

ON INTEGRATED CHANCE CONSTRAINTS IN ALM FOR PENSION FUNDS

BY

YOUSOUF A. F. TOUKOUROU AND FRANÇOIS DUFRESNE

ABSTRACT

We discuss the role of *integrated chance constraints* (ICC) as quantitative risk constraints in asset and liability management (ALM) for pension funds. We define two types of ICC: the *one period* integrated chance constraint (OICC) and the *multiperiod* integrated chance constraint (MICC). As their names suggest, the OICC covers only one period, whereas several periods are taken into account with the MICC. A multistage stochastic linear programming model is therefore developed for this purpose and a special mention is paid to the modeling of the MICC. Based on a numerical example, we first analyze the effects of the OICC and the MICC on the optimal decisions (asset allocation and contribution rate) of a pension fund. By definition, the MICC is more restrictive and safer compared to the OICC. Second, we quantify this MICC safety increase. The results show that although the optimal decisions from the OICC and the MICC differ, the total costs are very close, showing that the MICC is definitely a better approach since it is more prudent.

KEYWORDS

Pension funds, modeling, asset liability management, multistage stochastic programming, linear programming, integrated chance constraint.

1. INTRODUCTION

A pension fund is any plan, fund or scheme, established by a company, governmental institution or labor union, which provides retirement incomes. The actuarial present value of current and future payments constitutes the total liability of the fund. The pension fund receives contributions from its active members and/or the employer. This money (considered as the total wealth or total asset) is invested in a wide range of assets. The asset allocation is made in such a way that it guarantees, to a certain extent, the payments of future obligations. That is not so trivial: assets yield random returns and future benefits are not known

with certainty. An asset liability management study (ALM) provides a rich theoretical background to address that issue. Its goal is to determine the adequate asset allocations and contribution rates in order to guarantee the payment of current and future pensions.

The use of ALM methods has a long tradition in pension funds. At the beginning, it has started with *deterministic methods*, whose goal is to determine the adequate allocation that protects the fund from any unexpected movements in cash flows and/or interest rate. The future cash flows are estimated and assumed to be certain; the wealth is mainly allocated to bonds considered as risk free and the assets are deterministic. Bonds are chosen in such a way that their related incomes correspond to yearly pension payments. The models are essentially based on immunization and cash-flow matching; see, for example, Koopmans (1942) and Redington (1952). The drawbacks of different deterministic methods are examined in Hiller and Schaack (1990). Two key points should be emphasized here. First, it is difficult (if not impossible) to find bonds that enables a perfect match for a realistic pension fund problem. Second, these approaches could be costly since the yields on bonds are low.

A possible way to reduce the cost of the pension fund is to invest in riskier assets. The *mean-variance* portfolio of Markowitz (1952) provides a good compromise between the high yield and the pension funds level of risk. In this respect, the liability is evaluated deterministically and the decision maker has to determine the lowest risk portfolios in order to meet the estimated liability. This approach is known as the *asset only* method. The integration of the liability in the traditional mean-variance problem has led to the *surplus optimization* theory. The surplus is generally defined as the total asset minus the total liability. As interest rate and future payments are random, the total liability is a random variable influenced by many factors. The factors can be actuarial or economic. The economic risks are often related to inflation, salary growth and discount rates. As these risks also have an effect on the total amount of asset, it is important to include the random liability as part of the ALM model and not only as the target wealth for the pension fund portfolio optimization. More specifically, the correlations between asset and liability are thus considered in the determination of the optimal portfolio. For example, the assets, with higher covariance to liability, tend to reduce the risk exposition of the pension fund, see Sharpe and Tint (1990) and Keel and Müller (1995). Leibowitz and Henriksson (1988) showed that an asset such as cash, which should typically reduce the riskiness of an all-asset portfolio, may actually increase the riskiness of a portfolio that includes liability. Including liability in the optimal asset allocation decision has a long story in finance theory. We refer the readers to Sharpe and Tint (1990), Elton and Gruber (1992), Leibowitz *et al.* (1992) among others. The multiperiod ALM case has been discussed in Fama (1970), Hakansson (1970), Hakansson (1974) and Merton (1969).

The pension fund problem is a long-term problem with a horizon span of approximately 30 years. Hence, its model should be dynamic. Furthermore, regulations often impose many types of constraints. Those matters are hardly taken into account by the surplus optimization methods. In practice, *Monte-Carlo*

methods are commonly used due to their ability to incorporate the above issues. Initially, they consisted on defining a set of feasible allocations and contribution rates, and choosing the best one in some sense. The choice is based on the simulation of the future paths. Due to the technical innovations, these methods have significantly evolved with the work of Wilkie (1995) and Ahlgrim *et al.* (2005) concerning economic scenario generation. Møller and Steffensen (2007) provide different tools for valuing the pension fund liabilities. Recent years have also seen the emergence of methods known as *stochastic programming* (SP).

Based on a scenario approach, SP gives a flexible and powerful tool for ALMs. Its importance lies in its ability to easily incorporate various types of constraints (Zenios 2006). Moreover, assets and liabilities are all influenced by many sources of risk and the risk aversion is accommodated; the framework has a long time horizon split into subperiods (*multistage*); the portfolio can be rebalanced dynamically at the beginning of each subperiod; all these are incorporated in a single and consistent structure while satisfying operational or regulatory restrictions and policy requirements. SP for ALM is rooted with the work of Kusy and Ziemba (1986) who showed, based on a five-year period application to the Vancouver City Saving Credit Union, that SLP is theoretically and operationally superior to a corresponding deterministic linear programming (LP) model. The authors have proved that the effort required for the implementation of ALM and its computational requirements are comparable to those of the deterministic model. For a Japanese insurance company, Carino *et al.* (1994) developed a model that enables the decision makers to include risk management tools as well as including the complex regulations imposed by the Japanese insurance laws and practices. Over the two years of experiment, the resulting investment strategy has been fruitful as it has yielded extra income of 42 basis points (US\$79 million). More recently, Geyer and Ziemba (2008)—for the Austrian pension fund of the electronics firm Siemens—has implemented a model that allows specific features such as state-dependent correlation matrices and fat-tailed asset return distributions. Considering a Finnish pension company, Hilli *et al.* (2007) focus on the modeling of the stochastic factors and analyzes the obtained numerical solution. Dert (1995) pioneered the inclusion of *chance constraints* (CC) in multistage recourse models for pension funds. Chance constrained programs often lead to integer programming for which, it may be difficult to determine a tractable solution. As an alternative to CC, Hanveld *et al.* (2010) proposed the *integrated chance constraints* (ICC), whose feasibility set is more handleable as it does not require integer programming. The literature of SP in ALM also includes Consigli and Dempster (1998), Bogentoft *et al.* (2001), Drijver (2005), Faleh (2011), Aro and Pennanen (2013) among others.

The ALM model in this paper is a MSP, for which, we minimize the total funding cost under risk, legal, budget, regulatory and operating constraints. The total funding cost is composed of regular and remedial contributions. Regular contribution constitutes a certain proportion (contribution rate) of the total salary whereas remedial contribution is an additional financial support provided by the employer (or a sponsor) whenever the solvency target is in

question. More specifically, we focus on the risk constraints, which are of ICC type in this work. The ICC is computationally of great interest; in particular when a quantitative risk measure is preferable. We define the *funding ratio* as the ratio of total asset over total liability. Our goal is to meet a certain funding ratio, called here *target funding ratio*, at the end of each subperiod. For a predefined target funding ratio, the ICC put an upper bound on the expected shortfall, i.e. the expected amount by which the goal is not attained. Haneveld *et al.* (2010) and Drijver (2005) pioneered the application of ICC in ALM for pension fund. However, the risk parameter considered in their models is neither scale free, nor time dependent. Our model is close to Haneveld *et al.* (2010) with the particularity that the risk parameter is a linear function of total liability. Then, it becomes invariant with respect to the size of the fund as well as time dependent. We define two types of ICC: the *one period* integrated chance constraint (OICC) and the *multiperiod* integrated chance constraint (MICC). As their names suggest, the OICC covers only one period whereas several periods are taken into account with the MICC. A multistage stochastic linear program (SLP) is therefore developed for this purpose and a special mention is paid to the modeling of the MICC.

The rest of the paper is organized as follows. In Section 2, the theoretical background, the dynamics and the ALM optimization problem are extensively detailed. In Section 3, OICC and MICC are introduced and their SLP reformulations are derived as well. In Section 4, a numerical example is examined from the perspective of a defined benefit fund that invests in stocks, real estate, bonds, deposits and cash. All numerical results are implemented using the solver CPLEX in the mathematical programming language AMPL. We first analyze the effect of the risk parameter on the optimal decisions. This section finishes by a brief comparison of the two ICCs.

2. SETTINGS

2.1. Multistage recourse models

In this section, we describe the classical architecture of the multiperiod decision framework. The model's setup presented here resembles mostly to Haneveld *et al.* (2010).

Since we aim for strategic decisions, we model the ALM process over a number of years and one set of decisions is taken each year. We discretize time accordingly so that the model has a (finite) number of one-year time periods. Consequently, we assume that the ALM model has a horizon of T years from now, split in T subperiods of one year each. The resulting years are denoted by an index t , where time $t = 0$ is the current time. By year t ($t = 1, \dots, T$), we mean the span of time $[t - 1, t)$. We define

$$\mathcal{T}_t := \{t, t + 1, \dots, T\}.$$

We assume that uncertain parameters (e.g. asset returns) can be modeled as random variables with known distributions. At each time $t \in \mathcal{T}_0$, the pension fund is allowed to make decisions (corresponding to yearly corrections), based on the actual knowledge of parameters. During each one-year period, a realization of the corresponding random parameters becomes known (e.g. assets return during that year). That is, the concept underlying our model is the following sequence of decisions and observations:

$$\begin{array}{ccccccccc} \text{decide} & & \text{observe} & & \text{decide} & & & & \text{observe} & & \text{decide} & & \text{observe} \\ X_0 & \rightsquigarrow & \omega_1 & \rightsquigarrow & X_1 & \rightsquigarrow & \dots & \rightsquigarrow & \omega_{T-1} & \rightsquigarrow & X_{T-1} & \rightsquigarrow & \omega_T \end{array}$$

where X_t is the vector of decision variables at time $t \in \mathcal{T}_0$, and vector $\omega_t, t \in \mathcal{T}_1$ models all economic events which are the source of uncertainty and risk for the pension fund management, which, in our case, are asset returns as well as random contributions and liability streams. Time t is assumed to be the end of the financial year t . We assume that a financial year coincides with a calendar year. At time $t \in \mathcal{T}_0$, decisions X_t are taken with full knowledge of the past $[0, t]$ but with only probabilistic information about the future $(t, T]$.

Uncertainty in the model is expressed through a finite number S of sample paths spanning from $t = 0$ until $t = T$ called scenarios. That is, we assume the random variable follows a discrete distribution with S possible outcomes. Each scenario represents a sequence of possible realizations of all uncertain parameters in the model. As explained above, ω_t is the stochastic vector process whose values are revealed in year t . Then, the set of all scenarios is the set of all realizations $\omega^s := (\omega_1^s, \dots, \omega_T^s), s \in \mathcal{S} := \{1, \dots, S\}$ of $\omega := (\omega_1, \dots, \omega_T)$. Scenario s has a probability p^s , where $p^s > 0$ and $\sum_{s=1}^S p^s = 1$. It represents a description of possible future, starting just after $t = 0$. If we assume that we can observe the “state of the world” at time $t, (0 < t < T)$, then there is a unique history of realizations of $(\omega_1, \dots, \omega_{t-1})$ leading to that state, but the future as seen from time t may unfold in several ways. That is, there are several distinct scenarios which share a common history up to time t . A suitable representation of the set of scenarios is given by a scenario tree (see Figure 1). In respect to Figure 1, we define the node as the possible outcome of the stochastic vector ω_t at a given time $t \in \mathcal{T}_0$. Each path of ω_t from $t = 0$ to $t = 3$ represents one scenario; each node of the scenario tree has multiple successors, in order to model the process of information being revealed progressively through time. By convention, the scenarios are numbered top-down by their end node. The arcs in the tree denote realizations in one time period. We assume here that, for a specific decision time $t \in \mathcal{T}_0$, the numbers of realizations in one time period descending from the current nodes are identical.

For example in Figure 1, we have a 3-year horizon scenario tree with 40 scenarios. Over the first period starting from time 0 to time 1, there are five possible realizations. From each of these realizations, we have four possible outcomes over the second year; each of them is a conditional realization as it depends on

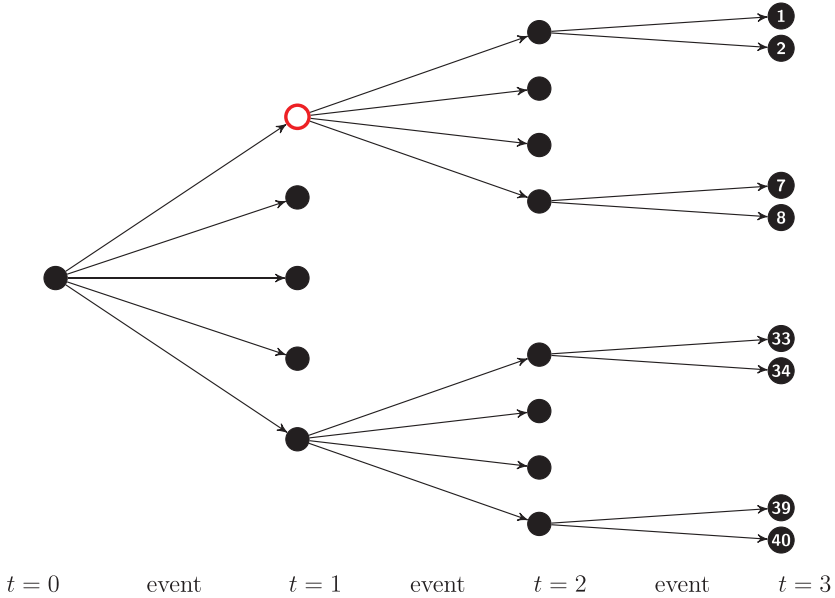


FIGURE 1: A scenario tree with 40 scenarios and 66 nodes. (Color online)

the preceding node. Over the third period, each of the second period observations can lead to two possible outcomes. All this gives a branching structure of $1 - 5 - 4 - 2$ and leading to a total of $S = 5 \times 4 \times 2 = 40$ possible scenarios.

A multistage recourse model is an optimization problem defined on such a scenario tree. Considering the remaining future represented by the subtree rooted at (t, s) , optimal decisions are taken for each node (t, s) of the event tree, given the information available at that point. Optimality is defined in terms of current costs plus expected future costs, which are computed with respect to the appropriate conditional distributions, van der Vlerk *et al.* (2003).

Ideally, one would like to make different decisions for every path at every $t \in \mathcal{T}_0$, but this would lead to undesirable anticipativity in the model. The simplest way to avoid this is to make one single decision at each time t for all paths by adding explicit constraints. That is, for any two different scenarios s_1 and s_2 ($s_1, s_2 \in \mathcal{S}$ and $s_1 \neq s_2$) having the same history up to time $t \in \mathcal{T}_0$, we enforce $X_t^{s_1} = X_t^{s_2}$, where X_t^s is the decision X_t under scenario s . For example, at the empty circle of Figure 1, $X_1^1 = X_1^2 = \dots = X_1^8$.

2.2. Dynamics

2.2.1. *Assets.* In this paper, we are considering a buy and hold model applied to a DB plan in which one seeks to minimize the expected cost of funding. In this respect, dynamics for both assets and liabilities should clearly be specified.

At initial time $t = 0$, the exact levels of wealth and liability are available to the decision maker who has to decide, each period, how to rearrange his portfolio

in order to cover liabilities and, at the same time, to achieve high returns on the financial market. The higher the returns are, the lower the contribution rate could be. Let denote A_t the total amount of wealth at time $t \in \mathcal{T}_0$. The total wealth is allocated into d classes of assets and in cash. Let $k \in \mathcal{K} := \{1, \dots, d\}$ denote the asset class index. At each decision time $t \in \mathcal{T}_0$, a specified amount of $H_{k,t}$ is allocated to asset k and C_t is the cash amount. We can write

$$A_t = \sum_{k=1}^d H_{k,t} + C_t.$$

Through buying and selling, the investor restructures his portfolio at each time t . Once the t^{th} stage decision is made, the holdings $H_{k,t}$ can be calculated. The shares in the portfolio are then kept constant till the next decision time. The value of $H_{k,t}$ is affected by the returns on the market. Let define $\xi_{k,t} := 1 + r_{k,t}$ where $r_{k,t}$ is the random rate of return on asset class k over year t .

Over year t , the pension fund pays benefits to its non-active members and receives contributions from its active members or/and the employer (also called the sponsor). Benefits regroup pensions which are paid to retirees, disability and death annuities or lump sum, whereas contributions are composed of yearly payments from all the active members and/or sponsor to the plan. When it appears that the plan is unfunded according to its solvency target, the sponsor may finance the deficit. As in van der Vlerk *et al.* (2003), we name this funding here as remedial contribution. We then assume that whenever the solvency target is not fulfilled, a remedial contribution in cash can be obtained from the sponsor. In practice, it does not really work that way. For example in van der Vlerk *et al.* (2003), the remedial is only provided after two consecutive periods of underfunding. We will see in the model description that the parameters are set such that the remedial contribution variable is non-zero only under some conditions. In general for DB plans, future benefits and liabilities depend on company policy regulation and can be estimated whereas yearly contribution is defined as a certain proportion of the yearly salary. Asset allocation and contribution rate are defined with respect to the level of future benefits and liabilities (e.g. Switzerland). Gottlieb and Whiston (2005) provides a rich source of information concerning different types of pension plans and features. During year t , let Ben_t and W_t denote, respectively, the total amount of benefits paid and the level of salary. The variable cr_t is the decided contribution rate for year $t + 1$. For returns and cash-flow variables, index t means that payments occur over year t but cash-flows are accounted at the end of year. Accordingly, the total asset dynamic is modeled as

$$A_t = \sum_{k=1}^d H_{k,t-1} \xi_{k,t} + C_{t-1} (1 + r_f) + cr_{t-1} W_t - Ben_t + Z_t = \sum_{k=1}^d H_{k,t} + C_t, \quad (1)$$

for $t \in \mathcal{T}_1$, where r_f is the risk free interest rate and Z_t is the remedial contribution at time t . Before receiving the remedial contribution at time t , the

total wealth is defined as A_t^* and one can write

$$A_t^* = \sum_{k=1}^d H_{k,t-1} \xi_{k,t} + C_{t-1} (1 + r_f) + cr_{t-1} W_t - \text{Ben}_t = A_t - Z_t. \tag{2}$$

Asset returns $(\xi_t)_{t=1}^T := (\xi_{1,t} \cdots \xi_{k,t} \cdots \xi_{d,t})_{t=1}^T$, pension payments Ben_t and salary W_t are modeled as stochastic processes on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t=1}^T, \mathbb{P})$. Obviously, at each decision time, A_t is a random variable whose distribution depends, on a first hand, on ξ_t , W_t and Ben_t , and on the second hand, on asset allocations before t . At a specific date t , the variable A_t is known as it can be observed. The initial wealth is defined by \bar{A}_0 and known at initial time $t = 0$. According to (1), total wealth A_{t-1} at time $t - 1$ is allocated into the d classes of assets and cash. Each asset k generates a return $\xi_{k,t}$ over period $[t - 1, t]$. The initial wealth plus accumulated interest at the end of period will be augmented by the balance of external flows: contributions minus pension payments. This latter can be either positive or negative depending on the difference between contributions and benefits paid. A negative balance could be due to the fact that a company is no more hiring new employees. This may happen for various reasons: runoff, economic difficulties, etc. Obviously, the total contributions $cr_{t-1} W_t$ will decrease considerably as the total salaries decrease whereas Ben_t will tend to increase as people leave the fund. When at time t the total asset cannot fulfill the pension fund solvency target, it may obtain a remedial contribution Z_t .

In order to be as close as possible to realities on the financial market, one has to consider costs of trading activities. Therefore, we include proportional transaction costs $\bar{c}^B := (\bar{c}_1^B \dots \bar{c}_k^B \dots \bar{c}_d^B)$ and $\bar{c}^S := (\bar{c}_1^S \dots \bar{c}_k^S \dots \bar{c}_d^S)$ for purchases and sales, respectively. Inclusion of transaction costs will lead to some changes in asset dynamics. Thus, (1) is then replaced by

$$A_T = \sum_{k=1}^d H_{T-1,k} \xi_{T,k} + C_{T-1} (1 + r_f) + cr_{T-1} W_T - \text{Ben}_T = A_T^* \tag{3}$$

over period $[T - 1, T]$ and when $t \in \mathcal{T}_1 \setminus \{T\}$,

$$\begin{aligned} A_t &= \sum_{k=1}^d H_{k,t-1} \xi_{k,t} + C_{t-1} (1 + r_f) + cr_{t-1} W_t - \text{Ben}_t + Z_t \\ &\quad - \sum_{k=1}^d (\bar{c}_k^B B_{k,t} + \bar{c}_k^S S_{k,t}) \\ &= \xi_t H_{t-1} + C_{t-1} (1 + r_f) + cr_{t-1} W_t - \text{Ben}_t + Z_t - (\bar{c}^B B_t + \bar{c}^S S_t) \\ &= A_t^* + Z_t - (\bar{c}^B B_t + \bar{c}^S S_t) \\ &= \mathbf{e} \cdot H_t + C_t \end{aligned} \tag{4}$$

where $\mathbf{e} := 1 \ 1 \ \dots \ 1$ is a $(1 \times d)$ vector. Vectors $H_t := (H_{1,t} \ \dots \ H_{k,t} \ \dots \ H_{d,t})^\top$, $B_t := (B_{1,t} \ \dots \ B_{k,t} \ \dots \ B_{d,t})^\top$ and $S_t := (S_{1,t} \ \dots \ S_{k,t} \ \dots \ S_{d,t})^\top$ of dimension $(d \times 1)$ each, are, respectively, amount of asset hold, bought and sold at each decision time $t \in \mathcal{T}_0$. In fact, (4) is obtained by subtracting transaction costs in the first equality of (1). At time T , no more asset is bought or sold: $B_T = S_T = 0$; the value of the portfolio is determined by adding all values of assets including the last period returns and external flows. This justifies why there is no transaction cost in (3). The reader should notice that variables cr_t, Z_t, B_t, S_t and H_t are all decision variables. We denote by \bar{H}_k , the initial holding in asset k , $k \in \mathcal{K}$ and $\bar{H} := (\bar{H}_1 \ \dots \ \bar{H}_k \ \dots \ \bar{H}_d)^\top$ is a $d \times 1$ vector. \bar{C}_0 is the initial cash amount. The first stage asset allocation is determined by

$$H_0 = \bar{H} + B_0 - S_0$$

with total asset

$$A_0 = \mathbf{e} \cdot \bar{H} + \bar{C}_0 + Z_0 - (\bar{c}^B B_0 + \bar{c}^S S_0) = \bar{A}_0 + Z_0 - (\bar{c}^B B_0 + \bar{c}^S S_0) = \mathbf{e} \cdot H_0 + C_0.$$

For $t \geq 1$,

$$H_t = \xi_t H_{t-1} + B_t - S_t$$

defines the dynamic of holding assets between two consecutive decision times. For any given (k, t) , whenever $S_{k,t} > 0$, $B_{k,t} = 0$ and vice-versa. Transaction costs also influence the cash dynamics. Buying an amount x_k of asset k requires $x_k (1 + \bar{c}_k^B)$ of cash and selling the same amount of asset k results in $x_k (1 - \bar{c}_k^S)$ of cash. Initially,

$$C_0 = \bar{C}_0 + Z_0 - (\mathbf{e} + \bar{c}^B) B_0 + (\mathbf{e} - \bar{c}^S) S_0$$

and for $t \geq 1$,

$$C_t = C_{t-1} (1 + r_f) + cr_{t-1} W_t - \text{Ben}_t + Z_t - (\mathbf{e} + \bar{c}_k^B) B_t + (\mathbf{e} - \bar{c}_k^S) S_t$$

where we assume that $cr_{t-1} W_t, \text{Ben}_t$ and Z_t come in cash.

2.2.2. Liability and external flows. As we consider a DB plan, total liabilities is the discounted expected value of future pre-defined payments. At a given time t , it represents the amount the fund has to reimburse if it has to close at that time. Its value has to be estimated with appropriate rules taking into account actuarial risks, pension fund provisions, and other relevant factors for the employer’s line of business. Let L_t denote the total amount of liabilities at time t .

All quantitative models considered in this paper will be applied to the planning problem of a large and stable pension fund. We can then assume that the fund keeps the same structure and number of members over the study period. Liability, contributions and benefits are therefore invariant with respect to actuarial risk over the period under study. Actuarial risks regroup the random events

that affect the number of members into the fund. However, those variables are yearly indexed with the general increase of wages w_t . For $t \in \mathcal{T}_1$, we have:

$$L_t = L_{t-1} (1 + w_t); \quad W_t = W_{t-1} (1 + w_t) \quad \text{and} \quad \text{Ben}_t = \text{Ben}_{t-1} (1 + \kappa w_t) \quad (5)$$

and their initial values L_0 , W_0 and Ben_0 known at $t = 0$; $\kappa \geq 0$ is a model parameter. In practice, the pension payments Ben_t are often indexed with a certain rate which is a function of the inflation rate. In order to reduce the complexity of our model, we assume that this indexation rate is a certain proportion of the salary increase as this latter is highly positively correlated to the inflation. From the above definitions, uncertainty, represented by vector $((1 + w_t), \xi_t)_{t=1}^T$, affects both assets and liabilities. As often in the literature (e.g. Kouwenberg (2001)), we use a vector autoregressive model (VAR model) such that:

$$h_t = c + \Omega h_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, \Sigma),$$

$$h_t := (\ln(1 + w_t) \ln(\xi_{1t}) \cdots \ln(\xi_{kt}) \cdots \ln(\xi_{dt}))^\top, \quad (6)$$

$$t \in \mathcal{T}_1$$

where h_t is a $\{(d + 1) \times 1\}$ vector of continuously compounded rate, c the $\{(d + 1) \times 1\}$ vector of coefficients, Ω the $\{(d + 1) \times (d + 1)\}$ matrix of coefficients, ϵ_t the $\{(d + 1) \times 1\}$ vector of error term and Σ the $\{(d + 1) \times (d + 1)\}$ covariance matrix. The parameter estimation of this model requires time series analysis. For example in Kouwenberg (2001), annual observations of the total asset returns and the general wage increase from 1956 to 1994 are used to estimate the coefficients of the VAR model. The resulting estimates will serve in constructing the scenario tree which constitutes the workhorse of multistage stochastic programs.

2.3. The ALM problem

The total cost of funding is the sum of regular ($\sum cr_{t-1} W_t$) and remedial ($\sum Z_t$) contributions over the studied period. In this study, we are looking for the investment strategy H_t , contribution rate cr_t and remedial contribution Z_t for which the total expected cost of funding is minimized. The optimization is made under risk, legal, budget, regulatory and operating constraints. The constraints and objective of the ALM study will be presented in this section.

We denote by symbol $\mathbb{E}_t(x)$ the conditional expectation of random variable x with respect to the natural filtration \mathcal{F}_t whereas $\mathbb{P}\{E\}$ denotes the occurrence probability of event E . At each decision time t , the optimization problem consists in minimizing the total expected costs under the constraints considered in the following subsections. To simplify the notation, we omit the scenario index s .

2.3.1. *Risk constraints.* The pension fund wants to guarantee the participants a certain amount of pension. But the members also depend on the pension fund

to actually provide for their needs in the future. Therefore, the safety of the portfolio is of paramount concern. This safety is translated in risk constraints.

A pension fund has long term obligations, up to decades, and therefore, its planning horizon is large, too. The main goal of an ALM is to find acceptable allocations which guarantee the solvency of the fund during the planning horizon. In general, solvency is measured by the funding ratio F_t (also called cover ratio) that we define for a given time t by

$$F_t := \frac{A_t^*}{L_t}.$$

Underfunding occurs when the funding ratio is less than one. The assertion $F_t \leq 1$ is equivalent to saying that the surplus at time t , i.e. $A_t^* - L_t$, is negative. When this occurs, the shortfall could be provided by the fund’s sponsor or any other external contribution. That is the remedial contribution as in Haneveld *et al.* (2010). Depending on how the random vector $\omega_t := ((1 + w_t), \xi_t)$, $t \in \mathcal{T}_1$, behaves, F_t may change over time. Therefore, the pension fund rebalances its assets portfolio and redefines its contribution rate in order to control the funding ratio. The higher F_t is, the healthier the fund is. However, the decision maker would like to avoid as much as possible the changes in contribution rates. We will see in the model description that the parameters can be set in order to limit those variations.

The long term objective of the pension fund consists in fulfilling both long and short (one year) term constraints. We define two types of funding ratio risk constraints in this paper. Their goal is to constrain the funding ratio to be larger, on average, than a predefined minimum γ , $\gamma \geq 0$. Namely, the expected shortfall $\mathbb{E}_{h-1} (A_h^* - \gamma L_h)^-$, $h > t$, is required to be less than a certain amount β_t known at time t . Here, $(a)^- := \max\{-a, 0\}$ is the negative part of $a \in \mathbb{R}$. Also in order to simplify understanding, the expression expected shortfall is used to name $\mathbb{E}_{h-1} (A_h^* - \gamma L_h)^-$. That is slightly different from its definition in actuarial science where γ has to be equal to one. The one period risk constraint (OICC¹) is expressed by

$$\mathbb{E}_t (A_{t+1}^* - \gamma L_{t+1})^- \leq \beta_t, \quad t \in \mathcal{T}_0 \setminus \{T\}. \tag{7}$$

and for the multiperiod (MICC) approach,

$$\mathbb{E}_{h-1} (A_h^* - \gamma L_h)^- \leq \beta_t, \quad h \in \mathcal{T}_{t+1} \text{ and } t \in \mathcal{T}_0 \setminus \{T\} \tag{8}$$

where β_t and γ are parameters defined by the pension fund. According to the short term approach (7), at each decision time t , the expected shortfall over the following period should be smaller than a certain amount β_t . Notice that, at time t , the short term risk only controls the expected shortfall of $(A_{t+1}^* - \gamma L_{t+1})^-$ over the following one-year period.

When we want to control the expected shortfall over the whole remaining period up to maturity, the risk constraint (8) is a good measure of long term

risk (multiperiod). That is, at time t , Equation (8) means that the one period expected shortfall $\mathbb{E}_{h-1} (A_h - \gamma L_h)^-$ should be smaller than β_t at any future node with $h \in \mathcal{T}_{t+1}$. Equation (8) can be rewritten as

$$\max_{h \in \mathcal{T}_{t+1}} \mathbb{E}_{h-1} (A_h^* - \gamma L_h)^- \leq \beta_t, \quad t \in \mathcal{T}_0 \setminus \{T\} \quad (9)$$

meaning at time t , that, the highest one year expected shortfall, over the remaining periods to maturity, has to be smaller than the amount β_t . Parameter β_t is set by the pension fund at time t as a function of the total assets and liabilities at that time; e.g. $\beta_t := f(A_t, L_t) = \alpha A_t$, $0 \leq \alpha \leq 1$. Readers should notice that when $T = 1$, (7) is equivalent to (8).

In SP, constraints such as (7) and (8), i.e. bounding an expected shortfall, are named ICC. They were proposed by Haneveld (1986) as a quantitative alternative for *chance constraints* (CC). In Section 3, both ICC and CC will be discussed more specifically. Successful applications of the ICC in ALM for pension fund can be found in Drijver (2005) and Haneveld *et al.* (2010). The authors assumed that $\beta_t := \beta$ is unchanged over the studying period and in their numerical illustrations, the ICC is only applied to the first stage. In such case, one can prove that the OICC and the MICC are equivalent. Instead, we remove that assumption in our work. Therefore, we define:

$$\beta_t := \alpha L_t \quad (10)$$

where α , $0 \leq \alpha \leq 1$, denotes a scale free risk parameter. Until now, we are not aware of any implementation of the MICC (8) in a multistage framework. The feasible set of the MICC formulation is contained in the feasible set of the OICC formulation. Obviously, the two constraints cannot be implemented in the same model at the same time. There comes the other particularity of this paper: we analyze the multiperiod risk constraint and then measure how conservative it is comparing to the one-period approach.

2.3.2. Other constraints. Risk constraints are important, but institutional and legal rules regarding pension fund operations in general are also relevant. As stated in Pflug *et al.* (1998), institutional and legal rules are designed to restrict the risk of losses which would adversely affect pensioners. Hence, the following restrictions are integrated to the model.

First, the fund is not allowed to sell an asset that is not owned. This is the *not short selling assets* constraints and can be expressed by

$$\begin{aligned} H_{k,t} &\geq 0, \\ B_{k,t} &\geq 0, \\ S_{k,t} &\geq 0 \text{ for } k \in \mathcal{K} \text{ and } t \in \mathcal{T}_0. \end{aligned}$$

The no short selling constraint goes with the *not borrowing cash* constraint expressed by

$$C_t \geq 0, \quad t \in \mathcal{T}_0.$$

Second, at any time, the fund should dispose a minimum amount in cash in order to pay eventual claims such as death benefits or pensions. This can be called *liquidity constraint* and is formulated in our model as

$$C_t (1 + r_f) + \mathbb{E}_t (cr_t W_{t+1} - Ben_{t+1}) \geq 0$$

which means that, on average, the cash allocation C_t at time t should be sufficient to cover the eventual negative value of the cash flow balance over period $[t, t + 1]$. Notice that the term *liquidity constraint* used here may have a different meaning in another context, e.g. Fonseca *et al.* (2007) in a macroeconomic framework.

Third, the fund is subject to *portfolio constraints* imposed by the legislator in order to keep a minimum control on its risk exposure. It consists on bounding the holding in asset k by setting upper and lower bounds, u_k and l_k , respectively, on $H_{k,t}$. That is

$$l_k A_t \leq H_{k,t} \leq u_k A_t, \quad k \in \mathcal{K}, \quad t \in \mathcal{T}_0 \setminus \{T\}. \tag{11}$$

For example in Switzerland,² the amount allocated to stocks should not exceed 50% of total wealth. In such case, $l_{\text{stocks}} = 0$ and $u_{\text{stocks}} = 0.5 A_t$ and constraint (11) is equivalent to

$$0 \leq H_{\text{stocks},t} \leq 0.5 A_t, \quad t \in \mathcal{T}_0 \setminus \{T\}.$$

These bounds are also applicable to cash C_t , and we obtain

$$l_c A_t \leq C_t \leq u_c A_t, \quad t \in \mathcal{T}_0 \setminus \{T\}. \tag{12}$$

Notice that, in Equations (11) and (12), upper and lower bounds can also be time dependent.

Finally in an asset allocation problem, dynamics and budget constraints, already defined in Section 2.2 are unavoidable. If they were left out, the optimization program would be unbounded. The constraints presented in this subsection are common in any ALM SP implementation; see e.g. Kusy and Ziemba (1986), Carino *et al.* (1994), Consigli and Dempster (1998), Bogentoft *et al.* (2001) and Dert (1995) among others.

2.3.3. The optimization problem. As we are considering a DB fund, it is natural to assume that the aim of the fund is to minimize its costs while controlling the risk. Thus, the ALM model is a dynamic decision making optimization tool to minimize the total expected cost under risk and operating constraints. Decisions are taken at the beginning of each one-year period. Accordingly, the ALM model is developed as a multiperiod decision problem, for which, we are asked

to come up with an optimal asset allocation, contribution rate and remedial contribution at the beginning of each year. Moreover, penalty costs are assigned to the undesirable events: remedial contributions, and yearly absolute variation of contribution rates. All these components together constitute the objective function:

$$\min_{H, cr, Z} \mathbb{E}_0 \left[\sum_{t=0}^{T-1} v_{t+1} (cr_t W_{t+1} + \lambda_z Z_{t+1}) + \sum_{t=0}^{T-2} v_{t+1} \lambda_{\Delta_{cr}} \Delta_{cr_t} W_{t+1} \right] \quad (13)$$

where $\Delta_{cr_t} := | cr_{t+1} - cr_t |$ is the absolute variation of contribution rate from year t to $t+1$, v_t is the discount factor for a cash flow in year t , λ_z and $\lambda_{\Delta_{cr}}$ are, respectively, penalty parameters for remedial contribution and absolute variation of contribution rate. The variables cr_t and Δ_{cr_t} are bounded:

$$cr^l \leq cr_t \leq cr^u \quad \text{and} \quad \underline{\Delta_{cr}} \leq \Delta_{cr_t} \leq \bar{\Delta_{cr}}, \quad t \in \mathcal{T}_0 \setminus \{T\}$$

where cr^l , $\underline{\Delta_{cr}}$ are the lower bounds and cr^u , $\bar{\Delta_{cr}}$ the upper bounds of cr_t , Δ_{cr_t} , respectively. The optimal decisions have to lead to a funding ratio greater than a certain minimum \bar{F} (sometimes called target funding ratio) at the end of period of study T :

$$F_T = \frac{A_T}{L_T} \geq \bar{F}.$$

The entire ALM model, with objective and constraints, can be found in Appendix 1. An optimization program such as (13) is often referred to as a *here and now* problem. Uncertainty, characterized by $\omega_t = ((1 + w_t), \xi_t)$, $t \in \mathcal{T}_1$, is approached by scenarios. Therefore, we define $\tilde{\omega}$ with a finite number S of possible realizations $\tilde{\omega}^s := (\tilde{\omega}_1^s, \dots, \tilde{\omega}_T^s)$, $s \in \mathcal{S} := \{1, \dots, S\}$, from $t = 0$ to $t = T$ with relative probability p^s .

The objective (13) is obviously linear as it can be rewritten as a linear combination of decision variables. We can also notice that dynamics and constraints (except risk constraints that have to be rewritten in a linear form for the stochastic program) presented in Sections 2.2 and 2.3 are all linear in decision variables. If the risk constraints OICC (7) and MICC (8) were written in a linear form, the ALM problem would be a SLP, theoretically solvable by any SLP software depending on its size. In the next section, we will show how they can be turned into linear. The books Kall and Mayer (2011), Shapiro *et al.* (2009) and Birge and Louveaux (2011) provide good resources to deal with such problems. When the size is big, resolution may require heuristic methods. Size is big means that number of asset classes is large or/and time horizon is long or/and number of scenarios is large. Decisions variables are H_t, B_t, S_t, C_t, cr_t and Z_t for $t \in \mathcal{T}_0$; but only first stage values H_0, B_0, S_0, C_0, cr_0 and Z_0 are crucial to the decision maker, since, almost surely, a true realization of the random data will be different from the set of generated scenarios.

By definition, the pension fund risk problem is often a shortfall problem. In such models, the relevant measure of risk for the firm is the expected amount

(if any) by which goals are not met, Carino *et al.* (1994). The model considered in this paper has a general DB ALM structure such as explored in Haneveld *et al.* (2010) and Ziemba and Mulvey (1998). Its main particularity consists in the integration of ICC by the way of OICC (7) and MICC (8). A successful implementation of constraints (7) in ALM for a DB fund can be found in van der Vlerk *et al.* (2003) and Haneveld *et al.* (2010). In their works, the optimization problem is solved assuming that the parameter β_t is constant: $\beta_t = \beta$. Furthermore, remedial contribution is provided only when funding ratio falls short in two consecutive years. Implementing this latter condition has led to the use of binary variables. The authors proposed a heuristic solution to the problem.

As we will explain in Section 3, the parameter β is not scale free. A certain value of β does not have equivalent meaning for two different pension funds. It can be too low for a certain fund, whereas too high for another one. In addition, the pension fund actual situation should be taken into account. Our paper is an extension of Haneveld *et al.* (2010). As a novelty, we assume that the risk parameter β_t vary with respect to time t and is defined as a proportion α of the actual level of liability at time t , see equality (10). Roughly speaking, on average, the total asset should cover a proportion of magnitude $(1 - \alpha)$ of liability at any time. In our model, remedial contribution can however be provided at anytime where solvency is in question, avoiding the use of binary variables, and indirectly, the need of heuristics. Penalty parameter λ_z punishes the abuse of remedial contributions.

The main features of this study turns around the following points:

- As in Haneveld *et al.* (2010) where optimal decision is analyzed for different values of their risk parameter β , we first measure the effect of our risk parameter α on the decisions H_0 , cr_0 and Z_0 ; this with respect to the OICC. In addition, for a fixed value α , the influence of the initial funding ratio is also explored.
- Second, as a safer alternative to the OICC, we propose the MICC (constraint (8)), and we then measure how constraining it is, compared to the OICC. In constraint (8), index h is a decision time index, and we are not aware of any implementation of such constraint in ALM. The OICC considered in the first item is actually extended to a multiperiod risk constraint, reinforcing the long-term aspect of the pension fund's ALM.

In the rest of this paper, OICC (resp. MICC) will stand for the one period (resp. multiperiod) ICC itself as well as the ALM model with the OICC (resp. MICC).

3. INTEGRATED CHANCE CONSTRAINT

3.1. Definition

The most important constraints, of course, deal with the goal of the pension fund: in all circumstances, the funding ratio should remain high enough for

avoiding financial distress. This latter is expressed in terms of shortfall constraints, which are of ICC type in this paper.

For the sake of clarity, we define the generic linear function $G : \mathbb{R}^d \times \Xi \rightarrow \mathbb{R}^m$ such that

$$G(X, \omega) := BX - D$$

where $X \in \mathcal{X}$ is an d -vector of decision variables, $\mathcal{X} \subset \mathbb{R}^d$ is a polyhedral and closed set and $\omega := (B, D) : \Omega \rightarrow \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^m$ is a random parameter on the probability space (Ω, \mathcal{F}, P) . The support of ω is defined as the smallest closed set $\Xi \subset \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^m$ having the property $P(\omega \in \Xi) = 1$. For $i \in \mathcal{I} := \{1, \dots, m\}$, the vector B is of dimension $\mathbb{R}^m \times \mathbb{R}^d$ such as $B := (B_1 \cdots B_i \cdots B_m)^\top$ with $B_i \in \mathbb{R}^d$, whereas $D := (D_1 \cdots D_i \cdots D_m)^\top$ with $D_i \in \mathbb{R}$. As supposed in our SP model, we assume that $\omega = (B, D)$ has a finite number S of possible realizations $\omega^s = (B^s, D^s)$, $s \in \mathcal{S} = \{1, \dots, S\}$ with respective probability p^s . The ICC is given by

$$\mathbb{E}(BX - D)^- = \sum_{s=1}^S p^s (B^s X - D^s)^- \leq \beta, \tag{14}$$

and has been introduced by Haneveld (1986) as an alternative to chance constraints (Charnes *et al.* (1958)). However, Haneveld *et al.* (2010), van der Vlerk *et al.* (2003) and Drijver (2005) have pioneered its application to ALM for pension funds and since then, it has been implemented in practice.

With the ICCs, the feasible set is a polyhedron (convex) as it contains only continuous variables. Usually, the ICCs can be implemented easily using an appropriate software and are very attractive from an algorithm point of view. Haneveld and Van Der Vlerk (2002) propose a faster algorithm for big size problems.

Without loss of generality, we assume $m > 1$. Therefore, Equation (14) can be rewritten as

$$\mathbb{E}\{(B_i X - D_i)^-, i \in I\} \leq \beta.$$

When index i is a decision stage index with conditional expectation at stage i , we obtain a multistage program and variable X becomes stage dependent (X_i). That is, at stage $j \in I \setminus \{m\}$:

$$\mathbb{E}_i (B_{i+1} X_i - D_{i+1})^- \leq \beta_j, \quad i \in \{j, j + 1, \dots, m - 1\} \tag{15}$$

which is equivalent to the MICC (8) for $I = \mathcal{T}_0$ and $B_{h+1} X_h - D_{h+1} = A_{h+1} - \gamma L_{h+1}$. At time t , that is

$$\mathbb{E}_h (A_{h+1}^* - \gamma L_{h+1})^- \leq \beta_t, \quad h \in \mathcal{T}_t \setminus \{T\}. \tag{16}$$

The parameter β_t is then set at time t and will remain applicable until T . As decision is taken at each stage, the MICC inequality (8) shows a collection of inequality (16) going from $t = 0$ to $t = T - 1$. Similarly, when $m = 1$, one can prove that Equation (14) leads to the OICC (7).

3.2. OICC and MICC: Scenario tree interpretation

Section 2.1 briefly explains our scenario tree model. We recall that the node (t, s) corresponds to a certain scenario s at decision time t . To avoid anticipativity, we have to consider that many pairs (t, s) might correspond to the same node on the scenario tree picture. For example in Figure 1, the nodes $(1, 1), (1, 2), \dots, (1, 8)$ correspond graphically to the empty red circle. At each node (t, s) , the fund’s manager has to rebalance the asset portfolio and fix the contribution rate. These decisions are taken considering the actual scenario and possible future paths as well as the risk constraints.

3.2.1. *OICC.* In principle, considering a certain node (t, s) , the OICC constraint (7) would be implemented as follows:

$$\mathbb{E}_{t,s} (A_{t+1}^{*s} - \gamma L_{t+1}^s)^- := \sum_{s' \in \mathcal{S}} p_{t,s}^{s'} (A_{t+1}^{*s'} - \gamma L_{t+1}^{s'})^- \leq \alpha L_t^s \tag{17}$$

where $p_{t,s}^{s'}$ stands for the conditional probability to reach node $(t + 1, s')$ going from (t, s) and $p_{t,s}^{s'} = 0$ for any scenario s' of $t + 1$ not descending from (t, s) . As in van der Vlerk *et al.* (2003), we include the linear inequality (17) in every subproblem (t, s) , $t < T$ of our multistage recourse model. At (t, s) , they reflect the short-term risk constraint, stating that the expected funding shortfall over the following period $(t + 1)$ is at most αL_t^s . In other words, on average, the pension fund should be able to cover the proportion $(1 - \alpha)$ of its total liability. The increase of α will relax the feasibility set of the optimization problem.

3.2.2. *MICC.* Considering the node (t, s) , the MICC constraint can be formulated in the following way:

$$\mathbb{E}_{h-1,s} (A_h^{*s} - \gamma L_h^s)^- \leq \alpha L_t^s, \quad h \in \mathcal{T}_{t+1} \text{ and } t \in \mathcal{T}_0 \setminus \{T\} \tag{18}$$

with

$$\mathbb{E}_{h-1,s} (A_h^{*s} - \gamma L_h^s)^- = \sum_{s' \in \mathcal{S}} p_{h-1,s}^{s'} (A_h^{*s'} - \gamma L_h^{s'})^- .$$

Under (18), at each node (t, s) , decisions are taken such that the descending nodes’ one-period expected shortfall are smaller than αL_t^s (defined at current node). Such constraint permits to have a certain control of the cover ratio over the whole remaining periods: $[t + 1, T]$, whereas (17) only covers one period: $[t, t + 1]$. For example, at initial time $t = 0$, the minimum cost is defined such that the expected shortfall at any node in the tree (as descendant of the initial node) is smaller than $\beta_0 = \alpha L_0^s$ as in Haneveld *et al.* (2010):

$$\sum_{s' \in \mathcal{S}} p_{t,s}^{s'} (A_{t+1}^{*s'} - \gamma L_{t+1}^{s'})^- \leq \beta_0, \quad t \in \mathcal{T}_0 \setminus \{T\}, s \in \mathcal{S} .$$

Furthermore, at each node (t, s) , $t \in \mathcal{T}_1 \setminus \{T\}$, $s \in \mathcal{S}$, we add the restriction:

$$\sum_{s' \in \mathcal{S}} p_{t,s}^{s'} \left(A_{t+1}^{*s'} - \gamma L_{t+1}^{s'} \right)^- \leq \alpha L_t^s.$$

That is how we implement (18) at initial node. If we repeat the same procedure at each node of the tree, we can then propose a simpler SP reformulation:

Proposition 3.1. *Constraint (18) is equivalent to the following statement:*

At each node (t, s) , $t < T$, $s \in \mathcal{S}$

$$\sum_{s' \in \mathcal{S}} p_{t,s}^{s'} \left(A_{t+1}^{*s'} - \gamma L_{t+1}^{s'} \right)^- \leq \min_{0 \leq t' \leq t} \alpha L_{t'}^s. \tag{19}$$

That is, at a given node (t, s) , $t < T$, $s \in \mathcal{S}$, the expected shortfall over the next period should be less or equal to the smallest value of $\alpha L_{t'}^s$ calculated over the preceding nodes (t', s) , $t' \leq t$. This is based on the fact that, in the multiperiod framework, decision taken at node (t, s) is influenced by the history of ω_t^s up to time t , in particular β_t^s at preceding nodes. Inequality (19) is linear and describes a polyhedral set. The proof of Proposition 3.1 is straightforward when we go backward in time starting from nodes $(T - 1, s)$, see Appendix 2 for an example based sketch of proof. At each node (t, s) , as we know the history of β_t^s up to time t , one can determine the smallest $\beta_{t'}^s$, $t' \leq t$. Therefore, implementation of MICC consists in including the linear constraint (19) at each node (t, s) .

4. NUMERICAL ILLUSTRATIONS

This section contains computational results for the SP model. Let us recall that we are dealing with a DB pension fund whose objective is to minimize the total expected costs under constraints. The study will focus on risk constraints that are of ICC type. First, based on the OICC, the effects of risk parameter and cover ratio on the optimal decisions are analyzed. Prima facie, the MICC appears to be a safer and more restrictive than OICC. Based on the same analysis as before, the cost of conservativeness is subsequently measured.

For this study, consider a hypothetical large pension fund that may invest into $d = 4$ classes of asset ordered by level of risk:

1. Deposits.
2. Bonds.
3. Real estate.
4. Stocks.

We are aware of the fact that the number of assets is often much larger in practice. That said, only four classes of assets are considered here in order to reduce the complexity of the model. After investing in these asset classes, the rest is

TABLE 1
DATA ON THE ASSET CLASSES.

Asset classes	k	l_k	u_k	\bar{c}	Initial investments
Cash	—	0	1	0	4,950
Deposits	1	0	0.5	0.00150	16,500
Bonds	2	0.1	1	0.00150	38,500
Real estate	3	0	0.30	0.00425	17,600
Stocks	4	0	0.50	0.00425	32,450

TABLE 2
VALUES OF THE OTHER DETERMINISTIC PARAMETERS.

$\lambda_Z = 350$	$\lambda_{\Delta_{cr}} = 1$	$r_f = 0.008$	$v_t = (1 + r_f)^{-t}$
$\Delta_{cr} = -0.08$	$cr^l = -0.08$	$\bar{\Delta}_{cr} = 0.05$	$cr^u = 0.3$
$\bar{A}_0 = 110,000$	$\gamma = 1.05$	$\bar{F} = 1.05$	

held in cash. The deterministic properties of asset classes are described in Table 1. Investment limits are defined with respect to practical rules of liquidity and diversification; transaction costs are taken from Haneveld *et al.* (2010) with $\bar{c}^S = \bar{c}^B = \bar{c}$, whereas the initial investments are defined considering general statistics of pension fund’s assets allocation in Switzerland, see Towers Towers Watson (2003) (where we assume that “real estate” corresponds to “other assets”). The portfolio constraints are defined in term of proportion and all amounts are assumed to be in thousands of Swiss francs. The values of the other deterministic parameters are shown in Table 2.

The time horizon $T = 5$ is split into five periods of one year each. Consequently, the considered ALM model has five stages, allowing for decisions at $t = 0$ (now) up to time $t = 4$. The random vector ω_t follows a VAR process, approximated in our case by a multistage scenario tree. In the following considerations, we first present the descriptive statistics of our model. Then, we discuss the numerical results obtained from our study.

4.1. Scenarios

The implementation of the scenario tree requires a careful specification of the VAR process. For this purpose, we use the estimation results obtained in Kouwenberg (2001). More specifically, the author estimates this process based on annual observations of the total asset returns and the general wage increase from 1956 to 1994. Table 3 displays descriptive statistics of the time series, whereas Table 4 shows the estimated correlation matrix of the residuals. Future returns for financial planning models can be constructed by sampling from the error distribution of the VAR model and applying the estimated equations of Table 5. We refer to Kouwenberg (2001) for further details on

TABLE 3
STATISTICS, TIME SERIES 1956–1994 KOUWENBERG (2001).

Assets	Statistics			
	Mean	S.D.	Skewness	Kurtosis
Wages	0.061	0.044	0.434	2.169
Deposits	0.055	0.025	0.286	2.430
Bonds	0.061	0.063	0.247	3.131
Real estate	0.081	0.112	-0.492	7.027
Stocks	0.102	0.170	0.096	2.492

TABLE 4
RESIDUAL CORRELATIONS OF VAR-MODEL, KOUWENBERG KOUWENBERG (2001).

Assets	Wages	Deposits	Bonds	Real estate	Stocks
Wages	1				
Deposits	0.227	1			
Bonds	-0.152	-0.268	1		
Real estate	-0.008	-0.179	0.343	1	
Stocks	-0.389	-0.516	0.383	0.331	1

TABLE 5
COEFFICIENT OF THE VAR MODEL, KOUWENBERG (2001).

$\ln(1 + \text{wages}_t)$	$= 0.018 + 0.693 \ln(1 + \text{wages}_{t-1}) + e_{1t}$	$\sigma_{1,t} = 0.030$
	(2.058) (5.789)	
$\ln(1 + \text{deposits}_t)$	$= 0.020 + 0.644 \ln(1 + \text{deposits}_{t-1}) + e_{2t}$	$\sigma_{2,t} = 0.017$
	(2.865) (5.448)	
$\ln(1 + \text{bonds}_t)$	$= 0.058 + e_{3t}$	$\sigma_{3,t} = 0.060$
	(6.241)	
$\ln(1 + \text{real estate}_t)$	$= 0.072 + e_{5t}$	$\sigma_{5,t} = 0.112$
	(4.146)	
$\ln(1 + \text{stocks}_t)$	$= 0.086 + e_{6t}$	$\sigma_{6,t} = 0.159$
	(3.454)	

this model estimation and for building the tree as well. For this purpose, we specify a branching structure of 1 – 10 – 6 – 6 – 4 – 4. This scenario tree has one initial node at time 0 and 10 succeeding nodes at time 1, . . . , resulting in $10 \times 6 \times 6 \times 4 \times 4 = S = 5,760$ path from 0 to 5, each with probability $p^S = \frac{1}{5,760}$.

4.2. Numerical results

This section presents the outputs of our study. All numerical results were implemented using the solver CPLEX in the mathematical programming language AMPL. The ALM models are formulated as large LP-problems with 616, 321 variables. In the model with the OICC, there are 995, 347 constraints and 3, 041, 032 non-zeros in the constraint matrix, whereas they are, respectively, 1, 002, 317 and 3, 105, 602 in the MICC. On average, the solution times are 381 s and 448 s, respectively, for OICC and MICC.

As a result of the ALM analysis, we are supposed to provide the first stage optimal decisions: a contribution rate, a remedial contribution and asset allocation that minimize the total cost. In the first part of this section, we analyze the effects of the risk parameter α and the initial funding ratio F_0 on the optimal decision. The optimization is made with respect to OICC. The values of α ranges from 0 to 0.085, whereas the initial funding ratio $F_0 = \frac{\bar{A}_0}{L_0}$ vary from 0.5 to 1.5. In order to vary F_0 , we change the initial liability L_0 accordingly, as \bar{A}_0 is specified from Table 2. In the second part, we compare the OICC to the MICC.

4.2.1. *OICC.* In order to analyze the impact of the risk parameter α , we fix the value of the initial cover ratio. That is

$$L_0 := 120,000 \Rightarrow F_0 = \frac{\bar{A}_0}{L_0} = 0.9166, \Rightarrow \textit{under covered}.$$

In what follows, the letter *O* at the top of a symbol stands for OICC, whereas *M* is related to MICC. Figure 2 shows the evolution of the contribution rate cr_0 . The value of cr_0 is particularly high as the institution is underfunded. We observe that when $\alpha \leq \alpha_*^O := 0.025$ (*O* at the top stands for OICC), the contribution rate is at its maximum: $cr^u = 0.3$ as specified in Table 2. From 0.025, cr_0 decreases linearly until it reaches the value 0.27 at $\alpha = \bar{\alpha}^O := 0.04$, and remains unchanged thereafter. According to the objective (13), the total cost is composed of the total contribution and of the total remedial contribution, these, over the period under study. Remedial contribution should be seen as an external financial support that may come from the sponsor of the pension fund. Figure 3 displays the allocation of the total costs into the two types of contribution. The proportion of remedial contribution linearly decreases from 9% at $\alpha = 0$ to reach 0% at $\alpha_*^O = 0.025$ and stays constant for $\alpha \geq \alpha_*^O$. Indeed, Figures 2 and 3 help understand how the ALM model parameters have been defined. It is conventional to assume that, from a certain level of risk and for a fix cover ratio, the sponsor will no more provide any financial support to the fund. In this model, the penalty parameter λ_z has been set such that the total remedial contribution is zero for $\alpha \geq \alpha_*^O$. Consequently, cr_0 decreases from $\alpha = \alpha_*^O$. It remains equal to 0.27 for $\alpha \geq \bar{\alpha}^O$ due to the target cover ratio at maturity: $F_5 \geq \bar{F} = 1.05$.

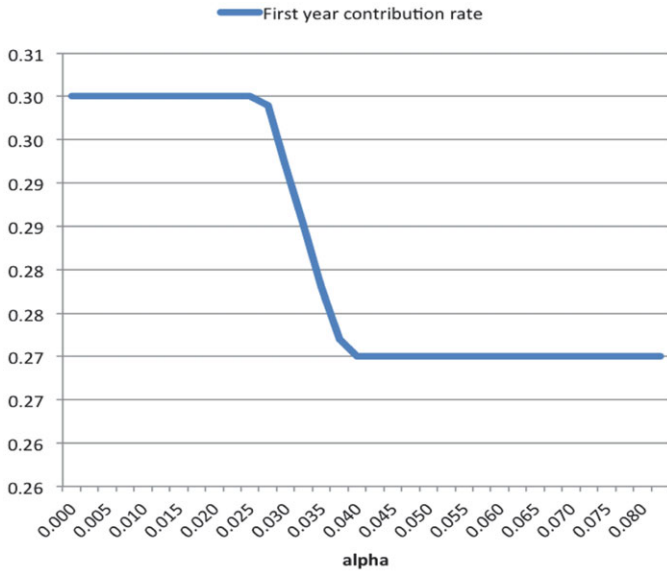


FIGURE 2: OICC: Contribution rate at $t = 0$ as function of α . (Color online)

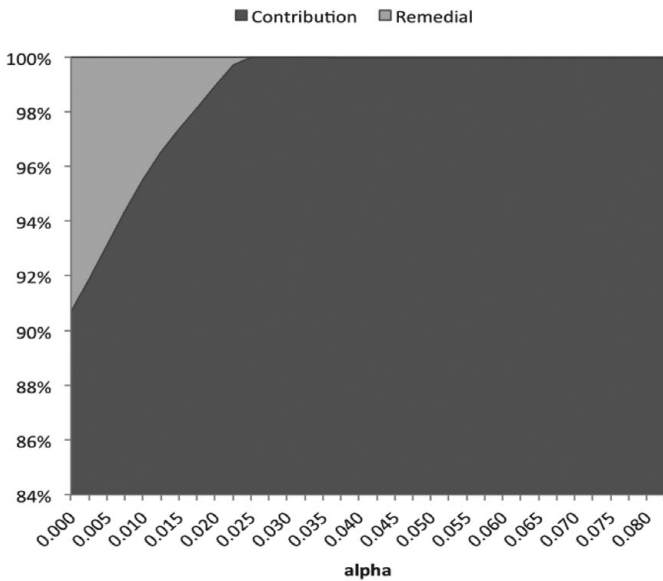


FIGURE 3: OICC: Initial cost allocation in function of α .

Figure 4 describes the optimal asset allocation for different values of α . Assets are ordered with respect to their level of risk. For small values of the risk parameter, the proportion of riskier assets (stocks and real estate) tends to

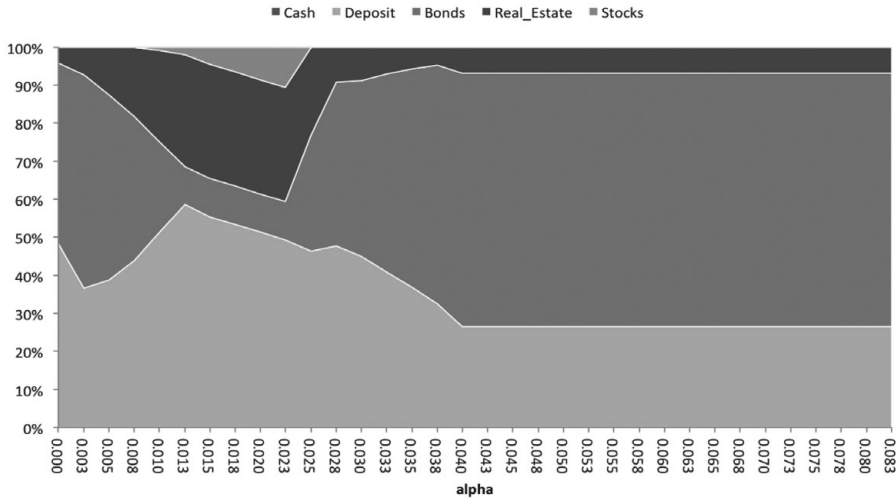


FIGURE 4: OICC: Asset allocation at $t = 0$ in function of α .

increase with an increase of α . When it approaches α_*^O , as the remedial contribution is already low, the decision maker starts reducing the risk exposition of its portfolio. However, the proportion of bonds is increased in order to improve the performance of the asset portfolio. The risk exposition is then progressively reduced until $\alpha = \bar{\alpha}^O$, from which, it remains unchanged thereafter. The value $\bar{\alpha}^O$ can be seen as the smallest value of α , above what, the OICC influence is no more significant, i.e. contribution rate, remedial contribution and asset allocation stay constant.

Next, we consider the effect of the initial funding ratio on the first stage optimal decision. We vary the value of L_0 so that the initial funding ratio F_0 lies between 0.5 and 1.5, and we assume $\alpha = 0.035$. Figure 5 displays the evolution of the contribution rate cr_0 , whereas Figure 6 shows how the total cost is distributed into regular and remedial contributions. As explained earlier, it is conventional to assume that, above a certain funding ratio F_0^{O*} , the remedial contribution is zero and contribution rate decreases as well. From Figures 5 and 6, it can be seen that the ALM model is set such that $F_0^{O*} := 0.9$.

Figure 7 describes the behavior of the first stage optimal asset allocation with respect to F_0 . When $F_0 < F_0^{O*}$, the optimal asset allocation is stable: approximately 30% in riskier assets. From $F_0 = F_0^{O*}$, the cover ratio is high enough to no more obtain remedial contribution and to reduce the contribution rate. However, the decision maker has to act in a riskier way in order to meet pension fund liabilities. As a result, the risk exposition increases up to 50% at $F_0 = 1.275$. An important target of our model is to guarantee a funding ratio $F_5 \geq \bar{F}$ by minimizing the total cost and risk level. For larger values of F_0 , with a higher chance to fulfill the condition $F_5 \geq \bar{F}$, the contribution rate and the risk exposition

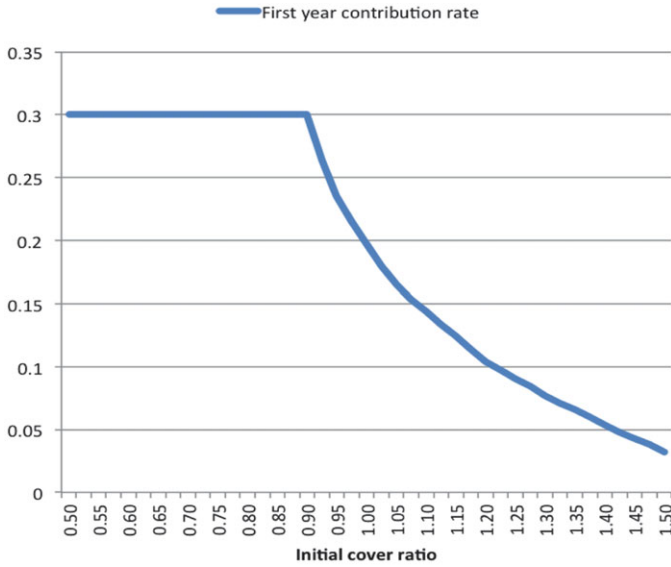


FIGURE 5: OICC: Contribution rate at $t = 0$ in function of F_0 . (Color online)

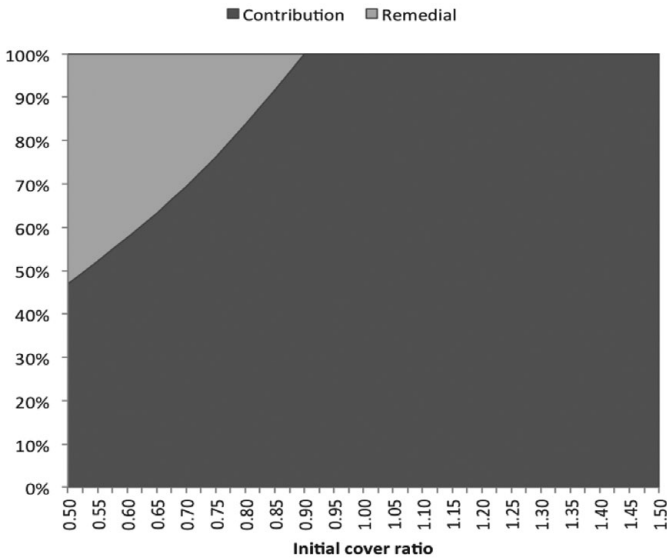


FIGURE 6: OICC: Initial cost allocation in function of F_0 .

(i.e. proportion of higher risk assets) decrease. We recall that the objective is not to maximize the wealth, but to minimize the total cost. Thus, the wealthier the fund is, the more prudent the allocation will be.

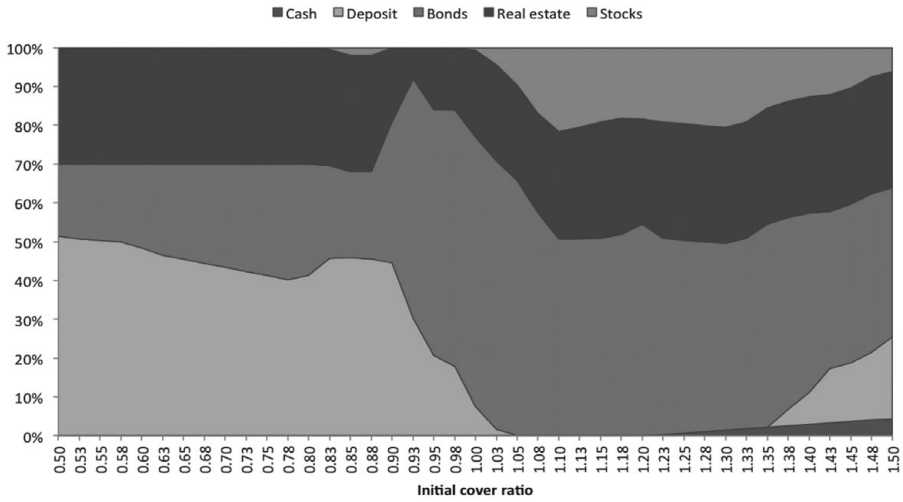


FIGURE 7: OICC: Asset allocation at $t = 0$ in function of F_0 .

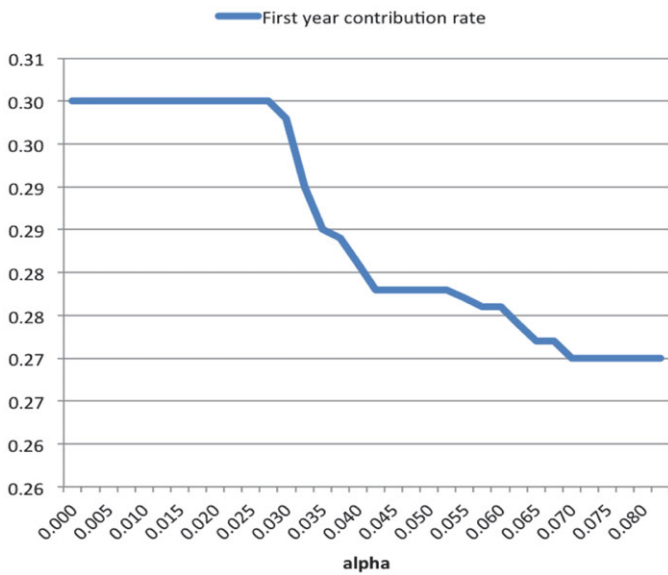


FIGURE 8: MICC: Contribution rate at $t = 0$ in function of α . (Color online)

4.2.2. *MICC*. In this section, we first present the results of the ALM model with the MICC, and second, we compare with the OICC. For the MICC analysis, assumptions are similar to the ones made for the model with OICC. Figures 8–13 display the results of the analysis. The effect of the risk parameter α is measured in Figures 8–10, whereas Figures 11–13 analyze the initial funding

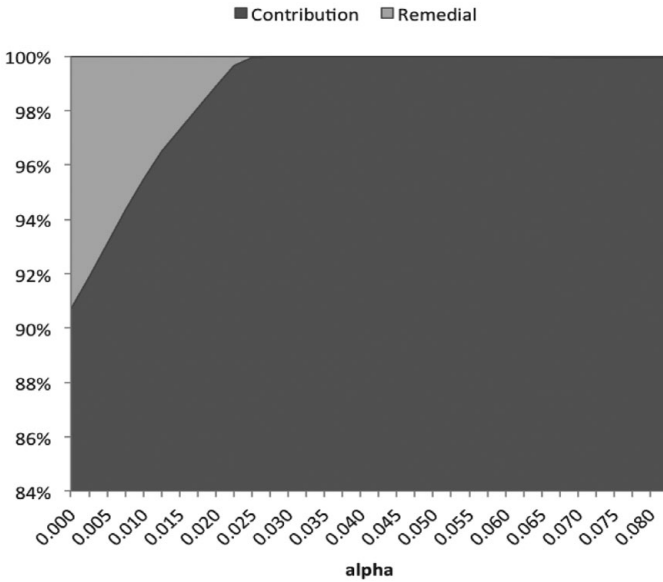


FIGURE 9: MICC: Initial cost allocation in function of alpha.

ratio impact. Although the first stage optimal decisions are different, they behave similarly. The parameters α_*^M , $\bar{\alpha}^M$ and F_0^{M*} (respectively α_*^O , $\bar{\alpha}^O$ and F_0^{O*} for OICC) slightly differ:

$$\alpha_*^M := 0.027; \quad \bar{\alpha}^M := 0.07; \quad F_0^{M*} := 0.9.$$

The feasible set of the MICC formulation is contained in the feasible set of the OICC formulation. That is why the above parameters are greater or equal to their analogues in OICC. Notice that, according to Figures 4 and 10 and for $\alpha \geq \bar{\alpha}^O$ in OICC and $\alpha \geq \bar{\alpha}^M$ in MICC, the asset allocations are *exactly* the same; showing that when alpha is above $\bar{\alpha}^O$ (resp. $\bar{\alpha}^M$), the OICC (resp. MICC) has no influence on the initial ALM model. In general, the optimal decisions obtained from OICC and MICC slightly differ. For example, when $\alpha = 0.05$ and $F_0 = 0.9166$, the first stage optimal decision of the MICC is

$$H_0 := (0.41 \ 0.59 \ 0 \ 0)^\top; \quad cr_0 = 0.279 \quad \text{and} \quad Z_0 = 0,$$

whereas for the OICC:

$$H_0 := (0.26 \ 0.66 \ 0.08 \ 0)^\top; \quad cr_0 = 0.270; \quad \text{and} \quad Z_0 = 0.$$

Asset allocations are calculated as percentage of the total asset. According to the above example, the decisions related to the OICC approach are riskier than the ones of MICC, especially regarding the asset allocation. In what follows, we

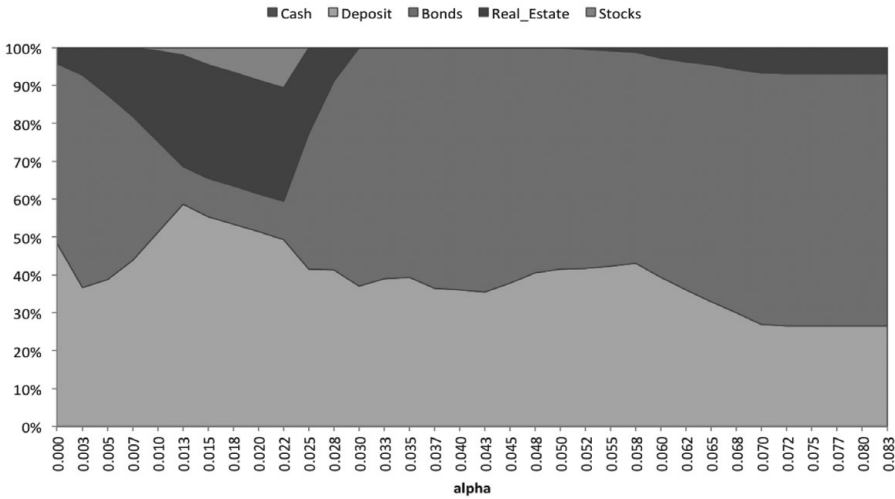


FIGURE 10: MICC: Asset allocation at $t = 0$ in function of α .

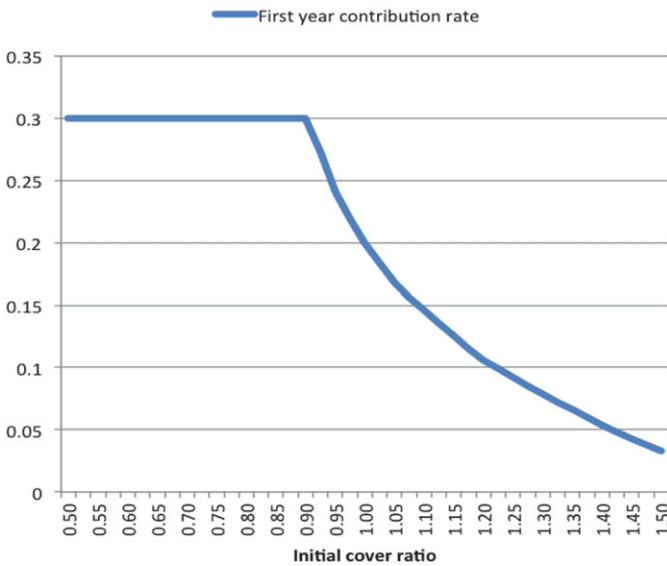


FIGURE 11: MICC: Contribution rate at $t = 0$ in function of F_0 . (Color online)

will try to quantify the cost of this risk reduction. For a pension fund, this can be done by measuring the difference in term of the total cost (regular contribution + remedial contribution). Hence, Figure 14 compares the total costs of OICC and MICC, whereas Figure 15 displays the contribution rate difference, all this with respect to α . When $\alpha \leq \alpha_*^O = 0.025$ or $\alpha \geq \bar{\alpha}^M = 0.07$, the contribution rate and total cost are equal for both models. For $\alpha_*^O \leq \alpha \leq \bar{\alpha}^M$, OICC and MICC slightly differ, i.e. MICC costs more for a maximum variation of 2, 000

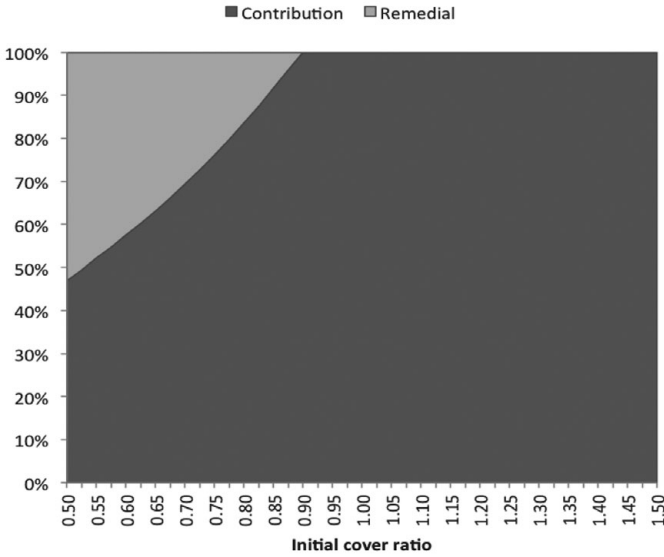


FIGURE 12: MICC: Initial cost allocation in function of F_0 .

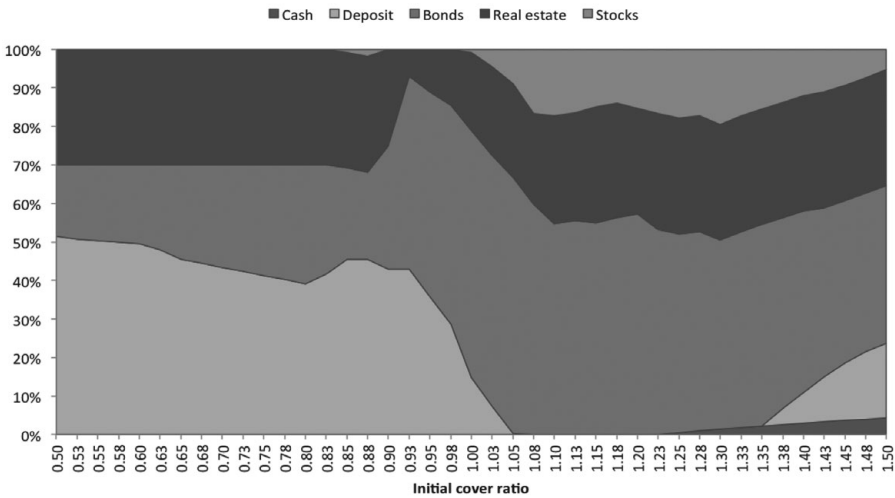


FIGURE 13: MICC: Asset allocation in function of F_0 .

(less than 2% of total asset) and 1.5%, respectively, for the total cost and the contribution rate. Consequently, although being more conservative, the MICC has to be considered as a serious contender for optimal ALM for the two following reasons:

- it is safer, and
- the cost of this safety is less than 2% of total asset.

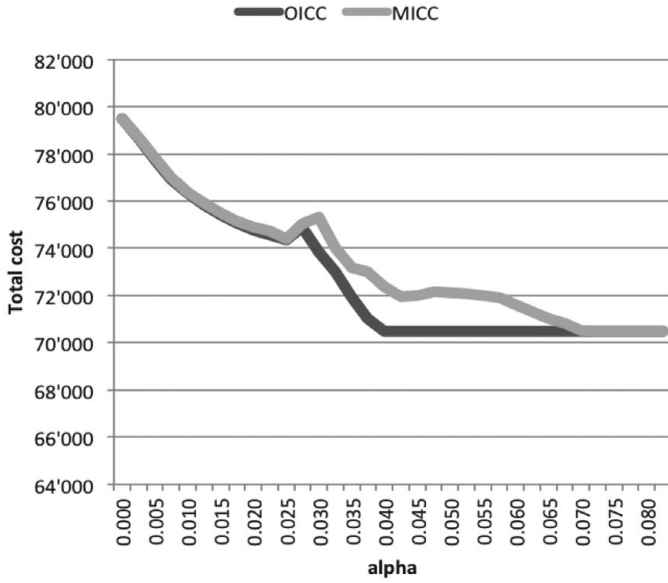


FIGURE 14: Comparison of OICC and MICC in function of α : Total cost.

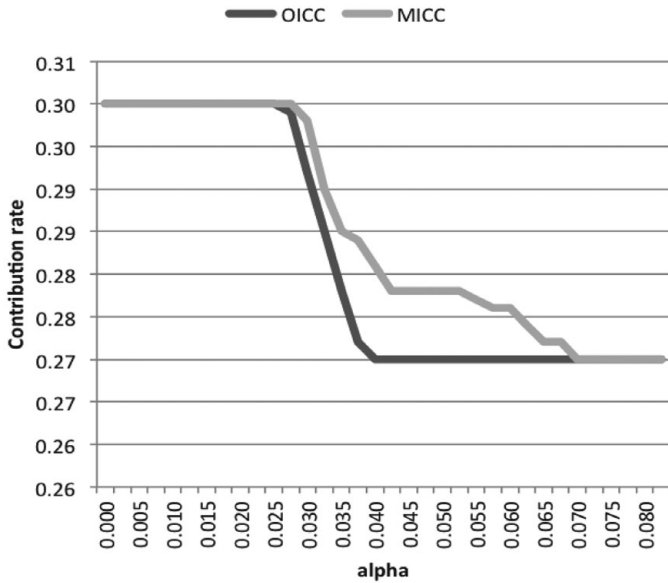


FIGURE 15: Comparison of OICC and MICC in function of α : Contribution rate cr_0 .

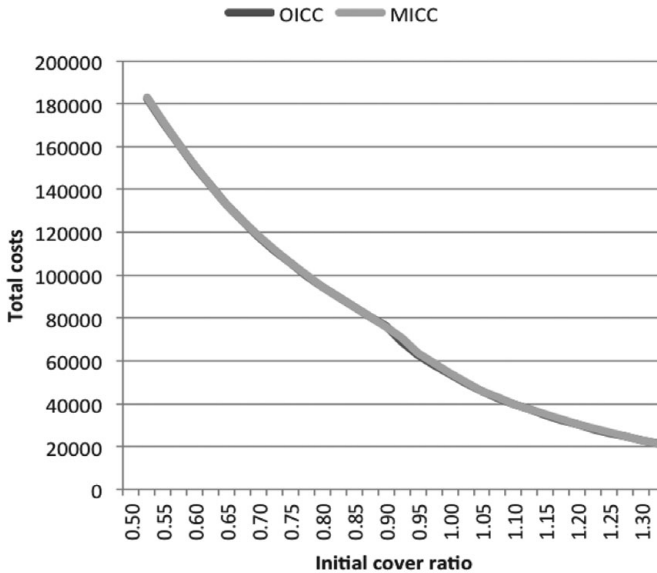


FIGURE 16: Comparison of OICC and MICC in function of F_0 : Total cost.

Figures 16 and 17, which compare the effect of F_0 on the OICC and on the MICC, confirm that result.

5. CONCLUSION

In this paper, we considered the effects of ICC on an ALM model for a defined benefit pension fund. ICC are appropriate for modeling risk constraints, in particular when a quantitative risk measure is preferable. At each decision time, the ICCs put an upper bound on the one period expected shortfalls. In particular, the upper limit considered here is a fixed proportion of total liability making the risk parameter scale free as well as time dependent.

First, the ALM model describing the influences of the ICCs on the optimal decisions is implemented. The application of the model to our numerical example show that it fairly reproduces the behavior of the decision maker, especially on the contribution rate, remedial contribution and wealth allocation.

Second, we defined two types of ICC: the one period OICC and the MICC. MICC is basically more restrictive and more cautious than OICC. Based on our illustrative example, the cost generated by the rise in security with the MICC is quantified. Although the optimal decisions from the OICC and the MICC are not the same, the total costs are very close, showing that the MICC could be a good alternative.

However, in further considerations, it would be interesting to analyze the impact of the first stage decisions on the other stages in order to conclude which

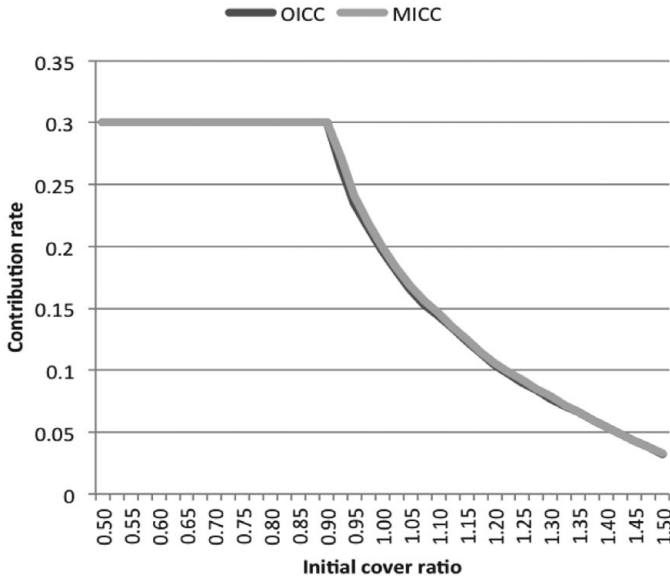


FIGURE 17: Comparison of OICC and MICC in function of F_0 : Contribution rate cr_0 .

approach is better. In addition, the results mainly rely on the chosen scenario tree. The analysis of the results stability regarding the scenario tree would also be of great interest.

ACKNOWLEDGEMENTS

The authors acknowledge partial support from the project RARE-318984 (an FP7 Marie Curie IRSES Fellowship). The authors would also like to thank Stefan Thonhauser (TU-Graz, Austria) and Joël Wagner (Department of Actuarial Science, HEC, University of Lausanne) for their time, their advice, their technical supports and all the interesting discussions during the preparation of this contribution.

NOTES

1. OICC (resp. MICC) stands for One period Integrated Chance Constraint (resp. Multiperiod Integrated Chance Constraint) which will be more clearly defined in Section 3.
2. OPP2 of April 18th, 1984, Art 55-b, (As of January 1st, 2012).

REFERENCES

- AHLGRIM, K.C., DÁRCY, S.P. and GORVETT, R.W. (2005) Modeling financial scenarios: A framework for the actuarial profession. In *Proceedings of the Casualty Actuarial Society*, volume 92, pp. 177–238. Casualty Actuarial Society Arlington, VA.
- AHMED, S. and SHAPIRO, A. (2008) Solving chance-constrained stochastic programs via sampling and integer programming. In *2008 Tutorials in Operations Research: State-of-the-Art Decision-*

- Making Tools in the Information-Intensive Age* (eds. Z.-L. Chen and S. Raghavan), pp. 261–269. Catonsville, MD: INFORMS.
- ARO, H. and PENNANEN, T. (2013) Liability-driven investment in longevity risk management. In *Optimal Financial Decision Making under Uncertainty* (eds. G. Consigli, D. Kuhn and P. Brandimarte), pp. 121–136. Cham, Switzerland: Springer International Publishing.
- ARTZNER, P., DELBAEN, F., EBER, J.-M. and HEATH, D. (1999) Coherent measures of risk. *Mathematical Finance*, **9**(3), 203–228.
- BIRGE, J.R. and LOUVEAUX, F.V. (2011) *Introduction to Stochastic Programming*. New York: Springer-Verlag.
- BOGENTOFT, E., ROMEIJN, H.E. and URYASEV, S. (2001) Asset/liability management for pension funds using CVaR constraints. *Journal of Risk Finance*, **3**(1), 57–71.
- CARINO, D.R., KENT, T., MYERS, D.H., STACY, C., SYLVANUS, M., TURNER, A.L., WATANABE, K. and ZIEMBA, W.T. (1994) The Russell-Yasuda Kasai model: An asset/liability model for a Japanese insurance company using multistage stochastic programming. *Interfaces*, **24**(1), 29–49.
- CHARNES, A., COOPER, W.W. and SYMONDS, G.H. (1958) Cost horizons and certainty equivalents: An approach to stochastic programming of heating oil. *Management Science*, **4**(3), 235–263.
- CONSIGLI, G. and DEMPSTER, M.A.H. (1998) Dynamic stochastic programming for asset-liability management. *Annals of Operations Research*, **81**, 131–162.
- DELT, C. (1995) *Asset Liability Management for Pension Funds: A Multistage Chance Constrained Programming Approach*. Erasmus Universiteit Rotterdam.
- DRIJVER, S.J. (2005) *Asset Liability Management for Pension Funds Using Multistage Mixed-Integer Stochastic Programming*. Groningen, The Netherlands: University of Groningen.
- ELTON, E.J. and GRUBER, M.J. (1992) Optimal investment strategies with investor liabilities. *Journal of Banking & Finance*, **16**(5), 869–890.
- FALEH, A. (2011) Un modèle de programmation stochastique pour l'allocation stratégique d'actifs d'un régime de retraite partiellement provisionné. Working Paper. Available at: <https://hal.archives-ouvertes.fr/hal-00561965/document>.
- FAMA, E.F. (1970) Multiperiod consumption-investment decisions. *The American Economic Review*, **60**, 163–174.
- FONSECA, R., MICHAUD, P.-C. and SOPRASEUTH, T. (2007) Richesse, contrainte de liquidité et coût de création d'entreprise. Technical report, Centre d'Études des Politiques Économiques (EPEE), Université d'Evry Val d'Essonne.
- GEYER, A. and ZIEMBA, W.T. (2008) The innovest Austrian pension fund financial planning model InnoALM. *Operations Research*, **56**(4), 797–810.
- GOTTLIEB, L. C. and WHISTON, B. A. (2005) *Morneau Sobeco Handbook of Canadian Pension and Benefit Plans*, 13th edition. Toronto: CCH Canadian Limited.
- HAKANSSON, N.H. (1970) Optimal investment and consumption strategies under risk for a class of utility functions. *Econometrica: Journal of the Econometric Society*, 587–607.
- HAKANSSON, N.H. (1974) Convergence to isoelastic utility and policy in multiperiod portfolio choice. *Journal of Financial Economics*, **1**(3), 201–224.
- HANEVELD, W.K.K. (1986) *Duality in Stochastic Linear and Dynamic Programming*, Lecture Notes in Economics and Mathematical Systems, volume 274. Berlin: Springer-Verlag.
- HANEVELD, W.K.K. and VAN DER VLERK, M.H. (2002) *Integrated Chance Constraints: Reduced Forms and An Algorithm*. Groningen, The Netherlands: University of Groningen.
- HANEVELD, W.K.K., STREUTKER, M.H. and VAN DER VLERK, M.H. (2010) An alm model for pension funds using integrated chance constraints. *Annals of Operations Research*, **177**(1), 47–62.
- HILLER, R.S. and SCHAACK, C. (1990) A classification of structured bond portfolio modeling techniques. *The Journal of Portfolio Management*, **17**(1), 37–48.
- HILLI, P., KOIVU, M., PENNANEN, T. and RANNE, A. (2007) A stochastic programming model for asset liability management of a Finnish pension company. *Annals of Operations Research*, **152**(1), 115–139.
- KALL, P. and MAYER, J. (2011) *Stochastic Linear Programming: Models, Theory, and Computation*. Berlin: Springer-Verlag.

- KEEL, A. and MÜLLER, H.H. (1995) Efficient portfolios in the asset liability context. *ASTIN Bulletin*, **25**(01), 33–48.
- KOOPMANS, K.C. (1942) *The Risk of Interest Fluctuations in Life Insurance Companies*. Philadelphia: Penn Mutual Life Insurance Company.
- KOUWENBERG, R. (2001) Scenario generation and stochastic programming models for asset liability management. *European Journal of Operational Research*, **134**(2), 279–292.
- KUSY, M.I. and ZIEMBA, W.T. (1986) A bank asset and liability management model. *Operations Research*, **34**(3), 356–376.
- LEIBOWITZ, M.L. and HENRIKSSON, R.D. (1988) Portfolio optimization within a surplus framework. *Financial Analysts Journal*, **44**(2), 43–51.
- LEIBOWITZ, M.L., KOGELMAN, S. and BADER, L.N. (1992) Asset performance and surplus control: A dual-shortfall approach. *The Journal of Portfolio Management*, **18**(2), 28–37.
- LUEDTKE, J. (2013) A branch-and-cut decomposition algorithm for solving chance-constrained mathematical programs with finite support. *Mathematical Programming*, **146**(1–2), 1–26.
- LUEDTKE, J., AHMED, S. and NEMHAUSER, G.L. (2010) An integer programming approach for linear programs with probabilistic constraints. *Mathematical Programming*, **122**(2), 247–272.
- MARKOWITZ, H. (1952) Harry Markowitz. *Portfolio selection*, *Journal of Finance*, **7**(1), 77–91.
- MERTON, R.C. (1969) Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, **51**(3), 247–257.
- MØLLER, T. and STEFFENSEN, M. (2007) *Market-Valuation Methods in Life and Pension Insurance*. Cambridge: Cambridge University Press.
- NEMIROVSKI, A. and SHAPIRO, A. (2006) Convex approximations of chance constrained programs. *SIAM Journal on Optimization*, **17**(4), 969–996.
- PLUG, G.CH., SWIETANOWSKI, A. and PAEUG, G.CH. (1998) Dynamic asset allocation under uncertainty for pension fund management. *Control and Cybernetics*, **28**(4), 755–777.
- PRÉKOPA, A. (1998) Programming under probabilistic constraint with discrete random variable. In *New Trends in Mathematical Programming* (eds. F. Giannessi, S. Komlósi and T. Rapcsák), pp. 235–255. Dordrecht: Springer Science+Business Media.
- RAIKE, W.M. (1970) Dissection methods for solutions in chance constrained programming problems under discrete distributions. *Management Science*, **16**(11), 708–715.
- REDINGTON, F.M. (1952) Review of the principles of life-office valuations. *Journal of the Institute of Actuaries*, **78**(3), 286–340.
- ROCKAFELLAR, R.T. and URYASEV, S. (2002) Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance*, **26**(7), 1443–1471.
- RUSZCZYŃSKI, A. (2002) Probabilistic programming with discrete distributions and precedence constrained knapsack polyhedra. *Mathematical Programming*, **93**(2), 195–215.
- SHAPIRO, A., DENTCHEVA, D. and RUSZCZYŃSKI, A. (2009) *Lectures on Stochastic Programming: Modeling and Theory*, volume 9. Philadelphia: Society for Industrial and Applied Mathematics and the Mathematical Planning Society.
- SHARPE, W.F. and TINT, L.G. (1990) Liabilities—a new approach. *The Journal of Portfolio Management*, **16**(2), 5–10.
- TANNER, M.W. and NTAIMO, L. (2010) IIS branch-and-cut for joint chance-constrained stochastic programs and application to optimal vaccine allocation. *European Journal of Operational Research*, **207**(1), 290–296.
- TOWERS WATSON. (2003) *Global Pensions Asset Study*. Available at: <https://towerswatson.com/en/Insights/IC-Types/Survey-Research-Results/2013/01/Global-Pensions-Asset-Study-2013>.
- VAN DER VLERK, M.H., HANEVELD, W.K.K. and STREUTKER, M.H. (2003) *Integrated Chance Constraints in an ALM Model for Pension Funds*. Berlin: Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät II, Institut für Mathematik.
- WILKIE, A.D. (1995) More on a stochastic asset model for actuarial use. *British Actuarial Journal*, **1**(05), 777–964.
- ZENIOS, S.A. (2006) *Practical Financial Optimization: Decision Making for Financial Engineers*. Oxford: Blackwell Publishing.
- ZIEMBA, W.T. and MULVEY, J.M. (1998) *Worldwide Asset and Liability Modeling*, volume 10. Cambridge: Cambridge University Press.

YOUSSEF A. F. TOUKOUROU (Corresponding author)

*Department of Actuarial Science
Faculty of Business and Economics
University of Lausanne UNIL-Dorigny
CH-1015 Lausanne, Switzerland
E-Mail: toukourouyoussef@yahoo.fr*

FRANÇOIS DUFRESNE

*Department of Actuarial Science
Faculty of Business and Economics
University of Lausanne UNIL-Dorigny
CH-1015 Lausanne, Switzerland
E-Mail: francois.dufresne@unil.ch*

APPENDICES

Appendix A: The ALM program description

We start this section by defining indices, variables and parameters of the model. Second, the ALM model with the objective and the constraints are also displayed.

Indices

t	time index, $t = 0, 1, \dots, T$
s	scenarios index, $s = 1, \dots, S$
k	index of asset classes, $k = 1, \dots, d$

Decision variables

Z_t^s	remedial contribution by the sponsor at time t in scenario s
C_t^s	total cash amount at the beginning of year t in scenario s
$H_{k,t}^s$	value of the investments hold in asset class k , at the beginning of year t and in scenario s
$B_{k,t}^s$	value of the asset class k , bought at the beginning of year t and in scenario s
$S_{k,t}^s$	value of the asset class k , sold at the beginning of year t and in scenario s
cr_t^s	contribution rate for year $t + 1$ in scenario s
A_t^s	total asset value at time t in scenario s

- A_t^{*s} total asset value just before the asset allocation and the remedial contribution at time t in scenario s
- $\Delta_{cr_t}^s$ variation (increase or decrease) of contribution rate from year t to $t + 1$ in scenario s

Random parameters

- $r_{k,t}^s$ random rate of return on asset class k over year t in scenario s
and $\xi_{k,t} := 1 + r_{k,t}$
- W_t^s random total wages of active participants in year t in scenario s
- L_t^s random value of liabilities at time t in scenario s
- Ben_t^s random total benefit payments to active participants in year t in scenario s

Deterministic parameters

- T time horizon
- S number of scenarios
- d number of asset classes
- α risk parameter defined by either the sponsor or the regulator
- \bar{c}_k^B proportional transaction cost for purchasing an asset class k
- \bar{c}_k^S proportional transaction cost for purchasing an asset class k
- l_k lower bound on the proportion of asset class k in the total asset portfolio
- u_k upper bound on the proportion of asset class k in the total asset portfolio
- l_c lower bound on the proportion of cash k in the total asset portfolio
- u_c upper bound on the proportion of cash k in the total asset portfolio
- cr^l lower bound on the contribution rate
- cr^u upper bound on the contribution rate
- Δ_{cr} lower bound on the yearly absolute variation of the contribution rate

- $\bar{\Delta}_{cr}$ upper bound on the yearly absolute variation of the contribution rate
- \bar{F} target funding ratio
- γ lower bound on the funding ratio
- r_f risk free interest rate
- λ_z penalty parameter for remedial contribution
- $\lambda_{\Delta_{cr}}$ penalty parameter for absolute variation of contribution rate
- $\bar{H}_{k,0}$ value of the initial allocation in asset class k
- \bar{C}_0 initial cash amount

Objective

The objective of the model is to determine the asset allocation, contribution rate and remedial contributions that minimize the total cost defined as follows:

$$\min_{H, cr, Z} \mathbb{E}_0 \left[\sum_{t=0}^{T-1} v_{t+1} (cr_t W_{t+1} + \lambda_z Z_{t+1}) + \sum_{t=0}^{T-2} v_{t+1} \lambda_{\Delta_{cr}} \Delta_{cr_t} W_{t+1} \right],$$

under the following constraints. In the model description, anytime we use indices s and/or k is equivalent to saying for any $s = 1, \dots, S$ and/or for any $k = 1, \dots, d$.

Budget constraints and total value of the assets

$$A_0 = \sum_{k=1}^d \bar{H}_{k,0} + \bar{C}_0 + Z_0 - \sum_{k=1}^d (\bar{c}_k^B B_{k,0} + \bar{c}_k^S S_{k,0}) = \sum_{k=1}^d H_{k,0} + C_0,$$

$$A_T^s = \sum_{k=1}^d H_{T-1,k}^s \xi_{T,k}^s + C_{T-1}^s (1 + r_f) + cr_{T-1}^s W_T^s - \text{Ben}_T^s = A_T^{*s},$$

$$A_t^s = \sum_{k=1}^d H_{k,t-1}^s \xi_{k,t}^s + C_{t-1}^s (1 + r_f) + cr_{t-1}^s W_t^s - \text{Ben}_t^s + Z_t^s - \sum_{k=1}^d (\bar{c}_k^B B_{k,t}^s + \bar{c}_k^S S_{k,t}^s); \quad t = 1, \dots, T - 1$$

$$A_t^s = A_t^{*s} + Z_t^s - \sum_{k=1}^d (\bar{c}_k^B B_{k,t}^s + \bar{c}_k^S S_{k,t}^s) = \sum_{k=1}^d H_{k,t}^s + C_t^s; \quad t = 1, \dots, T - 1.$$

Asset classes dynamics

$$\begin{aligned}
 H_{k,0} &= \bar{H}_{k,0} + B_{k,0} - S_{k,0} \\
 H_{k,t}^s &= \xi_{k,t}^s H_{k,t-1}^s + B_{k,t}^s - S_{k,t}^s; \quad t = 1, \dots, T.
 \end{aligned}$$

Cash dynamics

$$\begin{aligned}
 C_0 &= \bar{C}_0 + Z_0 - \sum_{k=1}^d (1 + \bar{c}_k^B) B_{k,0} + \sum_{k=1}^d (1 - \bar{c}_k^S) S_{k,0} \\
 C_t^s &= C_{t-1}^s (1 + r_f) + cr_{t-1}^s W_t^s - Ben_t^s + Z_t^s - \sum_{k=1}^d (1 + \bar{c}_k^B) B_{k,t}^s \\
 &\quad + \sum_{k=1}^d (1 - \bar{c}_k^S) S_{k,t}^s; \quad t = 1, \dots, T.
 \end{aligned}$$

Not short selling assets and not borrowing cash constraints

$$H_{k,t}^s \geq 0; \quad B_{k,t}^s \geq 0; \quad S_{k,t}^s \geq 0; \quad C_t^s \geq 0; \quad t = 0, \dots, T.$$

Liquidity constraints

$$C_t^s (1 + r_f) + \mathbb{E}_{t,s} (cr_t^s W_{t+1} - Ben_{t+1}) \geq 0; \quad t = 0, \dots, T - 1.$$

Portfolio constraints

$$\begin{aligned}
 l_k A_t^s &\leq H_{k,t}^s \leq u_k A_t^s; \quad t = 0, \dots, T - 1, \\
 l_c A_t^s &\leq C_t^s \leq u_c A_t^s; \quad t = 0, \dots, T - 1.
 \end{aligned}$$

Constraints on contribution rates

$$cr^l \leq cr_t^s \leq cr^u \quad \text{and} \quad \underline{\Delta}_{cr} \leq \Delta_{cr_t}^s \leq \bar{\Delta}_{cr}; \quad t = 0, \dots, T - 1.$$

The decision variables are subject to the non-anticipativity constraints. The ICCs defined in Section 2.3.1 control the risk-level of the model. They also have to be included in the model as explained in Section 3.2.

Appendix B: Proposition 3.1, an example based sketch of proof

We consider the event tree of Figure B1 with a time horizon $T = 3$ and a branching structure of $1 - 5 - 4 - 2$, leading to $S = 5 \times 4 \times 2 = 40$ scenarios. A node is a possible outcome of the stochastic event at a given time. The starting and

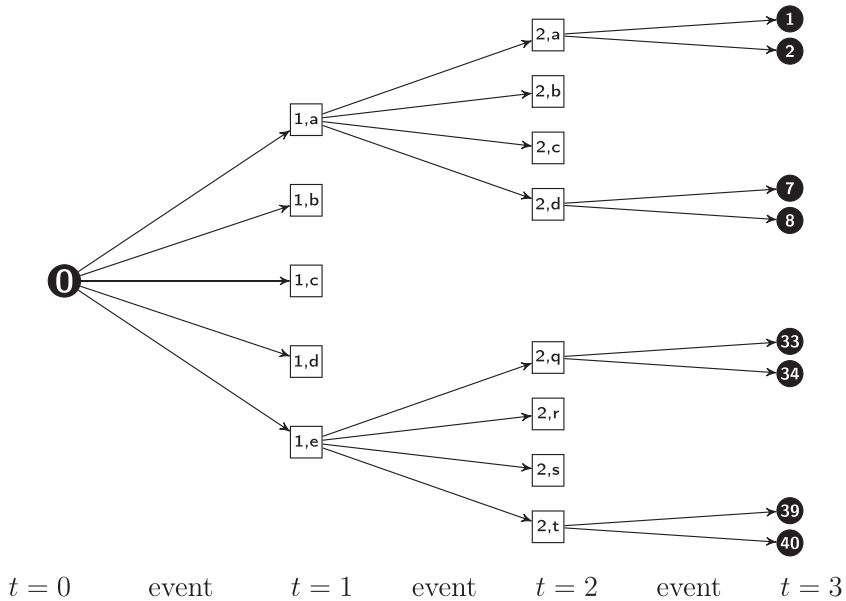


FIGURE B1: A scenario tree with 40 scenarios and 66 nodes.

ending nodes of the tree are round whereas the others are rectangular. At each $t \in \{2, 3\}$, the rectangular nodes are name according to time and following a top-down alphabetic order. For example, the rectangle $(1, a)$ describe the outcome at the first node of time 1. At each node, the economical values such as total asset, total liability and expected shortfall can be determined. For sake of clarity, we recall that, due to non-anticipativity, the node $(1, a)$ is equivalent to the node $(1, s')$, $s' \in \{1, \dots, 8\}$ as described before, the node $(1, b)$ is equivalent to the node $(1, s')$, $s' \in \{9, \dots, 16\}$ and so on for the other nodes. The other notations used here are similar to the ones in the paper. We define

$$\Lambda_{(t,s)} := \sum_{s' \in \mathcal{S}} p_{t,s}^{s'} \left(A_{t+1}^{*s'} - \gamma L_{t+1}^{s'} \right)^{-},$$

where $p_{t,s}^{s'}$ stands for the conditional probability to reach node $(t + 1, s')$ going from (t, s) and $p_{t,s}^{s'} = 0$ for any scenario s' of $t + 1$ not descending from (t, s) . The MICC defined in Equation (18) is then

$$\{ \Lambda_{h-1,s}, h \in \mathcal{T}_{t+1} \} \leq \beta_{t,s}, t \in \{0, 1, 2\}.$$

The value $\beta_{t,s} := \alpha L_t^s$ is the ICC upper limit computed at time t in scenario s . According to this set of constraints,

