

THE RIGHT ANGLE TO LOOK AT ORTHOGONAL SETS

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Abstract. If X and Y are orthogonal hyperdefinable sets such that X is simple, then any group G interpretable in $X \cup Y$ has a normal hyperdefinable X -internal subgroup N such that G/N is Y -internal; N is unique up to commensurability. In order to make sense of this statement, local simplicity theory for hyperdefinable sets is developed. Moreover, a version of Schlichting's Theorem for hyperdefinable families of commensurable subgroups is shown.

§1. Introduction. Two definable sets X and Y in some structure are said to be *orthogonal* if every definable subset of $X \times Y$ is a finite union of *rectangles*, i.e., of subsets of the form $U \times V$ with $U \subseteq X$ and $V \subseteq Y$ definable. It follows that if X and Y are orthogonal groups, every definable subgroup H of $X \times Y$ has a subgroup of finite index of the form $U \times V$ with $U \leq X$ and $V \leq Y$ subgroups: As H is a finite union of rectangles, one can find a maximal definable rectangle $U \times V \subseteq H$ containing the identity $1 = (1_X, 1_Y)$. As H also contains

$$(U \times V)^{-1}(U \times V) = (U^{-1} \times V^{-1})(U \times V) = U^{-1}U \times V^{-1}V \supseteq U \times V,$$

we obtain $U^{-1}U = U$ and $V^{-1}V = V$ by maximality, so $U \leq X$ and $V \leq Y$ are subgroups; moreover $U \times V$ is unique. Any other maximal rectangle contained in H can be translated to contain 1, and must thus be a coset of $U \times V$. So $U \times V$ has finite index.

However, the situation is considerably more complicated for a group G definable, or more generally interpretable, in $X \cup Y$, as it need not be a direct product of a group interpretable in X and a group interpretable in Y . In fact, an example by Berarducci and Mamino [2, Example 1.2] shows that G need not have *any* subgroup interpretable in either X or Y . However, they prove [2, Theorem 7.1] that if X is superstable of finite and definable Lascar rank, then any group G interpretable in $X \cup Y$ has a normal subgroup N interpretable in X , such that G/N is interpretable in Y .

In this paper we shall generalize their result to the case where X is merely simple. In this context, definability has to be replaced by type-definability, as even for a definable group the tools of simplicity theory in general only yield type-definable subgroups. In fact, we even have to study *hyperdefinable* groups, since the quotient G/N , for N type-definable, will be of that form. We therefore put ourselves in the

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hyperdefinable context and assume right from the start that our orthogonal sets X and Y are merely hyperdefinable. To this end, we shall include a quick development of hyperdefinability in Section 2, and of local simplicity theory for hyperdefinable sets in Section 6. Moreover, the general theory only yields N unique up to commensurability. Since there is *a priori* no simple hyperdefinable set containing all conjugates of N , we cannot use the usual locally connected component from simplicity theory [7, Definition 4.5.15]. We therefore show a completely general version of Schlichting's Theorem for a hyperdefinable family of commensurable subgroups in Section 7. Note that we do recover the theorem by Berarducci and Mamino even for general supersimple (and definable) X (Corollary 8.2).

Another problem is that of parameters. The usual hypothesis would be that of *stable embedding*, i.e., that every hyperdefinable subset of X is hyperdefinable with parameters in X . We shall circumvent this issue by only ever considering parameters from $X \cup Y$, as orthogonality automatically yields stable embeddedness of X and of Y in $X \cup Y$.

We shall work in a big κ -saturated and strongly κ -homogeneous monster model \mathfrak{M} , where κ is bigger than any cardinality we wish to consider. We shall not usually distinguish between elements and tuples.

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§2. Hyperimaginaries.

DEFINITION 2.1. • A *countable equivalence relation* is an equivalence relation given by the conjunction of countably many formulas (and hence only using countably many parameters and variables); it is *over* a set A of parameters if the formulas only use parameters from A .

- A *hyperimaginary* (element) of *type* E is the class a_E of some tuple a (of the right length) modulo a countable equivalence relation E over \emptyset .
- A hyperimaginary e is *definable* over some set B of hyperimaginaries if every automorphisms of the monster model which fixes B pointwise fixes also e ; it is *bounded* over B if its orbit under the group of automorphisms fixing B pointwise has bounded size (smaller than the saturation degree of the monster model). The *hyperimaginary definable closure* $\text{dcl}^{\text{heq}}(B)$ of B is the set of hyperimaginaries definable over B ; the *hyperimaginary bounded closure* $\text{bdd}(B)$ of B is the set of all hyperimaginaries bounded over B . Clearly, both dcl^{heq} and bdd are idempotent operators.¹
- Two hyperimaginaries are *equivalent* if they are interdefinable.
- If e is hyperimaginary, a *representative* for e is any real (or imaginary) tuple a with $e \in \text{dcl}^{\text{heq}}(a)$.

REMARK 2.2. (1) If E is an arbitrary type-definable equivalence relation over \emptyset (given by an intersection of arbitrary size, on tuples of arbitrary length), it is easy to see that E is equivalent to a conjunction of subintersections E_i ,

¹By compactness, an imaginary element in $\text{bdd}(B)$ is already in the algebraic closure $\text{acl}(B)$. So there is no need for a superscript bdd^{heq} .

each one defining a countable equivalence relation on a countable sub-tuple x_i . So

$$a E b \iff \forall i a_i E_i b_i.$$

This means that an automorphism fixes a_E if and only iff it fixes $(a_i)_{E_i}$ for all i , and we can replace a_E by the sequence $((a_i)_{E_i})_i$ of hyperimaginaries.

- (2) If E_A is a countable equivalence relation over A , we consider the countable equivalence relation

$$xx' F yy' \iff (x' = y' \wedge x' \models \text{tp}(A) \wedge x E_{x'} y) \vee xx' = yy',$$

where we only consider the countable subset of A actually occurring in the definition of E_A , and $E_{x'}$ is the result of substituting x' for A in the definition of E_A . Then for any a the class a_{E_A} is fixed by an automorphism fixing A iff and only iff $(aA)_F$ is fixed, and we can use the hyperimaginary $(aA)_F$ instead of a_{E_A} .

- (3) If E is a countable partial type over A which defines an equivalence relation on some partial type π over A , then by compactness there is a countable subtype π_0 such that E defines an equivalence relation on π_0 . Then

$$x F y \iff (\pi_0(x) \wedge \pi_0(y) \wedge x E y) \vee x = y$$

is a countable equivalence relation over A extending E .

If $E = (E_i : i \in I)$ is a sequence of countable equivalence relations over \emptyset and $a = (a_i : i \in I)$ is a sequence of tuples of the right length, we put $a_E = ((a_i)_{E_i} : i \in I)$, and we say that a_E is a tuple of hyperimaginaries. Similarly, we write $a E b$ if $a_i E_i b_i$ for all $i \in I$. Note that Remark 2.2 justifies that we restrict to countable equivalence relations over \emptyset in Definition 2.1: Indeed, any other equivalence class one might wish to consider is just a tuple of hyperimaginaries.

DEFINITION 2.3. Let a_E and b_F be tuples of hyperimaginaries. The type $\text{tp}(a_E/b_F)$ is given by all partial types over b of the form

$$\exists yz [x E y \wedge z F b \wedge \varphi(y, z)]$$

true of a , where φ is a parameter-free formula. It is easy to see that (in the monster model) two tuples of hyperimaginaries of type E have the same type over b_F if and only if they are conjugate by an automorphism fixing b_F .

Note that the type of a hyperimaginary over b_F is just a maximal E -invariant partial real type over b invariant under automorphisms fixing b_F . For any two representatives of b_F , any two such types are equivalent. We shall say that a partial type $\pi(y)$ is a partial E -type if $\pi(y)$ is E -invariant.

DEFINITION 2.4. A set X is *hyperdefinable* over some parameters A if it is of the form Y/E , where Y is a type-definable set in countably many variables and E a countable equivalence relation on Y , both over A . We denote by X_A^{heq} the collection of all hyperimaginaries in the definable closure of A and some tuple from X . If $A = \emptyset$ it is omitted.

For the rest of the paper, all tuples and parameter sets are hyperimaginary, unless stated otherwise. We shall not distinguish between elements and tuples of elements from a set.

§3. Orthogonality.

DEFINITION 3.1. Let X, Y be A -hyperdefinable sets in some structure \mathfrak{M} . We say that X and Y are *orthogonal* over A , denoted $X \perp_A Y$, if for any tuples a from X and b from Y , the partial type $\text{tp}(a/A) \cup \text{tp}(b/A)$ determines $\text{tp}(ab/A)$. If $A = \emptyset$ it will be omitted.

REMARK 3.2. Note that we do not require X (or Y) to be *stably embedded*, i.e., that every hyperdefinable subset of X be hyperdefinable with parameters in $A \cup X$. In a stable theory, every hyperdefinable subset is stably embedded, but this need not hold in general. We shall compensate for the lack of stable embeddedness by restricting our additional parameters to $X_A^{\text{heq}} \cup Y_A^{\text{heq}}$.²

EXAMPLE 3.3. If \mathfrak{M}_1 and \mathfrak{M}_2 are two structures and $\mathfrak{N} = \mathfrak{M}_1 \times \mathfrak{M}_2$ with a predicate X for \mathfrak{M}_1 and a predicate Y for \mathfrak{M}_2 , then X and Y are orthogonal in \mathfrak{N} over \emptyset (and in fact over any set of parameters).

REMARK 3.4. If X and Y are orthogonal type-definable sets over A and $Z \subseteq X^k \times Y^\ell$ is relatively A -definable, then Z is a finite union of rectangles $A_i \times B_i$, where $A_i \subseteq X^k$ and $B_i \subseteq Y^\ell$ are relatively A -definable.

PROOF. For any $z = (x, y) \in Z$ we have that

$$\text{tp}(x/A) \cup \text{tp}(y/A) \vdash (x, y) \in Z.$$

By compactness there are relatively A -definable subsets $A_z \subseteq X^k$ in $\text{tp}(x/A)$ and $B_z \subseteq Y^\ell$ in $\text{tp}(y/A)$ with $A_z \times B_z \subseteq Z$. Again by compactness, finitely many of these rectangles suffice to cover Z . \dashv

REMARK 3.5. $X \perp_A X$ if and only if $X \subseteq \text{dcl}^{\text{heq}}(A)$.

PROOF. For any $x, x' \in X$ we have

$$\text{tp}(x/A) \cup \text{tp}(x'/A) \vdash \text{tp}(x, x'/A).$$

If $x \notin \text{dcl}^{\text{heq}}(A)$ choose $x' \equiv_A x$ with $x' \neq x$. Then $xx' \equiv_A xx$, a contradiction. \dashv

For the rest of this section, X and Y will be orthogonal \emptyset -hyperdefinable sets. We note first that orthogonality is preserved under adding parameters from $X^{\text{heq}} \cup Y^{\text{heq}}$, and interpretation:

PROPOSITION 3.6. If $X' \subseteq X^{\text{heq}}$ and $Y' \subseteq Y^{\text{heq}}$ are hyperdefinable over some parameters $A \subseteq X^{\text{heq}} \cup Y^{\text{heq}}$, then $X' \perp_A Y'$.

PROOF. Suppose $A = (a, b)$ with $a \in X^{\text{heq}}$ and $b \in Y^{\text{heq}}$, and consider tuples $a' \in X'$ and $b' \in Y'$. Choose representatives $\tilde{a}, \tilde{a}' \in X$ of a, a' and $\tilde{b}, \tilde{b}' \in Y$ of b, b' . Then $\text{tp}(\tilde{a}\tilde{a}') \cup \text{tp}(\tilde{b}\tilde{b}') \vdash \text{tp}(\tilde{a}\tilde{a}'\tilde{b}\tilde{b}')$.

Now if $a'' \equiv_A a'$ and $b'' \equiv_A b'$, we can find A -conjugates $\tilde{a}\tilde{a}''$ of $\tilde{a}\tilde{a}'$ and $\tilde{b}\tilde{b}''$ of $\tilde{b}\tilde{b}'$ such that $a''\tilde{a}\tilde{a}'' \equiv_A a'\tilde{a}\tilde{a}'$ and $b''\tilde{b}\tilde{b}'' \equiv_A b'\tilde{b}\tilde{b}'$. By orthogonality of X and Y , we obtain $\tilde{a}''\tilde{a}\tilde{b}''\tilde{b} \equiv \tilde{a}'\tilde{a}\tilde{b}'\tilde{b}$, whence $a''ab''b \equiv a'ab'b$, and thus $a''b'' \equiv_A a'b'$. \dashv

PROPOSITION 3.7. X is stably embedded in $X \cup Y$: For tuples $a \in X^{\text{heq}}$ and $b \in Y^{\text{heq}}$, every ab -hyperdefinable subset X' of X^{heq} is hyperdefinable over a .

²See Proposition 3.6. In a stable theory, $X \perp_A Y$ implies $X \perp_B Y$ for any $B \supseteq A$ (full orthogonality).

PROOF. If $\Phi(x, a, b)$ hyperdefines X' and $\Psi(y) = \text{tp}(b)$, put

$$\Phi'(x, a) = \exists y [\Psi(y) \wedge \Phi(x, a, y)].$$

Clearly $\Phi(x, a, b) \vdash \Phi'(x, a)$. Conversely, suppose $a' \models \Phi'(x, a)$, and choose $b' \models \Psi$ with $a' \models \Phi(x, a, b')$. By orthogonality $a'ab \equiv a'ab'$, whence $a' \models \Phi(x, a, b)$, and $\Phi'(x, a)$ hyperdefines X' . \dashv

We put $\text{dcl}_X^{\text{heq}}(A) = \text{dcl}^{\text{heq}}(A) \cap X^{\text{heq}}$ and $\text{bdd}_X(A) = \text{bdd}(A) \cap X^{\text{heq}}$.

COROLLARY 3.8. Suppose $a \in X^{\text{heq}}$ and $b \in Y^{\text{heq}}$. Then

$$\text{dcl}_X^{\text{heq}}(a, b) = \text{dcl}_X^{\text{heq}}(a) \quad \text{and} \quad \text{bdd}_X(a, b) = \text{bdd}_X(a).$$

PROOF. Immediate from Proposition 3.7: If X' is a singleton (resp. bounded) subset of X^{heq} hyperdefinable over ab containing some element $e \in \text{dcl}_X^{\text{heq}}(ab)$ (resp. $e \in \text{bdd}_X(ab)$), then X' is hyperdefinable already over a . \dashv

§4. Weak elimination of hyperimaginaries. In this section, X and Y will be \emptyset -hyperdefinable sets.

DEFINITION 4.1. Let Z be \emptyset -hyperdefinable. We say that Z has *weak elimination of hyperimaginaries with respect to X^{heq} and Y^{heq}* if for every $z \in Z^{\text{heq}}$ there is some $x \in \text{bdd}_X(z)$ and $y \in \text{bdd}_Y(z)$ with $z \in \text{dcl}^{\text{heq}}(xy)$.

For the rest of this section, X and Y will be orthogonal over \emptyset .

THEOREM 4.2. The set $X \cup Y$ has weak elimination of hyperimaginaries with respect to X^{heq} and Y^{heq} .

PROOF. Consider $z \in (X \cup Y)^{\text{heq}}$, say $z = (x, y)_E$ for some tuples $x \in X$, $y \in Y$ and countable equivalence relation E over \emptyset . For $x' \equiv x$ consider the hyperdefinable equivalence relation $E_{x'}$ on $\text{tp}(y)$ given by

$$y E_{x'} y' \iff (x', y) E (x', y').$$

Then $E_{x'}$ is \emptyset -hyperdefinable by Proposition 3.7, and does not depend on the choice of $x' \equiv x$. Similarly, for $y' \equiv y$ the equivalence relation

$$x E_{y'} x' \iff (x, y') E (x', y')$$

on $\text{tp}(x)$ does not depend on $y' \equiv y$. Clearly $z \in \text{dcl}^{\text{heq}}(x_{E_{y'}}, y_{E_{x'}})$.

We claim that $x_{E_{y'}}$ is bounded over z . If not, there is an indiscernible sequence $(x_i, y_i : i < \omega)$ in $\text{tp}(x, y/z)$ with $\neg x_i E_{y'} x_j$ for $i \neq j$. By orthogonality, for $i < j$,

$$\text{tp}(x_i, x_j) \cup \text{tp}(y_i, y_j) \vdash \text{tp}((x_i, y_i), (x_j, y_j)).$$

But $\text{tp}(x_i, x_j) = \text{tp}(x_i, x_k)$ for $i < k < j$, whence

$$\text{tp}((x_i, y_i), (x_j, y_j)) = \text{tp}((x_i, y_i), (x_k, y_j)).$$

Now $(x_i, y_i) E (x, y) E (x_j, y_j)$ holds since $z = (x, y)_E$. Hence

$$(x_k, y_j) E (x_i, y_i) E (x_j, y_j).$$

Thus $x_k E_{y'} x_j$, a contradiction.

Hence $x_{E_{y'}} \in \text{bdd}_X(z)$; similarly $y_{E_{x'}} \in \text{bdd}_Y(z)$. \dashv

COROLLARY 4.3. *For any set A of parameters, $\text{bdd}_{XY}(A)$ and $\text{bdd}_X(A) \cup \text{bdd}_Y(A)$ are interdefinable. Moreover, for $aA \subset (X \cup Y)^{\text{heq}}$ we have $\text{tp}(a/\text{bdd}_{XY}(A)) \vdash \text{tp}(a/\text{bdd}(A))$.*

PROOF. Clearly $\text{bdd}_X(A) \cup \text{bdd}_Y(A) \subseteq \text{bdd}_{XY}(A)$.

For the converse inclusion, let $z \in \text{bdd}_{XY}(A)$. By Theorem 4.2 there is $x \in \text{bdd}_X(z)$ and $y \in \text{bdd}_Y(z)$ with $z \in \text{dcl}^{\text{eq}}(xy)$. So

$$z \in \text{dcl}^{\text{heq}}(\text{bdd}_X(\text{bdd}(A)), \text{bdd}_Y(\text{bdd}(A))) = \text{dcl}^{\text{heq}}(\text{bdd}_X(A), \text{bdd}_Y(A)).$$

For the second assertion, let B be a set of representatives for $\text{bdd}(A)$ and F a type-definable equivalence relation such that B_F is equivalent to $\text{bdd}(A)$. Then equality of E -type over $\text{bdd}(A)$ is a bounded equivalence relation E_B type-definable over B , given by

$$x E_B y \iff \bigwedge_{\varphi \text{ a } B\text{-formula}} [(\varphi(x, B) \rightarrow \exists y'z [y E y' \wedge BFz \wedge \varphi(y', z)]) \wedge (\varphi(y, B) \rightarrow \exists x'z [x E x' \wedge BFz \wedge \varphi(x', z)])].$$

As E_B is invariant under any A -automorphism, it is in fact type-definable over A . By Remark 2.2 the class a_{E_B} is interdefinable with a tuple

$$((aA_i)_{E_i} : i \in I) \in \text{bdd}_{XY}(A),$$

where the $A_i \subseteq A$ are countable. Since the partial type $(xA_i) E_i (aA_i)$ is in $\text{tp}(a/\text{bdd}_{XY}(A))$ for all $i \in I$, we get the result. \dashv

COROLLARY 4.4. *If $X' \subset \text{bdd}(X)$ and $Y' \subset \text{bdd}(Y)$ are \emptyset -hyperdefinable, then $X' \perp_{\text{bdd}_{XY}(\emptyset)} Y'$.*

PROOF. It is clearly sufficient to show $X \perp_{\text{bdd}_{XY}(\emptyset)} Y'$. Given $x \in X$ and $y \in Y'$, consider $y_0 \in Y$ with $y \in \text{bdd}(y_0)$, and put $\bar{y} = \text{bdd}_Y(y_0)$. Now if

$$x' \equiv_{\text{bdd}_{XY}(\emptyset)} x \quad \text{and} \quad y' \equiv_{\text{bdd}_{XY}(\emptyset)} y,$$

choose \bar{y}' with $\bar{y}'y' \equiv_{\text{bdd}_{XY}(\emptyset)} \bar{y}y$. As $X \perp_{\text{bdd}_{XY}(\emptyset)} Y$ by Lemma 3.6, we have $x\bar{y} \equiv_{\text{bdd}_{XY}(\emptyset)} x'\bar{y}'$. Since

$$\text{bdd}_{XY}(\bar{y}) \in \text{dcl}^{\text{heq}}(\text{bdd}_X(\bar{y}), \text{bdd}_Y(\bar{y})) = \text{dcl}^{\text{heq}}(\text{bdd}_X(\emptyset), \bar{y})$$

by Corollary 3.8, we obtain

$$x\text{bdd}_{XY}(\bar{y}) \equiv x'\text{bdd}_{XY}(\bar{y}') \quad \text{and} \quad \text{bdd}_{XY}(\bar{y})y \equiv \text{bdd}_{XY}(\bar{y}')y'.$$

(Note that $\text{bdd}_{XY}(\emptyset)$ is part of the tuples on either side, so we do not have to work over it.) Choose x'' with $x\text{bdd}_{XY}(\bar{y})y \equiv x''\text{bdd}_{XY}(\bar{y}')y'$. Then

$$x''\text{bdd}_{XY}(\bar{y}') \equiv x\text{bdd}_{XY}(\bar{y}) \equiv x'\text{bdd}_{XY}(\bar{y}'),$$

so $\text{tp}(x''/\text{bdd}_{XY}(\bar{y}')) = \text{tp}(x'/\text{bdd}_{XY}(\bar{y}'))$. By Corollary 4.3 we obtain $\text{tp}(x''/\text{bdd}(\bar{y}')) = \text{tp}(x'/\text{bdd}(\bar{y}'))$. As $y \in \text{bdd}(\bar{y})$, we get in particular

$$x'y' \equiv x''y' \equiv xy.$$

The result follows. \dashv

EXAMPLE 4.5. We do need $\text{bdd}_{XY}(\emptyset)$ in Corollary 4.4, as we might take $X' = X \times \text{bdd}_Y(\emptyset)$. Then $X' \not\perp Y$ unless $\text{bdd}_Y(\emptyset) = \text{dcl}^{\text{heq}}(\emptyset)$.

§5. Internality and analysability.

DEFINITION 5.1. Let X and Y be hyperdefinable sets over A . We say that X is (almost) Y -internal if there is some parameter set B such that for every $a \in X$ there is a tuple $b \in Y$ with $a \in \text{dcl}^{\text{heq}}(Bb)$ (or $a \in \text{bdd}(Bb)$, respectively).³

If the parameters B can be chosen in some set Z , we say that X is (almost) Y -internal within Z .

We say that X is Y -analysable (within Z) if for all $a \in X$ there is a sequence $(a_i : i < \alpha)$ such that $\text{tp}(a_i/A, a_j : j < i)$ is Y -internal (within Z) for every $i < \alpha$, and $a \in \text{bdd}(A, a_i : i < \alpha)$.

For the rest of the section, X and Y will be hyperdefinable orthogonal sets over \emptyset .

PROPOSITION 5.2. If an \emptyset -hyperdefinable set X' is X -analysable within $X \cup Y$, then X' is almost X -internal within $\text{bdd}_Y(\emptyset)$; if X' is X -internal within $X \cup Y$, then X' is X -internal within $\text{bdd}_Y(\emptyset)$.

PROOF. We first show that if X' is (almost) X -internal within $X \cup Y$, then it is (almost) X -internal within $\text{bdd}_Y(\emptyset)$. So suppose $\bar{a} \in X$ and $\bar{b} \in Y$ are such that for every $c_E \in X'$ there is a tuple $a \in X$ with $c_E \in \text{bdd}(\bar{a}\bar{b}a)$. Let $\Phi(x, \bar{a}\bar{b}a)$ be the E -type $\text{tp}(c_E/\bar{a}\bar{b}a)$. Then for every symmetric formula $\psi(x, y) \in E$ there is $n_\psi < \omega$ such that a maximal ψ -antichain in Φ has size n_ψ , and a formula $\phi_\psi(x, \bar{a}\bar{b}a) \in \Phi$ such that every ψ -antichain in ϕ_ψ has size at most n_ψ . Consider the type-definable relation F on $\text{tp}(\bar{a}\bar{b}a)$ given by

$$(\bar{a}'\bar{b}'a') F (\bar{a}''\bar{b}''a'') \Leftrightarrow \bigwedge_{\psi \in E} [\forall x(\phi_\psi(x, \bar{a}'\bar{b}'a') \rightarrow \exists x'[\Phi(x', \bar{a}''\bar{b}''a'') \wedge x \psi^2 x']) \wedge \forall x'(\phi_\psi(x', \bar{a}''\bar{b}''a'') \rightarrow \exists x[\Phi(x, \bar{a}'\bar{b}'a') \wedge x \psi^2 x'])],$$

where $x \psi^2 x'$ means $\exists x'' [\psi(x, x'') \wedge \psi(x'', x')]$. Then $(\bar{a}'\bar{b}'a') F (\bar{a}''\bar{b}''a'')$ holds if and only if $\Phi(x, \bar{a}'\bar{b}'a')$ and $\Phi(x, \bar{a}''\bar{b}''a'')$ contain the same points modulo E : If they contain the same points modulo E , for every ψ in E let $(x_i : i < n_\psi)$ be a ψ -antichain in $\Phi(x, \bar{a}'\bar{b}'a')$, and choose $(x'_i : i < n_\psi)$ in $\Phi(x, \bar{a}''\bar{b}''a'')$ with $x_i E x'_i$ for all $i < n_\psi$. Then whenever x satisfies $\phi_\psi(x, \bar{a}'\bar{b}'a')$ there is $i < n_\psi$ with $\psi(x, x_i)$, whence $x \psi^2 x'_i$. By symmetry the converse also holds, so $(\bar{a}'\bar{b}'a') F (\bar{a}''\bar{b}''a'')$. On the other hand, if there is x such that $\Phi(x, \bar{a}'\bar{b}'a')$ but $\neg x E x'$ for all x' with $\Phi(x', \bar{a}''\bar{b}''a'')$, by compactness there is $\psi' \in E$ such that $\neg \psi'(x, x')$ for all x' with $\Phi(x', \bar{a}''\bar{b}''a'')$. Then any $\psi \in E$ with $\psi^2 \vdash \psi'$ witnesses $\neg(\bar{a}'\bar{b}'a') F (\bar{a}''\bar{b}''a'')$.

It follows that F is an equivalence relation, and any automorphism fixes $(\bar{a}\bar{b}a)_F$ if and only if it permutes the set $C \subset X'$ of E -classes in $\Phi(x, \bar{a}\bar{b}a)$. In particular $(\bar{a}\bar{b}a)_F \in \text{dcl}^{\text{heq}}(C)$ and $C \subseteq \text{bdd}((\bar{a}\bar{b}a)_F)$, whence in particular $c_E \in \text{bdd}((\bar{a}\bar{b}a)_F)$. Moreover, if X' is X -internal, then $C = \{c_E\}$ and $c_E \in \text{dcl}^{\text{heq}}(\bar{a}\bar{b}a)_F$.

By weak elimination of hyperimaginaries, there is $\tilde{a} \in \text{bdd}_X((\bar{a}\bar{b}a)_F)$ and $\tilde{b} \in \text{bdd}_Y((\bar{a}\bar{b}a)_F)$ with $(\bar{a}\bar{b}a)_F \in \text{dcl}^{\text{heq}}(\tilde{a}\tilde{b})$. Thus we are done if we can show $\tilde{b} \in \text{bdd}_Y(\emptyset)$.

³In simplicity theory, this is called *finite generation*; for *internality* we would require for every $a \in X$ the existence of some $B \downarrow_A a$ and tuple $b \in Y$ with $a \in \text{dcl}^{\text{heq}}(Bb)$.

Suppose $\tilde{b} \notin \text{bdd}_Y(\emptyset)$. Then there is an \emptyset -conjugate \tilde{b}' of \tilde{b} outside $\text{bdd}(\tilde{a}\tilde{b})$; if σ is an automorphism mapping \tilde{b}' to \tilde{b} , put $\tilde{a}'\tilde{b}' = \sigma(\tilde{a}\tilde{b})$. Then $\tilde{b} \notin \text{bdd}(\tilde{a}'\tilde{b}')$. On the other hand, since $\tilde{a}'\tilde{b}' \equiv \tilde{a}\tilde{b}$, for every $e \in C$ there is $a_e \in X$ with $e \in \text{bdd}(\tilde{a}'\tilde{b}'a_e)$. Therefore

$$\tilde{b} \in \text{bdd}_Y((\tilde{a}\tilde{b}a)_F) \subseteq \text{bdd}_Y(C) \subseteq \text{bdd}_Y(\tilde{a}', \tilde{b}', a_e : e \in C),$$

whence $\tilde{b} \in \text{bdd}_Y(\tilde{b}')$ by Corollary 3.8, a contradiction.

Now assume that $x \in X'$ and $(x_i : i < \alpha)$ is an X -analysis of x within $X \cup Y$. We show inductively on $i \leq \alpha$ that $\text{tp}(x_j : j < i)$ is X -internal within $\text{bdd}_Y(\emptyset)$. So suppose $\text{tp}(x_j : j < k)$ is X -internal within $\text{bdd}_Y(\emptyset)$ for all $k < i$. If i is limit, then clearly $\text{tp}(x_j : j < i)$ is X -internal within $\text{bdd}_Y(\emptyset)$. If $i = k + 1$, then by the result for internality $\text{tp}(x_k/x_j : j < k)$ is X -internal within $\text{bdd}_Y(x_j : j < k)$ and there is $a \in X$ with

$$x_k \in \text{dcl}^{\text{heq}}(a, \text{bdd}_Y(x_j : j < k), x_j : j < k).$$

Or, by X -internality of $\text{tp}(x_j : j < k)$ within $\text{bdd}_Y(\emptyset)$ there is $a' \in X$ with $(x_j : j < k) \in \text{dcl}^{\text{heq}}(a', \text{bdd}_Y(\emptyset))$. Then by Corollary 3.8

$$\text{bdd}_Y(x_j : j < k) \subseteq \text{bdd}_Y(a', \text{bdd}_Y(\emptyset)) = \text{bdd}_Y(\emptyset),$$

and $x_k \in \text{bdd}(a, a', \text{bdd}_Y(\emptyset))$. So $\text{tp}(x_j : j < i)$ is X -internal within $\text{bdd}_Y(\emptyset)$, and $\text{tp}(x)$ is almost X -internal within $\text{bdd}_Y(\emptyset)$. ⊣

COROLLARY 5.3. *Let X' and Y' be \emptyset -hyperdefinable. If X' is almost X -internal within $X \cup Y$ and Y' is almost Y -internal within $X \cup Y$, then $X' \perp_{\text{bdd}_{XY}(\emptyset)} Y'$.*

PROOF. Proposition 3.6 and Corollary 4.3 yield $X \perp_{\text{bdd}_{XY}(\emptyset)} Y$. By Proposition 5.2 we have

$$X' \subset \text{bdd}(X, \text{bdd}_{XY}(\emptyset)) \quad \text{and} \quad Y' \subset \text{bdd}(Y, \text{bdd}_{XY}(\emptyset)).$$

Hence $X' \perp_{\text{bdd}_{XY}(\emptyset)} Y'$ by Corollary 4.4. ⊣

COROLLARY 5.4. *If an \emptyset -hyperdefinable set Z is almost X - and almost Y -internal within $X \cup Y$, then it is bounded.*

PROOF. We have $Z \perp_{\text{bdd}_{XY}(\emptyset)} Z$ by Corollary 5.3, so Z is bounded by Remark 3.5. ⊣

COROLLARY 5.5. *If $Z \subseteq (X \cup Y)^{\text{heq}}$ is \emptyset -hyperdefinable and almost X -internal within $X \cup Y$, then it is X -internal within $\text{bdd}_Y(\emptyset)$.*

PROOF. Let $z \in Z$. By weak elimination of hyperimaginaries there is $x \in \text{bdd}_X(z)$ and $y \in \text{bdd}_Y(z)$ with $z \in \text{dcl}^{\text{heq}}(xy)$. Then $\text{tp}(y)$ is Y -internal since $y \in Y^{\text{heq}}$, but also almost X -internal, as $y \in \text{bdd}(z)$ and $\text{tp}(z)$ is almost X -internal. So $y \in \text{bdd}_Y(\emptyset)$ by Corollary 5.4. ⊣

Again Example 4.5 shows that we need $\text{bdd}_Y(\emptyset)$ in Corollaries 5.3 and 5.5.

§6. Local simplicity.

DEFINITION 6.1. Let $A \subseteq B$, and $\pi(x, B)$ be a partial type over B . We say that $\pi(x, B)$ does not divide over A if for any indiscernible sequence $(B_i : i < \omega)$ in $\text{tp}(B/A)$ the partial type

$$\bigcup_{i < \omega} \pi(x, B_i)$$

is consistent. Clearly, $\text{tp}(a/B)$ divides over A if and only if $\text{tp}(a_0/B)$ does so for some finite subtuple $a_0 \subseteq a$.

EXAMPLE 6.2. If $\text{tp}(a) \perp \text{tp}(b)$, then $\text{tp}(a/b)$ does not divide over \emptyset .

We now define the appropriate version of local rank. We follow Ben Yaacov’s terminology [1, Definition 1.4], more general than [7, Definition 4.3.5].

DEFINITION 6.3. Let $\pi(x)$, $\Phi(x, y)$ and $\Psi(y_1, \dots, y_k)$ be partial types in (at most) countably many variables.

(1) Ψ is a k -inconsistency witness for Φ if

$$\models \forall y_1 \dots y_k \neg \exists x [\Psi(y_1, \dots, y_k) \wedge \bigwedge_{i=1}^k \Phi(x, y_i)].$$

(2) Let Ψ be a k -inconsistency witness for Φ . The local (Φ, Ψ) -rank $D(\cdot, \Phi, \Psi)$ is defined on partial types in x as follows:

- $D(\pi(x), \Phi, \Psi) \geq 0$ if $\pi(x)$ is consistent.
- $D(\pi(x), \Phi, \Psi) \geq n + 1$ if there is a sequence $(a_i : i < \omega)$ such that $\models \Psi(\bar{a})$ for any k -tuple $\bar{a} \subset (a_i : i < \omega)$, and $D(\pi(x) \wedge \Phi(x, a_i), \Phi, \Psi) \geq n$ for all $i < \omega$.

If $D(\pi, \Phi, \Psi) \geq n$ for all $n < \omega$, we put $D(\pi, \Phi, \Psi) = \infty$.

An inconsistency witness is a k -inconsistency witness, for some $k < \omega$.

REMARK 6.4. Note that $D(\pi(x, a), \Phi, \Psi) \geq n$ is a closed condition on a , and $D(\text{tp}(x/a), \Phi, \Psi) \geq n$ is a closed condition on x over a . By compactness and Ramsey’s theorem, we may require $(a_i : i < \omega)$ to be indiscernible in Definition 6.3 (2).

LEMMA 6.5. Let Ψ be an inconsistency witness for Φ , and π a partial type over A . Then $D(\pi, \Phi, \Psi)$ is infinite if and only if for every linear order I there are elements $(b_i, a_i^j : i \in I, j < \omega)$ such that $\models \Psi(\bar{a})$ for all $\bar{a} \subset (a_i^j : j < \omega)$ of the right length, $b_i \models \pi \wedge \bigwedge_{k < i} \Phi(x, a_k^0)$, and $(a_i^j : j < \omega)$ is indiscernible over $A \cup \{b_k a_k^0 : k < i\}$, for all $i \in I$. Moreover, we may require $(b_i a_i^0 : i \in I)$ to be indiscernible.

PROOF. If the condition is satisfied, we can take $I = \omega$. Then for all $n \in \omega$ the partial type $\pi \wedge \bigwedge_{i \leq n} \Phi(x, a_i^0)$ is satisfied by b_n and hence non-empty. So

$$\begin{aligned} D(\pi, \Phi, \Psi) &> D(\pi \wedge \Phi(x, a_0^0), \Phi, \Psi) > D(\pi \wedge \Phi(x, a_0^0) \wedge \Phi(x, a_1^0), \Phi, \Psi) \\ &> \dots > D(\pi \wedge \Phi(x, a_0^0) \wedge \dots \wedge \Phi(x, a_n^0), \Phi, \Psi) \geq 0. \end{aligned}$$

Hence $D(\pi, \Phi, \Psi) > n$ for all $n < \omega$, and $D(\pi, \Phi, \Psi) = \infty$.

For the converse, by compactness it is sufficient to consider finite I . We show by induction that if $D(\pi, \Phi, \Psi) \geq n$, then the condition is satisfied for I of size n . For $n = 0$ there is nothing to show. Suppose $D(\pi, \Phi, \Psi) \geq n + 1$. Then by definition there is a sequence $(a_0^j : j < \omega)$ whose subsequences satisfy Ψ , and such that $D(\pi \wedge \Phi(x, a_0^j), \Phi, \Psi) \geq n$ for all $j < \omega$. By Remark 6.4 we may assume that $(a_0^j : j < \omega)$ is indiscernible over A . Choose $b_0 \models \pi \wedge \Phi(x, a_0^0)$. By inductive hypothesis for the partial type $\pi \wedge \Phi(x, a_0^0)$ over $A \cup \{b_0, a_0^0\}$, there are $(b_i, a_i^j : 1 \leq i \leq n, j < \omega)$ such that $\models \Psi(\bar{a})$ for all $\bar{a} \subset (a_i^j : j < \omega)$ of the right length,

$$b_i \models \pi \wedge \Phi(x, a_0^0) \wedge \bigwedge_{1 \leq k \leq i} \Phi(x, a_k^0),$$

and $(a_i^j : j < \omega)$ is indiscernible over $A \cup \{b_0, a_0^0\} \cup \{b_k, a_k^0 : 1 \leq k < i\}$, for all $1 \leq i \leq n$, as required.

The final assertion follows by compactness and Ramsey’s theorem. ⊢

DEFINITION 6.6. Let I be an ordered set. A sequence $I = (a_i : i \in I)$ is *independent over A* , or *A -independent*, if $\text{tp}(a_i/A, a_j : j < i)$ does not divide over A for all $i \in I$. If $A \subseteq B$ and $p \in S(B)$, the sequence $(a_i : i \in I)$ is a *Morley sequence in p over A* if it is B -indiscernible, $a_i \models p$ and $\text{tp}(a_i/B, a_j : j < i)$ does not divide over A for all $i \in I$. If $A = B$, we simply call it a *Morley sequence in p* .

FACT 6.7 ([7, Corollary 3.2.5] or [5, Proposition 16.12]). *If $\text{tp}(b/cd)$ does not divide over d and $\text{tp}(a/cbd)$ does not divide over bd , then $\text{tp}(ab/cd)$ does not divide over d .*

For the rest of the section we fix a hyperdefinable set X over \emptyset . We call a type $p(x)$ an *X -type* if it implies $x \in X$.

The following theorem generalizes [7, Theorem 2.4.7] to the local hyperdefinable context. Note that in the classical development the forking properties for hyperimaginaries are deduced from the corresponding properties for representatives. Here we cannot do this, as the ambient theory may well not be simple. So we have to work with hyperimaginaries in X throughout.

THEOREM 6.8. *The following are equivalent:*

- (1) *Symmetry holds on X : For all $a, b, c \in X$, $\text{tp}(a/bc)$ does not divide over b if and only if $\text{tp}(c/ab)$ does not divide over b .*
- (2) *Transitivity holds on X : If $a, b, c, d \in X$, then $\text{tp}(a/bcd)$ does not divide over b if and only if $\text{tp}(a/bc)$ does not divide over b and $\text{tp}(a/bcd)$ does not divide over bc .*
- (3) *Local character holds on X : There is κ such that for all countable $a \in X$ and $A \subset X$ there is $A_0 \subseteq A$ with $|A_0| \leq \kappa$ such that $\text{tp}(a/A)$ does not divide over A_0 . In fact, we can take $\kappa = 2^{|T|}$.*
- (4) *$D(., \Phi, \Psi) < \infty$ for any partial X -type $\Phi(x, y)$ and inconsistency witness Ψ for Φ .*
- (5) *For any $A \subseteq B \subset X$, a partial X -type $\pi(x, B)$ does not divide over A if and only if there is a Morley sequence I in $\text{tp}(B/A)$ such that $\{\pi(x, B') : B' \in I\}$ is consistent.*

If any of these conditions is satisfied, then for all $A \subseteq B \subset X$ and $a \in X$ the type $\text{tp}(a/B)$ does not divide over A if and only if

$$D(\text{tp}(a/B), \Phi, \Psi) = D(\text{tp}(a/A), \Phi, \Psi)$$

for all (Φ, Ψ) . Moreover, Extension holds on X : For any partial X -type $\pi(x)$ over B , if π does not divide over A then it has a completion which does not divide over A .

PROOF. (1) \Rightarrow (2) Clearly, if $\text{tp}(a/bcd)$ does not divide over b , it does not divide over bc and $\text{tp}(a/bc)$ does not divide over b . Conversely, suppose that $\text{tp}(a/bcd)$ does not divide over bc and $\text{tp}(a/bc)$ does not divide over b . By symmetry, $\text{tp}(d/abc)$ does not divide over bc and $\text{tp}(c/ab)$ does not divide over b .

By Fact 6.7 $\text{tp}(cd/ab)$ does not divide over b , so again by symmetry $\text{tp}(a/bcd)$ does not divide over b .

(2) \Rightarrow (4) Suppose there is a partial X -type Φ and an inconsistency witness Ψ for Φ such that $D(x = x, \Phi, \Psi) = \infty$. Put $I = \{\pm 1, \pm(1 + \frac{1}{n}) : n > 0\}$ and choose a sequence $(b_i, a_i^j : i \in I, j < \omega)$ as given by Lemma 6.5. Let $A^- = \{b_i a_i^0 : i < -1\}$ and $A^+ = \{b_i a_i^0 : i > 1\}$. Then $\text{tp}(b_1/A^-A^+)$ does not divide over A^- and $\text{tp}(b_1/A^-A^+a_{-1})$ does not divide over A^-A^+ , since the former is finitely satisfiable in A^- and the latter in A^+ . However, $(a_{-1}^j : j < \omega)$ witnesses that $\Phi(x, a_{-1})$, and hence $\text{tp}(b_1/A^-A^+a_{-1})$, divides over A^- , contradicting transitivity.

(4) \Rightarrow (3) Assume (4). First, we note that for $A \subseteq B \subset X$, if

$$D(\text{tp}(a/B), \Phi, \Psi) = D(\text{tp}(a/A), \Phi, \Psi)$$

for all (Φ, Ψ) , then $\text{tp}(a/B)$ does not divide over A . This is obvious, as if some A -indiscernible sequence $(B_i : i < \omega)$ in $\text{tp}(B/A)$ witnesses dividing, we can take $\Phi(x, y) = \text{tp}(a, B)$ and $\Psi = \text{tp}(B_1, B_2, \dots, B_n)$ for $n < \omega$ sufficiently large. Then Ψ is an n -inconsistency witness (clearly, we may restrict to countable B), and

$$D(\text{tp}(a/B), \Phi, \Psi) < D(\text{tp}(a/A), \Phi, \Psi).$$

Given $\text{tp}(a/A)$ it is hence enough to take $A_0 \subseteq A$ big enough such that

$$D(\text{tp}(a/A), \Phi, \Psi) = D(\text{tp}(a/A_0), \Phi, \Psi)$$

for all (Φ, Ψ) . There are only $2^{|T|}$ such pairs, so we need at most that many parameters.

(3) \Rightarrow (4) Suppose $D(x = x, \Phi, \Psi) = \infty$. Then for any cardinal κ we can find an indiscernible sequence $(b_i, a_i^j : i \leq \kappa^+, j < \omega)$ as in Lemma 6.5. Since $\Phi(x, a_i^0)$ divides over $\{a_j^0 : j < i\}$ for all $i \leq \kappa^+$, the type $\text{tp}(b_{\kappa^+}/a_i^0 : i < \kappa^+)$ divides over any subset of its domain of cardinality $\leq \kappa$.

(4) \Rightarrow (5). Assume (4). Given $a_E \in X$ and $A \subseteq B = b_E \subset X$, for any pair (Φ, Ψ) and any formula $\varphi(y, b)$ we can adjoin either

$$\exists yz [x E y \wedge z E b \wedge \varphi(y, z)] \quad \text{or} \quad \exists yz [x E y \wedge z E b \wedge \neg \varphi(y, z)]$$

and preserve $D(\cdot, \Phi, \Psi)$ -rank. By compactness we can thus complete $\text{tp}(a_E/A)$ to an E -type p over B of the same $D(\cdot, \Phi, \Psi)$ -rank. In particular, no Φ -instance in p divides over A with Ψ as inconsistency witness. Coding finitely many pairs $(\Phi_i, \Psi_i : i < n)$ in a single one, one obtains an extension p such that no Φ_i -instance Ψ_i -divides for any $i < n$; by compactness we can do this for all pairs (Φ, Ψ) simultaneously and obtain an extension which does not divide over A . Take $B = X^{\mathfrak{M}} \supset A$ for some sufficiently saturated model \mathfrak{M} . Then a sequence $(a_i : i < \omega) \subset B$ such that $a_i \models p \upharpoonright_{(A, a_j : j < i)}$ is a Morley sequence in $\text{tp}(a/A)$.

This shows in particular that if $\pi(x, B)$ does not divide over A , then there is a Morley sequence I in $\text{tp}(B/A)$ such that $\{\pi(x, B') : B' \in I\}$ is consistent.

Conversely, suppose that $\pi(x, B)$ divides over A , as witnessed by an A -indiscernible sequence $(B_i : i < \omega)$ in $\text{tp}(B/A)$ with $\bigcup_{i < \omega} \pi(x, B_i)$ inconsistent. Take any Morley sequence I in $\text{tp}(B/A)$. By [7, Corollary 2.2.8] (which is shown there for real tuples, but transfers easily to hyperimaginaries) we may assume that

$B_i \widehat{I}$ is A -conjugate to I for all $i \in I$ and that $(B_i : i < \omega)$ is indiscernible over AI . If $\bar{\pi}(x) = \bigcup_{B' \in I} \pi(x, B')$ were consistent, then $(B_i : i < \omega)$ would witness that

$$D(\bar{\pi}(x) \wedge \pi(x, B_0), \pi(x, y), \Psi) < D(\bar{\pi}(x), \pi(x, y), \Psi)$$

for some inconsistency witness Ψ . But by A -conjugacy the two ranks must be equal, a contradiction.

(5) \Rightarrow (1) Let us first show *Extension*. If $A \subseteq B \subset X$ and $\pi(x, B)$ is a partial X -type which does not divide over A , let $(B_i : i < \alpha)$ be a very long Morley sequence in $\text{tp}(B/A)$. Consider any realization $a \models \bigwedge_{i < \alpha} \pi(x, B_i)$. Since α is large, there is an infinite subset $J \subset \alpha$ such that $\text{tp}(B_i/aA)$ is constant for $i \in J$. Put $p(x) = \text{tp}(a/AB)$, a completion of π . Then $(B_i : i \in J)$ witnesses that p does not divide over A .

Now given $a, b, c \in X$ such that $\text{tp}(a/bc)$ does not divide over b , let $B = X^{\mathfrak{M}} \ni bc$ for some sufficiently saturated model \mathfrak{M} , and p an extension of $\text{tp}(a/bc)$ to B which does not divide over b . Choose a sequence $(a_i : i < \omega) \subset B$ such that $a_i \models p \upharpoonright_{(bc, a_j : j < i)}$. This is a Morley sequence in $\text{tp}(a/bc)$ over b . Then $(a_i : i < \omega)$ is a Morley sequence in $\text{tp}(a/b)$, and $a_i \models \text{tp}(a/bc)$ for all $i < \omega$. Hence $\text{tp}(c/ba)$ does not divide over b , and symmetry holds.

Finally we show that if (1)–(5) hold and $\text{tp}(a/B)$ does not divide over A for $A \subseteq B \subset X$ and $a \in X$, then $D(\text{tp}(a/A), \Phi, \Psi) \geq n$ implies $D(\text{tp}(a/B), \Phi, \Psi) \geq n$ for all (Φ, Ψ) . For $n = 0$ this is obvious. So suppose $D(\text{tp}(a/A), \Phi, \Psi) \geq n + 1$. Then there is $(d_i : i < \omega)$ indiscernible over A such that $\bar{d} \models \Psi$ for all $\bar{d} \subset (d_i : i < \omega)$ of the right length, and $D(\text{tp}(a/A) \wedge \Phi(x, d_i), \Phi, \Psi) \geq n$ for all $i < \omega$. Let q be a completion of $\text{tp}(a/A) \wedge \Phi(x, d_0)$ with $D(q, \Phi, \Psi) \geq n$. Clearly, we may assume $a \models q$, and that $\text{tp}(d_0/aB)$ does not divide over aA . As $\text{tp}(a/B)$ does not divide over A , by symmetry and transitivity $\text{tp}(ad_0/B)$ does not divide over A , and $\text{tp}(a/d_0B)$ does not divide over d_0A . By induction hypothesis,

$$D(\text{tp}(a/d_0A), \Phi, \Psi) \geq n \text{ implies } D(\text{tp}(a/d_0B), \Phi, \Psi) \geq n.$$

As $\text{tp}(d_0/B)$ does not divide over A and $(d_i : i < \omega)$ is A -indiscernible, we may assume that it remains indiscernible over B . But then it witnesses

$$D(\text{tp}(a/B), \Phi, \Psi) \geq D(\text{tp}(a/d_0B), \Phi, \Psi) + 1 \geq n + 1. \quad \dashv$$

DEFINITION 6.9. An A -hyperdefinable set X is *simple (over A)* if it satisfies any of the conditions of Theorem 6.8 when we adjoin A to the language. If X is simple over A and $a, b, c \in X$, we shall say that a and c are independent over Ab , written $a \perp_{Ab} c$, if $\text{tp}(a/Abc)$ does not divide over Ab .

Note that we only allow tuples and parameters from $A \cup X$. If X is stably embedded, we can of course allow parameters from anywhere. It is immediate from the definition that if X is simple over A and $B \subset X$, then X is simple over AB .

REMARK 6.10. If X is merely hyperdefinable, it may be simple although no definable or even type-definable imaginary set in the ambient structure is simple.

If X is simple, it is now standard to extend the notions of dividing and independence to hyperimaginaries in X_A^{heq} . Moreover, we can develop basic simplicity

theory (canonical bases, the independence theorem, stratified ranks, generic types, stabilizers, see [5, 7]) within X_A^{heq} , replacing models \mathfrak{M} by subsets $X_A^{heq} \cap \mathfrak{M}^{heq}$.

PROPOSITION 6.11. *Let X and Y be orthogonal \emptyset -hyperdefinable sets such that X is simple over \emptyset . If $A \subset Y$ is a set of parameters, then X is simple over A , and over $\text{bdd}_Y(A)$. In particular, let Z be a set hyperdefinable over some parameters $A \subset X \cup Y$. If Z is X -internal within $X \cup Y$, then Z is simple over A ; if $Z \subseteq (X \cup Y)^{heq}$ is almost X -internal, then Z is simple as well.*

PROOF. Simplicity over A is obvious from orthogonality; simplicity over $\text{bdd}_Y(A)$ follows. Now if Z is X -internal within $X \cup Y$, then $Z \subseteq X_{\text{bdd}_Y(A)}^{heq}$ by Proposition 5.2, and must be simple as well; if $Z \subseteq (X \cup Y)^{heq}$ is almost X -internal within $X \cup Y$, it is X -internal within $\text{bdd}_Y(A)$ by Corollary 5.5. \dashv

§7. A hyperdefinable version of Schlichting’s Theorem. Recall that Schlichting’s Theorem [6], generalized by Bergman and Lenstra [3], states that if \mathfrak{H} is a family of uniformly commensurable subgroups of a group G , then there is a subgroup N commensurable with all groups in \mathfrak{H} (in fact a finite extension of a finite intersections of groups in \mathfrak{H}) which is invariant under all automorphisms of G which fix \mathfrak{H} setwise. Here two subgroups H and K are commensurable if their intersection has finite index in both H and in K ; uniformly commensurable means that there is a finite bound on these indices as H and K vary inside \mathfrak{H} .

We shall call two hyperdefinable subgroups G and H *commensurable* if the index of their intersection in both G and in H is bounded, i.e., less than the cardinality κ of the monster model. If G is a hyperdefinable group, a *hyperdefinable family of subgroups* is a family $\mathfrak{H} = \{H_a : a \models \pi\}$ for some partial types $\pi(y)$ and $\Phi(x, y)$ such that $H_a = \{x \in G : \models \Phi(x, a)\}$ is a subgroup of G for any $a \models \pi$. Note that a hyperdefinable family of commensurable subgroups is automatically uniformly commensurable by compactness, i.e., the index of the intersection $H \cap H^*$ in H (and by symmetry in H^*) is bounded independently of the choice of $H, H^* \in \mathfrak{H}$.

For hyperdefinable families of commensurable groups in a simple theory a version of Schlichting’s Theorem has been shown in [7, Theorem 4.5.13], generalizing a result of Hrushovski for theories of finite and definable S1-rank. Here we shall show it for hyperdefinable families of commensurable subgroups in any theory.

THEOREM 7.1. *Let G be a hyperdefinable group, \mathfrak{H} a hyperdefinable family of commensurable subgroups, and Γ a hyperdefinable group of automorphisms of G stabilizing \mathfrak{H} setwise. Then there is a Γ -invariant hyperdefinable subgroup N commensurable with any group in \mathfrak{H} ; moreover N is invariant under any model-theoretic automorphism stabilising \mathfrak{H} .*

PROOF. Suppose $\mathfrak{H} = \{\Phi(x, a) : a \models \pi\}$; clearly we may assume that Φ is closed under finite conjunctions. We put $H_a = \{g \in G : \models \Phi(g, a)\}$. As \mathfrak{H} is Γ -invariant, we have $\gamma H \in \mathfrak{H}$ for any $H \in \mathfrak{H}$.

Enumerate $\Phi = \{\phi_i : i < \alpha\}$ for some ordinal α ; for $i < \alpha$ put

$$\psi_i(x, x', y, \zeta) = \exists z [z \in \zeta^{-1}(x^{-1}x') \wedge \neg\phi_i(z, y)].$$

(Here ζ is a variable for elements from Γ , acting on elements from G . Thus $\zeta^{-1}(x^{-1}x')$ is an element of G depending on x, x' and ζ ; as it is hyperimaginary, it corresponds to a class of real tuples, and we demand that z be one of them.) Clearly ψ_i is (equivalent to) a partial type. Consider a hyperdefinable subgroup K of G . A complete $\psi_i(x, x', a, \gamma)$ -graph of size n in K is a set of elements $\{h_j : j < n\}$ of K such that for any $j \neq j'$ one has $\models \psi_i(h_j, h_{j'}, a, \gamma)$. The existence of such a graph implies in particular that the index of $K \cap \gamma H_a$ in K is at least n .

As \mathfrak{H} is a hyperdefinable family of commensurable subgroups, by compactness for every $i < \alpha$ there is an integer n_i such that for any $H \in \mathfrak{H}$ and $(\gamma, a) \models \Gamma \times \pi$ there is no complete $\psi_i(x, x', a, \gamma)$ -graph in H of size n_i . If K is a bounded intersection of groups in \mathfrak{H} , we put $i(K, a, \gamma) = (k_i : i < \alpha)$, where $k_i \leq n_i$ is the size of a maximal complete $\psi_i(x, x', a, \gamma)$ -graph in K , and call this the *index* of (a, γ) in K . We order the set of indices lexicographically.

Clearly, for $K_0 \leq K_1$ we have $i(K_0, a, \gamma) \leq i(K_1, a, \gamma)$; by compactness equality holds if and only if $K_0 \gamma H_a = K_1 \gamma H_a$: if $i(K_0, a, \gamma) = i(K_1, a, \gamma)$, then for any $i < \alpha$ let $(g_j^i : j < k_i)$ be a maximal complete $\psi_i(x, x', a, \gamma)$ -graph in K_0 . By equality of the index, this is also a maximal complete graph in K_1 . Then for any $g \in K_1$ and $i < \alpha$ there is some $g_j^i \in K_0$ with $\models \phi_i(\gamma^{-1}(g^{-1}g_j^i), a)$. By compactness, as Φ is closed under finite conjunctions, there is $g' \in K_0$ such that $\models \phi_i(\gamma^{-1}(g^{-1}g'), a)$ for all $i < \alpha$, that is $g^{-1}g' \in \gamma H_a$ and $g \in K_0 \gamma H_a$.

By compactness, for every bounded intersection K of groups in \mathfrak{H} there is some maximal index i such that for some $(\gamma, a) \models \Gamma \times \pi$ we have $i = i(K, a, \gamma)$; call this the index $i(K)$ of K . Since for $\gamma' \in \Gamma$ and $(\gamma, a) \models \Gamma \times \pi$ we have

$$i(\gamma'K, a, \gamma'\gamma) = i(K, a, \gamma),$$

we obtain $i(K) = i(\gamma K)$. As the set of indices is bounded and $i(K_0) \leq i(K_1)$ for $K_0 \leq K_1$, there is some bounded intersection K of groups in \mathfrak{H} such that $i(K)$ is minimal possible, say i_0 . We shall call K *strong* if $i(K) = i_0$.

If K is strong, we put

$$\mathfrak{H}(K) = \{H \in \mathfrak{H} : \exists (\gamma, a) \models \Gamma \times \pi [H = \gamma H_a \wedge i(K, a, \gamma) = i_0]\}.$$

Then $\gamma'K$ is also strong for any $\gamma' \in \Gamma$, and

$$\mathfrak{H}(\gamma'K) = \{\gamma'\gamma H_a : \gamma H_a \in \mathfrak{H}(K)\} = \gamma'(\mathfrak{H}(K)).$$

Now for $H \in \mathfrak{H}(K)$ the set $\bigcap_{g \in K} (KH)^g$ is a subgroup of G containing K ; it is hyperdefinable, as we only have to conjugate by a set of representatives of K/H , which is bounded. Then

$$N(K) = \bigcap_{H \in \mathfrak{H}(K)} \bigcap_{g \in K} (KH)^g$$

is a subgroup of G containing K ; it is hyperdefinable as it contains K and must have bounded index in $\bigcap_{g \in K} (KH)^g$ for any $H \in \mathfrak{H}(K)$, so is a bounded intersection.

If K_1 is strong and $K_0 \leq K_1$, then K_0 is again strong and $\mathfrak{H}(K_0) \subseteq \mathfrak{H}(K_1)$. Moreover $K_0H = K_1H$ for any $H \in \mathfrak{H}(K_0)$, whence

$$\bigcap_{g \in K_0} (K_0H)^g = \bigcap_{g \in K_1} (K_1H)^g,$$

and

$$K_1 \leq N(K_1) \leq N(K_0) \leq K_0H = K_1H.$$

It follows that there is an absolute bound on the index $|N(K) : K_1|$, independent of the choice of strong K . As for a bounded family $(K_i : i \in I)$ of strong subgroups the intersection $\bigcap_{i \in I} K_i$ is again strong, there is some strong K such that $N = N(K)$ is maximal possible. Then N is hyperdefinable, commensurable with all groups in \mathfrak{H} , and invariant under Γ and all model-theoretic automorphisms stabilizing \mathfrak{H} setwise. \dashv

COROLLARY 7.2. *Let G be a hyperdefinable group, and H a subgroup commensurable with all its G -conjugates. Then there is a normal hyperdefinable subgroup N commensurable with H .*

PROOF. We apply Theorem 7.1 to the family \mathfrak{H} of G -conjugates of H , with the action of $\Gamma = G$ by conjugation. \dashv

§8. Groups interpretable in orthogonal sets. Recall that two hyperdefinable subgroups H_1 and H_2 of some group G are *commensurable* if $H_1 \cap H_2$ has bounded index both in H_1 and in H_2 .

THEOREM 8.1. *Suppose X and Y are orthogonal \emptyset -hyperdefinable sets in a structure \mathfrak{M} , and G is an \emptyset -hyperdefinable group in $(X \cup Y)^{heq}$. If X is simple over \emptyset , there is an \emptyset -hyperdefinable normal X -internal subgroup N of G such that the quotient G/N is Y -internal. N is unique up to commensurability.*

PROOF. Let us first show uniqueness: If N' is a second such group, then $N/(N \cap N')$ and $N'/(N \cap N')$ are X -internal and Y -internal, and hence bounded by orthogonality of X and Y . Thus N and N' are commensurable.

By Theorem 4.2 every element $g \in G$ is of the form $(g_X, g_Y)_E$ for some $g_X \in X^{heq}$ and $g_Y \in Y^{heq}$, both bounded over g , and some type-definable equivalence relation E with bounded classes, depending on $\text{tp}(g)$. Hence $\text{tp}(g/g_Y)$ is X -internal and $\text{tp}(g/g_X)$ is Y -internal. Now if $h = (h_X, h_Y)_E$ and $gh = ((gh)_X, (gh)_Y)_E$, then

$$(gh)_X \in \text{bdd}_X(g_X, g_Y, h_X, h_Y),$$

whence $(gh)_X \in \text{bdd}_X(g_X, h_X)$ by Corollary 3.8. Similarly $(gh)_Y \in \text{bdd}_Y(g_Y, h_Y)$.

Now X is simple, as is $\text{tp}(g, h/g_Y, h_Y)$ for any $g, h \in G$ by Proposition 6.11. Hence we can consider $g, h \in G$ such that $g \perp_{g_Y, h_Y} h$. Then for any stratified local rank D

$$\begin{aligned} D(gh/(gh)_Y) &\geq D(gh/(gh)_Y, g_Y, h_Y, g) = D(h/g_Y, h_Y, g) \\ &= D(h/g_Y, h_Y) = D(h/h_Y), \end{aligned} \tag{†}$$

where the first equality holds as $(gh)_Y \in \text{bdd}_Y(g_Y, h_Y)$, and the last equality follows from $h \perp_{h_Y} g_Y$ by orthogonality (Example 6.2). Similarly

$$D(gh/(gh)_Y) \geq D(g/g_Y).$$

Now suppose G is a subset of $(X^m \times Y^n)/E$, where m, n are at most countable. Then if $g = (\bar{x}_g, \bar{y}_g)_E \in G$, we have $g \perp_{g_Y} \bar{y}_g$ by X -internality of $\text{tp}(g/g_Y)$ and orthogonality, and $g_Y \in \text{bdd}(\bar{y}_g)$, whence $D(g/g_Y) = D(g/\bar{y}_g)$. By compactness, there is a G -type $p((\bar{x}, \bar{y})_E)$ implying that $D((\bar{x}, \bar{y})_E/\bar{y})$ is maximal for all local

stratified ranks. But if $g, h \models p$ with $g \downarrow_{g_Y, h_Y} h$, then we must have equality in (\dagger) . Therefore g, h and gh are pairwise independent over $g_Y, h_Y, (gh)_Y$. Put $A = (g_Y, h_Y, (gh)_Y)$. Then $X' = \text{tp}(g/A)$ is X -internal and simple, as is $X'X'^{-1}$. We may therefore define

$$S_0 = \{g \in G : \exists x (x \equiv_A^{lstp} gx \equiv_A^{lstp} a \wedge x \downarrow_A g \wedge gx \downarrow_A g)\} \subseteq X'X'^{-1}$$

and the stabilizer $S = \text{stab}(g/A) = S_0^2$, an X -internal hyperdefinable subgroup of G .

Now [4, Lemme 1.2] (see [4, Remarque 1.3] for the extension from the stable to the simple context) states that whenever g, h and gh are pairwise independent over A , then g is generic in the coset Sg , and this coset is hyperdefinable over $\text{bdd}(A)$.

By orthogonality, $g \downarrow_{g_Y} h_Y, (gh)_Y$. This implies in particular

$$D(S) = D(Sg) = D(g/g_Y, h_Y, (gh)_Y) = D(g/g_Y) = D(p).$$

Suppose that S is not commensurable with S^h for some $h \in G$. Then SS^h is still X -internal, with

$$D(SS^h) \geq D(S) = D(p)$$

for every stratified local rank D , and for at least one such rank D_0 we have $D_0(SS^h) > D_0(p)$. Choose $g' \in SS^h$ with $D_0(g'/h) = D_0(SS^h)$. By pre-multiplying with a generic element of S and post-multiplying with a generic element of S^h , the inequality (\dagger) implies that we may assume $D(g'/h) \geq D(p)$ for every stratified local rank D . However, $\text{tp}(g'/h)$ is X -internal, so $g'_Y \in \text{bdd}_Y(g')$ implies $g'_Y \in \text{bdd}_Y(h)$ by Lemma 3.8 and Proposition 5.2. Thus

$$D(g'/g'_Y) \geq D(g'/h) \geq D(p) \quad \text{and} \quad D_0(g'/g'_Y) \geq D_0(g'/h) > D_0(p),$$

contradicting our choice of p . Hence S is commensurable with all its conjugates. By Corollary 7.2 there is a hyperdefinable normal subgroup N of G commensurable with S . So N is X -internal, and $D(N) = D(S) = D(p)$ for all stratified local ranks D . Now the same proof, with NZ instead of SS^h , shows that if Z is an X -internal hyperdefinable subset of G , then Z is covered by boundedly many cosets of N . In particular, for any $g' \in G$ the type $\text{tp}(g'/g'_Y)$ is covered by boundedly many cosets of N . But then $g'N \in \text{bdd}(g'_Y)$, and G/N is almost Y -internal, whence Y -internal by Corollary 5.5. \dashv

COROLLARY 8.2. *Suppose X and Y are orthogonal type-definable sets over \emptyset in a structure \mathfrak{M} , and G is a type-interpretable group over \emptyset in $(X \cup Y)^{eq}$. If X is simple over \emptyset , there is a normal X -internal subgroup N of G type-interpretable over \emptyset , such that the quotient G/N is Y -internal. N is unique up to commensurability. If X is definable and supersimple, then we can take N relatively interpretable.*

PROOF. The first part is obvious from Theorem 8.1, as a hyperdefinable subgroup of a type-interpretable group is again type-interpretable.

If X is definable and supersimple, N must be contained in a definable X -internal set \bar{X} by [7, Lemma 3.4.17]; note that \bar{X} will also be supersimple. So N is the intersection of definable supergroups by [7, Theorem 5.5.4], one of which, say N_0 , must be contained in \bar{X} by compactness. Then N_0 is X -internal. As above, N_0 must be commensurable with all its G -conjugates; moreover, commensurability is uniform

by compactness (or [7, Lemma 4.2.6]). By [7, Theorem 4.2.4] there is a relatively interpretable normal subgroup \bar{N} commensurable with N_0 . So \bar{N} is X -internal, and G/\bar{N} is Y -internal. \dashv

QUESTION 8.3. *If X and Y are orthogonal type-definable sets (or even definable sets), X is simple and G is a relatively definable group in $(X \cup Y)^{eq}$, can we find a relatively definable normal X -internal subgroup N such that G/N is Y -internal?*

QUESTION 8.4. *What can we say if neither X nor Y is simple? Is it true that in every hyperdefinable subgroup of $((X \cup Y)^{heq})$ there is a maximal normal hyperdefinable X -internal subgroup N_X , a maximal normal hyperdefinable Y -internal subgroup N_Y , an X -internal hyperdefinable local group G_X , a Y -internal hyperdefinable local group G_Y and a hyperdefinable locally bounded equivalence relation E on $G_X \times G_Y$ such that $G/(N_X N_Y)$ is isogenous, or even isomorphic, to $(G_X \times G_Y)/E$?*

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