Proceedings of the Royal Society of Edinburgh, **146A**, 23–58, *2016*
DOI:10.1017/S0308210515000311

Investigating the multiplicity and concentration behaviour of solutions for a quasi-linear Choquard equation via the penalization method

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(MS received 21 March 2014; accepted 8 October 2014)

We study the multiplicity and concentration behaviour of positive solutions for a quasi-linear Choquard equation

$$-\varepsilon^p \Delta_p u + V(x)|u|^{p-2}u = \varepsilon^{\mu-N}\left(\frac{1}{|x|^\mu} * F(u)\right) f(u) \quad \text{in } \mathbb{R}^N,$$

where Δ_p is the p-Laplacian operator, $1 < p < N$, V is a continuous real function on $\mathbb{R}^N$, $0 < \mu < N$, $F(s)$ is the primitive function of $f(s)$, ε is a positive parameter and $*$ represents the convolution between two functions. The question of the existence of semiclassical solutions for the semilinear case $p = 2$ has recently been posed by Ambrosetti and Malchiodi. We suppose that the potential satisfies the condition introduced by del Pino and Felmer, i.e. V has a local minimum. We prove the existence, multiplicity and concentration of solutions for the equation by the penalization method and Lyusternik–Schnirelmann theory and even show novel results for the semilinear case $p = 2$.

Keywords: Choquard equation; non-local nonlinearities; concentration; Lyusternik–Schnirelmann theory; variational methods

2010 *Mathematics subject classification:* Primary 35J50; 35J60; 35A15

1. Introduction and main results

In this paper, we consider the existence, multiplicity and concentration behaviour of positive solutions for the following class of generalized Choquard equation (here 'SNE' abbreviates 'semilinear nonlinear equation'):

$$\left.\begin{aligned} -\varepsilon^2 \Delta u + V(x)u = \varepsilon^{\mu-N}\left(\frac{1}{|x|^\mu} * |u|^q\right)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^{1,2}(\mathbb{R}^N), \quad u(x) > 0 \quad \text{for all } x \in \mathbb{R}^N, \end{aligned}\right\} \tag{SNE}$$

where $\varepsilon > 0$, $0 < \mu < N$ and the exponent q lies in a suitable range.

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This problem was motivated by some recent works related to the existence of standing waves of the nonlinear Schrödinger equation of the kind

$$\mathrm{i}\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}\Delta\Psi + W(x)\Psi - (K(x) * |\Psi|^q)|\Psi|^{q-2}\Psi \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

Here m is the mass of the bosons, $\hbar$ is the Planck constant, $W(x)$ is the external potential and $K(x)$ is the response function that possesses information on the mutual interaction between the bosons. This type of non-local equation is known to influence the propagation of electromagnetic waves in plasmas [7] and also plays an important role in the theory of Bose–Einstein condensation [16]. It is clear that $\Psi(x,t) = u(x)\mathrm{e}^{-(\mathrm{i}E/\hbar)t}$ solves the evolution equation (1.1) if, and only if, u solves

$$-\varepsilon^2\Delta u + V(x)u = (K(x) * |u|^q)|u|^{q-2}u \quad \text{in } \mathbb{R}^N \tag{1.2}$$

with $V(x) = W(x) - E$ and $\varepsilon^2 = \hbar^2/2m$.

When the response function $K(x)$ is equal to $\delta(x)$ and $q = 2$, the nonlinear response is local, and nonlinear Schrödinger equation (1.2) becomes

$$-\varepsilon^2\Delta u + V(x)u = g(u) \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

The semiclassical problems for the Schrödinger equation (1.2), i.e. the parameter ε goes to zero, describe the transition between quantum mechanics and classical mechanics. The study of existence and concentration of the semiclassical states of the Schrödinger equation (1.3) goes back to the pioneering work by Floer and Weinstein [21]. Since then it has been studied extensively under various hypotheses on the potential and the nonlinearity (see, for example, [3, 5, 6, 8–11, 18, 19, 21–23, 35, 36, 38, 40] and the references therein).

If the parameter ε is equal to 1 and the response function $K(x)$ is of Coulomb type (for example, $1/|x|^\mu$ and $F(u) = |u|^q/q$), then we arrive at the Choquard–Pekar equation,

$$-\Delta u + u = \left(\frac{1}{|x|^\mu} * |u|^q\right)|u|^{q-2}u \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

The case when $q = 2$ and $\mu = 1$ goes back to the description by Pekar in 1954 of the quantum theory of a polaron at rest [37] and the modelling in the 1976 work of Choquard of an electron trapped in its own hole, in a certain approximation to Hartree–Fock theory of one-component plasma [25].

Equation (1.4) was proposed by Penrose in 1996 as a model of self-gravitating matter [33] and is known in that context as the Schrödinger–Newton equation. Since then many efforts have been made to study the existence of non-trivial solutions for problem (1.4). In [27], by using critical point theory, Lions obtained a solution $u \in H^1(\mathbb{R}^3)$, $u \not\equiv 0$. For a general class of $K(x)$ and nonlinearity, Ackermann [1] proposed an approach to prove the existence of infinitely many geometrically distinct weak solutions to the problem with potential V periodic in x_i and 0 not in the spectrum of $-\Delta + V(x)$. Concerning the properties of the ground-state solutions, Ma and Zhao [28] considered the generalized Choquard equation (1.4) for $q \geqslant 2$; they proved that every positive solution is radially symmetric and monotone decreasing about some point, under the assumption that a certain set of real numbers, defined in terms of N, α and q, is non-empty. Under the same assumption,

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Cingolani *et al*. [14] gave some existence and multiplicity results in the electromagnetic case, and established the regularity and some decay of the ground states of (1.4) asymptotically at infinity. Moroz and Van Schaftingen [30] eliminated this restriction and showed the regularity, positivity and radial symmetry of the ground states for the optimal range of parameters, and also derived that these solutions decay asymptotically at infinity. Moroz and Van Schaftingen also established the existence of a ground state for the Choquard equation with general nonlinearity in [31].

The question of the existence of semiclassical solutions for the non-local problem (1.2) has been posed more recently [5, p. 29]. Consider the semilinear elliptic equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{\mu-N}\left(\frac{1}{|x|^\mu} * |u|^q\right)|u|^{q-2}u \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

It can be observed that if v is a solution of (1.5) for $x_0 \in \mathbb{R}^N$, then the new function defined by $u = v(x_0 + \varepsilon x)$ satisfies

$$-\Delta u + V(x_0 + \varepsilon x)u = \left(\frac{1}{|x|^\mu} * |u|^q\right)|u|^{q-2}u \quad \text{in } \mathbb{R}^N.$$

This suggests some convergence, as $\varepsilon \to 0$, of the family of solutions to a solution u_0 of the limit problem

$$-\Delta u + V(x_0)u = \left(\frac{1}{|x|^\mu} * |u|^q\right)|u|^{q-2}u \quad \text{in } \mathbb{R}^N. \tag{1.6}$$

It is expected that, in the semiclassical limit $\varepsilon \to 0$, the dynamics should be governed by the classical external potential $V(x)$. In particular, there should be a correspondence between semiclassical solutions of the equation and critical points of the potential.

In the case when $N = 3$, $\mu = 1$ and $F(u) = |u|^q$, $q = 2$, Wei and Winter [41] constructed families of solutions by a Lyapunov–Schmidt-type reduction when $\inf V > 0$, and Secchi [39] obtained the result when $V > 0$ and $\liminf |x| \to \infty$, $V(x)|x|^\gamma > 0$ for some $\gamma \in [0, 1)$. This method of construction depends on the existence, uniqueness and non-degeneracy up to translations of the positive solution of the limiting equation (1.4), which is a difficult problem that has only been fully solved in the case when $N = 3$, $\mu = 1$ and $p = 2$. In the presence of non-constant electric and magnetic potentials, Cingolani *et al*. [13] showed that there exists a family of solutions having multiple concentration regions that are located around the minimum points of the potential. Assuming that electric and magnetic potentials are compatible with the action of a group G of linear isometries of $\mathbb{R}^3$, Cingolani *et al*. [15] showed that there is a combined effect of the symmetries and the potential V on the number of semiclassical solutions. Yang and Ding [43] considered (1.2). By using variational methods, they were able to obtain the existence of solutions with non-negative potentials. Very recently, in an interesting paper, Moroz and Van Schaftingen [32] used variational methods to develop a novel non-local penalization technique to show that equation (SNE) has a family of solutions concentrated at the local minimum of V, with V satisfying some additional assumptions at infinity.

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Motivated by the above references, in the present work we study the existence, multiplicity and concentration behaviour of positive solutions of the generalized Choquard equation (SNE). Since the methods applied here can also be used to study a more general case, instead of studying the semilinear case we consider the generalized quasi-linear Choquard equation driven by the p-Laplacian operator and general nonlinearity and conclude directly that the multiplicity and concentration behaviour of positive solutions still hold for the purely Laplacian case with power-type nonlinearity. In fact, we shall study the quasi-linear Choquard equation of the form

$$\left.\begin{aligned} -\varepsilon^p\Delta_p u + V(x)|u|^{p-2}u = \varepsilon^{\mu-N}\left(\frac{1}{|x|^\mu} * F(u)\right)f(u) \quad \text{in } \mathbb{R}^N,\\ u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N) \quad \text{with } 1 < p < N,\\ u(x) > 0 \quad \text{for all } x \in \mathbb{R}^N, \end{aligned}\right\} \tag{QNE}$$

where $\varepsilon > 0$, $0 < \mu < N$, F is the primitive function of f and the p-Laplacian operator Δ_p is defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

Here we assume that $p > 1$ and general hypotheses, which are satisfied by a large class of nonlinearities that includes u^q for $q > p$, hold on f. Except for a very special situation treated by Wei and Winter [41], little is known regarding the non-degeneracy of the ground-state solutions of the semilinear Choquard equation

$$-\Delta u + u = \left(\int_{\mathbb{R}^N} \frac{1}{|x|^\mu} * F(u)\right)f(u) \quad \text{in } \mathbb{R}^N,$$

and it is impossible to apply the arguments in [41] to investigate the existence and concentration behaviour of the solutions. We also point out that, for the quasi-linear Choquard equation

$$-\Delta_p u + |u|^{p-2}u = \left(\int_{\mathbb{R}^N} \frac{1}{|x|^\mu} * F(u)\right)f(u) \quad \text{in } \mathbb{R}^N,$$

whether its positive solutions are non-degenerate, unique under translation, radially symmetric and monotone decreasing about some point, or decay asymptotically at infinity is still an open problem. So we cannot apply the Lyapunov–Schmidt-type reduction arguments or the arguments in [12, 13, 15] to obtain the multiplicity or concentration of the semiclassical states for the quasi-linear Choquard equation.

To overcome this difficulty, we adapt the penalization method developed by del Pino and Felmer [18, 19]. However, now that we are working with a class of non-local problems, we have to be careful in our use of this method. We have proved new estimates involving this class of problems, which permit this use of the penalization method. Note that our arguments are totally different from those in [32], because in the latter the penalization method is associated with some Hardy-type inequalities, which are avoided in our paper. Here, we are able to prove the existence, multiplicity and concentration behaviour around the minimum point set of potential V. Our approach uses some abstract minimax theorems, such as the

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Lyusternik–Schnirelmann category, to obtain the multiplicity of positive solutions. Relating to the concentration behaviour for the semiclassical solutions of the non-local quasi-linear problem, since we are working with the p-Laplacian operator, which is nonlinear, we use the Moser iteration method to overcome some technical difficulties in order to show some estimates about the non-local term (for more details see §§ 5 and 6). Finally, our results are also new for $p = 2$ (more precisely, for (1.5)).

The potential V is a continuous function satisfying assumptions (V_1)–(V_3):

$$V(x) \geqslant V_0 = \inf_{x \in \mathbb{R}^N} V(x) > 0 \quad \text{for all } x \in \mathbb{R}^N. \tag{V_1}$$

There is an open and bounded domain $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 < \inf_{x \in \partial\Lambda} V(x) \tag{V_2}$$

and

$$M = \{x \in \mathbb{R}^N : V(x) = V_0\} \subset \Lambda. \tag{V_3}$$

Assuming $0 < \mu < p$, in order to prove the existence of positive solutions, the nonlinearity $f\colon \mathbb{R}^+ \to \mathbb{R}$ is a function of C^1 class and verifies the following conditions:

$$\lim_{s \to 0} \frac{|f(s)|}{s^{p-1}} = 0; \tag{f_1}$$

there exists $p < q < \frac{1}{2}p^*(2 - (\mu/N))$ such that

$$\lim_{s \to \infty} \frac{f(s)}{s^{q-1}} = 0, \tag{f_2}$$

where $p^* = Np/(N-p)$ is the critical exponent. Moreover, we also assume that there holds

$$f'(s)s^2 - (p-1)f(s)s > 0 \quad \forall s > 0. \tag{f_3}$$

To comment on the assumptions on the nonlinearity f, we recall an important inequality due to Hardy, Littlewood and Sobolev [26], which will be frequently used.

Proposition 1.1 (Hardy–Littlewood–Sobolev inequality). *Let $s, r > 1$ and $0 < \mu < N$ with $1/s + \mu/N + 1/r = 2$. Let $f \in L^s(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(s, N, \mu, r)$, independent of f and h, such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu} \leqslant C(s, N, \mu, r)|f|_s |h|_r.$$

Remark 1.2. In general, $F(s) = |s|^q$ for some $q > 0$. By the Hardy–Littlewood–Sobolev inequality,

$$\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u)\right) F(u)$$

is well defined if $F(u) \in L^s(\mathbb{R}^N)$ for $s > 1$ defined by

$$\frac{2}{s} + \frac{\mu}{N} = 2.$$

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Now we are working with $u \in W^{1,p}(\mathbb{R}^N)$, we require that $sq \in [p, p^*]$, and since we consider the subcritical case it must hold that

$$\frac{p}{2}\left(2 - \frac{\mu}{N}\right) < q < \frac{p^*}{2}\left(2 - \frac{\mu}{N}\right).$$

Here we only consider the case $q > p$.

REMARK 1.3. From assumption (f_3), since f is a function of C^1, the following monotonicity conditions hold:

$$s \to \frac{f(s)}{s^{p-1}} \text{ is strictly increasing on } (0, +\infty), \tag{f_3'}$$

and, consequently,

$$0 < pF(s) \leqslant f(s)s \quad \forall s > 0, \tag{f_4}$$

where

$$F(t) = \int_0^t f(s)\,\mathrm{d}s.$$

It also implies that the nonlinearity f verifies the well-known Ambrosetti–Rabinowitz-type superlinear condition for non-local problem, since, for $\theta := 2p > p$,

$$0 < \theta F(s) \leqslant 2f(s)s \quad \forall s > 0. \tag{f_5}$$

Recall that if Y is a closed set of a topological space X, we denote by $\mathrm{cat}_X(Y)$ the Lyusternik–Schnirelmann category of Y in X, namely the smallest number of closed and contractible sets in X which cover Y.

Our main result, theorem 1.4, establishes the existence of a multiplicity of solutions to the non-local problem involving the Lyusternik–Schnirelmann category of the sets M and M_δ, where

$$M_\delta = \{x \in \mathbb{R}^N : \mathrm{dist}(x, M) \leqslant \delta\} \quad \text{for } \delta > 0.$$

THEOREM 1.4. *If $0 < \mu < p$, suppose that the nonlinearity f satisfies* (f_1)–(f_3) *with $p < q < (N-\mu)p/(N-p)$, and potential V satisfies assumptions* (V_1)–(V_3). *Then, for any $\delta > 0$, there exists ε_δ such that* (QNE) *has at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions, for any $0 < \varepsilon < \varepsilon_\delta$. Moreover, if u_ε denotes one of these positive solutions with global maximum $\eta_\varepsilon \in \mathbb{R}^N$, then*

$$\lim_{\varepsilon \to 0} V(\eta_\varepsilon) = V_0.$$

Here, we make a few observations about the restrictions on the parameter $0 < \mu < p$ and $p < q < (N-\mu)p/(N-p)$ in the above theorem. In fact, to obtain the existence results for the autonomous case $V(x) = V_0$ for all $x \in \mathbb{R}^N$, we can weaken the conditions on the nonlinearities by assuming there are $C_0 > 0$ and $q_1, q_2 > \frac{1}{2}p$ with

$$\frac{p}{2}\left(2 - \frac{\mu}{N}\right) < q_1 \leqslant q_2 < \frac{p^*}{2}\left(2 - \frac{\mu}{N}\right)$$

such that, for all $s \in \mathbb{R}$,

$$|f(s)| \leqslant C_0(|s|^{q_1-1} + |s|^{q_2-1}). \tag{f_1'}$$

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However, to obtain the multiplicity and concentration of the solutions for the quasi-linear non-local problems when V satisfies (V_1)–(V_3), we need to put further restrictions on the exponent q to adapt the penalization method introduced by del Pino and Felmer in [18]. We will treat the non-local part $(1/|x|^{\mu}) * F(u)$ as a bounded term and introduce the monotone condition (f_3) on f at the same time. Using this idea, we introduce the assumption $0 < \mu < p$ and the range of the exponent $p < q < (N-\mu)p/(N-p)$ to make the penalization method applicable.

As a particular case, if $p = 2$, we directly obtain the multiplicity and concentration results for semilinear Choquard-type equation, which also complement the results obtained in [13, 15, 32].

Corollary 1.5. *Consider the semilinear Choquard equation in* $\mathbb{R}^3$*:*

$$\left.\begin{aligned} -\varepsilon^2\Delta u + V(x)u = \varepsilon^{\mu-3}\left(\frac{1}{|x|^{\mu}} * F(u)\right)f(u),\\ u \in H^1(\mathbb{R}^3), \quad u > 0. \end{aligned}\right\} \tag{SNE$'$}$$

If $0 < \mu < 2$*, suppose that the nonlinearity* f *satisfies* (f_1)–(f_3) *with* $2 < q < 2(3-\mu)$*, and the potential function* V *satisfies assumptions* (V_1)–(V_3)*. Then, for any* $\delta > 0$*, there exists* ε_δ *such that* (SNE$'$) *has at least* $\operatorname{cat}_{M_\delta}(M)$ *positive solutions for any* $0 < \varepsilon < \varepsilon_\delta$*. Moreover, if* u_ε *denotes one of these positive solutions with* $\eta_\varepsilon \in \mathbb{R}^3$ *its global maximum, then*

$$\lim_{\varepsilon\to 0} V(\eta_\varepsilon) = V_0.$$

1.1. Notation

C and C_i denote positive constants. B_R denotes the open ball centred at the origin with radius $R > 0$.

If u is a mensurable function, we denote by u_+ and u_- its positive and negative part respectively, i.e.

$$u_+(x) = \max\{u(x), 0\} \quad \text{and} \quad u_-(x) = \max\{-u(x), 0\}.$$

$C_0^{\infty}(\mathbb{R}^N)$ denotes the space of the functions that are infinitely differentiable with compact support in $\mathbb{R}^N$.

$E := W^{1,p}(\mathbb{R}^N)$ is the usual Sobolev space with norm

$$\|u\| := \left(\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p)\right)^{1/p}.$$

$L^s(\mathbb{R}^N)$, for $1 \leqslant s \leqslant \infty$, denotes the Lebesgue space with the norms

$$|u|_s := \left(\int_{\mathbb{R}^N} |u|^s\right)^{1/s}.$$

From the assumptions on V, it follows that

$$\|u\|_\varepsilon := \left(\int_{\mathbb{R}^N} (|\nabla u|^p + V(\varepsilon x)|u|^p)\right)^{1/p}$$

is an equivalent norm in E.

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Let X be a real Banach space and let $I\colon X \to \mathbb{R}$ be a functional of class $\mathcal{C}^1$. We say that $(u_n) \subset X$ is a Palais–Smale (PS) sequence at c (henceforth denoted $(\mathrm{PS})_c$) for I if (u_n) satisfies

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as } n \to \infty.$$

Moreover, I satisfies the PS condition at c if any PS sequence at c possesses a convergent subsequence. If $\mathcal{N}$ is a C^1-manifold of X and $I\colon \mathcal{N} \to \mathbb{R}$ is a C^1-functional, we say that $I|_{\mathcal{N}}$ satisfies $(\mathrm{PS})_c$ if any sequence $(u_n) \subset \mathcal{N}$ such that

$$I(u_n) \to c \quad \text{and} \quad \|I'(u_n)\|_* \to 0$$

contains a convergent subsequence. Here, denote by $\|I'(u_n)\|_*$ the norm of the derivative of I restricted to $\mathcal{N}$ at the point u.

2. The limit problem

As far as we know, there are no results about the p-Laplacian equation with nonlocal nonlinearities. Thus, we need to show some results about the limit problem. Our intention in this section is to show some results involving the quasi-linear problem. To this end, we begin our study by considering the quasi-linear problem

$$\left.\begin{aligned} -\Delta_p u + A|u|^{p-2}u = \left(\frac{1}{|x|^\mu} * F(u)\right) f(u) \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \quad 1 < p < N, \end{aligned}\right\} \tag{2.1}$$

where A is a positive constant.

In order to find positive solutions, we shall henceforth consider $f(s) = 0$ for all $s \leqslant 0$.

The corresponding energy functional associated with problem (2.1) is defined by

$$L_A(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + A|u|^p) - \frac{1}{2}\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u)\right) F(u).$$

From the growth assumptions on f and remark 1.2, the Hardy–Littlewood–Sobolev inequality implies that L_A is well defined on E and belongs to C^1, with its derivative given by

$$L_A'(u)\varphi = \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla\varphi + A|u|^{p-2}u\varphi) - \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u)\right) f(u)\varphi \quad \forall u, \varphi \in E.$$

Therefore, it is easy to see that all the solutions of (2.1) correspond to critical points of the energy functional, L_A.

Next, let us denote by $\mathcal{N}_A$ the Nehari manifold associated to L_A, given by

$$\mathcal{N}_A = \{u \in E\colon u \neq 0,\ L_A'(u)u = 0\}$$

or

$$\mathcal{N}_A = \left\{u \in E\colon u \neq 0,\ \int_{\mathbb{R}^N} (|\nabla u|^p + A|u|^p) = \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u)\right) f(u)u\right\}.$$

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From assumptions (f_1) and (f_2), for any $\xi > 0$ there exists $C_\xi > 0$ such that

$$|F(s)| \leqslant \xi |s|^p + C_\xi |s|^q.$$

By using the Hardy–Littlewood–Sobolev inequality, for $u \in \mathcal{N}_A$, we have

$$\|u\|^p \leqslant C(\|u\|^{2p} + \|u\|^{p+q} + \|u\|^{2q}).$$

Thus, there exists $\alpha > 0$ such that

$$\|u\| \geqslant \alpha \quad \forall u \in \mathcal{N}_A. \tag{2.2}$$

Setting

$$J_A(u) = \int_{\mathbb{R}^N} (|\nabla u|^p + A|u|^p) - \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u)\right) f(u)u,$$

we have $\langle J_A'(u), u\rangle < 0$ and, from remark 1.3, standard arguments show that $\mathcal{N}_A$ is a complete manifold of codimension 1 in E.

The following lemma is a revised version of the corresponding lemma in [4], which we sketch here for the reader's convenience.

LEMMA 2.1. *L_A satisfies the mountain pass geometry, that is,*

(1) *there exist $\rho, \delta_0 > 0$ such that $L_A|_S \geqslant \delta_0 > 0$ for all $u \in S = \{u \in E \colon \|u\| = \rho\}$,*

(2) *there exist $r > 0$ and e with $\|e\| > r$ such that $L_A(e) < 0$.*

Proof.
(1) From the growth assumptions on f and the Hardy–Littlewood–Sobolev inequality, we derive

$$L_A(u) \geqslant \frac{1}{p}\|u\|^p - C(\|u\|^{2p} + \|u\|^{2q}).$$

Since $q > p$, (1) follows if we choose ρ small enough.

(2) Fixing $u_0 \in E$ with $u_0^+(x) = \max\{u_0(x), 0\}$, we set

$$g(t) = \mathfrak{F}\left(\frac{tu_0}{\|u_0\|}\right) > 0 \quad \text{for } t > 0,$$

where

$$\mathfrak{F}(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u)\right) F(u). \tag{2.3}$$

By the Ambrosetti–Rabinowitz condition (f_2),

$$\frac{g'(t)}{g(t)} \geqslant \frac{2p}{t} \quad \text{for all } t > 0.$$

Integrating this over $[1, s\|u_0\|]$ with $s > 1/\|u_0\|$, we find

$$\mathfrak{F}(su_0) \geqslant \mathfrak{F}\left(\frac{u_0}{\|u_0\|}\right)\|u_0\|^{2p} s^{2p}.$$

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Therefore,

$$L_A(su_0) \leqslant C_1 s^p - C_2 s^{2p} \quad \text{for } s > \frac{1}{\|u_0\|},$$

and (2) holds for $e = su_0$ and s large enough. □

THEOREM 2.2. *Suppose that* (f_1)–(f_3) *hold with* $p < q < (N-\mu)p/(N-p)$. *Then, for any* $A > 0$, *problem (2.1) has a positive ground-state solution.*

Proof. By the mountain pass theorem without the PS condition, there exists a PS sequence $(u_n) \subset E$ such that

$$L'_A(u_n) \to 0, \qquad L_A(u_n) \to m_A,$$

where the minimax value m_A can be characterized by

$$0 < m_A := \inf_{u \in E \setminus \{0\}} \max_{t \geqslant 0} L_A(tu) = \inf_{u \in \mathcal{N}_A} L_A(u). \tag{2.4}$$

It is easy to see that (u_n) is bounded in E. Moreover, we claim that there exist $r, \delta > 0$ and a sequence $(y_n) \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |u_n|^p \geqslant \delta.$$

If the above claim does not hold for (u_n), by Lions's result we must have that

$$u_n \to 0 \in L^s(\mathbb{R}^N) \quad \text{for } p < s < p^*.$$

Recalling that, for any $\xi > 0$ there exists $C_\xi > 0$ such that

$$|F(s)| \leqslant \xi |s|^p + C_\xi |s|^q \quad \forall s \geqslant 0,$$

it follows from the Hardy–Littlewood–Sobolev inequality that

$$\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(u_n)\right) f(u_n) u_n \to 0,$$

leading to $\|u_n\| \to 0$, which contradicts (2.4), showing that the claim holds. Fixing $v_n = u_n(\cdot - y_n)$, we derive

$$\int_{B_r(0)} |v_n|^p \geqslant \tfrac{1}{2}\delta. \tag{2.5}$$

Since L_A and L'_A are both invariant by translation, it also holds that

$$L'_A(v_n) \to 0 \quad \text{and} \quad L_A(v_n) \to m_A.$$

Observing that (v_n) is also bounded, we may assume $v_n \rightharpoonup v$ in E, $v_n(x) \to v(x)$ almost everywhere (a.e.) in $\mathbb{R}^N$, $v_n \to v$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ and $v \neq 0$ by (2.5).

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For $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned}\int_{\mathbb{R}^N} (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v, \nabla v_n - \nabla v)\varphi \\ = -\int_{\mathbb{R}^N} (|\nabla v|^{p-2}\nabla v, \nabla v_n - \nabla v)\varphi \\ - \int_{\mathbb{R}^N} (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v, \nabla \varphi)(v_n - v) \\ - A\int_{\mathbb{R}^N} |v_n|^{p-2}(v_n - v)\varphi + \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(v_n)\right) f(v_n)(v_n - v)\varphi.\end{aligned} \tag{2.6}$$

It is easy to see that

$$\left.\begin{aligned}\int_{\mathbb{R}^N} (|\nabla v|^{p-2}\nabla v, \nabla v_n - \nabla v)\varphi \to 0, \\ \int_{\mathbb{R}^N} (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v, \nabla\varphi)(v_n - v) \to 0\end{aligned}\right\} \tag{2.7}$$

and

$$\int_{\mathbb{R}^N} |v_n|^{p-2}(v_n - v)\varphi \to 0 \tag{2.8}$$

as n goes to ∞. For the non-local term, from the growth condition, we know that $F(v_n)$ is bounded in $L^{2N/(2N-\mu)}(\mathbb{R}^N)$. Moreover, since $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$, the continuity of F gives $F(v_n(x)) \to F(v(x))$ a.e. in $\mathbb{R}^N$. Therefore, $F(v_n)$ must converge weakly to $F(v)$ in $L^{2N/(2N-\mu)}(\mathbb{R}^N)$. Using the Hardy–Littlewood–Sobolev inequality, we know the convolution term

$$\frac{1}{|x|^\mu} * w(x) \in L^{2N/\mu}(\mathbb{R}^N)$$

for all $w \in L^{2N/(2N-\mu)}(\mathbb{R}^N)$; this is a linear bounded operator from $L^{2N/(2N-\mu)}(\mathbb{R}^N)$ to $L^{2N/\mu}(\mathbb{R}^N)$. Consequently,

$$\frac{1}{|x|^\mu} * F(v_n(y)) \rightharpoonup \frac{1}{|x|^\mu} * F(v(y)) \quad \text{in } L^{2N/\mu}(\mathbb{R}^N).$$

Using the fact f has a subcritical growth, we have

$$\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(v_n)\right) f(v_n)(v_n - v)\varphi \to 0. \tag{2.9}$$

The following inequalities are very useful [17]. For $p \geqslant 2$ and $\xi, \eta \in \mathbb{R}^N$,

$$\left.\begin{aligned}\langle |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\rangle \geqslant d_1|\xi - \eta|^p, \\ ||\xi|^{p-2} - |\eta|^{p-2}| \leqslant d_2(|\xi|^{p-2} + |\eta|^{p-2})|\xi - \eta|.\end{aligned}\right\} \tag{2.10}$$

For $1 < p < 2$ and $\xi, \eta \in \mathbb{R}^N$,

$$\left.\begin{aligned}\langle |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\rangle \geqslant d_3(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2, \\ ||\xi|^{p-2} - |\eta|^{p-2}| \leqslant d_4|\xi - \eta|^{p-1}.\end{aligned}\right\} \tag{2.11}$$

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Here d_1, d_2, d_3 and d_4 are some constants. For $1 < p \leqslant 2$, from (2.6)–(2.9) and (2.11), we know that

$$\begin{aligned}\biggl(\int_{\mathbb{R}^N} |\nabla v_n - \nabla v|^p \varphi\biggr)^{2/p} &\leqslant \biggl(\int_{\mathbb{R}^N} \frac{|\nabla v_n - \nabla v|^2}{(|\nabla v_n| + |\nabla v|)^{2-p}}\varphi\biggr)\biggl(\int_{\mathbb{R}^N} (|\nabla v_n + |\nabla v|^2)^p \varphi\biggr)^{(2-p)/p} \\ &\leqslant C\int_{\mathbb{R}^N} \langle |\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v, \nabla v_n - \nabla v\rangle \varphi \to 0. \qquad (2.12)\end{aligned}$$

Similarly, we can prove the same local convergence property for the case $p \geqslant 2$. The above limits imply that for some subsequence of (v_n) we have

$$\nabla v_n(x) \to \nabla v(x) \quad \text{a.e. } x \in \mathbb{R}^N.$$

These limits allow us to conclude that $L'_A(v) = 0$. Using the definition of m_A together with Fatou's lemma, we also deduce that $L_A(v) = m_A$. Moreover, by choosing v_- as a test function, and recalling that $f(s) = 0$ for all $s \leqslant 0$ and $L'_A(v)v_- = 0$, a direct computation gives $v \geqslant 0$. Now, if $\mu < p$ and $p < q_1 \leqslant q_2 < (N-\mu)p/(N-p)$, we claim that $v \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$. Indeed, setting

$$K(x) := \frac{1}{|x|^\mu} * F(v) = \int_{\mathbb{R}^N} \frac{F(v)}{|x-y|^\mu},$$

we first claim there exists $C > 0$ such that

$$|K(x)| \leqslant C \quad \forall x \in \mathbb{R}^N. \qquad (2.13)$$

Since, for some C_0,

$$|F(v)| \leqslant C_0(|v|^p + |v|^q)$$

holds, we derive

$$\begin{aligned}|K(x)| &= \biggl|\int_{\mathbb{R}^N} \frac{F(v)}{|x-y|^\mu}\biggr| \\ &= \biggl|\int_{|x-y|\leqslant 1} \frac{F(v)}{|x-y|^\mu}\biggr| + \biggl|\int_{|x-y|\geqslant 1} \frac{F(v)}{|x-y|^\mu}\biggr| \\ &\leqslant C_0\int_{|x-y|\leqslant 1} \frac{|v|^p + |v|^q}{|x-y|^\mu} + C_0\int_{|x-y|\geqslant 1} (|v|^p + |v|^q) \\ &\leqslant C_0\int_{|x-y|\leqslant 1} \frac{|v|^p + |v|^q}{|x-y|^\mu} + C.\end{aligned}$$

Here, we have used the fact that $q \in (p, p^*)$. Choosing

$$t \in \biggl(\frac{N}{N-\mu}, \frac{N}{N-p}\biggr] \quad \text{and} \quad s \in \biggl(\frac{N}{N-\mu}, \frac{Np}{(N-p)q}\biggr],$$

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it follows from the Hölder inequality that

$$\int_{|x-y|\leqslant 1} \frac{|v|^p}{|x-y|^\mu} \leqslant \left(\int_{|x-y|\leqslant 1} |v|^{tp}\right)^{1/t} \left(\int_{|x-y|\leqslant 1} \frac{1}{|x-y|^{t\mu/(t-1)}}\right)^{(t-1)/t}$$
$$\leqslant C_1 \left(\int_{|r|\leqslant 1} |r|^{N-1-t\mu/(t-1)}\,\mathrm{d}r\right)^{(t-1)/t}$$

and

$$\int_{|x-y|\leqslant 1} \frac{|v|^q}{|x-y|^\mu} \leqslant \left(\int_{|x-y|\leqslant 1} |v|^{sq}\right)^{1/s} \left(\int_{|x-y|\leqslant 1} \frac{1}{|x-y|^{s\mu/(s-1)}}\right)^{(s-1)/s}$$
$$\leqslant C_2 \left(\int_{|r|\leqslant 1} |r|^{N-1-s\mu/(s-1)}\,\mathrm{d}r\right)^{(s-1)/s}.$$

Since both $N-1-t\mu/(t-1) > -1$ and $N-1-s\mu/(s-1) > -1$, there exists $C_3 > 0$ such that

$$\int_{|x-y|\leqslant 1} \frac{|v|^p + |v|^q}{|x-y|^\mu} \leqslant C_3 \quad \forall x \in \mathbb{R}^N,$$

proving (2.13).

From the above arguments, v is a solution of the quasi-linear problem

$$-\Delta_p v + A|v|^{p-2}v = K(x)f(v) \quad \text{in } \mathbb{R}^N$$

with $K \in L^\infty(\mathbb{R}^N)$ and f is a function with subcritical growth. Adapting some arguments found in [4,24], we can show that there exists $C > 0$ such that $\|v\|_{L^\infty} < C$ and v decays to zero as $|x| \to \infty$. Then, by regularity theory, there exists $\alpha \in (0,1)$ such that $v \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. Now, we can apply Harnack's inequality to conclude that $v(x) > 0$ in $\mathbb{R}^N$. □

LEMMA 2.3. *Suppose that* (f_1)–(f_3) *hold with* $p < q < (N-\mu)p/(N-p)$. *Let* u *be the solution obtained in theorem 2.2. Then there exist* $C, \beta > 0$ *such that*

$$|u(x)| \leqslant C\exp(-\beta|x|) \quad \forall x \in \mathbb{R}^N.$$

Proof. Since $K(x) := (1/|x|^\mu) * F(u) \leqslant C$, using assumption ($\mathrm{f}_1$) and the fact that the solution u decays uniformly to zero as $|x| \to +\infty$, we can take $\rho_0 > 0$ such that

$$\left(\frac{1}{|x|^\mu} * F(u)\right) f(u(x))u^{1-p}(x) \leqslant \tfrac{1}{2}A$$

for all $|x| \geqslant \rho_0$. Consequently,

$$-\Delta_p u(x) + \tfrac{1}{2}Au^{p-1}(x) = \left(\frac{1}{|x|^\mu} * F(u)\right) f(u(x)) - \tfrac{1}{2}Au^{p-1} \leqslant 0$$

for all $|x| \geqslant \rho_0$. Let s and T be positive constants such that

$$(p-1)s^p < \tfrac{1}{2}A \quad \text{and} \quad u(x) \leqslant T\exp(-s\rho_0) \text{ for all } |x| = \rho_0.$$

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Hence, the function $\psi(x) = T\exp(-s|x|)$ satisfies

$$-\Delta_p \psi + \tfrac{1}{2}A\psi^{p-1} \geqslant (\tfrac{1}{2}A - (p-1)s^p)\psi^{p-1} > 0$$

for all $x \neq 0$. Since $p > 1$,

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geqslant 0$$

for all $x, y \in \mathbb{R}^N$. We now take as a test function $\eta = \max\{u - \psi, 0\} \in W_0^{1,p}(|x| > \rho_0)$. Hence, combining these estimates yields

$$\begin{aligned} 0 &\geqslant \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla \eta + \tfrac{1}{2}Au^{p-1}\eta) \\ &\geqslant \int_{\mathbb{R}^N} [(|\nabla u|^{p-2}\nabla u - |\nabla \psi|^{p-2}\nabla \psi)\nabla \eta + \tfrac{1}{2}A(u^{p-1} - \psi^{p-1})\eta] \\ &\geqslant \tfrac{1}{2}A \int_{\{x \in \mathbb{R}^N :\, u \geqslant \psi\}} (u^{p-1} - \psi^{p-1})(u - \psi)\,\mathrm{d}x \geqslant 0 \end{aligned}$$

for all $|x| > \rho_0$. Therefore, the set $\Omega := \{x \in \mathbb{R}^N : |x| > \rho_0 \text{ and } u \geqslant \psi(x)\}$ is empty. □

3. The penalized problem

By changing variables, it is possible to see problem (QNE) is equivalent to the perturbed problem

$$\left.\begin{aligned} -\Delta_p u + V(\varepsilon x)|u|^{p-2}u = \left(\frac{1}{|x|^\mu} * F(u)\right)f(u) \quad \text{in } \mathbb{R}^N, \\ u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), \qquad u(x) > 0 \quad \text{for all } x \in \mathbb{R}^N. \end{aligned}\right\} \tag{QNE*}$$

Moreover, without loss of generality, we can assume that

$$V(0) = \min_{x \in \mathbb{R}^N} V(x) = V_0.$$

In what follows, the energy functional associated with (QNE*) is given by

$$J_\varepsilon(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + V(\varepsilon x)|u|^p) - \mathfrak{F}(u),$$

where $\mathfrak{F}$ was given in (2.3).

In order to overcome the lack of compactness of the problem (QNE), we shall adapt the penalization method introduced by del Pino and Felmer in [18] to treat the non-local problems. So, we choose $\ell > 2$ to be determined later, and take $a > 0$ to be the unique number such that $f(a)/a^{p-1} = V_0/\ell$. We set

$$\hat{f} := \begin{cases} f(s) & \text{if } s \leqslant a, \\ \dfrac{V_0}{\ell}s^{p-1} & \text{if } s \geqslant a. \end{cases}$$

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Let $0 < t_a < a < T_a$ and take a function $\eta \in C_0^\infty(\mathbb{R}, \mathbb{R})$ such that

(η_1) $\eta(s) \leqslant \hat{f}(s)$ for all $s \in [t_a, T_a]$,

(η_2) $\eta(t_a) = \hat{f}(t_a)$, $\eta(T_a) = \hat{f}(T_a)$, $\eta'(t_a) = \hat{f}'(t_a)$ and $\eta'(T_a) = \hat{f}'(T_a)$,

(η_3) the map $s \to \eta(s)/s^{p-1}$ is non-decreasing for all $s \in [t_a, T_a]$.

By using the above functions we can define $\tilde{f} \in C^1(\mathbb{R}, \mathbb{R})$ as follows:

$$\tilde{f} := \begin{cases} \hat{f}(s) & \text{if } s \notin [t_a, T_a], \\ \eta(s) & \text{if } s \in [t_a, T_a]. \end{cases}$$

Letting $\mathcal{X}_\Lambda$ denote the characteristic function of the set Λ, we introduce the penalized nonlinearity $g \colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by setting

$$g(x, s) := \mathcal{X}_\Lambda(x) f(s) + (1 - \mathcal{X}_\Lambda(x)) \tilde{f}(s). \tag{3.1}$$

Note that, by (f_1)–(f_4) and (η_1)–(η_3), it is easy to check that $g(x, s)$ satisfies the following properties:

$$\lim_{|s| \to 0} \frac{g(x, s)}{s^{p-1}} = 0. \tag{g_1}$$

There exists $p < q < p^*/2(2 - \mu/N)$ such that

$$\lim_{s \to \infty} \frac{g(x, s)}{s^{q-1}} = 0. \tag{g_2}$$

Moreover,

$$0 < 2pG(x, s) := 2p \int_0^s g(x, \tau)\, \mathrm{d}\tau \leqslant 2g(x, s)s \quad \text{for all } x \in \Lambda,\ s > 0, \tag{g_3)$_{\mathrm{i}}$$

and

$$0 < pG(x, s) \leqslant g(x, s)s \leqslant \frac{V_0}{\ell} s^p \quad \text{for all } x \in \mathbb{R}^N \setminus \Lambda,\ s > 0. \tag{g_3)$_{\mathrm{ii}}$$

Finally, we also have that

$$s \to \frac{g(x, s)s}{s^{p/2}} \text{ and } \frac{G(x, s)}{s^{p/2}} \text{ are both increasing for all } x \in \mathbb{R}^N,\ s > 0. \tag{g_4}$$

Remark 3.1. It is easy to check that if u_ε is a positive solution of the equation

$$\left.\begin{aligned} -\varepsilon^p \Delta_p u + V(x)|u|^{p-2}u = \varepsilon^{\mu - N}\left(\frac{1}{|x|^\mu} * G(x, u)\right) g(x, u) \quad \text{in } \mathbb{R}^N, \\ u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), \qquad u(x) > 0 \quad \text{for all } x \in \mathbb{R}^N \end{aligned}\right\} \tag{APE}$$

such that $u_\varepsilon(x) \leqslant t_a$ for all $x \in \mathbb{R}^N \setminus \Lambda$, then $g(x, u_\varepsilon) = f(u_\varepsilon)$ and therefore u_ε is also a solution of problem (QNE).

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In view of the remark above, we deal in the following with the penalized problem

$$\left.\begin{aligned} -\Delta_p u + V(\varepsilon x)|u|^{p-2}u = \left(\frac{1}{|x|^{\mu}} * G(\varepsilon x, u)\right) g(\varepsilon x, u) \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \qquad u(x) > 0 \quad \text{for all } x \in \mathbb{R}^N, \end{aligned}\right\} \tag{APE*}$$

and we shall look for solutions u_ε of problem (APE*) verifying

$$u_\varepsilon(x) \leqslant t_a \quad \text{for all } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon, \tag{3.2}$$

where

$$\Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}.$$

In what follows, the energy functional associated with (APE*) is given by

$$I_\varepsilon(u) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + V(\varepsilon x)|u|^p) - \Sigma_\varepsilon(u),$$

where

$$\Sigma_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(\frac{1}{|x|^{\mu}} * G(\varepsilon x, u)\right) G(\varepsilon x, u).$$

Moreover, the Nehari manifold associated to I_ε will be denoted by $\mathcal{N}_\varepsilon$, i.e.

$$\mathcal{N}_\varepsilon = \left\{ u \in E : u \neq 0, \ \int_{\mathbb{R}^N} (|\nabla u|^p + V(\varepsilon x)|u|^p) = \int_{\mathbb{R}^N} \left(\frac{1}{|x|^{\mu}} * G(\varepsilon x, u)\right) g(\varepsilon x, u)u \right\}.$$

Note that $0 < 2pG(x, s) \leqslant 2g(x, s)s$ always holds for all $x \in \mathbb{R}^N$, $s > 0$.

Consequently, for any non-negative function $u \neq 0 \in E$, similar arguments to those explored in the proof of lemma 2.1 can again be used to show that

$$I_\varepsilon(tu) \leqslant C_1 t^p - C_2 t^{2p} \quad \text{for } t > \frac{1}{\|u\|}.$$

Then, using the monotone condition (g$_4$), there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\varepsilon$ and $I_\varepsilon(t_u u) = \max_{t \geqslant 0} I_\varepsilon(tu)$. Setting

$$\mathcal{Y}_\varepsilon(u) = \left\{ \int_{\mathbb{R}^N} (|\nabla u|^p + V(\varepsilon x)|u|^p) - \int_{\mathbb{R}^N} \left(\frac{1}{|x|^{\mu}} * G(\varepsilon x, u)\right) g(\varepsilon x, u)u \right\},$$

we shall show that $\mathcal{N}_\varepsilon$ is a complete manifold of codimension 1 in E. In fact, note that $\tilde{f}(s) \leqslant f(s)$ for all $s > 0$. By using the Hardy–Littlewood–Sobolev inequality again, for $u \in \mathcal{N}_\varepsilon$, we obtain

$$\|u\|^p \leqslant C(\|u\|^{2p} + \|u\|^{2q}).$$

Then, there exists $\alpha_0 > 0$ such that

$$\|u\| \geqslant \alpha_0 \quad \forall u \in \mathcal{N}_\varepsilon. \tag{3.3}$$

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From (f_3) and (g_2),

$$
\begin{aligned}
\mathcal{Y}'_\varepsilon(u)u &= p\|u\|^p - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{1}{|x-y|^\mu}\right)(g(\varepsilon y,u)u(y)g(\varepsilon x,u)u(x) \\
&\qquad + G(\varepsilon y,u)g'(\varepsilon x,u)u^2(x) \\
&\qquad + G(\varepsilon y,u)g(\varepsilon x,u)u(x)) \\
&\leqslant -\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G(\varepsilon x,u)\right)[g(\varepsilon x,u)u + g'(\varepsilon x,u)u^2] \\
&= -\int_{\Lambda_\varepsilon \cup \{u<t_a\}}\left(\frac{1}{|x|^\mu} * G(\varepsilon x,u)\right)\{f(u)u + f'(u)u^2\} \\
&\quad - \int_{(\mathbb{R}^N\setminus\Lambda_\varepsilon)\cap\{t_a\leqslant u\leqslant T_a\}}\left(\frac{1}{|x|^\mu} * G(\varepsilon x,u)\right)\{\eta(u)u + \eta'(u)u^2\} \\
&\quad - \frac{V_0 p}{\ell}\int_{(\mathbb{R}^N\setminus\Lambda_\varepsilon)\cap\{u\geqslant T_a\}}\left(\frac{1}{|x|^\mu} * G(\varepsilon x,u)\right)|u|^p \\
&\leqslant -p\int_{\Lambda_\varepsilon \cup \{u<t_a\}}\left(\frac{1}{|x|^\mu} * G(\varepsilon x,u)\right)f(u)u \\
&\quad - p\int_{(\mathbb{R}^N\setminus\Lambda_\varepsilon)\cap\{t_a\leqslant u\leqslant T_a\}}\left(\frac{1}{|x|^\mu} * G(\varepsilon x,u)\right)\eta(u)u \\
&\quad - \frac{V_0 p}{\ell}\int_{(\mathbb{R}^N\setminus\Lambda_\varepsilon)\cap\{u\geqslant T_a\}}\left(\frac{1}{|x|^\mu} * G(\varepsilon x,u)\right)|u|^p. \qquad (3.4)
\end{aligned}
$$

Since $u \neq 0$, at least one of the following three cases must hold:

$$
\begin{aligned}
\operatorname{supp} u \cap \{\Lambda_\varepsilon \cup \{u < t_a\}\} &\neq \emptyset, \\
\operatorname{supp} u \cap \{(\mathbb{R}^N \setminus \Lambda_\varepsilon) \cap \{t_a \leqslant u \leqslant T_a\}\} &\neq \emptyset, \\
\operatorname{supp} u \cap \{(\mathbb{R}^N \setminus \Lambda_\varepsilon) \cap \{u \geqslant T_a\}\} &\neq \emptyset.
\end{aligned}
$$

Thus,

$$\mathcal{Y}'_\varepsilon(u)u < 0,$$

which means the manifold $\mathcal{N}_\varepsilon$ is a natural constraint for I_ε.

4. Palais–Smale condition for the penalized problem

In this section, we shall establish a convergence criterion for the PS sequences of the energy functional of the penalized problem (APE*). Recall that the energy functional associated with (APE*) is given by

$$I_\varepsilon(u) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + V(\varepsilon x)|u|^p) - \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G(\varepsilon x,u)\right)G(\varepsilon x,u)$$

and the Nehari manifold $\mathcal{N}_\varepsilon$ is

$$\mathcal{N}_\varepsilon = \left\{u \in E\colon u \neq 0,\ \int_{\mathbb{R}^N}(|\nabla u|^p + V(\varepsilon x)|u|^p) = \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G(\varepsilon x,u)\right)g(\varepsilon x,u)u\right\}. \qquad (4.1)$$

The same arguments as in lemma 2.1 prove the ensuing result.

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Lemma 4.1. *The penalized functional I_ε has a PS sequence $(u_n) \subset E$ at m_ε, where the minimax value m_ε is defined by*

$$m_\varepsilon := \inf_{u \in E \setminus \{0\}} \max_{t \geqslant 0} I_\varepsilon(tu) = \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u), \tag{4.2}$$

that is,

$$I'_\varepsilon(u_n) \to 0 \quad \text{and} \quad I_\varepsilon(u_n) \to m_\varepsilon.$$

Moreover, there exists a constant $\kappa > 0$, independent of ε, ℓ and a, such that $m_\varepsilon < \kappa$ for all ε small.

Proof. We only need to check I_ε satisfies the mountain pass geometry.

Claim 4.2. *There exist $\rho, \delta_0 > 0$ such that $I_\varepsilon|_S \geqslant \delta_0 > 0$ for all $u \in S = \{u \in E \colon \|u\| = \rho\}$.*

From the growth assumptions (g_1) and (g_2) on $g(x, s)$ and the Hardy–Littlewood–Sobolev inequality, we derive

$$I_\varepsilon(u) \geqslant \frac{1}{p}\|u\|^p - C(\|u\|^{2p} + \|u\|^{2q}).$$

The claim follows if we choose ρ small enough.

Claim 4.3. *There exist $r > 0$ and e with $\|e\| > r$ such that $I_\varepsilon(e) < 0$.*

Take a positive function $\psi \in E$ with $\operatorname{supp} \psi \subset \varLambda$ and observe that

$$G(\varepsilon x, \psi) = F(\psi).$$

Similar to the arguments in lemma 2.1, there exist two positive constants C_1, C_2 independent of ε, M, a, such that

$$I_\varepsilon(s\psi) \leqslant C_1 s^p - C_2 s^{2p} \quad \text{for } s > \frac{1}{\|\psi\|},$$

showing that claim 4.3 is true for $e = s\psi$ and s large enough.

Using the mountain pass theorem without the PS condition, we get the existence of a $(\mathrm{PS})_{m_\varepsilon}$ sequence $(u_n) \subset E$ with

$$m_\varepsilon := \inf_{u \in E \setminus \{0\}} \max_{t \geqslant 0} I_\varepsilon(tu) = \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u).$$

Recalling that $\operatorname{supp} \psi \subset \varLambda$, it is easy to check the existence of constant $\kappa > 0$, independent of ε, ℓ, a, such that $m_\varepsilon < \kappa$ for all ε small. □

Before to prove our next result, we need to fix the ensuing notation:

$$\mathcal{B} := \left\{ u \in W^{1,p}(\mathbb{R}^N) \colon \|u\|^p \leqslant \frac{2p}{p-1}(\kappa + 1) \right\} \tag{4.3}$$

and

$$\tilde{K}_\varepsilon(u)(x) := \frac{1}{|x|^\mu} * G(\varepsilon x, u).$$

With the above notation, we are able to show the following result.

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LEMMA 4.4. *Suppose that* (f_1)–(f_3) *occur with* $p < q < (N-\mu)p/(N-p)$. *Then there exists* ℓ_0 *such that*

$$\frac{\sup_{u\in\mathcal{B}} |\tilde{K}_\varepsilon(u)(x)|_{L^\infty(\mathbb{R}^N)}}{\ell_0} < \frac{1}{2} \quad \textit{for all } \varepsilon.$$

Proof. Note that

$$|G(\varepsilon x, u)| \leqslant |F(u)| \leqslant C(|u|^p + |u|^q) \quad \text{for all } \varepsilon.$$

Repeating the arguments used in the proof of (2.13), we find a positive constant C_0 such that

$$\sup_{u\in\mathcal{B}} |\tilde{K}_\varepsilon(u)(x)|_{L^\infty(\mathbb{R}^N)} \leqslant C_0. \tag{4.4}$$

From this, there exists $\ell_0 > 0$ such that

$$\frac{\sup_{u\in\mathcal{B}} |\tilde{K}_\varepsilon(u)(x)|_{L^\infty(\mathbb{R}^N)}}{\ell_0} \leqslant \frac{C_0}{\ell_0} \leqslant \frac{1}{2}. \tag{4.5}$$

□

Now take $a > 0$ to be the unique number such that $f(a)/a^{p-1} = V_0/\ell_0$ and consider the penalized problem with nonlinearity defined in (3.1).

LEMMA 4.5. *Assume* (V_1)–(V_3) *and* (f_1)–(f_3) *with* $p < q < (N-\mu)p/(N-p)$ *and let* (u_n) *be a* $(\mathrm{PS})_c$ *sequence for* I_ε *with* $c \in [m_\varepsilon, \kappa]$. *Then, for each* $\zeta > 0$ *there exists* $R = R(\zeta) > 0$, *verifying*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R} (|\nabla u_n|^p + V(\varepsilon x)|u_n|^p) < \zeta.$$

Proof. Once we obtain that (u_n) is a PS sequence of I_ε at c, it is easy to see (u_n) is bounded. In fact,

$$\begin{aligned} I_\varepsilon(u_n) - \frac{1}{2p} I'_\varepsilon(u_n)u_n &= \left(\frac{1}{2} - \frac{1}{2p}\right) \int_{\mathbb{R}^N} (|\nabla u_n|^p + V(\varepsilon x)|u_n|^p) \\ &\quad + \frac{1}{2p} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, u_n)\right) g(\varepsilon x, u_n)u_n \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, u_n)\right) G(\varepsilon x, u_n) \\ &\geqslant \left(\frac{1}{2} - \frac{1}{2p}\right) \int_{\mathbb{R}^N} (|\nabla u_n|^p + V(\varepsilon x)|u_n|^p). \end{aligned}$$

Consequently, there exists $n_0 \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^N} (|\nabla u_n|^p + V_0|u_n|^p) \leqslant \frac{2p}{p-1}(\kappa + 1), \quad n \geqslant n_0,$$

and so, by lemma 4.4,

$$\frac{\sup_{n\geqslant n_0} |\tilde{K}(u_n)(x)|_{L^\infty(\mathbb{R}^N)}}{\ell_0} \leqslant \frac{1}{2}.$$

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For $R > 0$, let $\eta_R \in C^\infty(\mathbb{R}^N)$ be such that $\eta_R(x) = 0$ if $x \in B_{R/2}(0)$ and $\eta_R(x) = 1$ if $x \notin B_R(0)$, with $0 \leqslant \eta_R(x) \leqslant 1$ and $|\nabla \eta_R(x)| \leqslant C/R$, where C is a constant independent of R. Note that

$$\int_{\mathbb{R}^N} \eta_R(|\nabla u_n|^p + V(\varepsilon x)|u_n|^p) = I'_\varepsilon(u_n)(u_n\eta_R) + \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, u_n)\right) g(\varepsilon x, u_n) u_n \eta_R - \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} u_n \nabla u_n \nabla \eta_R.$$

Since $(u_n\eta_R)$ is bounded in E, it follows that $I'_\varepsilon(u_n)(u_n\eta_R) = o_n(1)$. For $n \geqslant n_0$ and $\varepsilon > 0$ fixed, let $R > 0$ be large enough such that $\Lambda_\varepsilon \subset B_{R/2}(0)$, using $(\mathrm{g}_3)_{\mathrm{ii}}$ with ℓ_0 obtained in lemma 4.4, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_{R/2}} &(|\nabla u_n|^p + V(\varepsilon x)|u_n|^p) \\ &= \int_{\mathbb{R}^N\setminus B_{R/2}} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, u_n)\right) g(\varepsilon x, u_n) u_n + \frac{C}{R}\|u_n\|^p + o_n(1) \\ &\leqslant \int_{\mathbb{R}^N\setminus B_{R/2}} \frac{\sup_{n\geqslant n_0} |\tilde{K}(u_n)(x)|_{L^\infty(\mathbb{R}^N)}}{\ell_0} V_0 |u_n|^p + \frac{C}{R}\|u_n\|^p + o_n(1).\end{aligned}$$

Combining (4.5) with the boundedness of (u_n) and taking $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R} (|\nabla u_n|^p + V(\varepsilon x)|u_n|^p) < \zeta.$$

□

LEMMA 4.6. *Under the conditions of lemma 4.5, the functional I_ε satisfies the $(\mathrm{PS})_c$ condition for all $c \in [m_\varepsilon, \kappa]$.*

Proof. Since (u_n) is also bounded, we may assume $u_n \rightharpoonup u$ in E, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ and $u_n \to u$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$. Setting

$$\Psi_n = \int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla u_n - \nabla u) + \int_{\mathbb{R}^N} V(\varepsilon x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u), \quad (4.6)$$

we have

$$\Psi_n = I'_\varepsilon(u_n)u_n - I'_\varepsilon(u_n)u + \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, u_n)\right) g(\varepsilon x, u_n)(u_n - u) + o_n(1).$$

If $\Psi_n \to 0$ as $n \to \infty$, then arguments such as those in § 2 lead to

$$u_n \to u \quad \text{in } E.$$

To see why $\Psi_n \to 0$, we begin by observing that

$$I'_\varepsilon(u_n)u_n = I'_\varepsilon(u_n)u = o_n(1).$$

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Next, we shall prove that the following limit holds:

$$\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*G(\varepsilon x,u_n)\right)g(\varepsilon x,u_n)(u_n-u)=o_n(1).$$

Note that $G(\varepsilon x,u_n)$ is bounded in $L^{2N/(2N-\mu)}(\mathbb{R}^N)$. Moreover, since $u_n(x)\to u(x)$ a.e. in $\mathbb{R}^N$, and thus $G(\varepsilon x,u_n(x))\to G(\varepsilon x,u(x))$, we obtain that $G(\varepsilon x,u_n)$ must converge weakly to $G(\varepsilon x,u)$ in $L^{2N/(2N-\mu)}(\mathbb{R}^N)$. Using the Hardy–Littlewood–Sobolev inequality, we know the convolution term

$$\frac{1}{|x|^\mu}*w(x)\in L^{2N/\mu}(\mathbb{R}^N)\quad\text{for all } w(x)\in L^{2N/(2N-\mu)}(\mathbb{R}^N)$$

is a linear bounded operator from $L^{2N/(2N-\mu)}(\mathbb{R}^N)$ to $L^{2N/\mu}(\mathbb{R}^N)$. Consequently,

$$\frac{1}{|x|^\mu}*G(\varepsilon x,u_n)\rightharpoonup\frac{1}{|x|^\mu}*G(\varepsilon x,u)\quad\text{in } L^{2N/\mu}(\mathbb{R}^N).$$

Since g has a subcritical growth, by Sobolev's compact embedding, for any fixed $R>0$, it holds that

$$\int_{B_R}\left(\frac{1}{|x|^\mu}*G(\varepsilon x,u_n)\right)g(\varepsilon x,u_n)(u_n-u)\to 0.$$

Using the growth condition and the boundedness of $((1/|x|^\mu)*G(\varepsilon x,u_n))$, we get

$$\int_{\mathbb{R}^N\setminus B_R}\left(\frac{1}{|x|^\mu}*G(\varepsilon x,u_n)\right)|g(\varepsilon x,u_n)u_n|\leqslant C_1\int_{\mathbb{R}^N\setminus B_R}|u_n|^p.$$

From lemma 4.5 and the Sobolev embedding theorem, for each $\zeta>0$ there exists $R=R(\zeta)>0$ such that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_R}\left(\frac{1}{|x|^\mu}*G(\varepsilon x,u_n)\right)|g(\varepsilon x,u_n)u_n|\leqslant C_2\zeta.$$

Similarly, using the Hölder inequality, we can also prove that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_R}\left(\frac{1}{|x|^\mu}*G(\varepsilon x,u_n)\right)|g(\varepsilon x,u_n)u|\leqslant C_3\zeta.$$

In conclusion,

$$\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*G(\varepsilon x,u_n)\right)g(\varepsilon x,u_n)(u_n-u)\to 0.$$

□

Corollary 4.7. *Under the conditions of lemma 4.5, $I_\varepsilon|_{\mathcal{N}_\varepsilon}$ satisfies the* $(\mathrm{PS})_c$ *condition for all $c\in[m_\varepsilon,\kappa]$.*

Proof. Let $(u_n)\subset\mathcal{N}_\varepsilon$ be a sequence such that $I_\varepsilon(u_n)\to c$ and $\|I'_\varepsilon(u_n)\|_*\to 0$. Then, there exists $(\lambda_n)\subset\mathbb{R}$ such that

$$I'_\varepsilon(u_n)=\lambda_n\mathcal{Y}'_\varepsilon(u_n)+o_n(1).$$

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From (3.4),

$$\begin{aligned}\mathcal{Y}'_\varepsilon(u_n)u_n \leqslant -p\int_{\Lambda_\varepsilon\cup\{u_n<t_a\}}\left(\frac{1}{|x|^\mu}*G(\varepsilon x,u_n)\right)f(u_n)u_n \\ - p\int_{(\mathbb{R}^N\setminus\Lambda_\varepsilon)\cap\{t_a\leqslant u_n\leqslant T_a\}}\left(\frac{1}{|x|^\mu}*G(\varepsilon x,u_n)\right)\eta(u_n)u_n \\ - \frac{V_0 p}{\ell_0}\int_{(\mathbb{R}^N\setminus\Lambda_\varepsilon)\cap\{u_n\geqslant T_a\}}\left(\frac{1}{|x|^\mu}*G(\varepsilon x,u_n)\right)|u_n|^p \\ \leqslant -p\int_{\Lambda_\varepsilon}\left(\frac{1}{|x|^\mu}*G(\varepsilon x,u_n)\right)f(u_n)u_n.\end{aligned}$$

Since

$$0 = I'_\varepsilon(u_n)u_n = \lambda_n\mathcal{Y}'_\varepsilon(u_n)u_n + o_n(1),$$

our goal is to show $\lambda_n\to 0$. Otherwise, it must hold that $\mathcal{Y}'_\varepsilon(u_n)u_n\to 0$, and then

$$\int_{\Lambda_\varepsilon}\left(\frac{1}{|x|^\mu}*G(\varepsilon x,u_n)\right)f(u_n)u_n\to 0.$$

From the definition of the Nehari manifold, we have

$$\begin{aligned}\int_{\mathbb{R}^N}(|\nabla u_n|^p+V(\varepsilon x)|u_n|^p) &= \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*G(\varepsilon x,u_n)\right)g(\varepsilon x,u_n)u_n \\ &= \int_{\Lambda_\varepsilon}\left(\frac{1}{|x|^\mu}*G(\varepsilon x,u_n)\right)f(u_n)u_n \\ &\quad + \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}\left(\frac{1}{|x|^\mu}*G(\varepsilon x,u_n)\right)g(\varepsilon x,u_n)u_n \\ &= \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}\left(\frac{1}{|x|^\mu}*G(\varepsilon x,u_n)\right)g(\varepsilon x,u_n)u_n + o_n(1) \\ &\leqslant \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}\frac{\sup_{u\in\mathcal{B}}|\tilde{K}(u)(x)|_{L^\infty(\mathbb{R}^N)}}{\ell_0}V_0|u_n|^p + o_n(1) \\ &\leqslant \frac{1}{2}\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}V_0|u_n|^p + o_n(1),\end{aligned}$$

leading to

$$\int_{\mathbb{R}^N}(|\nabla u_n|^p+V(\varepsilon x)|u_n|^p)\to 0,$$

which is a contradiction. Thus, $\lambda_n\to 0$ and (u_n) is a $(\mathrm{PS})_c$ sequence of I_ε. Now, the corollary follows from lemma 4.6. □

5. Solutions for the penalized problem

In this section, we shall prove the existence and multiplicity of solutions. We begin showing the existence of the positive ground-state solution for (APE*).

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Theorem 5.1 (existence of ground-state solution). *Suppose that the nonlinearity f satisfies* (f_1)–(f_3) *with $p < q < (N-\mu)p/(N-p)$ and that the potential function V satisfies assumptions* (V_1)–(V_3)*. Then, for any $\varepsilon > 0$, problem* (APE*) *has a positive ground-state solution u_ε.*

Proof. Similarly to lemma 2.1, it follows that I_ε also satisfies the mountain pass geometry. Let

$$m_\varepsilon := \inf_{u \in E\backslash\{0\}} \max_{t \geqslant 0} I_\varepsilon(tu) = \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u).$$

Then, we know there exists a PS sequence at m_ε, i.e.

$$I'_\varepsilon(u_n) \to 0 \quad \text{and} \quad I_\varepsilon(u_n) \to m_\varepsilon.$$

Thus, by lemma 4.6, the existence of ground-state solution u_ε is guaranteed. Moreover, by choosing $u_{\varepsilon-}$ as a test function and recalling that $g(x,s) = 0$ for all $s \leqslant 0$ and $I'_\varepsilon(u_\varepsilon)u_{\varepsilon-} = 0$, a direct computation gives $u_\varepsilon \geqslant 0$. Repeating the arguments in § 2, we have $u_\varepsilon \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$, $\alpha \in (0,1)$. By applying Harnack's inequality, we can conclude that $u_\varepsilon(x) > 0$ in $\mathbb{R}^N$. □

Next, we shall show the existence of multiple solutions and study the behaviour of their maximum points in relation to the set M.

Let $\delta > 0$ be fixed and let w be a ground-state solution of problem (2.1) with $A = V_0$. Define η to be a smooth non-increasing cut-off function in $[0,\infty)$ such that

$$\eta(s) = \begin{cases} 1 & \text{if } 0 \leqslant s \leqslant \frac{1}{2}\delta, \\ 0 & \text{if } s \geqslant \delta. \end{cases}$$

For any $y \in M$, let us define

$$\Psi_{\varepsilon,y}(x) = \eta(|\varepsilon x - y|)w\left(\frac{\varepsilon x - y}{\varepsilon}\right),$$

with $t_\varepsilon > 0$ satisfying

$$\max_{t \geqslant 0} I_\varepsilon(t\Psi_{\varepsilon,y}) = I_\varepsilon(t_\varepsilon \Psi_{\varepsilon,y}),$$

and let us define $\Phi_\varepsilon \colon M \to \mathcal{N}_\varepsilon$ by $\Phi_\varepsilon(y) = t_\varepsilon \Psi_{\varepsilon,y}$. By construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$.

Lemma 5.2. *The function Φ_ε has the following limit:*

$$\lim_{\epsilon \to 0} I_\varepsilon(\Phi_\varepsilon(y)) = m_{V_0} \quad \textit{uniformly in } y \in M.$$

Proof. Suppose by contradiction that the lemma is false. Then, there exist $\delta_0 > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0$ such that

$$|I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - m_{V_0}| \geqslant \delta_0. \tag{5.1}$$

Considering the change of variable $z = (\varepsilon x - y_n)/\varepsilon_n$, if $z \in B_{\delta/\varepsilon_n}(0)$, it follows that $\varepsilon_n z \in B_\delta(0)$ and $\varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda$. Since $G(\varepsilon x, s) = F(s)$ in Λ, we

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deduce that

$$I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = \frac{t^p_{\varepsilon_n}}{p}\int_{\mathbb{R}^N} |\nabla(\eta(|\varepsilon_n z|)w(z))|^p + \frac{t^p_{\varepsilon_n}}{p}\int_{\mathbb{R}^N} V(\varepsilon_n z + y_n)|(\eta(|\varepsilon_n z|)w(z))|^p - \mathfrak{F}(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z)).$$

From Lebesgue's theorem,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla\Psi_{\varepsilon_n,y_n}|^p + V(\varepsilon_n x)|\Psi_{\varepsilon_n,y_n}|^p)\,\mathrm{d}x = \int_{\mathbb{R}^N}(|\nabla w|^p + V_0|w|^p)\,\mathrm{d}x$$

and

$$\lim_{|n|\to\infty}\mathfrak{F}(\Psi_{\varepsilon_n,y_n}) = \mathfrak{F}(w).$$

Since $t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}\in\mathcal{N}_{\varepsilon_n}$, it is easy to see the sequence $t_{\varepsilon_n}\to 1$. In fact, from

$$t^p_{\varepsilon_n}\int_{\mathbb{R}^N}|\nabla\Psi_{\varepsilon_n,y_n}|^p + V(\varepsilon_n x)|\Psi_{\varepsilon_n,y_n}|^p = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})f(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}}{|x-y|^\mu},$$

we derive

$$\|w\|^p = \lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})f(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}}{t^p_{\varepsilon_n}|x-y|^\mu}.$$

Now, using the fact that w is a ground-state solution of problem (2.1) together with remark 1.3, we get that $t_{\varepsilon_n}\to 1$. Now, note that

$$\lim_{n\to\infty}\mathfrak{F}(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z)) = \mathfrak{F}(w(z)).$$

So we get $\lim_{n\to\infty} I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = m_{V_0}$, which contradicts (5.1), finishing the proof of the lemma. □

For any $\delta > 0$, let $\rho = \rho(\delta) > 0$ be such that $M_\delta \subset B_\rho(0)$. Let $\chi\colon \mathbb{R}^N\to\mathbb{R}^N$ be defined as

$$\chi(x) = \begin{cases} x & \text{for } |x|\leqslant\rho,\\ \dfrac{\rho x}{|x|} & \text{for } |x|\geqslant\rho.\end{cases}$$

Finally, let us consider $\beta_\varepsilon\colon\mathcal{N}_\varepsilon\to\mathbb{R}^N$ given by

$$\beta_\varepsilon(u) = \frac{\int_{\mathbb{R}^N}\chi(\varepsilon x)|u|^p}{\int_{\mathbb{R}^N}|u|^p}.$$

From the above notation, we have the following lemma.

Lemma 5.3. *The function Φ_ε verifies the ensuing limit*

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(\Phi_\varepsilon(y)) = y \quad \textit{uniformly in } y\in M.$$

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Proof. If it is not true, then there exist $\delta_0 > 0$, $y_n \in M$ and $\varepsilon_n \to 0$ such that

$$|\beta_\varepsilon(\Phi_{\varepsilon_n}(y_n)) - y_n| \geqslant \delta_0 > 0 \quad \forall n \in \mathbb{N}. \tag{5.2}$$

Using the definitions of Φ_{ε_n} and β_{ε_n}, we obtain

$$\beta_\varepsilon(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^N}[\chi(\varepsilon z + y_n) - y_n]|\eta(|\varepsilon_n z|)w(z)|^p}{\int_{\mathbb{R}^N}|\eta(|\varepsilon_n z|)w(z)|^p}.$$

Lebesgue's theorem implies

$$|\beta_\varepsilon(\Phi_{\varepsilon_n}(y_n)) - y_n| \to 0,$$

which contradicts (5.2). □

Proposition 5.4. *Let $\varepsilon_n \to 0$. Let $(u_n) \subset \mathcal{N}_{\varepsilon_n}$ be a sequence verifying $I_{\varepsilon_n}(u_n) \to m_{V_0}$. Then, there exists $\tilde{y}_n \in \mathbb{R}^N$, such that $v_n = u_n(x + \tilde{y}_n)$ has a convergent subsequence in E. Moreover, up to a subsequence, $y_n \to y \in M$, where $y_n = \varepsilon_n \tilde{y}_n$.*

Proof. Since $u_n \in \mathcal{N}_{\varepsilon_n}$ and $I_{\varepsilon_n}(u_n) \to m_{V_0}$, we have that (u_n) is bounded in E. Thus, there are $r, \delta > 0$ and $y_n \in \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |u_n|^p \, \mathrm{d}x \geqslant \delta. \tag{5.3}$$

If (5.3) does not hold, again using Lions's lemma, we have that

$$u_n \to 0 \text{ in } L^s(\mathbb{R}^N) \quad \text{for } p < s < p^*.$$

Once we obtain that $u_n \in \mathcal{N}_{\varepsilon_n}$, the Hardy–Littlewood–Sobolev inequality gives $u_n \to 0$ in E, which is a contradiction because $I_{\varepsilon_n}(u_n) \to m_{V_0} > 0$. Thus, (5.3) holds. Setting $v_n(x) = u_n(x + \tilde{y}_n)$, up to a subsequence if necessary, we can assume $v_n \rightharpoonup v \not\equiv 0$ in E. Let $t_n > 0$ be such that $\tilde{v}_n = t_n v_n \in \mathcal{N}_{V_0}$. Then,

$$m_{V_0} \leqslant L_{V_0}(\tilde{v}_n) = L_{V_0}(t_n u_n) \leqslant I_\varepsilon(t_n u_n) \leqslant I_\varepsilon(u_n) \to m_{V_0},$$

and so

$$L_{V_0}(\tilde{v}_n) \to m_{V_0} \quad \text{and} \quad (\tilde{v}_n) \subset \mathcal{N}_{V_0}.$$

Then $(\tilde{v}_n)$ is a minimizing sequence, and by Ekeland's variational principle [42] we may also assume it is a bounded $(\mathrm{PS})_{m_{V_0}}$ sequence. Thus, for some subsequence, $\tilde{v}_n \rightharpoonup \tilde{v}$ weakly in E with $\tilde{v} \neq 0$ and $L'_{V_0}(\tilde{v}) = 0$. Furthermore, using results found in [2, 29], we derive that

$$L_{V_0}(\tilde{v}_n - \tilde{v}) \to m_{V_0} - L_{V_0}(\tilde{v}), \qquad L'_{V_0}(\tilde{v}_n - \tilde{v}) \to 0.$$

Since

$$\begin{aligned} m_{V_0} &= \lim_{n\to\infty} L_{V_0}(\tilde{v}_n) \\ &= \lim_{n\to\infty}\left(\frac{1}{p}\mathfrak{F}'(\tilde{v}_n)\tilde{v}_n - \mathfrak{F}(\tilde{v}_n)\right) \geqslant \left(\frac{1}{p}\mathfrak{F}'(\tilde{v})\tilde{v} - \mathfrak{F}(\tilde{v})\right) \\ &= L_{V_0}(\tilde{v}) \geqslant m_{V_0}, \end{aligned}$$

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it follows that

$$L_{V_0}(\tilde{v}_n - \tilde{v}) \to 0, \qquad L'_{V_0}(\tilde{v}_n - \tilde{v}) \to 0.$$

Consequently,

$$\tilde{v}_n \to \tilde{v} \quad \text{in } E.$$

Thus, from $\tilde{v}_n = t_n v_n \in \mathcal{N}_{V_0}$, we see that (t_n) is bounded, and so we can assume that, for some subsequence, $t_n \to t_0 > 0$ and, consequently, $v_n \to v$ in E.

Now, we shall show that $(y_n) = (\varepsilon_n \tilde{y}_n)$ has a subsequence satisfying $y_n \to y \in M$. We begin by claiming that (y_n) is bounded in $\mathbb{R}^N$. Indeed, suppose by contradiction that (y_n) is not bounded. Then, there exists a subsequence, still denoted by (y_n), such that $|y_n| \to \infty$. Once we obtain that

$$\int_{\mathbb{R}^N} (|\nabla u_n|^p + V(\varepsilon x)|u_n|^p) = \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, u_n) \right) g(\varepsilon x, u_n) u_n,$$

and $I_{\varepsilon_n}(u_n) \to m_{V_0}$, we can infer that $u_n \in \mathcal{B}$ for n large enough. Then, by (4.4), there exists $C_0 > 0$ satisfying

$$\sup_{n \in \mathbb{N}} \left| \frac{1}{|x|^\mu} * G(\varepsilon x, u_n) \right|_{L^\infty(\mathbb{R}^N)} < C_0.$$

Consider $R > 0$ such that $\Lambda \subset B_R(0)$. Without loss of generality we may assume that $|y_n| > 2R$. Thus, for any $z \in B_{R/\varepsilon_n}(0)$,

$$|\varepsilon_n z + y_n| \geqslant |y_n| - |\varepsilon_n z| > R. \tag{5.4}$$

By the change of variables $x \mapsto z + \tilde{y}_n$, using the fact that $V(\varepsilon x) \geqslant V_0$ and (5.4), we get

$$\begin{aligned}
\int_{\mathbb{R}^N} & (|\nabla v_n|^p + V_0|v_n|^p) \\
&\leqslant C_0 \int_{\mathbb{R}^N} g(\varepsilon z + y_n, v_n) v_n \\
&= C_0 \int_{B_{R/\varepsilon_n}(0)} g(\varepsilon z + y_n, v_n) v_n + C \int_{\mathbb{R}^N \setminus B_{R/\varepsilon_n}(0)} g(\varepsilon z + y_n, v_n) v_n \\
&\leqslant C_0 \int_{B_{R/\varepsilon_n}(0)} \tilde{f}(v_n) v_n + C_0 \int_{\mathbb{R}^N \setminus B_{R/\varepsilon_n}(0)} f(v_n) v_n.
\end{aligned}$$

Since $\tilde{f}(s) \leqslant (V_0/\ell_0) s^{p-1}$ and $v_n \to v$ in E, we obtain

$$\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v_n|^p + V_0|v_n|^p) \leqslant \int_{\mathbb{R}^N \setminus B_{R/\varepsilon_n}(0)} f(v_n) v_n = o_n(1).$$

Taking $n \to \infty$ leads to a contradiction. Thus, up to a subsequence, $y_n \to y \in \mathbb{R}^N$.

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It remains to check that $y \in M$. Arguing by contradiction again, we suppose that $V(y) > V_0$. Then, recalling that $\tilde{v}_n \to \tilde{v}$ in E, we can use Fatou's lemma to obtain

$$\begin{aligned} m_{V_0} &= L_{V_0}(\tilde{v}) \\ &< \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla \tilde{v}|^p + V(y)|\tilde{v}|^p) - \mathfrak{F}(\tilde{v}) \\ &\leqslant \liminf_{n\to\infty} \left\{ \frac{1}{p}\int (|\nabla \tilde{v}_n|^p + V(\varepsilon_n z + y_n)|\tilde{v}_n|^p) - \mathfrak{F}(\tilde{v}_n) \right\} \\ &\leqslant \liminf_{n\to\infty} I_{\varepsilon_n}(t_n u_n) \\ &\leqslant \liminf_{n\to\infty} I_{\varepsilon_n}(u_n) \\ &= m_{V_0}, \end{aligned}$$

which is absurd. □

Let $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a positive function verifying $h(\varepsilon) \to 0$ as $\varepsilon \to 0$ and let $\hat{\mathcal{N}}_\varepsilon := \{u \in \mathcal{N}_\varepsilon \colon I_\varepsilon(u) \leqslant m_{V_0} + h(\varepsilon)\}$. From lemma 5.3, it follows that $\hat{\mathcal{N}}_\varepsilon \neq \emptyset$.

Lemma 5.5. *Let $\delta > 0$ and $M_\delta = \{x \in \mathbb{R}^N \colon \operatorname{dist}(x, M) \leqslant \delta\}$. Then*

$$\lim_{\varepsilon\to 0} \sup_{u\in\hat{\mathcal{N}}_\varepsilon} \inf_{y\in M_\delta} |\beta_\varepsilon(u) - y| = 0.$$

Proof. Let $\varepsilon_n \to 0$. For each $n \in \mathbb{N}$, there exists $(u_n) \subset \hat{\mathcal{N}}_\varepsilon$, such that

$$\inf_{y\in M_\delta} |\beta_{\varepsilon_n}(u_n) - y| = \sup_{u\in\hat{\mathcal{N}}_{\varepsilon_n}} \inf_{y\in M_\delta} |\beta_{\varepsilon_n}(u) - y| + o_n(1).$$

Since $(u_n) \subset \hat{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, it follows that

$$m_{V_0} \leqslant m_{\varepsilon_n} \leqslant I_{\varepsilon_n}(u_n) \leqslant m_{V_0} + h(\varepsilon_n),$$

which means that

$$I_{\varepsilon_n}(u_n) \to m_{V_0} \quad \text{and} \quad (u_n) \subset \mathcal{N}_\varepsilon.$$

From proposition 5.4, there exists a sequence $\tilde{y}_n \in \mathbb{R}^N$ such that $v_n(x) = u_n(x+\tilde{y}_n)$ has a convergent subsequence in E. Moreover, up to a subsequence, $y_n \to y \in M$, where $y_n = \varepsilon_n \tilde{y}_n$. Therefore,

$$\begin{aligned} \beta_\varepsilon(u_n) &= \frac{\int_{\mathbb{R}^N} \chi(\varepsilon z)|u_n|^p}{\int_{\mathbb{R}^N} |u_n|^p} \\ &= \frac{\int_{\mathbb{R}^N} \chi(\varepsilon z + y_n)|u_n(z+\tilde{y}_n)|^p}{\int_{\mathbb{R}^N} |u_n(z+\tilde{y}_n)|^p} \\ &= y_n + \frac{\int_{\mathbb{R}^N} [\chi(\varepsilon z + y_n) - y_n]|v_n(z)|^p}{\int_{\mathbb{R}^N} |v_n(z)|^p} \to y \in M. \end{aligned}$$

Consequently, there exists $y_n \in M_\delta$ such that

$$\lim_{n\to\infty} |\beta_\varepsilon(u_n) - y_n| = 0,$$

finishing the proof of the lemma. □

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Theorem 5.6 (multiplicity of solutions). *Suppose that the nonlinearity* f *satisfies* (f_1)–(f_3) *with* $p < q < (N-\mu)p/(N-p)$ *and the potential function* V *satisfies assumptions* (V_1)–(V_3)*. Then, for any* $\delta > 0$*, there exists* $\varepsilon_\delta > 0$ *such that problem* (APE*) *has at least* $\mathrm{cat}_{M_\delta}(M)$ *positive solutions, for any* $0 < \varepsilon < \varepsilon_\delta$*.*

Proof. We fix a small $\varepsilon > 0$. Then, by lemmas 5.2 and 5.5, $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopic to the inclusion map $\mathrm{id}\colon M \to M_\delta$, and so

$$\mathrm{cat}_{\hat{N}_\varepsilon}(\hat{N}_\varepsilon) \geqslant \mathrm{cat}_{M_\delta}(M).$$

Since that functional I_ε satisfies the $(\mathrm{PS})_c$ condition for $c \in [m(V_0), m(V_0) + h(\varepsilon)]$, by the Lyusternik–Schnirelmann theory of critical points [29], we can conclude that I_ε has at least $\mathrm{cat}_{M_\delta}(M)$ critical points on $\mathcal{N}_\varepsilon$. Since the manifold $\mathcal{N}_\varepsilon$ is a natural constraint for I_ε, I_ε has at least $\mathrm{cat}_{M_\delta}(M)$ critical points in E. By repeating the arguments explored in § 2, we deduce $u_\varepsilon \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$, $\alpha \in (0,1)$. Applying Harnack's inequality, we can conclude that $u_\varepsilon(x) > 0$ in $\mathbb{R}^N$. □

6. Solutions for the original equation

In this section, our main goal is to show that the solutions u_ε obtained in theorem 5.6 are indeed solutions for problem (QNE*). We will use the Moser iteration technique [34] to prove

$$u_\varepsilon(x) \leqslant t_a \quad \text{for all } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon,$$

where

$$\Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}.$$

For completeness, we shall sketch the proof here.

Lemma 6.1. *Let* (ε_n) *be a sequence where* $\varepsilon_n \to 0$ *as* $n \to 0$ *and* $(x_n) \subset \bar{\Lambda}_{\varepsilon_n}$*. If* u_{ε_n} *is a solution of problem* (APE*) *in theorem 5.6, then, up to a subsequence,* $v_n := u_{\varepsilon_n}(\cdot + x_n)$ *converges uniformly on compact subsets of* $\mathbb{R}^N$*.*

Proof. For each $n \in \mathbb{N}$ and $L > 0$, let

$$v_{L,n} = \begin{cases} v_n(x), & v_n(x) \leqslant L, \\ L, & v_n(x) \geqslant L, \end{cases}$$

$$z_{L,n} = v_{L,n}^{p(\beta-1)} v_n \quad \text{and} \quad w_{L,n} = v_n v_{L,n}^{\beta-1}$$

with $\beta > 1$ to be determined later. Note that v_n satisfies

$$-\Delta_p v + V_\varepsilon(x)|v|^{p-2}v = \left(\frac{1}{|x|^\mu} * G_\varepsilon(v)\right) g_\varepsilon(v) \quad \text{in } \mathbb{R}^N,$$
$$v \in W^{1,p}(\mathbb{R}^N), \qquad v(x) > 0 \quad \text{for all } x \in \mathbb{R}^N,$$

where $V_\varepsilon(x) = V(\varepsilon x + x_n)$, $g_\varepsilon(v) = g(\varepsilon x + x_n, v)$ and $G_\varepsilon(v) = G(\varepsilon x + x_n, v)$.

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Taking $\varphi = z_{L,n}$ as a test function, we obtain

$$\int_{\mathbb{R}^N} v_{L,n}^{p(\beta-1)}|\nabla v_n|^p = -p(\beta-1)\int_{\mathbb{R}^N} v_{L,n}^{p\beta-p-1}v_n|\nabla v_n|^{p-2}\nabla v_n\nabla v_{L,n} + \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G_\varepsilon(v_n)\right)g_\varepsilon(v_n)v_n v_{L,n}^{p(\beta-1)} - \int_{\mathbb{R}^N} V_\varepsilon(x)|v_n|^p v_{L,n}^{p(\beta-1)}.$$

Since

$$\int_{\mathbb{R}^N} v_{L,n}^{p\beta-p-1}v_n|\nabla v_n|^{p-2}\nabla v_n\nabla v_{L,n} = \int_{\{v_n\cdot L\}} v_{L,n}^{p(\beta-1)}|\nabla v_n|^p \geqslant 0,$$

we get

$$\int_{\mathbb{R}^N} v_{L,n}^{p(\beta-1)}|\nabla v_n|^p \leqslant \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G_\varepsilon(v_n)\right)g_\varepsilon(v_n)v_n v_{L,n}^{p(\beta-1)} - \int_{\mathbb{R}^N} V_0|v_n|^p v_{L,n}^{p(\beta-1)}.$$

Since (v_n) is bounded in E, there exists $C_0 > 0$ such that

$$\sup_{n\in\mathbb{N}}\left|\frac{1}{|x|^\mu} * G_\varepsilon(v_n)\right|_{L^\infty(\mathbb{R}^N)} < C_0.$$

From assumptions (f_1) and (f_2), for any $\xi > 0$ there exists $C_\xi > 0$ such that

$$|g_\varepsilon(v_n)| \leqslant \xi|v_n|^{p-1} + C_\xi|v_n|^{q-1}.$$

Thus, choosing $0 < \xi$ small enough, we obtain a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^N} v_{L,n}^{p(\beta-1)}|\nabla v_n|^p \leqslant C_1\int_{\mathbb{R}^N} v_n^q v_{L,n}^{p(\beta-1)}. \tag{6.1}$$

On the other hand, by the Sobolev embedding we get

$$\begin{aligned}|w_{L,n}|_{p^*}^p &\leqslant C_2\int_{\mathbb{R}^N}|\nabla(v_n {v_{L,n}}^{\beta-1})|^p \\ &\leqslant C_3(\beta-1)^p\int_{\mathbb{R}^N} v_n^p v_{L,n}^{p(\beta-2)}|\nabla v_{L,n}|^p + C_3\int_{\mathbb{R}^N} v_{L,n}^{p(\beta-1)}|\nabla v_n|^p \\ &= C_3(\beta-1)^p\int_{\{v_n\leqslant L\}} v_{L,n}^{p(\beta-1)}|\nabla v_n|^p + C_3\int_{\mathbb{R}^N} v_{L,n}^{p(\beta-1)}|\nabla v_n|^p \\ &\leqslant C_4\beta^p\int_{\mathbb{R}^N} v_{L,n}^{p(\beta-1)}|\nabla v_n|^p.\end{aligned} \tag{6.2}$$

From (6.1) and (6.2), Hölder's inequality and the boundedness of (v_n) imply

$$\begin{aligned}|w_{L,n}|_{p^*}^p &\leqslant C_5\beta^p\int_{\mathbb{R}^N} v_n^p v_{L,n}^{p(\beta-1)} = C_5\beta^p\int_{\mathbb{R}^N} v_n^{q-p}w_{L,n}^p \\ &\leqslant C_5\beta^p\left(\int_{\mathbb{R}^N} v_n^{p^*}\right)^{(q-p)/p^*}\left(\int_{\mathbb{R}^N} {w_{L,n}}^{pp^*/(p^*-(q-p))}\right)^{(p^*-(q-p))/p^*} \\ &\leqslant C_6\beta^p|w_{L,n}|_{\alpha^*}^p\end{aligned} \tag{6.3}$$

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with

$$p < \alpha^* = \frac{pp^*}{p^* - (q-p)} < p^*,$$

whenever $w_{L,n} \in L^{\alpha^*}(\mathbb{R}^N)$.

Since $v_{L,n} \leqslant v_n$, we know that $w_{L,n} \in L^{\alpha^*}(\mathbb{R}^N)$ if $v_n^\beta \in L^{\alpha^*}(\mathbb{R}^N)$. If it is true, it follows from (6.3) that

$$\left(\int_{\mathbb{R}^N} v_{L,n}^{p^*(\beta-1)} v_n^{p^*}\right)^{p/p^*} \leqslant C_6 \beta^p \left(\int_{\mathbb{R}^N} (v_{L,n}^{(\beta-1)} v_n)^{\alpha^*}\right)^{p/\alpha^*} \leqslant C_6 \beta^p |v_n|_{\beta\alpha^*}^{\beta p}.$$

Applying Fatou's lemma in L, we get

$$|v_n|_{\beta p^*} \leqslant C_7^{1/\beta} \beta^{1/\beta} |v_n|_{\beta\alpha^*} < \infty, \tag{6.4}$$

if $v_n^{\beta\alpha^*} \in L^1(\mathbb{R}^N)$ holds.

Start the iteration by setting $\beta := p^*/\alpha^* > 1$. Since $v_n \in L^{p^*}(\mathbb{R}^N)$, the inequality (6.4) is then true. Note that if $\beta^2\alpha^* = \beta p^*$, then (6.4) also holds with β replaced by β^2. Consequently,

$$|v_n|_{p^*\beta^2} \leqslant C_7^{1/\beta^2} \beta^{2/\beta^2} |v_n|_{\alpha^*\beta^2} \leqslant C_7^{1/\beta+1/\beta^2} \beta^{1/\beta+2/\beta^2} |v_n|_{\beta\alpha^*}.$$

Iterating this process and using $\beta\alpha^* = p^*$, we obtain

$$|v_n|_{\beta^m\alpha^*} \leqslant C_7^{\sum_{t=1}^m \beta^{-t}} \beta^{\sum_{t=1}^m t\beta^{-t}} |v_n|_{p^*}.$$

Passing to the limit as $m \to \infty$, we have

$$|v_n|_\infty \leqslant C_8 \quad \forall n \in \mathbb{N}.$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $\xi > 0$. Equations (g_1) and (g_2) imply that

$$\left| V_{\varepsilon_n}(x) v_n^{p-1}(x) - \left(\frac{1}{|x|^\mu} * G_\varepsilon(v_n)\right) g_\varepsilon(v_n)(x)\right| \leqslant C \quad \forall n \in \mathbb{N}\ \forall x \in \Omega.$$

From a result due to Di Benedetto [20], for any compact set $K \subset \Omega$, there exists a constant $\bar{C}_{K,\Omega}$, depending only on C_8, N, p and $\operatorname{dist}(K, \partial\Omega)$, such that

$$|v_n|_{C_K^{0,\alpha}(\Omega)} \leqslant \bar{C}_{K,\Omega} \quad \forall n \in \mathbb{N}$$

with some $0 < \alpha < 1$. Then (v_n) possesses a convergent subsequence in $C^0_{\text{loc}}(\mathbb{R}^N)$, finishing the proof. □

PROPOSITION 6.2. *For any $\varepsilon > 0$, define*

$$m_\varepsilon^* := \sup\left\{\max_{\partial\Lambda_\varepsilon} u_\varepsilon : u_\varepsilon \in \hat{\mathcal{N}}_\varepsilon \textit{ is a solution of problem } (\mathrm{APE}^*)\right\}.$$

Then, m_ε^* is finite for ε small enough and $\lim_{\varepsilon\to 0^+} m_\varepsilon^* = 0$.

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Proof. Arguing by contradiction, we suppose that there exists $(\varepsilon_n) \subset \mathbb{R}^+$ such that $\varepsilon_n \to 0$ and $u_{\varepsilon_n} \in \hat{\mathcal{N}}_{\varepsilon_n}$ is a solution of (APE*) such that

$$b_{\varepsilon_n} = \max_{\partial \Lambda_{\varepsilon_n}} u_{\varepsilon_n} \to \infty.$$

Then, $u_{\varepsilon_n}(x_n) \geqslant b > 0$ for some $b > 0$ and $(x_n) \in \partial \Lambda_{\varepsilon_n}$.

Since $\varepsilon_n \to 0$, we have that $I_{\varepsilon_n}(u_n) \to m_{V_0}$. Once we obtain $u_{\varepsilon_n}(x_n) \geqslant b > 0$, by setting $v_n := u_{\varepsilon_n}(\cdot + x_n)$, it follows that $v_n \rightharpoonup v$ weakly in E with $v \neq 0$ by lemma 6.1. Let $t_n > 0$ be such that $\tilde{v}_n = t_n v_n \in \mathcal{N}_{V_0}$. Then,

$$L_{V_0}(\tilde{v}_n) \to m_{V_0} \quad \text{and} \quad (\tilde{v}_n) \subset \mathcal{N}_{V_0}.$$

Repeating the arguments in lemma 5.4, we derive

$$\tilde{v}_n \to \tilde{v} \quad \text{in } E.$$

Thus, from $\tilde{v}_n = t_n v_n \in \mathcal{N}_{V_0}$, (t_n) is bounded. Thus, we can assume that, for some subsequence, $t_n \to t_0 > 0$, and so $v_n \to v$ in E. Next, we shall show that $(\bar{x}_n) = (\varepsilon_n x_n)$ has a subsequence satisfying

$$\bar{x}_n \to \bar{x} \in M \quad \text{and} \quad V(\bar{x}) = V_0.$$

First we claim that $(\bar{x}_n)$ is bounded in $\mathbb{R}^N$, because $(\bar{x}_n) \subset \partial \Lambda$. Thus, up to a subsequence, $\bar{x}_n \to \bar{x} \in \partial \Lambda$, from which it follows that $V(\bar{x}) > V_0$. Then, recalling that $\tilde{v}_n \to \tilde{v}$ in E, we can use Fatou's lemma to obtain

$$\begin{aligned} m_{V_0} &= L_{V_0}(\tilde{v}) \\ &< \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla \tilde{v}|^p + V(\bar{x})|\tilde{v}|^p) - \mathfrak{F}(\tilde{v}) \\ &\leqslant \liminf_{n\to\infty} \left\{ \frac{1}{p} \int (|\nabla \tilde{v}_n|^p + V(\varepsilon_n z + \bar{x}_n)|\tilde{v}_n|^p) - \mathfrak{F}(\tilde{v}_n) \right\} \\ &= \liminf_{n\to\infty} I_{\varepsilon_n}(t_n u_n) \\ &\leqslant \liminf_{n\to\infty} I_{\varepsilon_n}(u_n) \\ &= m_{V_0}, \end{aligned}$$

which is absurd. Thus, $m_\varepsilon^* < +\infty$ for $\varepsilon > 0$ small enough.

To prove $\lim_{\varepsilon\to 0^+} m_\varepsilon^* = 0$, we suppose by contradiction that there exist $\varepsilon_n \to 0^+$ and $b > 0$ with

$$m_{\varepsilon_n}^* \geqslant b > 0.$$

Thus, for each $n \in \mathbb{N}$, there exists a solution $u_{\varepsilon_n} \in \hat{\mathcal{N}}_{\varepsilon_n}$ of (APE*) in such a way that

$$\max_{\partial \Lambda_{\varepsilon_n}} u_{\varepsilon_n} \geqslant m_{\varepsilon_n}^* - \tfrac{1}{2}b \geqslant \tfrac{1}{2}b \quad \forall n \in \mathbb{N}.$$

Hence, there exists a sequence $(x_n) \subset \partial \Lambda_{\varepsilon_n}$ such that

$$u_{\varepsilon_n}(x_n) \geqslant \tfrac{1}{2}b > 0 \quad \forall n \in \mathbb{N}.$$

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Repeating the same arguments employed in the first part of the proof, we get a contradiction. Therefore,

$$\lim_{\varepsilon\to 0^+} m_\varepsilon^* = 0.$$

□

6.1. Proof of theorem 1.4

We divide the proof into two parts.

6.1.1. *Proof of existence*

Proof of theorem 1.4. Given $\delta > 0$ such that $M_\delta \subsetneq \Lambda$, we can invoke theorem 5.6 to obtain, for any $\varepsilon \in (0, \varepsilon_\delta)$ fixed, $\operatorname{cat}_{M_\delta}(M)$ solution of (APE*). Taking ε_δ smaller if necessary, we can use lemma 6.2 to conclude that, if u_ε is one of these solutions, then

$$u_\varepsilon(x) < t_a \quad \text{for all } x \in \partial\Lambda_\varepsilon.$$

The rest of the proof is similar to [18], but we sketch it for completeness. The function $u_\varepsilon \in E$ solves the equation

$$-\Delta_p u + V(\varepsilon x)|u|^{p-2}u = \left(\frac{1}{|x|^\mu} * G(\varepsilon x, u)\right) g(\varepsilon x, u) \quad \text{in } \mathbb{R}^N.$$

Define

$$v_\varepsilon = \begin{cases} \max\{u_\varepsilon - t_a, 0\}, & x \in \mathbb{R}^N \setminus \Lambda_\varepsilon, \\ 0, & \text{elsewhere,} \end{cases}$$

and observe that $v_\varepsilon \in E$. Taking it as a test function in the above equation we get

$$\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} |\nabla v_\varepsilon|^p + c(x)v_\varepsilon^2 + t_a c(x) v_\varepsilon = 0, \tag{6.5}$$

where

$$c(x) := V(\varepsilon x)|u_\varepsilon(x)|^{p-2} - \left(\frac{1}{u_\varepsilon(x)}\right)\left(\frac{1}{|x|^\mu} * G(\varepsilon x, u_\varepsilon)\right) g(\varepsilon x, u_\varepsilon).$$

Since

$$\left|\frac{1}{|x|^\mu} * G(\varepsilon x, u_\varepsilon)\right|_{L^\infty(\mathbb{R}^N)} < C_0$$

and

$$\frac{g(\varepsilon x, u_\varepsilon)}{u_\varepsilon(x)} \leqslant \frac{V_0}{\ell_0} u_\varepsilon(x)^{p-2},$$

we see that

$$\left(\frac{1}{u_\varepsilon(x)}\right)\left(\frac{1}{|x|^\mu} * G(\varepsilon x, u_\varepsilon)\right) g(\varepsilon x, u_\varepsilon) \leqslant \tfrac{1}{2} V_0 u_\varepsilon(x)^{p-2},$$

since $C_0/M_0 \leqslant \frac{1}{2}$. Consequently, $c(x) \geqslant 0$ in $\mathbb{R}^N \setminus \Lambda_\varepsilon$, and all the terms in (6.5) are zero. In particular, $v_\varepsilon \equiv 0$. Thus, (3.2) holds and u_ε is a solution of (QNE*). The theorem is proved. □

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6.1.2. *Proof of concentration*

Lemma 6.3. *Let $v_n > 0$ be a solution of the following problem:*

$$-\Delta_p v_n + V_n(x)|v_n|^{p-2}v_n = \left(\int_{\mathbb{R}^N} \frac{F(v_n)}{|x-y|^\mu}\right) f(v_n) \quad \text{in } \mathbb{R}^N,$$
$$v_n \in W^{1,p}(\mathbb{R}^N) \quad \text{with } 1 < p < N,$$

where $V_n(x) = V(\epsilon_n x + \varepsilon_n \tilde{y}_n)$. Assume that (f$_1$)–(f$_3$) *hold with $\mu < p$ and $p < q < (N-\mu)p/(N-p)$. If $v_n \to v$ in E with $v \not\equiv 0$, then $v_n \in L^\infty(\mathbb{R}^N)$ and there exists $C > 0$ such that $|v_n|_\infty \leqslant C$ for all $n \in \mathbb{N}$. Furthermore,*

$$\lim_{|x|\to\infty} v_n(x) = 0 \quad \text{uniformly in } n \in \mathbb{N}.$$

Proof. Let (v_n) be a sequence of positive solutions satisfying $v_n \to v$ in E and define

$$K_n(x) := \int_{\mathbb{R}^N} \frac{F(v_n)}{|x-y|^\mu}.$$

We first claim there exists $C > 0$ such that

$$|K_n(x)| \leqslant C \quad \forall n \in \mathbb{N}. \tag{6.6}$$

Next we adapt some arguments in [4, 24] that are related to the Moser iteration method.

For any $R > 0$, $0 < r \leqslant \frac{1}{2}R$, let $\eta \in C^\infty(\mathbb{R}^N)$, $0 \leqslant \eta \leqslant 1$ with $\eta(x) = 1$ if $|x| \geqslant R$ and $|\nabla \eta| \leqslant 2/r$. Note that by (f$_1$) and the claim above, we obtain the following estimate: given $\xi > 0$, there exists C_ξ such that

$$|K_n(x) f(v_n)| \leqslant \xi |v_n(x)|^{p-1} + C_\xi |v_n(x)|^{p^*-1} \quad \forall x \in \mathbb{R}^N \text{ and } n \in \mathbb{N}. \tag{6.7}$$

For each $n \in \mathbb{N}$ and for $l > 0$, let

$$v_{l,n} = \begin{cases} v_n(x), & v_n(x) \leqslant l, \\ l, & v_n(x) \geqslant l, \end{cases}$$

and

$$z_{l,n} = \eta^p v_{l,n}^{p(\beta-1)} v_n \quad \text{and} \quad w_{l,n} = \eta v_n v_{l,n}^{\beta-1}$$

with $\beta > 1$ to be determined later.

Taking $z_{l,n}$ as a test function, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \eta^p v_{l,n}^{p(\beta-1)} |\nabla v_n|^p = {} & -p(\beta-1) \int_{\mathbb{R}^N} v_{l,n}^{p\beta-p-1} \eta^p v_n |\nabla v_n|^{p-2} \nabla v_n \nabla v_{l,n} \\ & + \int_{\mathbb{R}^N} K_n(x) f(v_n) \eta^p v_n v_{l,n}^{p(\beta-1)} - \int_{\mathbb{R}^N} V_n |v_n|^p \eta^p v_{l,n}^{p(\beta-1)} \\ & - p \int_{\mathbb{R}^N} \eta^{p-1} v_{l,n}^{p(\beta-1)} v_n |\nabla v_n|^{p-2} \nabla v_n \nabla \eta. \end{aligned}$$

By (6.7) and for ξ sufficiently small, we get

$$\int_{\mathbb{R}^N} \eta^p v_{l,n}^{p(\beta-1)} |\nabla v_n|^p \leqslant C_\xi \int_{\mathbb{R}^N} v_n^{p^*} \eta^p v_{l,n}^{p(\beta-1)} - p \int_{\mathbb{R}^N} \eta^{p-1} v_{l,n}^{p(\beta-1)} v_n |\nabla v_n|^{p-2} \nabla v_n \nabla \eta.$$

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Now, following the same arguments explored in [4], we find

$$|v_n|_{L^\infty(|x|\geqslant R)} \leqslant C|v_n|_{p^*(|x|>R/2)}.$$

Again using the convergence of (v_n) to v in $W^{1,p}(\mathbb{R}^N)$, for each $\gamma > 0$ fixed, there exists $R > 0$ such that $|v_n|_{L^\infty(|x|\geqslant R)} < \gamma$ for all $n \in N$. Thus,

$$\lim_{|x|\to\infty} v_n(x) = 0 \quad \text{uniformly in } n \in \mathbb{N},$$

which completes the proof of the lemma. □

LEMMA 6.4. *There exists $\delta > 0$ such that $|v_n|_\infty \geqslant \delta$ for all $n \in \mathbb{N}$.*

Proof. Since $v_n \to v \neq 0$ in E, there exist $(y_n) \subset \mathbb{R}^N$ and $\tilde{R}, \beta > 0$ such that

$$\int_{B_{\tilde{R}}(y_n)} |v_n|^p \geqslant \beta.$$

If $\|v_n\|_{L^\infty(\mathbb{R}^N)} \to 0$, then we get

$$\beta \leqslant \int_{B_{\tilde{R}}(y_n)} |v_n|^p \leqslant |v_n|_\infty^p |B_{\tilde{R}}(y_n)| \leqslant C|v_n|_\infty^p \to 0,$$

which is a contradiction. □

If u_{ε_n} is a solution of problem (QNE*), then $v_n(x) = u_{\varepsilon_n}(x + \tilde{y}_n)$ is a solution of the problem

$$-\Delta_p v_n + V_n(x)|v_n|^{p-2}v_n = \left(\int_{\mathbb{R}^N} \frac{F(v_n)}{|x-y|^\mu}\right) f(v_n) \quad \text{in } \mathbb{R}^N,$$
$$v_n \in E, \qquad v_n(x) > 0 \quad \forall x \in \mathbb{R}^N,$$

with $V_n(x) = V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)$ and $(\tilde{y}_n) \subset \mathbb{R}^N$ given in proposition 5.4. Moreover, up to a subsequence,

$$v_n \to v \quad \text{in } E \quad \text{and} \quad y_n \to y \quad \text{in } M,$$

where $y_n = \varepsilon_n \tilde{y}_n$. If b_n is a maximum point of v_n, we know it is a bounded sequence in $\mathbb{R}^N$. Thus, there exists $R > 0$ such that $b_n \in B_R(0)$. Thus, the global maximum of u_{ε_n} is $z_\varepsilon = b_n + \tilde{y}_n$ and

$$\varepsilon_n z_{\varepsilon n} = \varepsilon_n b_n + \varepsilon_n \tilde{y}_n = \varepsilon_n b_n + y_n.$$

From the boundedness of (b_n), we get the limit

$$\lim_{\varepsilon\to 0} \varepsilon_n z_\varepsilon = y,$$

which, together with the continuity of V, gives

$$\lim_{n\to\infty} V(\varepsilon_n z_{\varepsilon_n}) = V_0.$$

If u_ε is a positive solution of (QNE*), the function $w_\varepsilon(x) = u_\varepsilon(x/\varepsilon)$ is a positive solution of (QNE). Thus, the maximum points η_ε and z_ε of w_ε and u_ε, respectively, satisfy the equality $\eta_\varepsilon = \varepsilon z_\varepsilon$, from which it follows that

$$\lim_{|\varepsilon|\to\infty} V(\eta_\varepsilon) = V_0.$$

https://doi.org/10.1017/S0308210515000311 Published online by Cambridge University Press

Acknowledgements

C.O.A. is partly supported by CNPq/Brazil grant no. 301807/2013-2 and M.Y. is supported by NSFC (grant nos 11101374 and 11271331) and CNPq/Brazil grant no. 500001/2013-8.

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