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How do autodiffeomorphisms act on embeddings?

A. Skopenkov
Moscow Institute of Physics and Technology, and Independent University of Moscow, B. Vlasyevskiy 11, Moscow 119002, Russia
(skopenko@mccme.ru)

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We work in the smooth category. The following problem was suggested by E. Rees in 2002: describe the precomposition action of self-diffeomorphisms of $S^p \times S^q$ on the set of isotopy classes of embeddings $S^p \times S^q \to \mathbb{R}^m$.

Let $G\colon S^p \times S^q \to \mathbb{R}^m$ be an embedding such that

$$G|_{a\times S^q}\colon a \times S^q \to \mathbb{R}^m - G(b \times S^q)$$

is null-homotopic for some pair of different points $a, b \in S^p$. We prove the following statement: *if ψ is an autodiffeomorphism of $S^p \times S^q$ identical on a neighbourhood of $a \times S^q$ for some $a \in S^p$ and $p \leqslant q$ and $2m \geqslant 3p + 3q + 4$, then $G \circ \psi$ is isotopic to G.*

Let N be an oriented $(p+q)$-manifold and let f, g be isotopy classes of embeddings $N \to \mathbb{R}^m$, $S^p \times S^q \to \mathbb{R}^m$, respectively. As a corollary we obtain that under certain conditions *for orientation-preserving embeddings $s\colon S^p \times D^q \to N$ the S^p-parametric embedded connected sum $f\#_s g$ depends only on f, g and the homology class of $s|_{S^p\times 0}$.*

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1. Introduction and main results

1.1. Statements of the main results

This paper is on the classical knotting problem: *for an n-manifold N and a number m describe the set $E^m(N)$ of isotopy classes of embeddings $N \to \mathbb{R}^m$.* For recent surveys see [20, 30]; whenever possible we refer to these surveys and not to original papers. If the category (piecewise linear (PL) or smooth) is not mentioned, then the smooth category is tacitly meant. We denote by $[F]$ the isotopy class of an embedding F, except in § 2.1.

An interesting problem is to describe the 'precomposition' action of the group $\mathrm{Aut}(N)$ of autodiffeomorphisms of N on $E^m(N)$:[1]

$$E^m(N) \times \mathrm{Aut}(N) \to E^m(N) \quad \text{defined by} \quad ([F], \varphi) \mapsto [F] \circ \varphi := [F \circ \varphi].$$

[1] The set of submanifolds of $\mathbb{R}^m$, diffeomorphic to N, up to isotopy, is the quotient of $E^m(N)$ by this action. Action of the group $\mathrm{Aut}_+(N)$ of orientation-preserving autodiffeomorphisms of oriented N is analogously related to the set of oriented submanifolds of $\mathbb{R}^m$, orientably diffeomorphic to N.

For example, the action of $\mathrm{Aut}(S^1)$ on $E^3(S^1)$ factors through the widely studied 'change of the orientation' action. For the action of $\mathrm{Aut}(S^2)$ on $E^4(S^2)$ see [11, 14, 26]. For

$$N = T^{p,q} := S^p \times S^q$$

the problem was raised by E. Rees in 2002. We obtain a partial solution of this problem. The main results of this paper are theorems 1.4, 1.10, 1.11, 1.14, 1.16 and corollary 1.17. The remarks of this text are not used in the statements or proofs of the main results (except that remark 1.7(a) and (c) are used for corollary 1.17 and theorem 1.4, respectively).

Definition 1.1.

- Denote by $\sigma\colon S^q \to S^q$ the symmetry with respect to the hyperplane $x_1 = 0$.
- Denote by i_q the inclusion $S^q \to S^m$.
- Denote by # the embedded connected sum of embeddings or isotopy classes, or the connected sum of autodiffeomorphisms. (See [1, § 1] for an accurate definition of the embedded connected sum of embeddings analogous to [9, § 3]. By general position, in codimension at least 3 the embedded connected sum of embeddings defines the embedded connected sum of their isotopy classes.)
- Denote by 0_k the vector of k zero coordinates and $1_k := (1, 0_k) \in S^k$.
- Denote by $\mathrm{i} = \mathrm{i}_{p,q,m}\colon T^{p,q} \to S^m$ *the standard embedding* defined by $\mathrm{i}(x, y) := (y, 0_{m-p-q-1}, x)/\sqrt{2}$, or its restrictions.

Remark 1.2. (a) For the standard embedding $\mathrm{i}\colon T^{1,1} \to S^3$ and an autodiffeomorphism ψ of $T^{1,1}$ corresponding to a non-trivial element of $\mathrm{SL}_2(\mathbb{Z})$ we have $[\mathrm{i} \circ \psi] \neq [\mathrm{i}]$. This is proved by comparing $G|_{1_1 \times S^1}$, $G|_{S^1 \times 1_1}$ and $\mathrm{lk}(G(1_1 \times S^1), G(-1_1 \times S^1))$ for $G = \mathrm{i}, \mathrm{i} \circ \psi$.

For the action of $\mathrm{Aut}_+(T^{1,1})$ on $E^4(T^{1,1})$ see [12]. For the case of embeddings $T^{1,2} \to \mathbb{R}^6$, $T^{2,2} \to \mathbb{R}^7$ and $T^{1,3} \to \mathbb{R}^7$ the results of [6–8, 31, 32] could be useful.

(b) A group structure on $E^m(S^q)$ is defined in [9] for $m \geqslant q+3$. The group $E^m(S^q)$ is trivial for $2m \geqslant 3q+4$ [9].

(c) It would be interesting to know how composition with σ acts on $E^m(S^q)$. Composition with σ induces an automorphism of $E^m(S^q)$ for $m \geqslant q+3$ (this is proved analogously to theorem 1.11 below). For $m = q + 3 = 7$ the action is identical; see [32, symmetry remark and footnote, § 3].

(d) If $g \in E^m(S^q)$, $\psi \in \mathrm{Aut}_+(S^q)$ and $m \geqslant q+3$, then $g \circ \psi = g + [\mathrm{i}_q \circ \psi]$. The proof is obtained from the proof of lemma 1.6 below by changing S^p to a point.

(e) Analogously, take any $g \in E^m(T^{p,q})$ and $\psi \in \mathrm{Aut}_+(S^{p+q})$.

If $m \geqslant p+q+3$, then $g \circ (\mathrm{id}_{T^{p,q}} \# \psi) = g \# [\mathrm{i}_{p+q} \circ \psi]$, where $\mathrm{id}_{T^{p,q}} \# \psi$ is the 'connected sum' autodiffeomorphism of $T^{p,q}$. (In the notation of [1, § 1] the connected sum of the autodiffeomorphisms a and b is the autodiffeomorphism $a\#b\colon M\#N \to M\#N$ defined by $a\#b|_{M_0} = a|_{M_0}$, $a\#b|_{N_0} = b|_{N_0}$ and $a\#b|_{S^{p-1}\times I} = \mathrm{id}\, S^{p-1} \times I$.)

For $m \geqslant 2p+q+3$, $g\#[\mathrm{i}_q \circ \psi] = g$ if and only if $[\mathrm{i}_q \circ \psi] = [\mathrm{i}_q]$ by [4, proposition 5.6].

(f) For $q \geqslant 5$ the set of isotopy classes of $\mathrm{Aut}_+(S^q)$ can be identified with the group θ_{q+1} of homotopy spheres [5,37]. The 'composition with i_q' map $\partial\colon \theta_{q+1} \to E^m(S^q)$ is a homomorphism appearing in the exact sequence [9, (1.9)]. Hence, by [9, (1.9)] and [18, § 7.4] we have the following result.

Proposition 1.3.

(a) *For* $m - 3 = q \in \{7, 8, 9\}$ *there is* $\psi \in \mathrm{Aut}_+(S^q)$ *such that* $[\mathrm{i}_q \circ \psi] \neq [\mathrm{i}_q]$.

(b) *For* $q \in \{7, 8\}$, $m \geqslant q + 4$ *and any* $\psi \in \mathrm{Aut}_+(S^q)$ *we have* $[\mathrm{i}_q \circ \psi] = [\mathrm{i}_q]$.

Theorem 1.4. *For each integer* $l \geqslant 2$ *there is* $g \in E^{4l+3}(T^{1,2l+1})$ *and an autodiffeomorphism* ψ *of* $T^{1,2l+1}$ *identical on a neighbourhood of* $1_1 \times S^{2l+1}$ *and such that* $g \circ \psi \neq g$ *while* $[\mathrm{i} \circ \psi] = [\mathrm{i}]$.

Moreover, we can take as ψ *the autodiffeomorphism constructed from the non-trivial element of* $\pi_1(\mathrm{SO}_{2l+2})$ *as in remark 1.7(c).*

Remark 1.5. The definition of an 'S^p-parametric connected sum' abelian group structure on $E^m(T^{p,q})$ for $m \geqslant 2p + q + 3$ can be found in [34, § 2.1] and [23] (the definition is recalled in § 2.3).

Lemma 1.6. *If* $g \in E^m(T^{p,q})$, $m \geqslant 2p + q + 3$ *and* ψ *is an autodiffeomorphism of* $T^{p,q}$ *identical on a neighbourhood of* $S^p \times 1_q$, *then* $g \circ \psi = g + [\mathrm{i} \circ \psi]$.

Remark 1.7. (a) The group $E^m(T^{p,q})$ is trivial for $m \geqslant p + q + \max\{p, q\} + 2$ by the Haefliger unknotting theorem [30, theorem 2.8(b)]. Hence, *the group structure, lemmas 1.6 and 1.9, and theorems 1.10 and 1.11 below are only interesting for* $p \leqslant q$. So a reader might assume that $p \leqslant q$, although this condition is not required and so is not added to the statements of the above-listed results.

The lowest dimensional case when $2p + q + 3 \leqslant m < p + q + \max\{p, q\} + 2$ is $m = 8$, $p = 1$ and $q = 3$.

(b) The condition that ψ is identical on a neighbourhood of $S^p \times 1_q$ is essential in lemma 1.6 by theorem 1.4.

(c) Let $\varphi\colon S^q \to \mathrm{SO}_{p+1}$ be a smooth map that maps a neighbourhood of $1_q \in S^q$ to the identity. Define an automorphism

$$\overline{\varphi} \text{ of } T^{p,q} \quad \text{by} \quad \overline{\varphi}(a, b) := (\varphi(b)a, b).$$

It is clear that $\overline{\varphi}$ is identical on a neighbourhood of $S^p \times 1_q$.

(d) It follows from [25, corollary C] that for n odd and $n \neq 3, 5, 9$, there are pairs of embeddings $F_0, F_1\colon S^2 \times S^{n-2} \to \mathbb{R}^{2n-2}$ such that the normal bundle of F_0 is trivial, whereas the normal bundle of F_1 is not. Consequently, in this case the action of $\mathrm{Aut}_+(S^2 \times S^{n-2})$ on $E^{2n-2}(S^2 \times S^{n-2})$ is not transitive.

Definition 1.8. Define $D^p_+, D^p_- \subset S^p$ by equations $x_1 \geqslant 0$, $x_1 \leqslant 0$, respectively. Then $S^p = D^p_+ \cup D^p_-$. For an autodiffeomorphism α of S^q denote by

$$\hat{\alpha} := \mathrm{id}_{S^p} \times \alpha$$

the product autodiffeomorphism of $T^{p,q}$. An autodiffeomorphism ψ of $T^{p,q}$ is *symmetric* if

$$\psi(S^p \times D^q_\pm) = S^p \times D^q_\pm \quad \text{and} \quad \hat{\sigma} \circ \psi = \psi \circ \hat{\sigma}.$$

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Lemma 1.9. *If $m \geqslant 2p+q+2$ and ψ is a symmetric autodiffeomorphism of $T^{p,q}$, then* $[\mathrm{i}\circ\psi]=[\mathrm{i}]$.

Lemmas 1.6 and 1.9 imply the following result.

Theorem 1.10. *If $m \geqslant 2p+q+3$ and ψ is a symmetric autodiffeomorphism of $T^{p,q}$ identical on a neighbourhood of $S^p\times 1_q$, then $g\circ\psi = g$ for each $g\in E^m(T^{p,q})$.*

Theorem 1.11. *For $m \geqslant 2p+q+3$ the composition with a symmetric autodiffeomorphism of $T^{p,q}$ defines an automorphism of the group $E^m(T^{p,q})$.*

Remark 1.12.

(a) The property of being symmetric depends on the order of factors in $S^p\times S^q$, i.e. a symmetric autodiffeomorphism of $T^{p,q}$ need not be one of $T^{q,p}$.

(b) For a smooth map $\varphi\colon S^p\to \mathrm{SO}_q\subset \mathrm{SO}_{q+1}$ the autodiffeomorphism of $T^{p,q}$, defined analogously to remark 1.7(c) with p and q exchanged, is symmetric and is identical on a neighbourhood of $1_p\times S^q$, but is not necessarily identical on a neighbourhood of $S^p\times 1_q$.

(c) It would be interesting to know if there is a symmetric autodiffeomorphism of $T^{p,q}$ identical on a neighbourhood of $S^p\times 1_q$ but not isotopic to the identity. If there is not, theorem 1.10 is not interesting.

(d) It would be interesting to know if either of the conditions (that ψ is symmetric or identical on a neighbourhood of $S^p\times 1_q$) is essential in lemma 1.9 and theorems 1.10 and 1.11.

(e) Theorem 1.11 is not covered by theorem 1.10 for any symmetric orientation-reversing autodiffeomorphism of $T^{p,q}$.

(f) The proofs of lemma 1.6 and theorems 1.10 and 1.11 are not hard (see § 2). However, working with distinct embeddings having the same image requires care. Lemma 1.9 is an easy corollary of an unknotting theorem and smoothing theory (see § 2).

Definition 1.13. An embedding $G\colon T^{p,q}\to\mathbb{R}^m$ (or its isotopy class) is called *unlinked* if

$$G|_{1_p\times S^q}\colon 1_p\times S^q\to \mathbb{R}^m - g(\{-1_p\}\times S^q)$$

is null-homotopic. For example, i is unlinked and any $g\in E^m(T^{p,q})$ is unlinked for $m\geqslant 2q+2$.

Theorem 1.14. *Suppose that $p\leqslant q$, $g\in E^m(T^{p,q})$ is unlinked and ψ is an autodiffeomorphism of $T^{p,q}$ identical on a neighbourhood of $1_p\times S^q$.*

(a) *If $2m\geqslant 3p+3q+4$, then $g\circ\psi=g$.*

(b) *If $m\geqslant p+q+2+\max\{p,q/2\}$, then $g\circ\psi = g\#u$ for some $u\in E^m(S^{p+q})$.*

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Remark 1.15.

(a) Theorem 1.14(a) follows from theorem 1.14(b) because $E^m(S^n) = 0$ for $2m \geqslant 3n + 4$.

(b) The lowest dimensional case when theorem 1.14 holds but is not covered by known results is $m = 6$, $p = 1$ and $q = 2$. In this case $E^6(S^1 \times S^2)$ is described in [31] and $\mathrm{Aut}_+(S^1 \times S^2)$ contains a subgroup $\pi_1(\mathrm{SO}_3)$ (see remark 1.7(c)).

(c) Theorem 1.4 falls into the dimension assumption of theorem 1.14 and so shows that the unlinkedness assumption in theorem 1.14 is essential.

(d) Theorem 1.14(b) is obtained from lemma 2.2 below, an application of the Penrose–Whitehead–Zeeman–Irwin trick [28, theorem 2.4] and the Hudson 'concordance implies isotopy' result; see details after lemma 2.2.

(e) The PL analogues of lemmas 1.6 and 1.9 and theorems 1.10, 1.11, 1.14 and 1.16 are true. The proofs are the same (except that in lemma 2.3 we do not need smoothing).

(f) It would be interesting to describe the action of $\mathrm{Aut}(D^p \times S^q)$ on $E^m(D^p \times S^q)$. Analogues for $D^p \times S^q$ of our results on $T^{p,q}$ are correct and could be useful.

(g) We do not use any results on $\mathrm{Aut}(T^{p,q})$. However, the interested reader can find information in, for example, [19].

1.2. Application to an S^p-parametric embedded connected sum

In the rest of this paper N is a compact n-manifold and $f \in E^m(N)$.

For $m \geqslant n + 3$, an embedding $s\colon D^n \to \mathrm{Int}\, N$ and $g \in E^m(S^n)$ one can define an embedded connected sum $f\#_s g$ (analogous to the embedded connected sum on $E^m(S^n)$ [9]). A classical interesting question is *when does $f\#_s g$ depend only on f, g and the component of N containing $s(D^n)$?*[2]

If N is connected oriented, then $f\#_s g$ is independent on orientation-preserving s (because every two orientation-preserving embeddings of a disk into a connected oriented manifold are isotopic). So for $m \geqslant n+3$ a group structure on $E^m(S^n)$ [9] and an action $\#$ of $E^m(S^n)$ on $E^m(N)$ are defined. For descriptions of this action see [6, 8, 31, 32, 36] and [4, proposition 5.6].

For $m \geqslant n + p + 3$, an embedding $s\colon S^p \times D^{n-p}_- \to \mathrm{Int}\, N$ and $g \in E^m(T^{p,n-p})$ one can define an 'S^p-parametric embedded connected sum' $f\#_s g$ (see [23], [29, pp. 262–264] and [33, § 2]; see also the definition in § 2.3). This defines a group structure on $E^m(T^{p,n-p})$ [34, 35] and an action $\#_s$ of $E^m(T^{p,n-p})$ on $E^m(N)$ (because $(f\#_s g)\#_s g' = f\#_s(g+g')$ analogously to [34, § 3, proof of associativity in the proof of the group structure theorem 2.2]).

We study the following interesting questions (which are classical for $p = 0$): *when does $f\#_s g$ depend only on f, g and the isotopy (homotopy, homology) class of $s|_{S^p\times 0}$?*

A relation of these questions to action by autodiffeomorphisms is as follows.

[2] A sufficient condition for this is non-orientability or non-closedness of the component, or $[\mathrm{i}] \circ \sigma = [\mathrm{i}]$. We conjecture that this sufficient condition is not necessary. An analogous remark should be made for the $S^0 \times S^n$-analogue of an embedded connected sum discussed below.

THEOREM 1.16. *Let N be an n-manifold, $f \in E^m(N)$, $g \in E^m(T^{p,n-p})$, let $s\colon S^p \times D^{n-p}_- \to \operatorname{Int} N$ be an embedding and let ψ be a symmetric autodiffeomorphism of $T^{p,n-p}$. If $m \geqslant n+p+3$, then $f\#_s g = f\#_{s\circ\psi|_{S^p\times D^{n-p}_-}}(g\circ\psi)$.*

COROLLARY 1.17.

(I) *Let N be an oriented n-manifold, let $g \in E^m(T^{p,n-p})$ be unlinked, let $s\colon S^p \times D^{n-p} \to \operatorname{Int} N$ be an orientation-preserving embedding, let $f \in E^m(N)$ and $m \geqslant n+p+3$, $2m \geqslant 3n+4$.*

(a) *The sum $f\#_s g$ depends only on f, g and the isotopy class of $s|_{S^p\times 0}$.*

(b) *If N is $(2p+2-n)$-connected, then $f\#_s g$ depends only on f, g and the homotopy class of $s|_{S^p\times 0}$.*

(c) *If $p \geqslant 2$ and N is $(p-1)$-connected, then $f\#_s g$ depends only on f, g and the homology class of $s|_{S^p\times 0}$ in $H_p(N;\mathbb{Z})$.*

(II) *Modifications of statements (a), (b) and (c) hold with '$f\#_s g$' replaced by 'the class of $f\#_s g$ in $E^m(N)/\#$' and '$2m \geqslant 3n+4$' replaced by '$2m \geqslant 3n-p+4$'.*

REMARK 1.18. (a) Let N be an oriented $(p-1)$-connected n-manifold and let $n \geqslant 2p+2$. Let $\mathbb{Z}_{(k)}$ be $\mathbb{Z}$ for k even and $\mathbb{Z}_2$ for k odd. Then the Whitney invariant $W\colon E^{2n-p+1}(N) \to H_p(N;\mathbb{Z}_{(n-p-1)})$ is bijective. [3] 'The parametric connected sum' construction together with the explicit construction $\tau\colon \mathbb{Z}_{(n-p-1)} \to E^{2n-p+1}(T^{p,n-p})$ of embeddings (see [30, § 3.4] and [21]) give the inverse of W. That is, *for each $f \in E^{2n-p+1}(N)$ an action $\#$ of $H_p(N;\mathbb{Z}_{(n-p-1)})$ on $E^{2n-p+1}(N)$ is well defined by $\#([s]\otimes r)f := f\#_s\tau(r)$ and is free and transitive.*

Indeed, by [33, end of § 2],

$$W(f\#_s g) = W(f) + [s](W(g)\cap[S^p\times 1_q]), \quad \text{where } W(g) \in H_p(T^{p,n-p};\mathbb{Z}_{(n-p-1)}). \qquad (*)$$

Hence, $W(f\#_u g) = W((f\#_s g)\#_t g)$ when $[u] = [s]+[t] \in H_p(N;\mathbb{Z}_{(n-p-1)})$. Since W is injective, $f\#_u g = (f\#_s g)\#_t g$. This and $(f\#_s g)\#_s g' = f\#_s(g+g')$ imply that $\#$ is an action. By $(*)$, $\#$ is free. By the injectivity of W and $(*)$, $\#$ is transitive.

(b) Denote by $E^m_0(T^{p,n-p})$ the subgroup of unlinked embeddings in $E^m(T^{p,n-p})$. Under the assumptions of corollary 1.17(c), for $2m \geqslant 3n+4$ a map

$$E^m(N)\times H_p(N;\mathbb{Z})\times E^m_0(T^{p,n-p}) \to E^m(N) \quad \text{is well defined by} \quad (f,[s],g)\mapsto f\#_s g.$$

We conjecture that this map gives an action of $H_p(N;\mathbb{Z})\times E^m_0(T^{p,n-p})$ on $E^m(N)$.

(c) Corollary 1.17 is trivial for $p=0$. We conjecture that the assumption $p \geqslant 2$ is superfluous in corollary 1.17(c).

[3] If $F, F'\colon N \to$ are representatives of $f, f' \in E^{2n-p+1}(N)$ and $F = F'$ on the complement to a closed n-ball $B \subset N$, then $W(f,f') \in H_p(N-B;\mathbb{Z}_{(n-p-1)}) \cong H_p(N;\mathbb{Z}_{(n-p-1)})$ is defined to be the Alexander dual of the homology class of $F(B)\cup F'(B)$; see details in [30, § 2] and [24]. By definition, $W(f) = W(f,f')$ is a 'difference' between f and a certain chosen isotopy class f'.

(d) Fix a certain smooth triangulation of N. Embed N into $\mathbb{R}^M$ for some large M. Denote by ON a tubular neighbourhood and by $\nu_N\colon ON \to N$ the normal bundle of N in $\mathbb{R}^M$. A *stable normal framing* on a subset $X \subset N \subset \mathbb{R}^M$ is an embedding $\zeta\colon X \times D^{M-n} \to ON$ such that $\zeta(a \times D^{M-n}) = \nu_N^{-1}(a)$ for each $a \in X$.[4] An embedding $s\colon S^p \times D^{n-p} \to X$ is *ζ-good* if $s^*\zeta$ is the standard stable normal framing of S^p.

Let ζ be a stable normal framing of an open neighbourhood U in N of the $(p+1)$-skeleton (of the triangulation) of N. For ζ-good embeddings $s\colon S^p \times D^{n-p} \to \operatorname{Int} N$, the isotopy class $f\#_s g$ depends only on $f \in E^m(N)$, $g \in E^m(T^{p,n-p})$ and the isotopy class of $s|_{S^p\times 0}$. (Then analogously to corollary 1.17(b) and (c) one obtains analogous assertions for homotopy and homology classes of $s|_{S^p\times 0}$.)

Proof. Take two ζ-good embeddings $s, s'\colon S^p \times D^{n-p} \to N$ isotopic on $S^p \times 0$. The image of the isotopy between $s|_{S^p\times 0}$ and $s'|_{S^p\times 0}$ is $(p+1)$-dimensional. Hence, by general position we may assume that this image is disjoint with the dual $(n-p-2)$-skeleton. Thus we may assume that this image is contained in U. Hence, the isotopy can be extended to an isotopy of $S^p \times D^{n-p}$ between s and a ζ-good embedding $s''\colon S^p \times D^{n-p} \to N$ coinciding with s' on $S^p \times 0$. Since both s' and s'' are ζ-good, they are isotopic. Hence, s and s' are isotopic. $\square$

2. Proofs

2.1. Proof of theorem 1.4

Denote by $[a]$ the homotopy class of a map a. The following lemma is possibly known.

LEMMA 2.1. *For each $n \geqslant 2$ there is an autodiffeomorphism ψ of $T^{1,n-1}$ identical on a neighbourhood of $1_1 \times S^{n-1}$ for which $\operatorname{pr}_{S^{n-1}} \circ\psi$ is not homotopic to $\operatorname{pr}_{S^{n-1}}$.*

Proof. Let $\varphi\colon S^1 \to \mathrm{SO}_n$ be a homotopy non-trivial map that maps a neighbourhood of $1_1 \in S^1$ to the identity. Define an automorphism ψ of $T^{1,n-1}$ by $\psi(a,b) := (a, \varphi(a)b)$. Clearly, ψ is identical on a neighbourhood of $1_1 \times S^{n-1}$. Let SG_n be the space of maps $S^{n-1} \to S^{n-1}$ of degree $+1$, the base point being the identity. Identify by the exponential law $\pi_1(SG_n)$ and the set of maps $S^1 \times S^{n-1} \to S^{n-1}$ mapping $(1_1, x)$ to x for each $x \in S^{n-1}$, up to homotopy through such maps. Let $i\colon \pi_1(\mathrm{SO}_n) \to \pi_1(SG_n)$ be the inclusion-induced map. It is known that i is an isomorphism and $[\operatorname{pr}_{S^{n-1}} \circ\psi] = i[\varphi] \neq i[*] = [\operatorname{pr}_{S^{n-1}}]$, where $*$ is the constant map. So $\operatorname{pr}_{S^{n-1}} \circ\psi$ is not homotopic to $\operatorname{pr}_{S^{n-1}}$. $\square$

Construction of example from theorem 1.4. Define $n := 2l+2$. Take an autodiffeomorphism ψ of $T^{1,n-1}$ given by lemma 2.1. Let $v\colon S^{n-1} \to S^{n-1}$ be a unit length tangent vector field on S^{n-1} whose degree is $+1$. That is, v is a degree $+1$ map

[4] A stable normal framing over the 0-skeleton extendable to the 1-skeleton is equivalent to an orientation on N. A stable normal framing over the 2-skeleton is equivalent to a stable normal framing over the 1-skeleton extendable to the 2-skeleton (because $\pi_2(\mathrm{SO}) = 0$) and to a spin structure on N. A stable normal framing on the $(p-1)$-skeleton extendable to the p-skeleton is equivalent to a lift $N \to BO\langle p+1\rangle$ of the stable Gauss map $N \to BO$ [17].

such that $v(x) \perp x$ for each $x \in S^{n-1}$. Let G be the composition

$$T^{1,n-1} \xrightarrow{\hat{v}} T^{n-1,n-1} \xrightarrow{\mathrm{i}} \mathbb{R}^{2n-1}, \quad \text{where } \hat{v}(\mathrm{e}^{\mathrm{i}\theta}, x) := (v(x)\cos\theta + x\sin\theta, x).$$

Since $v(x) \perp x$ for each $x \in S^{n-1}$, the map $\hat{v}$ is well defined. Let g be the isotopy class of G.

By theorem 1.14(a), $[\mathrm{i}\circ\psi] = [\mathrm{i}]$. To prove that $g \circ \psi \neq g$ we need some preliminaries.

Definition of $L(F) \in \pi_n(S^{n-1})$ for an embedding $F\colon T^{1,n-1} \to \mathbb{R}^{2n-1}$ coinciding with G on $D^1_+ \times S^{n-1}$. Denote by $L'(F)$ the homotopy class of the composition

$$(F \circ (\sigma|_{D^1_+} \times \mathrm{id}_{S^{n-1}})) \cup G|_{D^1_- \times S^{n-1}} \colon T^{1,n-1} \to S^{2n-1} - G(1_1 \times S^{n-1}) \xrightarrow{h} S^{n-1},$$

where h is a homotopy equivalence of degree $+1$ (see [30, § 3] and [22]). For each F as above the restriction of the above composition to $1_1 \times S^{n-1}$ is homotopic to $\mathrm{pr}_{S^{n-1}}|_{1_1\times S^{n-1}}$. Consider the Barrat–Puppe exact sequence

$$\pi_n(S^{n-1}) \xrightarrow{\#} [T^{1,n-1}, S^{n-1}] \xrightarrow{r} [1 \times S^{n-1}, S^{n-1}],$$

where r is the restriction and $\#$ extends to the 'top cell' action of $\pi_n(S^{n-1})$ on $[T^{1,n-1}, S^{n-1}]$. It is well known that this action is free; see, for example, [27]. Hence there is a unique class $L(F) \in \pi_n(S^{n-1})$ such that $[L'(F)] = [\mathrm{pr}_{S^{n-1}}]\#L(F)$.

Definition of the map $\mu = \mu^m_{p,q}\colon \pi_{p+q}(S^{m-q-1}) \to E^m(T^{p,q})$ for $2m \geqslant 3p + 3q + 4$. For each $x \in \pi_{p+q}(S^{m-q-1})$ take a map x' such that

$$S^{p+q} \xrightarrow{x'} S^m - \bar{\mathrm{i}}(D^{p+1} \times S^q) \xrightarrow{h} S^{m-q-1}$$

represents x. Here h is a homotopy equivalence of degree $+1$ (see [30, § 3], [22]) and $\bar{\mathrm{i}}(x, y) := (y\sqrt{2 - |x|^2}, 0_{m-p-q-1}, x)/\sqrt{2}$. Since $2m \geqslant 3p + 3q + 4$, there is a unique up to isotopy embedding x'' homotopic to x'. Let $\mu(x)$ be the isotopy class of $\mathrm{i}\,\#x''$.

Proof that $L(F)$ is an isotopy invariant of F for $n \geqslant 6$ even.[5] By definition of sum (recalled before the proof of lemma 1.6 below) F is a representative of $\mu L(F) + g$, where $\mu = \mu^{2n-1}_{1,n-1}$. Hence it suffices to prove that μ is injective.

For $n \geqslant 6$ there is the following commutative (up to sign) diagram:

$$\begin{array}{ccccc}
\pi_n(S^{n-1}) & \xrightarrow[\Delta]{} & \pi_{n-1}(S^{n-2}) & & \\
\downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\Sigma} & & \\
E^{2n}(D^1 \times S^n) & \xrightarrow[\lambda]{} & \pi_n(S^{n-1}) & \xrightarrow[\mu]{} & E^{2n-1}(T^{1,n-1})
\end{array}$$

Here Δ is the map from the exact sequence of the 'forgetting the last vector' bundle $S^{n-2} \to V_{n,2} \to S^{n-1}$, the lower line is exact and τ is an isomorphism.

[5] It is not clear that $L(F)$ is preserved through an isotopy of F non-identical on $D^1_+ \times S^{n-1}$. We conjecture that $L(F)$ is an isotopy invariant of F for $n = 4$ (then theorem 1.4 holds for $l = 1$). For a proof one possibly needs the results of [7]. Note that $L(F) = \beta(F)$ for the more complicated β-invariant of [7].

The existence of such a diagram follows by lemma 5.1 and restriction lemma 5.2 of [35] for $p = 1$, $q = n - 1$ and $m = 2n - 1$, because $2(2n - 1) \geqslant 3n + 4 > 10$ for $n \geqslant 6$, so by the smooth version of [28, theorem 2.4] the map forg in [35, p. 15] is an isomorphism that 'respects' the map μ (the definitions of τ, λ [35, § 5] are not used here).

We have $\Delta\Sigma x = (1 - (-1)^n)x = 0$ for each $x \in \pi_{n-1}(S^{n-2})$ [15, 16]. Since $n < 2(n - 1) - 2$, the map Σ is an isomorphism. Hence $\Delta = 0$. Since both τ and Σ are isomorphisms, this implies that $\lambda = 0$. Hence, by exactness μ is injective. □

Proof that $g \circ \psi \neq g$ for $n \geqslant 6$ even. We may assume that ψ is identical on $D^1_+ \times S^{n-1}$. Hence $G \circ \psi = G$ on $D^1_+ \times S^{n-1}$. Thus $L(G \circ \psi)$ is defined. Clearly, $L'(G \circ \psi) = L'(G) \circ [\psi] = [\mathrm{pr}_{S^{n-1}} \circ \psi] \neq [\mathrm{pr}_{S^{n-1}}]$ by lemma 2.1. Hence $L(G \circ \psi) \neq 0$. Thus $g \circ \psi \neq g$. □

2.2. Proof of theorem 1.14(b)

The *self-intersection set* of a map $F: N \to \mathbb{R}^m$ from a space N is

$$\Sigma(F) := \{x \in N : |F^{-1}F(x)| > 1\}.$$

LEMMA 2.2. *Let $G: T^{p,q} \to \operatorname{Int} B^m$ be an unlinked embedding and let ψ be an autodiffeomorphism of $T^{p,q}$ identical on a neighbourhood of $1_p \times S^q$. Then there are a neighbourhood Δ of $1_p \times S^q$ and a homotopy $H: T^{p,q} \times I \to B^m \times I$ between $G \circ \psi$ and G such that $\Sigma(H) \subset (T^{p,q} - \Delta) \times I$ and $H|_{\Delta \times I}$ is the identical homotopy.*

Proof. We may assume that ψ is identical on $D^p_+ \times S^q$. The abbreviations

$$G_1, G_2: D^p_- \times S^q \to B^m - G(1_p \times S^q) \quad \text{of } G \text{ and of } G \circ \psi$$

coincide on the boundary. So together they form a map

$$G_{12} := G \circ (\sigma \times \mathrm{id}_{S^q} \cup \psi|_{D^p_- \times S^q}): T^{p,q} \to B^m - G(1_p \times S^q).$$

This map factors through the inclusion $G(D^p_- \times S^q) \to B^m - G(1_p \times S^q)$. Since G is unlinked, this inclusion is null-homotopic. Hence there is a null-homotopy H_{12} of G_{12}. Denote by $\operatorname{con} X := X \times I / X \times 1$ the cone of X. Take the composition

$$D^p_- \times S^q \times I = D^p \times I \times S^q \xrightarrow{\alpha \times \mathrm{id}_{S^q}} (\operatorname{con} S^p) \times S^q = \frac{T^{p,q} \times I}{\{S^p \times y \times 1\}_{y \in S^q}}$$
$$\xrightarrow{\beta} \frac{T^{p,q} \times I}{T^{p,q} \times 1} = \operatorname{con} T^{p,q} \xrightarrow{H_{12}} B^m - g(1_p \times S^q),$$

where we have the following:

- α is the contraction of $x \times I$ to $[x \times 0] \in \operatorname{con} S^p$ for each $x \in \partial D^p = S^{p-1}$; α maps $\partial(D^p \times I)$ to the base $[S^p \times 0]$ of the cone and $0 \times 1/2$ to the vertex $[S^p \times 0]$ of the cone;

- β is the contraction of the quotient of $T^{p,q} \times 1$ to the vertex of the cone.

There is a neighbourhood Δ of 1_p in S^p such that $G(\Delta \times S^q) \cap H_{12}(\operatorname{con} T^{p,q}) = \emptyset$.

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Let $D^p_k = D^p_+, D^p_-$ according to $k = 0, 1$, respectively. We have, for $k = 0, 1$,

$$D^p_- \times S^q \times k = D^p_- \times k \times S^q \xrightarrow{\alpha\times \mathrm{id}_{S^q}} [D^p_k \times 0] \times S^q = [D^p_k \times S^q \times 0] \xrightarrow{\beta} [D^p_k \times S^q \times 0],$$

$$H_{12}|_{D^p_+\times S^q\times 0} = G_1 \circ (\sigma \times \mathrm{id}_{S^q}) \quad \text{and} \quad H_{12}|_{D^p_-\times S^q\times 0} = G_2.$$

Hence the above composition $H_{12}\circ\beta\circ(\alpha\times\mathrm{id}_{S^q})$ is a homotopy between G_1 and G_2 relative to the boundary. The 'union' of this homotopy with the identical homotopy of $D^p_+ \times S^q$ is the required homotopy H. □

Proof of theorem 1.14(b). Denote a representative of g by G. Take Δ and H given by lemma 2.2. We may assume that $\Delta = D^p_+\times S^q$ and $\Sigma(H) \subset D^p_-\times S^q\times[\frac{1}{3}, \frac{2}{3}]$. Since $m \geqslant 2p+q+2$, we have $(p+1)+(p+q+1) < m+1$. Hence, by general position we may assume that $\Sigma(H)\cap D^p_-\times 1_q\times[\frac{1}{3}, \frac{2}{3}] = \emptyset$ and further that $\Sigma(H) \subset D^p_- \times D^q_- \times [\frac{1}{3}, \frac{2}{3}]$ (see [29, footnote 6]). This means that H is a proper *quasi-embedding* (see the definition in [28, § 2]). For $p = 0$ theorem 1.14(b) is trivial, so we may assume that $m + 1 \geqslant p + q + 1 + 3$. Also, $2(m + 1) \geqslant 3(p + q + 1) + 2 - p + 1$. Therefore, we can apply [28, theorem 2.4] to H. We obtain a PL concordance

$$F \text{ between } G \text{ and } G\circ\psi \quad \text{such that} \quad F = H \text{ on } (T^{p,q} - D^p_- \times D^q_-) \times [\tfrac{1}{3}, \tfrac{2}{3}].$$

Then F is a smooth embedding on this set. Denote by $U\colon S^{p+q} \to B^m\times 0$ a smooth embedding representing minus the complete obstruction in $E^m(S^{p+q})$ to smoothing of F [2, 3]. Change the concordance F by the boundary embedded connected sum with the cone (whose vertex is in $B^m \times (0, 1)$) over the embedding U. The obstruction to smoothing of the new concordance is zero. Therefore, $G \circ \psi$ is smoothly concordant to $G\#U$. Hence $g \circ \psi = g\#[U]$ [13]. □

2.3. Proof of lemmas 1.6 and 1.9, theorem 1.11 and corollary 1.17

In this subsection we omit the composition sign $\circ$, writing fg for $f \circ g$.

Let N be an n-manifold and let $s\colon S^p \times D^{n-p}_- \to N$ be an embedding. A map $f\colon N \to S^m$ is called *s-standardized* if

$$f(N - \mathrm{im}\, s) \subset \mathrm{Int}\, D^m_+ \quad \text{and} \quad fs = \mathrm{i}\,.$$

Denote by $i\colon S^p \times D^{n-p}_- \to T^{p,n-p}$ the inclusion. Except for the proof of theorem 1.16 at the end of this subsection, the reader may assume that $N = T^{p,n-p}$ and $s = i$.

Let $R\colon \mathbb{R}^m \to \mathbb{R}^m$ be the symmetry of $\mathbb{R}^m$ with respect to the hyperplane given by equations $x_1 = x_2 = 0$, i.e. $R(x_1, x_2, x_3, \dots, x_m) := (-x_1, -x_2, x_3, \dots, x_m)$. We also denote by R restrictions of this symmetry. Since $m \geqslant n + p + 2$, analogously to [34, standardization lemma 2.1] there are an s-standardized representative $\bar f$ of $f \in E^m(N)$ and an i-standardized representative $\bar g$ of $g \in E^m(T^{p,n-p})$. Then a representative $\bar h$ of $f\#_s g$ is defined by

$$\bar h(a) := \begin{cases} \bar f(a), & a \notin \mathrm{im}\, s,\\ R\bar g\hat R s^{-1}(a), & a \in \mathrm{im}\, s. \end{cases} \qquad (**)$$

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The two formulae agree on $\partial \operatorname{im} s$ because $\mathrm{i} = R\,\mathrm{i}\,\hat{R}$. Clearly, h is a smooth embedding, i.e. is injective, is differentiable and has non-degenerate derivative. For $m \geqslant n+p+3$,

- this gives a well-defined map $E^m(N) \times E^m(T^{p,n-p}) \to E^m(N)$ (analogously to [34, § 3, beginning of the proof of the group structure theorem 2.2]),[6] and
- $+ := \#_i$ gives a well-defined abelian group structure on $E^m(T^{p,n-p})$ [34, § 2.1].

Let R^t be the rotation of $\mathbb{R}^m$ whose restriction to the plane $\mathbb{R}^2 \times 0$ is the rotation through the angle $+\pi t$. This rotation leaves the orthogonal complement $0 \times \mathbb{R}^{m-2}$ fixed.

Proof of lemma 1.6. Let $\mathrm{i}_\psi := R\,\mathrm{i}\,\psi\hat{R}$. We may assume that ψ is identical on $S^p \times D^q_+$. This,

$$R(D^m_\pm) = D^m_\mp, \quad \hat{R}(S^p \times D^q_\pm) = S^p \times D^q_\mp \quad \text{and} \quad R\,\mathrm{i}\,\hat{R} = \mathrm{i}$$

imply that the embedding i_ψ is i-standardized. There is an isotopy $R^t\,\mathrm{i}\,\psi\widehat{R^t}$ between i_ψ and $\mathrm{i}\,\psi$. By the standardization lemma of [29,33] there is a standardized representative $\bar{g}$ of g. Then a representative $\bar{h}$ of $g+[\mathrm{i}]\psi$ is defined by $(**)$ for $\bar{f}$, $\bar{g}$, s replaced by $\bar{g}$, i_ψ, i, respectively. We have $\bar{h} = \bar{g} = \bar{g}\psi$ on $S^p \times D^q_+$ and $\bar{h} = Ri_\psi\hat{R} = \mathrm{i}\,\psi = \bar{g}\psi$ on $S^p \times D^q_-$. Hence $\bar{h} = \bar{g}\psi$. □

LEMMA 2.3. *Any two proper embeddings $S^p \times D^q \to B^m$ are properly isotopic for $m \geqslant 2p+q+2$.*

Proof. The pair $(S^p \times D^q, S^p \times \partial D^q)$ is $(q-1)$-connected. Since $m \geqslant 2p+q+2$, this pair is $(2(p+q)-m+1)$-connected. Therefore, any two proper embeddings $S^p \times D^q \to B^m$ are properly PL isotopic [13, theorem 10.2].

Obstructions to smoothing this isotopy (moving $S^p \times \partial D^q$ in ∂B^m) are in $H^j(S^p \times D^q; E^{m-p-q+j}(S^j))$ [2, 3, 10]. The only non-trivial obstruction could appear for $j = p$. Since $m-p-q \geqslant 2p+2$, we have $2(m-q) \geqslant 2(2p+2) \geqslant 3p+4$, so this obstruction is zero. □

In this subsection we denote by the same letter a symmetric autodiffeomorphism of $T^{p,q}$ and its restriction $S^p \times D^q_\pm \to S^p \times D^q_\pm$.

Lemma 1.9 is implied by lemma 1.9′(a).[7]

[6] In [32, definition 1.4 of the action b], essentially an action $b\colon H_p(N; \pi^S_{2n-p-1-m}) \to E^m(N)$ was constructed for a closed orientable $(p-1)$-connected n-manifold N and $2m \geqslant 3n+4-p$. There is a map (see [30, § 3.4] and [21])

$$\pi^S_{2n-p-1-m} = \pi_{n-p-1}(S^{m-n}) \xrightarrow{\mu} \pi_{n-p-1}(V_{m-n+p,p+1}) \xrightarrow{\tau} E^m(T^{p,n-p})$$

whose image consists of unlinked embeddings. We have $b([s] \otimes x, f) = f\#_s\tau\mu(x)$. The set of unlinked embeddings is $\operatorname{im}(\tau\mu)$ if either $p = 1$ or $m \geqslant 2n-p$. So the action b is the 'top cell part' of the map $([s], g, f) \mapsto f\#_s g$. The 'top cell part' is the same as the whole map if either $p = 1$ or $m \geqslant 2n-p$.

[7] Lemma 1.9 is used in the proof of theorem 1.11. So although lemma 1.9 for $m \geqslant 2p+q+3$ follows from theorem 1.11, lemma 1.9 for $m \geqslant 2p+q+3$ is not a corollary of theorem 1.11.

LEMMA 1.9′. *Let ψ be a symmetric autodiffeomorphism of $T^{p,q}$.*

(a) *If $m \geqslant 2p+q+2$, then there is an isotopy $H_t\colon S^m \to S^m$ of the identity map H_0 such that $H_1\,\mathrm{i}\,\psi = \mathrm{i}$ and $H_1(D^m_\pm) = D^m_\pm$.*

(b) *If H_t is an isotopy from (a), embedding $\bar g\colon T^{p,q} \to S^m$ is i-standardized and embedding $\bar f\colon N \to S^m$ is s-standardized, then*

- *embedding $H_1\bar f$ is $s\psi$-standardized,*
- *embedding $H_1\bar g\psi$ is i-standardized,*
- *embedding $\bar g_\psi := RH_1R\bar g\hat R\psi\hat R$ is i-standardized and isotopic to $\bar g\psi$.*

Proof. (a) By lemma 2.3 there is an isotopy between $\mathrm{i}\colon S^p\times D^q_+ \to D^m_+$ and $\mathrm{i}\,\psi\colon S^p\times D^q_+ \to D^m_+$. Since ψ is symmetric and being proper includes being orthogonal near the boundary, the symmetric extension of the above isotopy with respect to the hyperplane $x_1 = 0$ is a smooth isotopy. This extension is as required.

(b) We have

$$H_1\bar f s\psi = H_1\,\mathrm{i}\,\psi = \mathrm{i} \quad \text{on } S^p\times D^q_- \quad \text{and} \quad H_1\bar f(N - \operatorname{im} s) \subset H_1(D^m_+) = D^m_+.$$

Thus $H_1\bar f$ is $s\psi$-standardized.

Clearly, *an embedding $\bar g\colon T^{p,n-p} \to \mathbb{R}^m$ is i-standardized if and only if $\bar g\psi$ is $\psi^{-1}i$-standardized.* So the second bullet point follows from the first one.

Both ψ and $\hat R$ preserve $S^p\times D^q_\pm$, both H_1 and R preserve $D^m_\pm$, both $\bar g$ and $H_1\bar g\psi$ are i-standardized, and $\mathrm{i} = R\,\mathrm{i}\,\hat R = H_1\,\mathrm{i}\,\psi$. Hence embedding $\bar g_\psi = R(H_1(R\bar g\hat R)\psi)\hat R$ is i-standardized. Then $R^tH_tR^t\bar g\hat R^t\psi\hat R^t$ is an isotopy between $\bar g_\psi$ and $\bar g\psi$. □

Proof of theorem 1.11. Precomposition with ψ defines a self-bijection of $E^m(T^{p,q})$.

Take i-standardized representatives $\bar g$, $\bar g'$ of $g, g' \in E^m(T^{p,q})$. Then a representative $\bar h$ of $g'+g$ is defined by $(**)$ for $\bar f = \bar g'$, $\bar g = \bar g$ and $s = i$. Take an isotopy H_t given by lemma 1.9′(a). Then by lemma 1.9′(b) $H_1\bar g'\psi$ and $\bar g_\psi$ are i-standardized. Hence by lemma 1.9′(b) a representative $\bar h_\psi$ of $g'\psi + g\psi$ is defined by $(**)$ for $\bar f = H_1\bar g'\psi$, $\bar g = \bar g_\psi$ and $s = i$. We have $g'\psi + g\psi = (g'+g)\psi$ because

$$\bar h_\psi = \begin{cases} H_1\bar g'\psi = H_1\bar h\psi & \text{on } S^p\times D^q_+,\\ R\bar g_\psi\hat R = H_1R\bar g\hat R\psi = H_1\bar h\psi & \text{on } S^p\times D^q_-.\end{cases}$$

Thus, composition with ψ defines an automorphism of $E^m(T^{p,q})$. □

Proof of theorem 1.16. By the standardization lemma of [29,33] there are s-standardized and i-standardized representatives $\bar f$ of f and $\bar g$ of g, respectively. Then a representative $\bar h$ of $f\#_s g$ is defined by $(**)$. Take an isotopy H_t given by lemma 1.9′(a). Then by lemma 1.9′(b) $H_1\bar f$ is $s\psi$-standardized and $\bar g_\psi$ is i-standardized. Hence, by lemma 1.9′(b) a representative $\bar h_\psi$ of $f\#_{s\psi}g\psi$ is defined by $(**)$ for $\bar f$, $\bar g$ and s replaced by $H_1\bar f$, $\bar g_\psi$ and $s\psi$, respectively. We have $f\#_s g = f\#_{s\psi}g\psi$ because

$$\bar h_\psi = \begin{cases} H_1\bar f = H_1\bar h & \text{on } N - \operatorname{Int}\operatorname{im} s,\\ R\bar g_\psi\hat R\psi^{-1}s^{-1} = H_1R\bar g\hat Rs^{-1} = H_1\bar h & \text{on } \operatorname{im} s.\end{cases}$$

□

Proof of corollary 1.17. If $2p+2>n$, then $m\geqslant n+p+3\geqslant n+\max\{p,n-p\}+2$. Hence $g=[\mathrm{i}]$ by remark 1.7(a). Then $f\#_s g=f$. So we may assume that $2p+2\leqslant n$.

Part (a) follows by theorems 1.14 and 1.16 because $p\leqslant n-p$ and any orientation-preserving embedding $s\colon S^p\times D^{n-p}\to N$ extending given $s|_{S^p\times 0}$ is isotopic to the composition of one fixed such embedding with an autodiffeomorphism of $S^p\times D^{n-p}$ defined by $(a,b)\mapsto(a,\varphi(a)b)$ for a certain map $\varphi\colon S^p\to \mathrm{SO}_{n-p}$.[8]

Part (b) follows by (a) and the analogue of the Haefliger unknotting theorem [30, theorem 2.8.b] for embeddings $S^p\to N$ because $2n\geqslant 4p+4\geqslant 3p+4$.

Part (c) follows by (b) and the Hurewicz theorem because $n\geqslant 2p+2\geqslant 6 \implies p-1\geqslant 2p+2-n$. □

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[8] If either $n\geqslant 2p+2$ and $p\equiv 2,4,5,6 \mod 8$, or $p\geqslant\max\{2,n-2\}$, then $\pi_p(\mathrm{SO}_{n-p})=0$. If $m\geqslant 2n+2-p$ and $n\geqslant 2p$, then $E^m(T^{p,n-p})=0$ by remark 1.7(a). The dimension restriction in corollary 1.17 can be replaced by any condition of this footnote.

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