\mathcal{F}(\vec{X})}; x = f(\theta),\ f \in \mathcal{F}\left(\vec{X}\right)\right\}.$$

Further, a key role will be played by the well-known formula [6, Theorem 5.5.3] which states that for any Banach couple $(L^{p_0}(w_0), L^{p_1}(w_1))$, $p_0 \neq p_1$, of weighted L^p spaces on

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a measure space,

$$\left[L^{p_0}(w_0), L^{p_1}(w_1)\right]_\theta \cong L^{p_\theta}\left(w_0^{1-\theta} w_1^{\theta}\right), \quad \theta \in (0,1),$$

where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

We will now present the new proof of the characterisation of interpolating sequences for Hardy spaces H^p, $p \in [1,\infty)$. To this aim, we will need some auxiliary facts.

From now on, X is a Banach lattice on $\mathbb{T}$ such that $L^\infty(\mathbb{T}) \hookrightarrow X \hookrightarrow L^1(\mathbb{T})$. The following lemma can be derived by using the closed graph theorem:

Lemma 4. *Let E be a Banach sequence space such that $E \hookrightarrow \omega(\mathbb{N})$ and $T_\lambda(H(X)) \subset E$. Then T_λ is a bounded operator from $H(X)$ into E.*

Let us recall that e_j, $j \in \mathbb{N}$, denotes the standard unit vector.

Theorem 5. *Let $\lambda = \{\lambda_j\}$ be a sequence of distinct points in $\mathbb{D}$. Suppose that E is a Banach sequence space such that $E \hookrightarrow \omega(\mathbb{N})$ and there exists $M > 0$ such that for each $j \in \mathbb{N}$, $\|e_j\|_E \leqslant M \left\|\delta_{\lambda_j}\right\|_{H(X)^*}^{-1}$. If $T_\lambda(H(X)) = E$, then λ is uniformly separated.*

Proof. If $T_\lambda(H(X)) = E$, then we can apply Lemma 4 to conclude that $T_\lambda\colon H(X) \to E$ is a bounded linear surjection. By the open mapping theorem, there exists a constant $\gamma > 0$ such that

$$\gamma\{f \in E; \|f\|_E \leqslant 1\} \subset T_\lambda\left(\{f \in H(X); \|f\|_{H(X)} \leqslant 1\}\right).$$

In particular, for each positive integer k there exists a function $f_k \in H(X)$ with $T_\lambda f_k := \{f_k(\lambda_j)\} = M^{-1} e_k \|\delta_{\lambda_k}\|_{H(X)^*}$ and

$$\|f_k\|_{H(X)} \leqslant \gamma \left\|\left\{M^{-1} e_k \|\delta_{\lambda_k}\|_{H(X)^*}\right\}\right\|_E \leqslant \gamma.$$

For $n > k$, set

$$g_{n,k}(z) = f_k(z) \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1 - \overline{\lambda}_j z}{z - \lambda_j}, \quad z \in \mathbb{D}.$$

Clearly, $g_{n,k} \in H(X)$ and $\|g_{n,k}\|_{H(X)} = \|f_k\|_{H(X)} \leqslant \gamma$. Note also that for every $f \in H(X)$, we have

$$|f(z)| \leqslant \|\delta_z\|_{H(X)^*} \|f\|_{H(X)}, \quad z \in \mathbb{D}.$$

Notice that $f_k(\lambda_k) = M^{-1} \|\delta_{\lambda_k}\|_{H(X)^*}$ for each $k \in \mathbb{N}$. Therefore, for all $n > k$, we have

$$\prod_{\substack{j=1 \\ j \neq k}} \left| \frac{1 - \overline{\lambda}_j \lambda_k}{\lambda_k - \lambda_j} \right| = \frac{|g_{n,k}(\lambda_k)|}{M^{-1} \|\delta_{\lambda_k}\|_{H(X)^*}} \leqslant M \|g_{n,k}\|_{H(X)} \leqslant \gamma M.$$

Consequently, λ is uniformly separated and the proof is completed. □

Corollary 4. *Let X be an r.i. space X with a fundamental function ϕ_X and suppose that E is a Banach sequence space such that $E \hookrightarrow \omega(\mathbb{N})$ and there exists $M > 0$ such that for*

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each $j\in\mathbb{N}$, $\|e_j\|_E \leqslant M\phi_X(1-|\lambda_j|)$. *If* $\lambda=\{\lambda_j\}\subset\mathbb{D}$ *is a sequence of distinct points and* $T_\lambda(H(X))=E$, *then* λ *is uniformly separated.*

Proof. From [21, Lemma 1.2], it follows that for every $z\in\mathbb{D}$,

$$\frac{1}{4}\phi_X(1-|z|)^{-1}\leqslant\|\delta_z\|_{H(X)^*}\leqslant 2\phi_X(1-|z|)^{-1}.$$

Thus Theorem 5 applies. □

Theorem 6. *Let* $\lambda=\{\lambda_j\}$ *be a sequence of distinct points in* $\mathbb{D}$ *and let* $p\in[1,\infty)$. *Then*

$$\{\{f(\lambda_j)\};\, f\in H^p\}=\ell^p(\nu) \tag{7}$$

if and only if λ *is uniformly separated, where* ν *is the measure defined on* $2^{\mathbb{N}}$ *by* $\nu(\{j\})=1-|\lambda_j|^2$ *for each* $j\in\mathbb{N}$.

Proof. We claim that equation (7) implies that λ is uniformly separated. Indeed, it is well known that $\|\delta_z\|_{(H^p)^*}=\left(1-|z|^2\right)^{-1/p}$ for any $z\in\mathbb{D}$. This implies that for $E:=\ell^p(\nu)$, we have

$$\|e_j\|_E=\left(1-|\lambda_j|^2\right)^{1/p}=\|\delta_{\lambda_j}\|^{-1}_{(H^p)^*},\quad j\in\mathbb{N}.$$

Thus, if we assume that the Carleson operator T_λ given by $T_\lambda f=\{f(\lambda_j)\}$ for all $f\in H^p$ satisfies $T_\lambda(H^p)=\ell^p(\nu)$, then it follows from Theorem 5 that λ is uniformly separated.

To prove the reverse statement we assume that $\lambda=\{\lambda_j\}$ is uniformly separated. First we handle the case where $p\in(1,\infty)$. Define a linear mapping $S_\lambda\colon H^1\to\omega(\mathbb{N})$ by

$$S_\lambda g:=\{g(\lambda_j)\},\quad g\in H^1.$$

From Corollary 3 (ii), we conclude that $S_\lambda\colon H^1\to\ell^1(u)$ with $u:=\left\{\rho_j^{-1}\left(1-|\lambda_j|^2\right)\right\}$ is a bounded operator. Clearly, $S_\lambda\colon H^\infty\to\ell^\infty$ is also bounded. Thus

$$S_\lambda\colon\left(H^1,H^\infty\right)\to\left(\ell^1(u),\ell^\infty\right),$$

and in consequence, by the interpolation property,

$$S_\lambda\colon\left[H^1,H^\infty\right]_\theta\to\left[\ell^1(u),\ell^\infty\right]_\theta$$

is bounded for any $\theta\in(0,1)$. Using the interpolation formula established in [17] – that is, $\left[H^1,H^\infty\right]_\theta=H^{p_\theta}$ with $\frac{1}{p_\theta}=1-\theta$ – we deduce by taking $\theta=\frac{1}{p}$ that for $\frac{1}{q}:=1-\frac{1}{p}$,

$$S_\lambda\colon H^q\to\left[\ell^1(u),\ell^\infty\right]_\theta.$$

Since $\left[\ell^1(u),\ell^\infty\right]_\theta=\ell^q\left(u^{1-\theta}\right)=\ell^q\left(u^{1/q}\right)$, it follows that there exists a constant $\gamma>0$ such that for all $g\in H^q$, we have

$$\|S_\lambda g\|_{\ell^q\left(u^{1/q}\right)}\leqslant\gamma\|g\|_{H^q}.$$

Now observe that since $E:=\ell^p(\nu)\cong\ell^p\left(u^{1/p}\right)$, we have $E^\times\cong\ell^q\left(u^{-1/p}\right)$. Hence

$$\left\|\left\{\rho_j^{-1}\left(1-|\lambda_j|^2\right)g(\lambda_j)\right\}\right\|_{E^\times}\leqslant\gamma\|g\|_{H^q},\quad g\in H^q,$$

implies by Corollary 2 that

$$E \subset \{\{f(\lambda_j)\}; f \in H^p\}.$$

To prove the reverse inclusion, we note that $\rho_j \leqslant 1$ for each $j \in \mathbb{N}$. Then by Corollary 3(ii), the operator T_λ is bounded from H^1 into $\ell^1(\nu)$. In consequence,

$$T_\lambda\colon\ \left(H^1, H^\infty\right) \to \left(\ell^1(\nu), \ell^\infty\right).$$

Applying the interpolation formulas for the complex method generated by $\theta = 1 - \frac{1}{p}$, we obtain

$$\left[H^1, H^\infty\right]_\theta = H^p$$

and

$$\left[\ell^1(\nu), \ell^\infty\right]_\theta = \ell^p((1-|\lambda_j|^2)^{1-\theta}) = \ell^p(w).$$

This implies by the interpolation property that $T_\lambda\colon H^p \to \ell^p(\nu)$ is bounded, and so the required inclusion $T_\lambda(H^p) \subset \ell^p(\nu)$ follows.

In the case where $p = 1$, we apply Corollary 2 for $E = \ell^1(\nu)$. Since $E^\times = \ell^\infty\left(w^{-1}\right)$ with $w = \left\{1 - |\lambda_j|^2\right\}$, for $T_\lambda f := \{f(\lambda_j)\}$ we have $\ell^1(w) \subset T_\lambda\left(H^1\right)$ if and only if

$$\sup_{j \geqslant 1} \rho_j^{-1} |g(\lambda_j)| = \left\| \left\{ \rho_j^{-1} \left(1 - |\lambda_j|^2\right) g(\lambda_j) \right\} \right\|_{\ell^\infty(w^{-1})} \leqslant \gamma \|g\|_{H^\infty}$$

for all $g \in H^\infty$ with some $\gamma > 0$. This is clearly equivalent to $\inf_{j \geqslant 1} \rho_j > 0$, and hence the sequence λ is uniformly separated.

If λ is uniformly separated, then we notice that it follows from Corollary 3(ii), by $\rho_j \leqslant 1$ for all $j \in \mathbb{N}$, that $T_\lambda\left(H^1\right) \subset \ell^1(\nu)$. This completes the proof. □

5. Carleson operator

The new proof of the characterisation of interpolating sequences for H^p spaces (see Theorem 6) is an inspiration for considering a more general and abstract approach. For a given sequence $\lambda = \{\lambda_j\} \subset \mathbb{D}$, we define a linear map $T_\lambda\colon H(\mathbb{D}) \to \omega(\mathbb{N})$ by

$$T_\lambda f = \{f(\lambda_j)\}, \quad f \in H(\mathbb{D}).$$

In this section we address the question whether for a given Banach lattice X there exists a sequence space E such that T_λ is surjective as an operator from $H(X)$ onto E.

Let us remark, as we already have mentioned in the introduction, that this problem was solved by Carleson for H^∞ [9] and by Shapiro and Shields for H^p spaces, $p \in [1, \infty)$ [26] (see also Theorem 6). Their results state that $T_\lambda\colon H^p \to \ell^p(\nu)$ is surjective if and only if λ is uniformly separated – that is,

$$\inf_{k \in \mathbb{N}} \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \left| \frac{\lambda_j - \lambda_k}{1 - \overline{\lambda}_k \lambda_j} \right| > 0.$$

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We extend those classical results for a wide class of abstract Hardy spaces $H(X)$ generated by r.i. spaces X on $\mathbb{T}$. The main result of this section is contained in Theorem 7. However, to prove this general result we need some additional observations.

Let F be an exact interpolation functor. We restrict F to the class of couples of Banach function lattices on measure spaces. Let (X_0,X_1) be a couple of Banach function lattices on a σ-finite measure space (Ω,Σ,μ). An exact interpolation functor G is said to be the *Köthe dual functor to* F on a couple (X_0,X_1) if

$$G\left(X_0^{\times},X_1^{\times}\right) = F(X_0,X_1)^{\times}.$$

We also recall the following easily verified estimate [23, p. 372], which holds for any exact interpolation functor and any Banach couple $\vec{X} = (X_0,X_1)$:

$$\|x\|_{F(\vec{X})} \leqslant \psi_F\left(\|x\|_{X_0},\|x\|_{X_1}\right), \quad x \in X_0 \cap X_1, \tag{8}$$

where ψ_F is the characteristic function of the functor F.

Now we are ready to prove the main result of the paper.

Theorem 7. *Let $\lambda = \{\lambda_j\}$ be a sequence of distinct points in $\mathbb{D}$. Let X be minimal and maximal r.i. space on $\mathbb{T}$ such that $X = F(L^{p_0}(\mathbb{T}),L^{p_1}(\mathbb{T}))$ for some $p_0,p_1 \in [1,\infty]$, $p_0 < p_1$, where F is an exact interpolation functor for which the Köthe dual functor exists on all couples of L^p-spaces. Then*

$$\{\{f(\lambda_j)\}; f \in H(X)\} = F\left(\ell^{p_0}(\nu),\ell^{p_1}(\nu)\right) \tag{9}$$

if and only if λ is uniformly separated.

Proof. First we prove that λ being a uniformly separated sequence is a necessary condition.

With this aim, define $E := F(\ell^{p_0}(\nu),\ell^{p_1}(\nu))$ and observe that for each $j \in \mathbb{N}$, $e_j \in \ell^{p_0}(\nu) \cap \ell^{p_1}(\nu)$. Applying the interpolation estimate (8), we get

$$\|e_j\|_E \leqslant \psi_F(\|e_j\|_{\ell^{p_0}(\nu)},\|e_j\|_{\ell^{p_1}(\nu)}) = \psi_F((1-|\lambda_j|^2)^{1/p_0},(1-|\lambda_j|^2)^{1/p_1}),$$

where ψ_F is the characteristic function of F. Since $X = F(L^{p_0}(\mathbb{T}),L^{p_1}(\mathbb{T}))$, it follows by the interpolation property that

$$\phi_X(t) \sim \psi_F(t^{1/p_0},t^{1/p_1}), \quad t \in (0,1).$$

Thus there exists $M > 0$ such that

$$\|e_j\|_E \leqslant M\phi_X(1-|\lambda_j|^2), \quad j \in \mathbb{N}.$$

If we assume that $T_\lambda f = \{f(\lambda_j)\}$ for $f \in H(\mathbb{D})$ satisfies the equality $T_\lambda(H(X)) = E$, then by our estimate and Corollary 4, we see that λ is uniformly separated.

To prove the reverse statement, assume that $\lambda = \{\lambda_j\}$ is uniformly separated. Then by results of Carleson [9] and Shapiro and Shields [26], it follows that $T_\lambda(H^{p_k}) = \ell^{p_k}(\nu)$ for $k \in \{0,1\}$. Thus $T_\lambda\colon (H^{p_0},H^{p_1}) \to (\ell^{p_0}(\nu),\ell^{p_1}(\nu))$, and so by interpolation,

$$T_\lambda\colon F(H^{p_0},H^{p_1}) \to F\left(\ell^{p_0}(\nu),\ell^{p_1}(\nu)\right) = E$$

is a bounded operator. Applying a result from [29], we obtain $F(H^{p_0},H^{p_1}) = HF(L^{p_0},L^{p_1})$. In consequence, $H(X) = F(H^{p_0},H^{p_1})$, and by the interpolation property, we get that

$$T_\lambda\colon H(X) \to E$$

is bounded, and so $T_\lambda(H(X)) \subset E$.

We claim that $E \subset T_\lambda(H(X))$. To prove this, define a linear map $S_\lambda\colon H(\mathbb{D}) \to \omega(\mathbb{N})$ by the formula

$$S_\lambda g = \left\{\rho_j^{-1}\left(1-|\lambda_j|^2\right)g(\lambda_j)\right\}, \quad g \in H(\mathbb{D}).$$

From Corollary 1 it follows that $\ell^{p_k}(w_k) \subset T_\lambda(H^{p_k})$ for $k \in \{0,1\}$ is equivalent to the statement that the restriction of S_λ, which satisfies

$$S_\lambda\colon \left(H\left((L^{p_0})^\times\right), H\left((L^{p_1})^\times\right)\right) \to \left(\ell^{p_0}(\nu)^\times, \ell^{p_1}(\nu)^\times\right),$$

is a bounded operator. Then by interpolation, we conclude that if G is the Köthe dual functor to F,

$$S_\lambda\colon G\left(H\left((L^{p_0})^\times\right), H\left((L^{p_1})^\times\right)\right) \to G\left(\ell^{p_0}(\nu)^\times, \ell^{p_1}(\nu)^\times\right)$$

is a bounded operator. Applying again the result from [29], we obtain

$$G\left(H\left((L^{p_0})^\times\right), H\left((L^{p_1})^\times\right)\right) = H\left(G\left((L^{p_0})^\times, (L^{p_1})^\times\right)\right).$$

Thus we get

$$G\left(H\left((L^{p_0})^\times\right), H\left((L^{p_1})^\times\right)\right) = H\left(F(L^{p_0}, L^{p_1})^\times\right) = H\left(X^\times\right).$$

Since $G(\ell^{p_0}(\nu)^\times, \ell^{p_1}(\nu)^\times) = F(\ell^{p_0}(\nu), \ell^{p_1}(\nu))^\times = E^\times$, we conclude that

$$S_\lambda\colon H\left(X^\times\right) \to E^\times$$

is bounded. This yields the existence of $\gamma > 0$ such that

$$\left\|\left\{\rho_j^{-1}\left(1-|\lambda_j|^2\right)g(\lambda_j)\right\}\right\|_{E^\times} \leqslant \gamma\|g\|_{H(X^\times)}, \quad g \in H\left(X^\times\right).$$

Hence by Lemma 2 and Theorem 2 we get $E \subset T_\lambda(H(X))$. This completes the proof. □

We show that Theorem 7 can be directly applied to get the description of interpolating sequences for Hardy–Lorentz spaces. Before we give the proof of this result, we recall the definition of the Lorentz spaces $L^{p,q}$.

If (Ω,Σ,μ) is a σ-finite measure space, $p \in [1,\infty)$ and $q \in [1,\infty]$, then the *Lorentz space* $L^{p,q}(\Omega)$ is the space of all $f \in L^0(\Omega)$ such that

$$\|f\|_{p,q} := \left(\int_0^\infty \left[t^{1/p} f^{**}(t)\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q} < \infty,$$

with a natural modification when $q = \infty$. Here, for $f \in L^0(\Omega)$, f^{**} denotes the function given by $t \mapsto f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\mathrm{d}s$, $t > 0$, and f^* of $f \in L^0(\Omega)$ is given by the formula

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$f^*(t) = \inf\{s > 0;\ \mu_f(s) \leqslant t\}$, $t \geqslant 0$. It is well known that $L^{p,q}(\Omega)$ is an r.i. space, for which we have (see [5])

$$\|f\|_{p,q} \sim \left(\int_0^\infty \left[t^{1/p} f^*(t)\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q}.$$

Moreover, $L^{p,q}(\Omega)$ is always maximal and order-continuous, provided that $q \in [1,\infty)$. If this is the case, then we have (see [5])

$$\left(L^{p,q}(\Omega)\right)^\times = L^{p',q'}(\Omega),$$

where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. If $(\Omega,\Sigma,\mu) = \left(\mathbb{N}, 2^{\mathbb{N}}, \mu\right)$, then $L^{p,q}(\mathbb{N})$ is denoted by $\ell^{p,q}(\mu)$ and is called a *Lorentz sequence space*.

In the case where $\Omega = \mathbb{T}$ and μ is the Lebesgue measure m, the Hardy space $H(L^{p,q})$ is denoted by $H^{p,q}$ and is denominated a *Hardy–Lorentz space*. The following result contains the version of the Carleson result for Hardy–Lorentz spaces. In the proof, we will use the classical real interpolation method $(\cdot)_{\theta,q}$ (see [5]; for a more abstract approach, refer to the next section).

Theorem 8. *Let $\lambda = \{\lambda_j\}$ be a sequence of distinct points in $\mathbb{D}$ and let $p \in (1,\infty)$ and $q \in [1,\infty)$. Then*

$$\{\{f(\lambda_j)\};\ f \in H^{p,q}\} = \ell^{p,q}(\nu)$$

if and only if $\{\lambda_k\}$ is uniformly separated.

Proof. For a given $p \in (1,\infty)$, let $p_0, p_1 \in (1,\infty)$ such that $p_0 < p < p_1$. Furthermore, there exists $\theta \in (0,1)$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then it follows from [6, Theorem 5.3.1] that for the following spaces on (Ω,Σ,μ),

$$(L^{p_0}(\mu), L^{p_1}(\mu))_{\theta,q} = L^{p,q}(\mu).$$

Since, as noted before, $L^{p,q}(\mu)^\times = L^{p',q'}(\mu)$, where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$, we conclude that

$$\begin{aligned}\left(L^{p_0}(\mu)^\times, L^{p_1}(\mu)^\times\right)_{\theta,q'} &= \left(L^{p_0'}(\mu), L^{p_1'}(\mu)\right)_{\theta,q'} \\ &= L^{p',q'}(\mu) = \left(\left(L^{p_0}(\mu), L^{p_1}(\mu)\right)_{\theta,q}\right)^\times.\end{aligned}$$

This shows that if we take $F := (\cdot)_{\theta,q}$, then $G := (\cdot)_{\theta,q'}$ is an exact interpolation functor which is the Köthe dual to F with respect to a Banach couple $(L^{p_0}(\mu), L^{p_1}(\mu))$ for any measure space. Applying this fact to $(\mathbb{T}, m)$ and $\left(\mathbb{N}, 2^{\mathbb{N}}, \nu\right)$, we are in the position to apply Theorem 7. This completes the proof. □

6. Köthe dual functors

The aim of this section is to give general applications of Theorem 7, which deals with the so-called Köthe dual functors. This motivates us to study this notion in more detail. In particular, we prove that the abstract K-method and J-method are Köthe dual functors. This allows us to obtain the characterisation of interpolating sequences for interpolation spaces between $L^{p_0}(\mathbb{T})$ and $L^{p_1}(\mathbb{T})$ for some p_0, $p_1 \in [1,\infty]$, $p_0 < p_1$ (see Theorem 11). We apply this result for some particular spaces and obtain variants of the Carleson theorem for Hardy spaces generated by selected classes of important Banach lattices.

The most important methods of interpolation are the so-called K-method and J-method. For the convenience of the reader, we recall briefly the main definitions and facts relating to these concepts. If Φ is a Banach sequence lattice modelled on $\mathbb{Z}$ such that $\{\min(1,2^n)\}_{n=-\infty}^{\infty} \in \Phi$, then for every Banach couple $\vec{X} = (X_0,X_1)$ the *K-method space* $K_\Phi(\vec{X})$ is defined to be the Banach space of all $x \in X_0 + X_1$ such that $\left\{K(2^n,x;\vec{X})\right\} \in \Phi$ equipped with the norm

$$\|x\|_{K_\Phi(\vec{X})} = \left\|\left\{K(2^n,x;\vec{X})\right\}\right\|_\Phi.$$

Here K is the famous Petree *K-functional* defined for every $x \in X_0 + X_1$ by

$$K(t,x;\vec{X}) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1};\, x_0 + x_1 = x\}, \quad t > 0.$$

We also recall the definition of the J-method. For a given Banach couple $\vec{X} = (X_0,X_1)$ and $x \in X_0 \cap X_1$, define

$$J(t,x;\vec{X}) = \max\{\|x\|_{X_0}, t\|x\|_{X_1}\}, \quad t > 0.$$

Let $\left(\ell^1,\ell^1(2^{-n})\right)$ be a couple of sequence spaces and let E be an intermediate space with respect to $\left(\ell^1,\ell^1(2^{-n})\right)$. Denote by $J_E(\vec{X})$ the space of all $x \in X_0 \cap X_1$, which can be represented in the form

$$x = \sum_{n=-\infty}^{\infty} u_n \quad (\text{convergence in } X_0 + X_1),$$

equipped with the norm

$$\|x\|_{J_E(\vec{X})} = \inf\left\{\left\|\left\{J(2^n,u_n;\vec{X})\right\}\right\|_E;\, x = \sum_{n=-\infty}^{\infty} u_n\right\},$$

where $\left\{J(2^n,u_n;\vec{X})\right\} \in E$. It is well known (see, e.g., [7]) that the K-space $K_\Phi(\vec{X})$ and the *J-space* $J_E(\vec{X})$ are Banach spaces and the mappings $\vec{X} \mapsto K_\Phi(\vec{X})$ and $\vec{X} \mapsto J_E(\vec{X})$ are exact interpolation functors. The space Φ (resp., E) from the definition of a K-method (resp., J-method) is called a *parameter*.

We note that in the case $E = \ell^p\left(2^{-n\theta}\right)$, where $p \in [1,\infty]$ and $\theta \in (0,1)$, the K-space and J-space generated by E coincide – that is,

$$K_{\ell^p(2^{-n\theta})}(\vec{X}) = J_{\ell^p(2^{-n\theta})}(\vec{X}).$$

We point out that the constants of equivalence of norms in this formula are independent of $\vec{X}$ (see [6]). In what follows, these spaces are denoted by $\vec{X}_{\theta,p}$ (or $(X_0,X_1)_{\theta,p}$).

We also need the notion of a dual Banach couple (in the interpolation sense). Let $\vec{X} = (X_0,X_1)$ be a Banach couple and let X be an intermediate space with respect to $\vec{X}$.

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We define

$$X' := (X_0 \cap X_1, \|\cdot\|_X)^*.$$

The Banach spaces X_0' and X_1' form a Banach couple, which is denoted by $\vec{X}' = (X_0', X_1')$. We recall that the following isometric duality formulas hold (see [6, Theorem 2.7.1] or [7, Proposition 2.4.5]):

$$(X_0 \cap X_1)' \cong X_0' + X_1' \quad \text{and} \quad (X_0 + X_1)' \cong X_0' \cap X_1'.$$

To formulate the Köthe duality result for the K-method, we recall that a Banach couple (X_0, X_1) is *relatively complete* if the unit ball of the space $X_0 \cap X_1$ is a closed subset of the space $X_0 + X_1$.

If Φ is a parameter of the K-method, then by Φ^+ we denote the Banach sequence lattice modelled on $\mathbb{Z}$ and equipped with the norm

$$\|\xi\|_{\Phi^+} := \sup\left\{ \left| \sum_{n=-\infty}^{\infty} \xi_n \eta_{-n} \right| ; \|\{\eta_n\}\|_\Phi \leqslant 1 \right\}.$$

In the proof, we will need the following Köthe duality formulas (see, e.g., [19]). If (X_0, X_1) is a Banach couple of Banach function lattices on a measure space (Ω, Σ, μ), then

$$(X_0 \cap X_1)^\times \cong X_0^\times + X_1^\times \quad \text{and} \quad (X_0 + X_1)^\times \cong X_0^\times \cap X_1^\times.$$

We are ready to formulate and prove the Köthe duality result for K-method spaces. In what follows, X_k° denotes the closure of $X_0 \cap X_1$ in X_k, $k \in \{0,1\}$.

Theorem 9. *Let Φ be a parameter of the K-method and let $\vec{X} = (X_0, X_1)$ be a couple of Banach function lattices on a measure space (Ω, Σ, μ). Suppose that one of the following conditions is satisfied:*

(i) $K_\Phi(X_0, X_1) \hookrightarrow X_0^\circ + X_1^\circ$ *or*

(ii) *both X_0 and X_1 have the Fatou property.*

Then K_{Ψ^+} with $\Psi := K_\Phi\left(\ell^1, \ell^1(2^{-n})\right)$ is the Köthe dual functor of K_Φ on $\vec{X}$ – that is,

$$K_\Phi(X_0, X_1)^\times = K_{\Psi^+}\left(X_0^\times, X_1^\times\right).$$

Moreover, the constants of equivalence of norms in this formula are independent of $\vec{X}$.

Proof. We start with the formula

$$K_\Phi(Y_0, Y_1)' = K_{\Psi^+}(Y_0', Y_1'), \tag{10}$$

where the isomorphism constant does not exceed 18, provided that a Banach couple (Y_0, Y_1) is relatively complete or the parameter Φ is such that $K_\Phi(Y_0, Y_1) \hookrightarrow Y_0^\circ + Y_1^\circ$. This fact follows directly from [7, Theorems 3.5.9 and 3.7.2 and Proposition 3.1.17].

Now observe that ii yields

$$(X_0 \cap X_1)^{\times\times} \cong \left(X_0^\times + X_1^\times\right)^\times \cong X_0^{\times\times} \cap X_1^{\times\times} = X_0 \cap X_1.$$

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Clearly, the Fatou property of both X_0 and X_1 implies that (X_0,X_1) is a relatively complete couple. We claim that

$$K_{\Psi^+}\left(X_0^\times,X_1^\times\right)\hookrightarrow K_\Phi(X_0,X_1)^\times.$$

To show this, we apply the interpolation duality formula (10) for (X_0,X_1). We define a linear map σ by

$$\sigma x^\times(x)=\langle x,x^\times\rangle:=\int_\Omega xx^\times \mathrm{d}\mu,\quad x^\times\in X_0^\times+X_1^\times,\ x\in X_0\cap X_1.$$

Clearly, $\sigma\colon X_0^\times+X_1^\times\to X_0'+X_1'$, and moreover, $\sigma\colon\left(X_0^\times,X_1^\times\right)\to(X_0',X_1')$, with

$$\|\sigma\|_{X_k^\times\to X_k'}\leqslant 1,\quad k\in\{0,1\}.$$

Since K_{Ψ^+} is an exact interpolation functor, we obtain

$$\sigma\colon K_{\Psi^+}\left(X_0^\times,X_1^\times\right)\to K_{\Psi^+}(X_0',X_1')$$

with

$$\left\|\sigma x^\times\right\|_{K_{\Psi^+}(X_0',X_1')}\leqslant\left\|x^\times\right\|_{K_{\Psi^+}\left(X_0^\times,X_1^\times\right)},\quad x^\times\in K_{\Psi^+}\left(X_0^\times,X_1^\times\right).$$

From this and equation (10), it follows that

$$\sup\left\{\left|\langle x,x^\times\rangle\right|;x\in X_0\cap X_1,\|x\|_{K_\Phi(\vec{X})}\leqslant 1\right\}\leqslant\gamma\left\|x^\times\right\|_{K_{\Psi^+}\left(X_0^\times,X_1^\times\right)},$$

where $\gamma\leqslant 18$. Now, since $X_0\cap X_1$ is a Banach lattice, the left-hand side of this inequality equals $\|x^\times\|_{K_\Phi(\vec{X})^\times}$ (by order density of $X_0\cap X_1$ in $L^0(\mu)$), and so the claim is proved.

To finish the proof, we need to show that

$$K_\Phi(X_0,X_1)^\times\hookrightarrow K_{\Psi^+}\left(X_0^\times,X_1^\times\right).$$

Applying [7, Corollary 3.5.16], we deduce that the following formula holds:

$$K_\Phi(X_0,X_1)=J_\Psi(X_0,X_1),$$

where the isomorphism constant does not depend on $\vec{X}$. Thus it is enough to prove that

$$J_\psi(X_0,X_1)^\times\hookrightarrow K_{\Psi^+}\left(X_0^\times,X_1^\times\right).\tag{11}$$

First, we observe that for every $t>0$ and $x^\times\in X_0^\times+X_1^\times$, we have

$$\begin{aligned}K\left(t,x^\times;\left(X_0^\times,X_1^\times\right)\right)&=\left\|x^\times\right\|_{\left(X_0^\times+tX_1^\times\right)}=\left\|x^\times\right\|_{(X_0\cap t^{-1}X_1)^\times}\\&=\sup\left\{\frac{|\langle x,x^\times\rangle|}{J\left(t^{-1},x;\vec{X}\right)};x\in X_0\cap X_1,\ x\neq 0\right\}\end{aligned}$$

Fix $\varepsilon>0$. Given nonnegative $x^\times\in J_\Psi(X_0,X_1)^\times\hookrightarrow X_0^\times+X_1^\times$, we can find a sequence $\{x_n\}_{n=-\infty}^\infty$ with $x_n\in X_0\cap X_1$ such that $\langle x_n,x^\times\rangle\geqslant 0$ and

$$K\left(2^{-n},x^\times;\vec{X}^\times\right)\leqslant(1+\varepsilon)\frac{\langle x_n,x^\times\rangle}{J\left(2^n,x_n;\vec{X}\right)},\quad n\in\mathbb{Z}.\tag{12}$$

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Now fix $\eta = \{\eta_n\}$ in the unit ball of Ψ and define a sequence $\{u_n\}_{n=-\infty}^{\infty}$ in $X_0 \cap X_1$ by $u_n := J(2^n, x_n; \vec{X})^{-1}|\eta_n| x_n$ for each $n \in \mathbb{Z}$. Since $\eta \in \Psi \hookrightarrow \ell^1 + \ell^1(2^{-n})$, it holds $\sum_{n=-\infty}^{\infty} \min\{1, 2^{-n}\}|\eta_n| < \infty$.

This implies that for some $\gamma > 0$,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \|u_n\|_{X_0+X_1} &= \sum_{n=-\infty}^{0} \|u_n\|_{X_0+X_1} + \sum_{n=1}^{\infty} \|u_n\|_{X_0+X_1} \\ &\leqslant \sum_{n=-\infty}^{0} \frac{\|x_n\|_{X_0}}{J\left(2^n, x_n; \vec{X}\right)} |\eta_n| + \sum_{n=1}^{\infty} \frac{2^n \|x_n\|_{X_1}}{J\left(2^n, x_n; \vec{X}\right)} \frac{|\eta_n|}{2^n} \\ &\leqslant \sum_{n=-\infty}^{0} |\eta_n| + \sum_{n=1}^{\infty} \frac{|\eta_n|}{2^n} = \|\eta\|_{\ell_1 + \ell_1(2^{-n})} \leqslant \gamma \|\eta\|_{\Psi} \leqslant \gamma,\end{aligned}$$

and so the series $\sum_{n=-\infty}^{\infty} u_n$ converges in $X_0 + X_1$. We define

$$x_\eta := \sum_{n=-\infty}^{\infty} u_n.$$

Since $J\left(2^n, u_n; \vec{X}\right) = |\eta_n|$ for each $n \in \mathbb{Z}$, we have $x_\eta \in J_\Psi\left(\vec{X}\right)$ with

$$\|x_\eta\|_{J_\Psi(\vec{X})} \leqslant \|\eta\|_{\Phi}.$$

Combining our estimate with formula (12), we conclude that

$$\begin{aligned}\left|\sum_{n=-\infty}^{\infty} K\left(2^{-n}, x^\times; \vec{X}^\times\right) \eta_n\right| &\leqslant (1+\varepsilon) \sum_{n=-\infty}^{\infty} \langle x_n, x^\times \rangle J(2^n, x_n; \vec{X})^{-1} |\eta_n| \\ &= (1+\varepsilon) \left\langle \sum_{n=-\infty}^{\infty} J(2^n, x_n; \vec{X})^{-1} x_n |\eta_n|, x^\times \right\rangle \\ &= (1+\varepsilon) \left\langle \sum_{n=-\infty}^{\infty} u_n, x^\times \right\rangle \leqslant (1+\varepsilon) \|x_\eta\|_{J_\Psi(\vec{X})} \left\|x^\times\right\|_{J_\Psi(\vec{X})^\times} \\ &\leqslant (1+\varepsilon) \left\|x^\times\right\|_{J_\Psi(\vec{X})^\times}.\end{aligned}$$

Since $\varepsilon > 0$ and $\eta \in \Psi$ with $\|\eta\|_\Psi \leqslant 1$ were arbitrary, it follows that $x^\times \in K_{\Psi^+}\left(X_0^\times, X_1^\times\right)$ and

$$\left\|x^\times\right\|_{K_{\Psi^+}(X_0^\times, X_1^\times)} \leqslant \left\|x^\times\right\|_{J_\Psi(\vec{X})^\times}.$$

This yields the required continuous inclusion (11), which completes the proof. □

Theorem 9 implies the description of the Köthe dual functor of the functor J_E.

Theorem 10. *Let E be a parameter of the J-method and set $\Phi := J_E(\ell^\infty, \ell^\infty(2^{-n}))$. Then K_{Ψ^+} with $\Psi := K_\Phi\left(\ell^1, \ell^1(2^{-n})\right)$ is the Köthe dual functor of J_E on any couple (X_0, X_1)*

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of maximal Banach lattices on a measure space – that is,

$$J_E(X_0,X_1)^\times = K_{\Psi^+}\left(X_0^\times,X_1^\times\right),$$

where the constants of equivalence of norms are independent of (X_0,X_1).

Proof. From [7, Corollary 3.5.16], it follows for any relatively complete Banach couple (Y_0,Y_1) that

$$J_E(Y_0,Y_1) = K_\Phi(Y_0,Y_1),$$

where $\Phi := J_E(\ell^\infty,\ell^\infty(2^{-n}))$. Since any couple of maximal Banach lattices is relatively complete, the required statement follows from Theorem 9. □

Now we are ready to prove the main result of this section – namely, we characterise interpolating sequences of interpolation spaces with respect to the couple of weighted Lebesgue spaces. We note here that for some Banach couples all interpolation spaces are K-spaces, and so they can be parameterised by the parameters of the K-method. An important example of such couples was presented in [27], where it was proved that the couple $(L^{p_0}(w_0),L^{p_1}(w_1))$ of weighted Lebesgue spaces on any measure space is a uniform Calderón couple. Then by [7, Lemma 4.1.12], it follows that all interpolation spaces with respect to $(L^{p_0}(w_0),L^{p_1}(w_1))$ are K-spaces. Recall that the Banach couple $\vec{X} = (X_0,X_1)$ is called a *uniform Calderón couple* if there exists $\gamma = \gamma(\vec{X}) \geqslant 1$ such that for any $x,y \in X_0 + X_1$ satisfying the inequality

$$K(t,y;\vec{X}) \leqslant K(t,x;\vec{X}), \quad t > 0,$$

it follows that there exists an operator $T\colon \vec{X} \to \vec{X}$ with $\|T\|_{\vec{X}\to\vec{X}} \leqslant \gamma$ such that $y = Tx$.

Theorem 11. *Let* $\{\lambda_j\}$ *be a sequence of distinct points in* $\mathbb{D}$ *and let* X *be minimal and maximal r.i. space on* $\mathbb{T}$*, such that* X *is an interpolation space with respect to the couple* $(L^{p_0}(\mathbb{T}),L^{p_1}(\mathbb{T}))$ *for some* $p_0,p_1 \in [1,\infty]$, $p_0 < p_1$. *Then there exists a parameter* Φ *of the* K*-method such that*

$$\{\{f(\lambda_j)\}\,;\, f \in H(X)\} = K_\Phi\left(\ell^{p_0}(\nu),\ell^{p_1}(\nu)\right)$$

if and only if $\{\lambda_j\}$ *is uniformly separated.*

Proof. The couple $\vec{X} = (L^{p_0}(\mathbb{T}),L^{p_1}(\mathbb{T}))$ is the uniform Calderón couple of maximal spaces. Thus by the results already mentioned, there exists a parameter Φ of the K-method such that

$$X = K_\Phi\left(L^{p_0}(\mathbb{T}),L^{p_1}(\mathbb{T})\right).$$

Therefore, Theorems 7 and 9 apply and yield the desired conclusion. □

We conclude this section with the following surprising result, which is – in a sense – a reverse statement to Theorem 7.

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Theorem 12. *Let $\lambda = \{\lambda_j\}$ be a uniformly separated sequence in $\mathbb{D}$. Then for every separable maximal r.i. space E over a measure space $\left(\mathbb{N}, 2^{\mathbb{N}}, \nu_\lambda\right)$, there exists a separable maximal r.i. space X on $\mathbb{T}$ such that*

$$\{\{f(\lambda_j)\}; f \in H(X)\} = E,$$

and so the Carleson operator $T_\lambda\colon H(X) \to E$ is a bounded surjection.

Proof. As usual, we let $\nu := \nu_\lambda$. Since E is separable, it is an interpolation space between $\ell^1(\nu)$ and ℓ^∞. From [8] it follows that $\left(\ell^1(\nu), \ell^\infty\right)$ is a uniform Calderón–Mityagin couple. Thus by [7, Lemma 4.1.12], there exists a parameter Φ of the K-method such that

$$E = K_\Phi\left(\ell^1(\nu), \ell^\infty\right).$$

We consider an exact interpolation functor $F := K_\Phi$ chosen in such a way that X given by

$$X := F\left(L^1(\mathbb{T}), L^\infty(\mathbb{T})\right)$$

is separable maximal r.i. space on $\mathbb{T}$. For any couple of Banach lattices on a measure space, we have

$$F(X_0, X_1)^\times = K_\Phi(X_0, X_1)^\times.$$

Therefore, it follows from Theorem 9 that $G := K_{\Psi^+}$ with $\Psi := K_\Phi\left(\ell^1, \ell^1(2^{-n})\right)$ is the Köthe dual functor to F on couples of maximal Banach lattices. Now Theorem 7 yields

$$\{\{f(\lambda_j)\}; f \in H(X)\} = F\left(\ell^1(\nu), \ell^\infty\right) = K_\Phi\left(\ell^1(\nu), \ell^\infty\right) = E,$$

which completes the proof. □

7. Application for special Hardy spaces

In this final part of the paper, we apply the results obtained to some important examples of r.i. spaces and get a description of the interpolating sequences for Hardy–Lorentz and Hardy–Orlicz spaces. At first, we start with a brief *résumé* of the Calderón–Lozanovskiĭ construction, since it will be used throughout this section.

We recall that if $\vec{X} = (X_0, X_1)$ is a couple of Banach lattices on (Ω, Σ, μ) and $\rho\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ is a concave and positively homogeneous function, then the *Calderón–Lozanovskiĭ space* $\rho(\vec{X}) = \rho(X_0, X_1)$ consists of all $f \in L^0(\mu)$ such that $|f| \leqslant \kappa\rho(|f_0|, |f_1|)$ μ-almost everywhere on Ω for some $\kappa > 0$ and $f_k \in X_k$ with $\|f_k\|_{X_k} \leqslant 1$, $k \in \{0,1\}$. The space $\rho(\vec{X})$ is a Banach lattice equipped with the norm

$$\|f\|_{\rho(\vec{X})} = \inf\{\kappa > 0;\, |f| \leqslant \kappa\rho(|f_0|, |f_1|), \|f_0\|_{X_0} \leqslant 1, \|f_1\|_{X_1} \leqslant 1\}$$

(see [19, 20]). In the case of the power function $\rho(s,t) = s^{1-\theta}t^\theta$ with $\theta \in (0,1)$, $\rho\left(\overline{X}\right)$ is the well-known Calderón space denoted by $X_0^{1-\theta}X_1^\theta$ (see [8]).

We will use an easily verified formula that is true for any concave and positively homogeneous functions ρ, $p \in [1,\infty]$, and all weighted sequences $w_0 = \left\{w_n^0\right\}$, $w_1 = \left\{w_n^1\right\}$

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on $\mathbb{Z}$:

$$\rho(\ell^p(w_0), \ell^{p_1}) = \ell^p(w), \tag{13}$$

where $w = \{\rho_* (w_n^0, w_n^1)\}$ and $\rho_*(s,t) := 1/\rho(s^{-1}, t^{-1})$ for every $s,t > 0$.

We will also apply Lozanovskiĭ's duality formula from [20], which states that for any couple (X_0, X_1) of Banach lattices on a measure space (Ω, Σ, μ),

$$\rho(X_0, X_1)^\times = \widehat{\rho}(X_0^\times, X_1^\times),$$

up to equivalence of norms independent of ρ, where $\widehat{\rho}(s,t) = \inf_{a,b>0} \frac{as+bt}{\rho(a,b)}$ for all $s,t > 0$.

Let $\phi\colon [0,\infty) \to [0,\infty)$ be a concave function and (Ω, Σ, μ) be any measure space. We define the *Lorentz space* $\Lambda_\phi = \Lambda_\phi(\mu)$ to be the space of all $f \in L^0(\mu)$ such that $\int_0^\infty f^*(s) \mathrm{d}\phi(s) < \infty$. This is an r.i. space when equipped with the norm

$$\|f\|_{\Lambda_\phi(\mu)} = \int_0^\infty f^*(s) \mathrm{d}\phi(s).$$

In the following theorem we present a description of interpolating sequences for *Hardy–Lorentz spaces* $H(\Lambda_\phi)$ generated by separable Lorentz spaces $\Lambda_\phi(m)$ on $(\mathbb{T}, m)$.

Theorem 13. *Let $\lambda = \{\lambda_j\}$ be a sequence of distinct points in $\mathbb{D}$ and let $\phi\colon [0,\infty) \to [0,\infty)$ be a concave function. Then the Carleson operator T_λ is a bounded surjection from the Hardy–Lorentz space $H(\Lambda_\phi)$ onto the Lorentz space $\Lambda_\phi(\nu)$ if and only if λ is uniformly separated.*

Proof. If $\phi(0+) > 0$, then $\Lambda_\phi = L^\infty$ and so $H(\Lambda_\phi) = H^\infty$, from which the result follows by the Carleson theorem. In the case where $\phi(0+) = 0$, it follows that $\Lambda_\phi(m)$ is order-continuous and so it is a minimal and maximal space. Thus, by Lemma 2, $H(\Lambda_\phi)$ has the analytic Fatou property. We let $\psi(s,t) = \frac{t}{\phi(t/s)}$ for all $s,t \in (0,\infty)$ and take a concave function $\rho \sim \psi$. Then by equation (13),

$$\rho(\ell^1, \ell^1(2^{-n})) = \ell^1(w),$$

where $w := \{2^{-n}\phi(2^n)\}_{n=-\infty}^{\infty}$. This shows that $E := \ell^1(w)$ is the parameter of the J-method.

Using the standard properties of the Lorentz spaces (see [5]), it is not difficult to show that for any measure space (Ω, Σ, μ),

$$\Lambda_\phi(\mu) = J_E(L^1(\mu), L^\infty(\mu)),$$

where E is defined as before. In particular, we get that

$$\Lambda_\phi(m) = J_E(L^1(m), L^\infty(m)).$$

Now the characterisation of interpolating sequences for the Hardy–Lorentz spaces follows from the description of the Köthe dual functor to J_E (see Theorem 10) and Theorem 7. □

Should be Next, we discuss an application for Hardy spaces generated by an order-continuous part of the Marcinkiewicz space $M_\phi(m)$ on $(\mathbb{T}, m)$. If $\phi\colon [0,\infty) \to [0,\infty)$ is a concave function and (Ω, Σ, μ) is a measure space, then the *Marcinkiewicz space* $M_\phi(\mu)$

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consists of those $f \in L^0(\mu)$ for which $\sup\{\phi(t)f^{**}(t); t > 0\} < \infty$. This is an r.i. space equipped with the norm

$$\|f\|_{M_\phi(\mu)} = \sup\{\phi(t)f^{**}(t); t > 0\}.$$

The symbol $M_\phi^\circ(\mu)$ will denote the closure of simple functions in $M_\phi(\mu)$.

Theorem 14. *Let $\lambda = \{\lambda_j\}$ be a uniformly separated sequence of distinct points in $\mathbb{D}$ and assume that ϕ: $[0,\infty) \to (0,\infty)$ is a concave function continuous at 0. Then the Carleson operator T_λ is bounded from the Hardy–Marcinkiewicz space $H(M_\phi^\circ)$ into the space $M_\phi^\circ(\nu)$.*

Proof. We note that $\phi(0+) = 0$ implies $M_\phi \neq L^\infty$. Thus $M_\phi^\circ = (M_\phi)_a$ has an order-continuous norm. This implies by Lemma 2 that $H(M_\phi^\circ)$ has the analytic Fatou property. By the same argument as in the proof of Theorem 13, we have (with the same ρ and ψ)

$$\rho\left(\ell^\infty, \ell^\infty(2^{-n})\right) = \ell^\infty\left(\rho_*(1,2^{-n})\right) = \ell^\infty(w),$$

where $w = \{2^{-n}\phi(2^n)\}_{n=-\infty}^\infty$. Hence $\Phi = \ell^\infty(w)$ is the parameter of the K-method.

By applying the well-known formula for the K-functional for the couple $\left(L^1(\mu), L^\infty(\mu)\right)$ on any measure space (Ω,Σ,μ) – see, for example, [6, Theorem 5.2.1] – we get that for every $f \in L^1(\mu) + L^\infty(\mu)$,

$$K\left(t, f; \left(L^1(\mu), L^\infty(\mu)\right)\right) = \int_0^t f^*(s)\,\mathrm{d}s, \quad t > 0.$$

In consequence

$$M_\phi(\mu) = K_\Phi\left(L^1(\mu), L^\infty(\mu)\right).$$

Now we consider an exact interpolation functor $F := K_\Phi^\circ$ and apply the foregoing formula to obtain

$$F\left(L^1(\mu), L^\infty(\mu)\right) = M_\phi^\circ(\mu)$$

and

$$F\left(\ell^1(\nu), \ell^\infty\right) = M_\phi^\circ(\nu).$$

We finish in a similar fashion as at the end of the proof of Theorem 7 with $p_0 = 1$ and $p_1 = \infty$. □

We conclude this paper with the description of interpolating sequences for Hardy–Orlicz spaces (see [15]). Recall that if φ: $[0,\infty) \to [0,\infty)$ is an Orlicz function – that is, $\varphi(s) = 0$ if and only if $s = 0$, φ is nondecreasing convex and left continuous – then the Orlicz space L_φ over a measure space (Ω,Σ,μ) is defined to be the space of all $f \in L^0(\mu)$ with $\int_\Omega \varphi(\gamma|f|)\,\mathrm{d}\mu < \infty$ for some $\gamma > 0$, and it is equipped with the Luxemburg norm

$$\|f\|_{L_\varphi} = \inf\left\{\varepsilon > 0; \int_\Omega \varphi\left(\frac{|f|}{\varepsilon}\right)\mathrm{d}\mu \leqslant 1\right\}.$$

If $\Omega = \mathbb{N}$ and $\Sigma = 2^\Omega$, the Orlicz space $L_\varphi(\mu)$ is called an *Orlicz sequence space*, usually denoted by $\ell_\varphi(\mu)$. We note that the Orlicz space $L_\varphi(\mu)$ over a finite measure space (Ω,Σ,μ)

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is order continuous whenever φ satisfies the so-called Δ_2 *condition at infinity* (that is, there exist $\gamma > 0$ and $t_0 > 0$ such that $\varphi(2t) \leqslant \gamma\varphi(t)$ for all $t \geqslant t_0$). If $X = L_\varphi(\mathbb{T})$, the space $H(X)$ is denoted by H_φ.

We are ready to state and prove the following characterisation of interpolating sequences for Hardy–Orlicz spaces.

Theorem 15. *Let $\lambda = \{\lambda_j\}$ be a sequence of distinct points in $\mathbb{D}$ and L_φ be a minimal Orlicz space. Then the Carleson operator T_λ given by $T_\lambda f = \{f(\lambda_j)\}$, $f \in H(\mathbb{D})$, is a bounded surjection from H_φ onto $\ell_\varphi(\nu)$ if and only if λ is uniformly separated.*

Proof. Following [23], for any concave function $\rho\colon (0,\infty)\times(0,\infty) \to (0,\infty)$, we let $a_\rho = \{\rho(1,2^n)\}_{n=-\infty}^{\infty}$ and we define the lower functor ρ_ℓ by

$$\rho_\ell(X_0,X_1) := \left\{Ta_\rho\colon \left(\ell^\infty,\ell^\infty(2^{-n})\right) \to (X_0,X_1); T\colon \left(\ell^\infty,\ell^\infty(2^{-n})\right) \to (X_0,X_1)\right\}$$

equipped with the norm

$$\|x\|_{\rho_\ell(\vec{X})} := \inf\left\{\|T\|_{(\ell^\infty,\ell^\infty(2^{-n}))\to\vec{X}}; x = Ta_\rho\right\}.$$

The Ovchinnikov result from [23] states that for any couple $\vec{X} = (X_0,X_1)$ of maximal Banach lattices,

$$\rho_\ell(X_0,X_1) = \rho(X_0,X_1),$$

and the constants of equivalence of norms are independent of $\vec{X}$.

We combine this result with the Lozanovskiĭ formula on the Köthe duality to get

$$\rho_\ell(X_0,X_1)^\times = \widehat{\rho}\left(X_0^\times,X_1^\times\right) = (\widehat{\rho})_\ell\left(X_0^\times,X_1^\times\right).$$

Thus $(\widehat{\rho})_\ell$ is the Köthe dual functor of ρ_ℓ on $\vec{X}$.

Now we note that it is easy to verify that for any measure space (Ω,Σ,μ),

$$L_\varphi(\mu) \cong \rho\left(L^1(\mu),L^\infty(\mu)\right),$$

where

$$\rho(s,t) = \begin{cases} t\varphi^{-1}\left(\frac{s}{t}\right), & t > 0,\\ 0, & t = 0,\end{cases}$$

and φ^{-1} is the right continuous inverse of ϕ.

We consider the exact interpolation functor $F := (\rho_\ell)^\circ$. Then by the foregoing formula we have

$$F\left(L^1(\mathbb{T}),L^\infty(\mathbb{T})\right) = L_\varphi(\mathbb{T})^\circ \cong L_\varphi(\mathbb{T})$$

and

$$F\left(\ell^1(\nu),\ell^\infty\right) = \ell_\varphi(\nu)^\circ \cong \ell_\varphi(\nu).$$

To finish the proof, we apply Theorem 7 with $p_0 = 1$ and $p_1 = \infty$ to get the required equivalence. □

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