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RANDOM REAL BRANCHED COVERINGS OF THE PROJECTIVE LINE

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Abstract In this paper, we construct a natural probability measure on the space of real branched coverings from a real projective algebraic curve (X, c_X) to the projective line $(\mathbb{CP}^1, conj)$. We prove that the space of degree d real branched coverings having "many" real branched points (for example, more than $\sqrt{d}^{1+\alpha}$, for any $\alpha > 0$) has exponentially small measure. In particular, maximal real branched coverings – that is, real branched coverings such that all the branched points are real – are exponentially rare.

Keywords and phrases: real algebraic curves; branched coverings; random maps; Bergman kernel

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1. Introduction

Let (X, c_X) be a real algebraic curve – that is, a smooth complex curve equipped with an antiholomorphic involution c_X called the real structure. We denote by $\mathbb{R}X$ the real locus of X, that is, the set $\mathrm{Fix}(c_X)$ of fixed points of c_X. For example, the projective line $(\mathbb{CP}^1, conj)$ is a real algebraic curve whose real locus equals $\mathbb{RP}^1$.

The central objects of this paper are real branched coverings from X to $\mathbb{CP}^1$ – that is, the branched coverings $u : X \to \mathbb{CP}^1$ such that $u \circ c_X = conj \circ u$. Let us denote by $\mathcal{M}_d^{\mathbb{R}}(X)$ the set of degree d real branched coverings from X to $\mathbb{CP}^1$. The first purpose of the paper is to show that $\mathcal{M}_d^{\mathbb{R}}(X)$ has a natural probability measure μ_d induced by a compatible volume form ω of X (that is, $c_X^*\omega = -\omega$), which we fix once for all. This probability measure is the real analogue of the probability measure constructed in [2] on the space of branched coverings between a complex projective curve and $\mathbb{CP}^1$. Later in the introduction we will sketch the construction of the measure μ_d, which we will give in detail in Section 2.3.

By the Riemann–Hurwitz formula, the number of critical points, counted with multiplicity, of a degree d branched covering $u : X \to \mathbb{CP}^1$ equals $2d + 2g - 2$, where g is the genus of X. The probability measure μ_d allows us to ask the following question:

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What is the probability that all the critical points of a real branched covering $u \in \mathcal{M}_d^{\mathbb{R}}(X)$ are real?

In [1], it is proved that the expected number of real critical points is equivalent to $c\sqrt{d}$ as the degree d of the random branched covering goes to infinity. The constant c is explicit, given by $c = \sqrt{\frac{\pi}{2}}\mathrm{Vol}(\mathbb{R}X)$, where $\mathrm{Vol}(\mathbb{R}X)$ is the length of the real locus of X with respect to the Riemannian metric induced by ω. The main theorem of the paper is the following exponential rarefaction result for real branched coverings having "many" real critical points:

Theorem 1.1. *Let X be a real algebraic curve. Let $\ell(d)$ be a sequence of positive real numbers such that $\ell(d) \geq B \log d$ for some $B > 0$. Then there exist positive constants c_1 and c_2 such that the following holds:*

$$\mu_d\left\{u \in \mathcal{M}_d^{\mathbb{R}}(X), \#(Crit(u) \cap \mathbb{R}X) \geq \ell(d)\sqrt{d}\right\} \leq c_1 e^{-c_2\ell(d)^2}.$$

For example, for any fixed $\alpha > 0$, we can consider the sequence $\ell(d) = \sqrt{d}^{\alpha}$. Theorem 1.1 says that the space of real branched coverings having more than $\sqrt{d}^{1+\alpha}$ real critical points has exponentially small measure. In particular, maximal real branched converings (i.e., branched coverings such that all the critical points are real) are exponentially rare.

The probability measure on $\mathcal{M}_d^{\mathbb{R}}(X)$. The construction of the probability measure on $\mathcal{M}_d^{\mathbb{R}}(X)$ uses the fact that there is a natural map from $\mathcal{M}_d^{\mathbb{R}}(X)$ to the space of the degree d real holomorphic line bundle $\mathrm{Pic}_{\mathbb{R}}^d(X)$ (see Proposition 2.5). This map sends a degree d morphism u to the degree d line bundle $u^*\mathcal{O}(1)$. The fibre of this map over $L \in \mathrm{Pic}_{\mathbb{R}}^d(X)$ is the open dense subset of $\mathbb{P}(\mathbb{R}H^0(X,L)^2)$ given by (the class of) pairs of global sections without common zeros. In order to construct a probability measure on $\mathcal{M}_d^{\mathbb{R}}(X)$, we produce a family of probability measures $\{\mu_L\}_{L\in \mathrm{Pic}_{\mathbb{R}}^d(X)}$ on each space $\mathbb{P}(\mathbb{R}H^0(X,L)^2)$. The probability measure μ_L on $\mathbb{P}(\mathbb{R}H^0(X,L)^2)$ is the measure induced by the Fubini–Study metric associated with a real Hermitian product on $\mathbb{R}H^0(X,L)^2$. This Hermitian product is the natural $\mathcal{L}^2$-product induced by ω (see Section 2.1). This family of measures, together with the Haar probability measure on the base $\mathrm{Pic}_{\mathbb{R}}^d(X)$, gives rise to the probability measure μ_d on $\mathcal{M}_d^{\mathbb{R}}(X)$.

An example: the projective line. Let us consider the case $X = \mathbb{CP}^1$ equipped with the conjugation $conj([x_0 : x_1]) = [\bar{x}_0 : \bar{x}_1]$. Given two degree d real polynomials $P, Q \in \mathbb{R}_d^{hom}[X_0, X_1]$ without common zeros, we produce a degree d real branched covering $u_{PQ} : \mathbb{CP}^1 \to \mathbb{CP}^1$ by sending $[x_0 : x_1]$ to $[P(x_0,x_1) : Q(x_0,x_1)]$. We also remark that the pair $(\lambda P, \lambda Q)$ defines the same branched covering. Conversely, one can prove that any degree d real branched covering $u : \mathbb{CP}^1 \to \mathbb{CP}^1$ is of the form $u = u_{PQ}$ for some (class of) pair of polynomials (P,Q) without common zeros. This means that $\mathcal{M}_d^{\mathbb{R}}(\mathbb{CP}^1) = \mathbb{P}(\mathbb{R}_d^{hom}[X_0,X_1]^2 \setminus \Lambda_d)$, where Λ_d is the set of pairs of polynomials with at least one common zero. Consider the affine chart $\{x_1 \neq 0\}$, the corresponding coordinate $x = \frac{x_0}{x_1}$ and the polynomials $p(X) = P(X_0, 1)$ and $q(X) = Q(X_0, 1)$. Then one can see that a point $x \in \{x_1 \neq 0\}$ is a critical point of u_{PQ} if and only $p'(x)q(x) - q'(x)p(x) = 0$ (see Proposition 3.14).

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In the previous paragraph, we constructed a probability measure on this space by fixing a compatible volume form on source space, in this case $\mathbb{CP}^1$. Indeed, a compatible volume form induces an $\mathcal{L}^2$-scalar product on $\mathbb{R}_d^{hom}[X_0, X_1]$, which will induce a Fubini–Study volume form on $\mathbb{P}(\mathbb{R}_d^{hom}[X_0, X_1]^2)$ and then a probability on $\mathcal{M}_d^{\mathbb{R}}(\mathbb{CP}^1)$. If we equip $\mathbb{CP}^1$ with the Fubini–Study form, then the induced scalar product on $\mathbb{R}_d^{hom}[X_0, X_1]$ is the one which makes $\left\{\sqrt{\binom{d}{k}} X_0^k X_1^{d-k}\right\}_{0\leq k\leq d}$ an orthonormal basis. This scalar product was considered by Kostlan in [8] (see also [11]). It is the only scalar product invariant under the action of the orthogonal group $O(2)$ (which acts on the variables X_0 and X_1) and such that the standard monomials are orthogonal to each other.

About the proof. There are two main steps in the proof of our main theorem. First, we reduce our problem to the problem of the computation of the *Gaussian* measure of a cone $\mathcal{C}_{\ell(d)}$ which lies inside the space of pairs of global sections of a real holomorphic line bundle over X. This cone is defined by using the Wronskian of a pair of global sections, which plays a key role. Then we use peak-section theory to estimate some Markov moments related to this Wronskian. These moments, together with the Poincaré–Lelong formula, allow us to estimate the measure of the cone $\mathcal{C}_{\ell(d)}$.

Let us sketch the proof in more details. We fix a degree 1 real holomorphic line bundle F over X so that for any $L \in \mathrm{Pic}_{\mathbb{R}}^d(X)$ there exists a unique $E \in \mathrm{Pic}_{\mathbb{R}}^0(X)$ such that $L = F^d \otimes E$. Recall that any class of pairs of real global sections without common zeros $[\alpha:\beta] \in \mathbb{P}(\mathbb{R}H^0(X, F^d\otimes E)^2)$ defines a real branched covering $u_{\alpha\beta}$ by sending a point $x \in X$ to $[\alpha(x):\beta(x)] \in \mathbb{CP}^1$. Theorem 1.1 will follow from the estimate

$$\mu_{F^d\otimes E}\left\{[\alpha:\beta] \in \mathbb{P}(\mathbb{R}H^0(X, F^d\otimes E)^2), \#(\mathrm{Crit}(u_{\alpha\beta}) \cap \mathbb{R}X) \geq \ell(d)\sqrt{d}\right\} \leq c_1 e^{-c_2\ell(d)^2}, \quad (1)$$

where $\mu_{F^d\otimes E}$ is the probability measure induced by the Fubini-Study metric on $\mathbb{P}(\mathbb{R}H^0(X, F^d\otimes E)^2)$. Indeed, if we integrate inequality (1) along $\mathrm{Pic}_{\mathbb{R}}^0(X)$ we exactly obtain Theorem 1.1. To prove estimate (1), we will use the following two facts. First, a point x is a critical point of $u_{\alpha\beta}$ if and only if it is a zero of the Wronskian $W_{\alpha\beta} := \alpha \otimes \nabla\beta - \beta \otimes \nabla\alpha$. Second, the push-forward (with respect to the projectivisation) of the Gaussian measure on $\mathbb{R}H^0(X, F^d\otimes E)^2$ is exactly the probability measure $\mu_{F^d\otimes E}$. These two facts imply that estimate (1) is equivalent to the fact that the Gaussian measure of the cone

$$\mathcal{C}_{\ell(d)} := \left\{(\alpha,\beta) \in \mathbb{R}H^0(X, F^d\otimes E)^2, \#(\text{real zeros of } W_{\alpha\beta}) \geq \ell(d)\sqrt{d}\right\} \quad (2)$$

is bounded from above by $c_1 e^{-c_2\ell(d)^2}$.

In order to estimate the Gaussian measure of $\mathcal{C}_{\ell(d)}$, inspired by [5], we bound from above the moments of the random variable $(\alpha,\beta) \in \mathbb{R}H^0(X, F^d\otimes E)^2 \mapsto \log\left\|W_{\alpha\beta}(x)\right\|$, where x is a point in X such that $\mathrm{dist}(x, \mathbb{R}X)$ is bigger than $\frac{\log d}{\sqrt{d}}$ (see Proposition 3.15). This condition on the distance is natural, it is strictly related to peak-section theory (see [6, 12]) and it is the reason why we need the hypothesis on the growth of the sequence $\ell(d)$ in Theorem 1.1. The estimate of these moments uses two ingredients: the theory of peak sections and a comparison between the norms of two different evaluation maps (and more generally jet maps). Once these moments are estimated, the Markov inequality and

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Poincaré-Lelong formula give us the exponential rarefaction of the Gaussian measure of the cone (2).

Organisation of the paper. This paper is organised as follows. In Section 2.1 we introduce the main objects and notations. In Sections 2.2 and 2.3 we study the geometry of the manifold $\mathcal{M}_d^{\mathbb{R}}(X)$ and construct the probability measure μ_d on it.

The purpose of Section 3 is to prove Proposition 3.15 – that is, to estimate the moments of the random variable $(\alpha,\beta)\in\mathbb{R}H^0(X,F^d\otimes E)^2\mapsto\log\|(\alpha\otimes\nabla\beta-\beta\otimes\nabla\alpha)(x)\|$, for F and E respectively degree 1 and 0 real holomorphic line bundles. In order to do this, in Section 3.1 we introduce Gaussian measures on $\mathbb{R}H^0(X,F^d\otimes E)^2$, and in Section 3.2 we study jet maps at points $x\in X$ which are far from the real locus. Finally, in Section 4 we deduce Theorem 1.1 from the estimates established in Section 3.

2. Random real branched coverings

2.1. Background

Let (X,c_X) be a real algebraic curve – that is, a complex, projective, smooth curve equipped with an antiholomorphic involution c_X, called the real structure. We assume that the real locus $\mathbb{R}X:=\mathrm{Fix}(c_X)$ is nonempty. An example is $(\mathbb{CP}^1,conj)$, where $conj([x_0:x_1])=[\bar{x}_0:\bar{x}_1]$ is a real algebraic curve whose real locus is $\mathbb{RP}^1$. A real holomorphic line bundle $p:L\to X$ is a line bundle equipped with an antiholomorphic involution c_L such that $p\circ c_X=c_L\circ p$ and c_L is antiholomorphic in the fibres. We denote by $\mathbb{R}H^0(X;L)$ the real vector space of real holomorphic global sections of L – that is, sections $s\in H^0(X;L)$ such that $s\circ c_X=c_L\circ s$. Let $\mathrm{Pic}^d_{\mathbb{R}}(X)$ be the set of degree d real line bundles. It is a principal space under the action of the compact topological abelian group $\mathrm{Pic}^0_{\mathbb{R}}(X)$ and so it inherits a normalised Haar measure that we denote by dH (see, for example, [7]). Finally, recall that a real Hermitian metric h on L is a Hermitian metric on L such that $c_L^*h=\bar{h}$.

Proposition 2.1. *Let (X,c_X) be a real algebraic curve and let ω be a compatible volume form of mass 1 – that is, $c_X^*\omega=-\omega$ and $\int_X\omega=1$. Let $L\in Pic^d_{\mathbb{R}}(X)$ be a degree d real holomorphic line bundle over X; then there exists a unique real Hermitian metric h (up to multiplication by a positive real constant) such that $c_1(L,h)=d\cdot\omega$.*

Proof. For the existence and uniqueness of such a metric, see [2, Proposition 1.4]. The fact that the metric h is real follows from the following argument. Let us consider the Hermitian metric $\overline{c_L^*h}$ on L. Claim: its curvature equals $-d\cdot c_X^*\omega$. Indeed, for any $x\in X$ we consider a real meromorphic section s of L such that x and $c_X(x)$ are neither zero nor poles of s (such a section exists by the Riemann–Roch theorem). Then the curvature of $(L,\overline{c_L^*h})$ around x is $\partial\bar\partial\log\left(\overline{c_L^*h}\right)_x(s(x),s(x))=\partial\bar\partial\log h_{c_X(x)}(c_L(s(x)),c_L(s(x)))=\partial\bar\partial\log h_{c_X(x)}(s(c_X(x)),s(c_X(x)))=\partial\bar\partial c_X^*\log h(s,s)=-c_X^*\partial\bar\partial\log h(s,s)$, where the last equality is due to the antiholomorphicity of c_X. Then the claim follows from the fact that $\partial\bar\partial\log h(s,s)=d\cdot\omega$.

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Now consider the real Hermitian metric $(h\cdot\overline{c_L^* h})^{1/2}$. Its curvature equals

$$\frac{1}{2}\big(\partial\bar\partial\log h(s,s)+\partial\bar\partial\log\Big(\overline{c_L^* h}\Big)(s,s)\big)=\frac{1}{2}(d\cdot\omega-d\cdot c_X^*\omega)=d\cdot\omega,$$

where the last equality follows from the fact that ω is compatible with the real structure. By the uniqueness of the metric with curvature $d\cdot\omega$, this implies that $(h\cdot\overline{c_L^* h})^{1/2}$ is a multiple of h. We actually have the equality $(h\cdot\overline{c_L^* h})^{1/2}=h$, because for a real point $x\in\mathbb{R}X$ and a real vector $v\in\mathbb{R}L_x$ we get $(h_x(v,v)\cdot(\overline{c_L^* h})_x(v,v))^{1/2}=(h_x(v,v)\cdot h_x(v,v))^{1/2}=h_x(v,v)$. □

Definition 2.2. Let ω be a compatible volume form of mass 1, let $L\in \mathrm{Pic}^d_{\mathbb{R}}(X)$ be a degree d line bundle over X and let h be the real Hermitian metric given by Proposition 2.1. We define the $\mathcal{L}^2$-scalar product on $\mathbb{R}H^0(X;L)$ by

$$\langle\alpha,\beta\rangle_{\mathcal{L}^2}=\int_{x\in X}h_x(\alpha(x),\beta(x))\omega$$

for any pair of real holomorphic sections $\alpha,\beta\in\mathbb{R}H^0(X;L)$.

2.2. The space of real branched coverings

In this section we introduce and study the space of real branched coverings from a real algebraic curve (X,c_X) to $(\mathbb{CP}^1, conj)$.

Definition 2.3. We denote by $\mathcal{M}^{\mathbb{R}}_d(X)$ the space of all degree d real branched coverings $u:X\to\mathbb{CP}^1$, which are the branched coverings such that $u\circ c_X=conj\circ u$.

A natural way to define a degree d real branched covering is as follows. Consider a degree d real holomorphic line bundle $L\in\mathrm{Pic}^d_{\mathbb{R}}(X)$ and two real holomorphic sections $\alpha,\beta\in\mathbb{R}H^0(X,L)$ without common zeros. Then we can define the degree d real branched covering $u_{\alpha\beta}$ defined by

$$u_{\alpha\beta}:x\in X\mapsto[\alpha(x):\beta(x)]\in\mathbb{CP}^1.$$

Proposition 2.4. *Two pairs $(\alpha,\beta),(\alpha',\beta')$ of real holomorphic sections of L define the same real branched covering if and only if $(\alpha',\beta')=(\lambda\alpha,\lambda\beta)$ for some $\lambda\in\mathbb{R}^*$.*

Proof. The proof follows the proof of [2, Proposition 1.1]. □

Proposition 2.5. *There exists a natural map from $\mathcal{M}^{\mathbb{R}}_d(X)$ to the space $Pic^d_{\mathbb{R}}(X)$ of degree d real line bundles over X. This natural map is given by $u\in\mathcal{M}^{\mathbb{R}}_d(X)\mapsto u^*\mathcal{O}(1)\in Pic^d_{\mathbb{R}}(X)$. The fibre over $L\in Pic^d_{\mathbb{R}}(X)$ is the open subset of $\mathbb{P}(\mathbb{R}H^0(X;L)^2)$ given by (the class of) pair of sections (α,β) without common zeros.*

Proof. Given a degree d real branched covering $u:X\to\mathbb{CP}^1$, we get a degree d real line bundle $u^*\mathcal{O}(1)$ over X and the class of two real global holomorphic sections without common zeros $[u^*x_0:u^*x_1]\in\mathbb{P}(\mathbb{R}H^0(X;u^*\mathcal{O}(1))^2)$. On the other hand, given a degree d real line bundle $L\to X$ and two real holomorphic global sections without common zeros $(\alpha,\beta)\in\mathbb{R}H^0(X;L)^2$, the map $u_{\alpha\beta}:X\to\mathbb{CP}^1$ defined by $x\mapsto[\alpha(x):\beta(x)]$ is a degree d real branched covering satisying $u^*_{\alpha\beta}\mathcal{O}(1)=L$. Moreover, by Proposition 2.4, two pairs

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(α, β) and (α', β') of real holomorphic sections of L define the same real branched covering if and only if $(\alpha', \beta') = (\lambda\alpha, \lambda\beta)$ for some $\lambda \in \mathbb{R}^*$, hence the result. □

2.3. Probability on $\mathcal{M}_d^{\mathbb{R}}(X)$

Let X be a real algebraic curve equipped with a compatible volume form ω of total mass 1. In this section, we construct a natural probability measure on the space $\mathcal{M}_d^{\mathbb{R}}(X)$ of degree d real branched coverings from X to $\mathbb{CP}^1$.

Let $L \in \mathrm{Pic}_{\mathbb{R}}^d(X)$ be a degree d real line bundle equipped with the real Hermitian metric h given by Proposition 2.1. We recall that in Definition 2.2 we defined the $\mathcal{L}^2$-scalar product on the space $\mathbb{R}H^0(X;L)$ induced by the Hermitian metric h. This $\mathcal{L}^2$-scalar product induces a scalar product on the Cartesian product $\mathbb{R}H^0(X;L)^2$ and then a Fubini–Study metric on $\mathbb{P}(\mathbb{R}H^0(X;L)^2)$. Recall how the Fubini–Study metric is constructed. First we restrict the scalar product to the unit sphere of $\mathbb{R}H^0(X;L)^2$. The obtained metric is invariant under the action of $\mathbb{Z}/2\mathbb{Z}$, and the Fubini–Study metric is then the quotient metric on $\mathbb{P}(\mathbb{R}H^0(X;L)^2)$.

Definition 2.6. Let L be a real holomorphic line bundle over X. We denote by μ_L the probability measure on $\mathbb{P}(\mathbb{R}H^0(X;L)^2)$ induced by the normalised Fubini–Study volume form. Here, the Fubini–Study metric on $\mathbb{P}(\mathbb{R}H^0(X;L)^2)$ is the one induced by the Hermitian metric on L given by Proposition 2.1.

Proposition 2.7. *The probability measure μ_L over $\mathbb{P}(\mathbb{R}H^0(X;L)^2)$ does not depend on the choice of the multiplicative constant in front of the metric h given by Proposition 2.1.*

Proof. The proof follows the proof of [2, Proposition 1.7] □

Remark 2.8. For a real holomorphic line bundle L, we denote by Λ_L the space of pairs of sections $(s_0, s_1) \in \mathbb{R}H^0(X;L)^2$ with at least a common zero. By [1, Proposition 2.11], the set Λ_L has zero measure (it is a hypersurface), at least if the degree of L is large enough. This implies that μ_L induces a probability measure on $\mathbb{P}(\mathbb{R}H^0(X;L)^2 \setminus \Lambda_L)$, still denoted by μ_L.

Definition 2.9. We define the probability measure μ_d on $\mathcal{M}_d^{\mathbb{R}}(X)$ by the following equality:

$$\int_{\mathcal{M}_d^{\mathbb{R}}(X)} f \,\mathrm{d}\mu_d = \int_{L \in \mathrm{Pic}_{\mathbb{R}}^d(X)} \left(\int_{\mathcal{M}_d^{\mathbb{R}}(X,L)} f \,\mathrm{d}\mu_L \right) \mathrm{dH}(L)$$

for any $f \in \mathcal{M}_d^{\mathbb{R}}(X)$ measurable function. Here

- $\mathcal{M}_d^{\mathbb{R}}(X,L)$ is the fibre of the natural morphism $\mathcal{M}_d^{\mathbb{R}}(X) \to \mathrm{Pic}_{\mathbb{R}}^d(X)$ defined in Proposition 2.5,
- μ_L denotes (by a slight abuse of notation) the restriction to $\mathcal{M}_d^{\mathbb{R}}(X,L)$ of the probability measure on $\mathbb{P}(\mathbb{R}H^0(X,L)^2)$ defined in Definition 2.6 and
- dH denotes the normalised Haar measure on $\mathrm{Pic}_{\mathbb{R}}^d(X)$.

Remark 2.10. The probability measure μ_d of Definition 2.9 is the real analogue of the one constructed in the complex setting in [2] for the study of random branched coverings from a fixed Riemann surface to $\mathbb{CP}^1$. Also in the complex setting, a similar construction was considered by Zelditch in [14] in order to study large deviations of empirical measures of zeros on a Riemann surface.

Example 2.11. Let us consider the case $(X, c_X) = (\mathbb{CP}^1, conj)$, where $\mathbb{CP}^1$ is equipped with the Fubini–Study form ω_{FS}. For the projective line $\mathbb{CP}^1$, the unique degree d real line bundle is the line bundle $\mathcal{O}(d)$, which is naturally equipped with a real Hermitian metric h_d whose curvature equals $d\cdot\omega$. The space of real holomorphic global sections $\mathbb{R}H^0(\mathbb{CP}^1;\mathcal{O}(d))$ is isomorphic to the space of degree d homogeneous polynomials $\mathbb{R}_d^{\mathrm{hom}}[X_0,X_1]$, and the $\mathcal{L}^2$-scalar product coincides with the Kostlan scalar product (i.e., the scalar product which makes $\{\sqrt{\binom{d}{k}}X_0^kX_1^{d-k}\}_{0\le k\le d}$ an orthonormal basis; see [8, 11]). Then a random real branched covering $u:\mathbb{CP}^1\to\mathbb{CP}^1$ is given by the class of a pair of independent Kostlan polynomials.

3. Gaussian measures and estimates of higher moments

In this section, we introduce some Gaussian measures on the spaces $\mathbb{R}H^0(X;L)^2$ and $H^0(X;L)^2$, as in [1, 5, 6, 10]. We follow the notations of Section 2. In particular, (X, c_X) is a real algebraic curve whose real locus $\mathbb{R}X$ is not empty.

3.1. Gaussian measures

In this section, given any degree d real line $L\in\mathrm{Pic}_{\mathbb{R}}^d(X)$, we equip the cartesian product $\mathbb{R}H^0(X;L)^2$ of the space of real holomorphic section with a Gaussian measure γ_L. In order to do this, we fix a compatible volume form ω of total volume 1 (i.e., $c_X^*\omega=-\omega$ and $\int_X\omega=1$). Given $L\in\mathrm{Pic}_{\mathbb{R}}^d(X)$, we equip L by the real Hermitian metric h with curvature $d\cdot\omega$ (the metric h is unique up to a multiplicative constant; see Proposition 2.1).

In Definition 2.2, we defined an $\mathcal{L}^2$-Hermitian product on the space $\mathbb{R}H^0(X;L)$ of real global holomorphic sections of L denoted by $\langle\cdot,\cdot\rangle_{\mathcal{L}^2}$ and defined by

$$\langle\alpha,\beta\rangle_{\mathcal{L}^2}=\int_{x\in X}h_x(\alpha(x),\beta(x))\omega$$

for all α,β in $\mathbb{R}H^0(X;L)$.

Definition 3.1. The $\mathcal{L}^2$-scalar product on $\mathbb{R}H^0(X;L)^2$ induces a Gaussian measure γ_L on $\mathbb{R}H^0(X;L)^2$ defined by

$$\gamma_L(A)=\frac{1}{\pi^{N_d}}\int_{(\alpha,\beta)\in A}e^{-\|\alpha\|^2_{\mathcal{L}^2}-\|\beta\|^2_{\mathcal{L}^2}}\mathrm{d}\alpha\mathrm{d}\beta$$

for any open subset $A\subset\mathbb{R}H^0(X;L)^2$. Here $\mathrm{d}\alpha\mathrm{d}\beta$ is the Lebesgue measure on $(\mathbb{R}H^0(X;L)^2;\langle\cdot,\cdot\rangle_{\mathcal{L}^2})$ and N_d denotes the dimension of $\mathbb{R}H^0(X;L)$, which equals the complex dimension of $H^0(X;L)$.

Remark 3.2. If $d>2g-2$, where g is the genus of X, then $H^1(X;L)=0$ and then, by the Riemann–Roch theorem, we have $N_d=d+1-g$.

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Proposition 3.3 ([2, Proposition 1.12]). *Let f be a function on a Euclidian space $(V,\langle\cdot,\cdot\rangle)$ which is constant over the lines – that is, $f(v)=f(\lambda v)$ for all $v\in V$ and all $\lambda\in\mathbb{R}^*$. Denote by $d\gamma$ the Gaussian measure on V induced by $\langle\cdot,\cdot\rangle$, and by $d\mu$ the normalised Fubini–Study measure on the projectivization $\mathbb{P}(V)$. Then for all cones $A\subset V$, we have*

$$\int_A f\,d\gamma=\int_{\mathbb{P}(A)}[f]\,d\mu,$$

where $\mathbb{P}(A)$ is the projectivization of A and $[f]$ is the function on $\mathbb{P}(V)$ induced by f.

We will also be interested in the *complex* Gaussian measure on the space $H^0(X,L)^2$. Indeed, the Hermitian metric h on L defines an $\mathcal{L}^2$-Hermitian product on $H^0(X,L)$ by the formula

$$\langle\alpha,\beta\rangle_{\mathcal{L}^2}=\int_{x\in X}h_x(\alpha(x),\beta(x))\omega$$

for all α,β in $H^0(X;L)$.

Definition 3.4. The complex Gaussian measure $\gamma_L^{\mathbb{C}}$ on $H^0(X;L)^2$ is defined by

$$\gamma_L^{\mathbb{C}}(A)=\frac{1}{\pi^{2N_d}}\int_{(\alpha,\beta)\in A}e^{-\|\alpha\|^2_{\mathcal{L}^2}-\|\beta\|^2_{\mathcal{L}^2}}\mathrm{d}\alpha\mathrm{d}\beta$$

for any open subset $A\subset H^0(X;L)^2$. Here $\mathrm{d}\alpha\mathrm{d}\beta$ is the Lebesgue measure on $(H^0(X;L)^2;\langle\cdot,\cdot\rangle_{\mathcal{L}^2})$ and N_d denotes the complex dimension of $H^0(X;L)$.

3.2. Jet maps and peak sections

Let F and E be, respectively, degree 1 and 0 real holomorphic line bundles over X. We equip F and E by the real Hermitian metrics given by Proposition 2.1, which we denote by h_F and h_E. In particular, the real Hermitian metric $h_d:=h_F^d\otimes h_E$ on $F^d\otimes E$ is such that its curvature equals $d\cdot\omega$. Finally, recall that the space $H^0(X,F^d\otimes E)$ is endowed with the $\mathcal{L}^2$-Hermitian product

$$\langle\alpha,\beta\rangle_{\mathcal{L}^2}=\int_{x\in X}h_d(\alpha(x),\beta(x))\omega$$

defined by for any α,β in $H^0(X;F^d\otimes E)$.

Definition 3.5. For any $x\in X$, let H_x be the kernel of the map $s\in H^0(X,F^d\otimes E)\mapsto s(x)\in(F^d\otimes E)_x$. Similarly, we denote by H_{2x} the kernel of the map $s\in H_x\mapsto\nabla s(x)\in(F^d\otimes E)_x\otimes T^*_{X,x}$, where ∇ is any connection on $F^d\otimes E$ (indeed, if $s\in H_x$, then the value $\nabla s(x)$ does not depend on ∇). We define the following jet maps:

$$ev_x: s\in H^0(X,F^d\otimes E)/H_x\mapsto s(x)\in(F^d\otimes E)_x,$$
$$ev_{2x}: s\in H_x/H_{2x}\mapsto\nabla s(x)\in(F^d\otimes E)_x\otimes T^*_{X,x}.$$

Definition 3.5 has the following real analogue:

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Definition 3.6. For any point $x \in X$, we define the real vector spaces $\mathbb{R}H_x^0 = H_x^0 \cap \mathbb{R}H^0(X, F^d \otimes E)$ and $\mathbb{R}H_{2x}^0 = H_{2x}^0 \cap \mathbb{R}H^0(X, F^d \otimes E)$ and the real jet maps

$$ev_x^{\mathbb{R}} : s \in \mathbb{R}H^0(X, F^d \otimes E)/\mathbb{R}H_x^0 \mapsto s(x) \in (F^d \otimes E)_x,$$
$$ev_{2x}^{\mathbb{R}} : s \in \mathbb{R}H_x/\mathbb{R}H_{2x} \mapsto \nabla s(x) \in (F^d \otimes E)_x \otimes T^*_{X,x}.$$

By the fact that F is ample (recall that $\deg F = 1$), we get that for d large enough, the maps $ev_x^{\mathbb{R}}$, ev_x, $ev_{2x}^{\mathbb{R}}$ and ev_{2x} are invertible. The following proposition estimates the norms of these maps and of their inverses:

Proposition 3.7 ([6, Propositions 4 and 6])**.** *For any $B > 0$, there exist an integer d_B and a positive constant c_B such that for any $d \geq d_B$ and any point $x \in X$ with $dist(x, \mathbb{R}X) \geq B\frac{\log d}{\sqrt{d}}$, the maps $d^{-\frac{1}{2}} ev_x^{\mathbb{R}}$, $d^{-\frac{1}{2}} ev_x$, $d^{-1} ev_{2x}^{\mathbb{R}}$ and $d^{-1} ev_{2x}$ – as well as their inverses – have norms and determinants bounded from above by c_B.*

Remark 3.8. In [6, Propositions 4 and 6], the constant B equals 1 and the line bundle E is trivial. The same proof actually holds for any fixed $B > 0$ and any $E \in \mathrm{Pic}^0_{\mathbb{R}}(X)$. Indeed, the proof is based on the theory of peak sections and Bergman kernels, and this theory holds in this more general setting (see, for example, [4] or [9, Theorem 4.2.1]).

Using the $\mathcal{L}^2$-Hermitian product on $H^0(X, F^d \otimes E)$, we can identify $H^0(X, F^d \otimes E)/H_x$ with the orthogonal complement of H_x in $H^0(X, F^d \otimes E)$. Similarly, we identify the quotient H_x/H_{2x} with the orthogonal complement of H_{2x} in H_x. We then have an orthogonal decomposition

$$H^0(X, F^d \otimes E) = H^0(X, F^d \otimes E)/H_x \oplus H_x/H_{2x} \oplus H_{2x}.$$

Similarly, using the $\mathcal{L}^2$-scalar product on $\mathbb{R}H^0(X, F^d \otimes E)$, we have the orthogonal decomposition

$$\mathbb{R}H^0(X, F^d \otimes E) = \mathbb{R}H^0(X, F^d \otimes E)/\mathbb{R}H_x \oplus \mathbb{R}H_x/\mathbb{R}H_{2x} \oplus \mathbb{R}H_{2x}.$$

The map $ev_x \times ev_{2x}$ (resp., $ev_x^{\mathbb{R}} \times ev_{2x}^{\mathbb{R}}$) gives an isomorphism between $H^0(X, F^d \otimes E)/H_x \oplus H_x/H_{2x}$ (resp., $\mathbb{R}H^0(X, F^d \otimes E)/\mathbb{R}H_x \oplus \mathbb{R}H_x/\mathbb{R}H_{2x}$) and the fibre $(F^d \otimes E)_x \oplus (F^d \otimes E)_x \otimes T^*_{X,x}$.

Moreover, note that we have natural identifications $H^0(X, F^d \otimes E)/H_x \oplus H_x/H_{2x} = H_{2x}^{\perp}$ and $\mathbb{R}H^0(X, F^d \otimes E)/\mathbb{R}H_x \oplus \mathbb{R}H_x/\mathbb{R}H_{2x} = \mathbb{R}H_{2x}^{\perp}$. A direct consequence of Proposition 3.7 is the following:

Corollary 3.9. *For any $B > 0$, there exist an integer d_B and a positive constant c_B such that for any $d \geq d_B$ and any x with $dist(x, \mathbb{R}X) \geq B\frac{\log d}{\sqrt{d}}$, the map $(ev_x^{\mathbb{R}} \times ev_{2x}^{\mathbb{R}})^{-1} \circ (ev_x \times ev_{2x}) : H_{2x}^{\perp} \to \mathbb{R}H_{2x}^{\perp}$ has determinant bounded from above by c_B and from below by $1/c_B$.*

Definition 3.10. We denote by s_0 and s_1 the global holomorphic sections of $L^d \otimes E$ with unit $\mathcal{L}^2$-norm which generate, respectively, the orthogonal complement of H_x in $H^0(X, F^d \otimes E)$ and the orthogonal complement of H_{2x} in H_x. We call these sections the *peak sections* at x.

The point-wise estimates of the norms (with respect to the Hermitian metric h_d of curvature $d\cdot\omega$) of the peak sections are well known and strictly related to the estimates of the Bergman kernel along the diagonal (see [3, 9, 12, 13]). With a slight abuse of notation, we will denote by $\|\cdot\|$ any norm induced by h_d.

Lemma 3.11 ([2, Proposition 1.5]). *For any $x\in X$, let s_0 and s_1 be the peak sections defined in Definition 3.10. Then as $d\to+\infty$, we have the estimates $\|s_0(x)\|=\frac{\sqrt{d}}{\sqrt{\pi}}(1+O(d^{-1}))$ and $\|\nabla s_1(x)\|=\frac{d}{\sqrt{\pi}}(1+O(d^{-1}))$, where the error terms are uniform in $x\in X$.*

3.3. Wronskian and higher moments

Let F and E be, respectively, degree 1 and 0 real holomorphic line bundles over X. The purpose of this section is to prove Proposition 3.15, which gives key estimates of the higher moments of the random variable $(\alpha,\beta)\in\mathbb{R}H^0(X,F^d\otimes E)^2\mapsto\log\left\|\frac{\pi}{d^{3/2}}W_{\alpha\beta}(x)\right\|$, where $W_{\alpha\beta}$ is the Wronskian, given by the following:

Definition 3.12. Let ∇ be a connection on $F^d\otimes E$. For any pair of real holomorphic global sections $(\alpha,\beta)\in\mathbb{R}H^0(X,F^d\otimes E)^2$, we denote by $W_{\alpha\beta}$ the Wronskian $\alpha\otimes\nabla\beta-\beta\otimes\nabla\alpha$, which is a real holomorphic global section of $F^{2d}\otimes E^2\otimes T_X^*$.

Remark 3.13. The Wronskian $W_{\alpha\beta}$ does not depend on the choice of a connection on $F^d\otimes E$. Indeed, two connections ∇ and ∇' on $F^d\otimes E$ differ by a 1-form θ, and then we have

$$\begin{aligned}(\alpha\otimes\nabla\beta-\beta\otimes\nabla\alpha)-(\alpha\nabla'\beta-\beta\nabla'\alpha)&=\alpha\otimes(\nabla-\nabla')\beta-\beta\otimes(\nabla-\nabla')\alpha\\&=\alpha\otimes\beta\otimes\theta-\beta\otimes\alpha\otimes\theta=0.\end{aligned}$$

Proposition 3.14 ([2, Proposition 2.3]). *Let F and E be, respectively, degree 1 and 0 real line bundles over X and $(\alpha,\beta)\in\mathbb{R}H^0(X,F^d\otimes E)^2$ be a pair of sections without common zeros. A point $x\in X$ is a critical point of the map $u_{\alpha\beta}:x\in X\mapsto[\alpha(x):\beta(x)]\in\mathbb{CP}^1$ if and only if it is a zero of the Wronskian $W_{\alpha\beta}$ defined in Definition 3.12.*

Proposition 3.15. *Let X be a real algebraic curve equipped with a compatible volume form ω of total volume 1, and let $F\in Pic^1_{\mathbb{R}}(X)$. For any $B>0$, there exist an integer d_B and a constant c_B such that for any $E\in Pic^0_{\mathbb{R}}(X)$, any $m\in\mathbb{N}$, any $d\geq d_B$ and any point $x\in X$ with $dist(x,\mathbb{R}X)\geq B\frac{\log d}{\sqrt{d}}$, we have*

$$\int_{(\alpha,\beta)\in\mathbb{R}H^0(X,F^d\otimes E)^2}\left|\log\left\|\frac{\pi}{d^{3/2}}W_{\alpha\beta}(x)\right\|\right|^m d\gamma_d(\alpha,\beta)\leq c_B(m+1)!.$$

Here, $dist(\cdot,\cdot)$ is the distance in X induced by ω, γ_d is the Gaussian measure on $\mathbb{R}H^0(X,F^d\otimes E)^2$ constructed in Section 3.1 and $\|\cdot\|$ denotes the norm induced by the Hermitian metrics on F and E given by Proposition 2.1.

Proof. Let us consider the integral we want to estimate:

$$\int_{(\alpha,\beta)\in\mathbb{R}H^0(X,F^d\otimes E)^2}\left|\log\left\|\frac{\pi}{d^{3/2}}W_{\alpha\beta}(x)\right\|\right|^m \mathrm{d}\gamma_d(\alpha,\beta). \quad (3)$$

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First, remark that the function in integral (3) depends only on the 1-jet of the sections α and β. We will then write the orthogonal decomposition $\mathbb{R}H^0(X, F^d \otimes E) = \mathbb{R}H_{2x} \oplus \mathbb{R}H_{2x}^{\perp}$, where $\mathbb{R}H_{2x}$ is the space of real sections s such that $s(x) = 0$ and $\nabla s(x) = 0$. As the Gaussian measure is a product measure, after integration over the orthogonal complement of $\mathbb{R}H_{2x}^{\perp} \times \mathbb{R}H_{2x}^{\perp}$ we have that integral (3) is equal to

$$\int_{(\alpha,\beta)\in\mathbb{R}H_{2x}^{\perp}\times\mathbb{R}H_{2x}^{\perp}} \left|\log\left\|\frac{\pi}{d^{3/2}}W_{\alpha\beta}(x)\right\|\right|^m \mathrm{d}\gamma_d\,|_{\mathbb{R}H_{2x}^{\perp}\times\mathbb{R}H_{2x}^{\perp}}(\alpha,\beta). \tag{4}$$

Using the notations of Section 3.2, and in particular Definitions 3.5 and 3.6, let $J_d : H_{2x}^{\perp} \to \mathbb{R}H_{2x}^{\perp}$ be the map $(ev_x^{\mathbb{R}} \times ev_{2x}^{\mathbb{R}})^{-1} \circ (ev_x \times ev_{2x})$ and denote

$$I_d = J_d \times J_d : H_{2x}^{\perp} \times H_{2x}^{\perp} \to \mathbb{R}H_{2x}^{\perp} \times \mathbb{R}H_{2x}^{\perp}.$$

By the changing of variables given by the isomorphism I_d, we get

$$(4) = \int_{(\alpha,\beta)\in H_{2x}^{\perp}\times H_{2x}^{\perp}} \left|\log\left\|\frac{\pi}{d^{3/2}}W_{\alpha\beta}(x)\right\|\right|^m (I_d^{-1})_*(\mathrm{d}\gamma_d\,|_{\mathbb{R}H_{2x}^{\perp}\times\mathbb{R}H_{2x}^{\perp}})(\alpha,\beta). \tag{5}$$

By Corollary 3.9, the maps I_d and I_d^{-1} have determinants bounded from above by a constant which depends only on B. In particular, there exists a constant c_1, depending only on B, such that

$$(5) \le c_1 \int_{(\alpha,\beta)\in H_{2x}^{\perp}\times H_{2x}^{\perp}} \left|\log\left\|\frac{\pi}{d^{3/2}}W_{\alpha\beta}(x)\right\|\right|^m \mathrm{d}\gamma_d^{\mathbb{C}}\,|_{H_{2x}^{\perp}\times H_{2x}^{\perp}}(\alpha,\beta), \tag{6}$$

where $\gamma_d^{\mathbb{C}}$ is the complex Gaussian measure defined in Definition 3.4. In order to prove the result, we have to bound from above the quantity

$$\int_{(\alpha,\beta)\in H_{2x}^{\perp}\times H_{2x}^{\perp}} \left|\log\left\|\frac{\pi}{d^{3/2}}W_{\alpha\beta}(x)\right\|\right|^m \mathrm{d}\gamma_d^{\mathbb{C}}\,|_{H_{2x}^{\perp}\times H_{2x}^{\perp}}(\alpha,\beta). \tag{7}$$

Let s_0 and s_1 be the peak sections at x introduced in Definition 3.10, and write $\alpha = a_0\sigma_0 + a_1\sigma_1$ and $\beta = b_0\sigma_0 + b_1\sigma_1$. We then have

$$\|W_{\alpha\beta}(x)\| = |a_0b_1 - a_1b_0|\|(s_0 \otimes \nabla s_1 - s_1 \otimes \nabla s_0)(x)\| = |a_0b_1 - a_1b_0|\frac{d^{3/2}}{\pi}(1 + O(d^{-c_2(B)})),$$

where the last equality follows from Proposition 3.11. This implies that the integral in expression (7) equals

$$\begin{aligned}
&\int_{\substack{a=(a_0,a_1)\in\mathbb{C}^2\\ b=(b_0,b_1)\in\mathbb{C}^2}} \left|\log\left(|a_0b_1 - a_1b_0|\left\|\frac{\pi}{d^{3/2}}(s_0\otimes\nabla s_1 - s_1\otimes\nabla s_0)(x)\right\|\right)\right|^m \frac{e^{-|a|^2-|b|^2}}{\pi^4}\mathrm{d}a\mathrm{d}b\\
&= \int_{\substack{a=(a_0,a_1)\in\mathbb{C}^2\\ b=(b_0,b_1)\in\mathbb{C}^2}} \left|\log\left(|a_0b_1 - a_1b_0|\right)\right|^m \frac{e^{-|a|^2-|b|^2}}{\pi^4}\left(1 + O(d^{-c_3(B)})\right)\mathrm{d}a\mathrm{d}b\\
&\le 2\int_{\substack{a\in\mathbb{C}^2\\ b\in\mathbb{C}^2}} |\log|a_0b_1 - b_0a_1||^m \frac{e^{-|a|^2-|b|^2}}{\pi^4}\mathrm{d}a\mathrm{d}b,
\end{aligned} \tag{8}$$

where the last inequality holds for $d \ge d_B$, for some d_B large enough.

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In the remaining part of the proof, we will estimate the last integral in equation (8). In order to do this, for any $a=(a_0,a_1)$ we make a unitary trasformation of $\mathbb{C}^2$ (of coordinates b_0,b_1) by sending the vector $(1,0)$ to $v_a=\frac{1}{\sqrt{|a_0|^2+|a_1|^2}}(a_0,a_1)$ and the vector $(0,1)$ to $w_a=\frac{1}{\sqrt{|a_0|^2+|a_1|^2}}(-\bar{a}_1,\bar{a}_0)$. We will write any vector of $\mathbb{C}^2$ as a sum tv_a+sw_a, with $s,t\in\mathbb{C}$. Under this change of variables, the integral in equation (8) becomes

$$\leq 2\int_{\substack{a\in\mathbb{C}^2\\(s,t)\in\mathbb{C}^2}}|\log|s|\|a\||^m\frac{e^{-|a|^2-|s|^2-|t|^2}}{\pi^4}\mathrm{d}a\mathrm{d}s\mathrm{d}t=2\int_{\substack{a\in\mathbb{C}^2\\s\in\mathbb{C}}}|\log|s|\|a\||^m\frac{e^{-|a|^2-|s|^2}}{\pi^3}\mathrm{d}a\mathrm{d}s. \tag{9}$$

We pass to polar coordinates $a=re^{i\theta}$ for $\theta\in S^3$ and $r\in\mathbb{R}_+$, and $s=\rho e^{i\phi}$ for $\phi\in S^1$ and $\rho\in\mathbb{R}_+$, and we obtain

$$2\int_{\substack{a\in\mathbb{C}^2\\s\in\mathbb{C}}}|\log|s|\|a\||^m\frac{e^{-|a|^2-|s|^2}}{\pi^3}\mathrm{d}a\mathrm{d}s=8\int_{\substack{r\in\mathbb{R}_+\\\rho\in\mathbb{R}_+}}|\log\rho r|^m e^{-r^2-\rho^2}r^3\rho\mathrm{d}r\mathrm{d}\rho. \tag{10}$$

Writing $\log\rho r=\log\rho+\log r$, developing the binomial and using the triangular inequality, we obtain

$$(10)\leq 8\int_{\substack{r\in\mathbb{R}_+\\\rho\in\mathbb{R}_+}}\sum_{k=0}^{m}\binom{m}{k}|\log\rho|^k|\log r|^{m-k}e^{-r^2-\rho^2}r^3\rho\mathrm{d}r\mathrm{d}\rho. \tag{11}$$

Let us study the integrals $\int_{\rho\in\mathbb{R}_+}|\log\rho|^n e^{-\rho^2}\rho\mathrm{d}\rho$ and $\int_{r\in\mathbb{R}_+}|\log r|^n e^{-r^2}r^3\mathrm{d}r$. To compute these two integrals, we will use the following formula obtained by integration by parts:

$$\int(\log x)^n\mathrm{d}x=x\log x-n\int(\log x)^{n-1}\mathrm{d}x,\quad n>0. \tag{12}$$

- Computation of the integral $\int_{\rho\in\mathbb{R}_+}|\log\rho|^n e^{-\rho^2}\rho\mathrm{d}\rho$. We write

$$\int_{\rho\in\mathbb{R}_+}|\log\rho|^n e^{-\rho^2}\rho\mathrm{d}\rho=\int_{\rho=0}^{1}(-\log\rho)^n e^{-\rho^2}\rho\mathrm{d}\rho+\int_{\rho=1}^{\infty}(\log\rho)^n e^{-\rho^2}\rho\mathrm{d}\rho. \tag{13}$$

For the first term of this sum we have

$$\int_{\rho=0}^{1}(-\log\rho)^n e^{-\rho^2}\rho\mathrm{d}\rho\leq\frac{\sqrt{2}}{2}\int_{\rho=0}^{1}(-\log\rho)^n\mathrm{d}\rho=\frac{\sqrt{2}}{2}n!, \tag{14}$$

where we used first that $e^{-\rho^2}\rho\leq\frac{\sqrt{2}}{2}$ for $\rho\in[0,1]$ and then n times equation (12).

For the second term of the sum in equation (13), we use first the fact that $e^{-\rho^2}\rho\leq\frac{e^{-\frac{1}{\rho^2}}}{\rho^3}$ for any $\rho\geq1$ and then the change $t=1/\rho$, yielding

$$\begin{aligned}\int_{\rho=1}^{\infty}(\log\rho)^n e^{-\rho^2}\rho\mathrm{d}\rho&\leq\int_{\rho=1}^{\infty}(\log\rho)^n\frac{e^{-\frac{1}{\rho^2}}}{\rho^3}\mathrm{d}\rho\underset{t=1/\rho}{=}\\&-\int_{1}^{0}(\log(1/t))^n te^{-t}\mathrm{d}t=\int_{0}^{1}(-\log(t))^n te^{-t}\mathrm{d}t.\end{aligned} \tag{15}$$

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The last integral is the same as in inequality (14), so from expressions (14) and (15) we obtain

$$\int_{\rho=1}^{\infty} (\log \rho)^n e^{-\rho^2} \rho \mathrm{d}\rho \leq \frac{\sqrt{2}}{2} n!. \tag{16}$$

Putting expressions (14) and (16) into equation (13), we obtain

$$\int_{\rho \in \mathbb{R}_+} |\log \rho|^n e^{-\rho^2} \rho \mathrm{d}\rho \leq \sqrt{2} n!. \tag{17}$$

- Computation of the integral $\int_{r\in\mathbb{R}_+} |\log r|^n e^{-r^2} r^3 \mathrm{d}r$. As before, we write

$$\int_{r\in\mathbb{R}_+} |\log r|^n e^{-r^2} r^3 \mathrm{d}r = \int_{r=0}^{1} (-\log r)^n e^{-r^2} r^3 \mathrm{d}r + \int_{r=1}^{\infty} (\log r)^n e^{-r^2} r^3 \mathrm{d}r. \tag{18}$$

For the first term of the sum, we get

$$\int_{r=0}^{1} (-\log r)^n e^{-r^2} r^3 \mathrm{d}r \leq (-1)^n \frac{\sqrt{2}}{\sqrt{3}} \int_{r=0}^{1} (\log r)^n \mathrm{d}r = \frac{\sqrt{2}}{\sqrt{3}} n!, \tag{19}$$

where the first inequality follows from $e^{-r^2} r^3 \leq \frac{\sqrt{2}}{\sqrt{3}}$, for $r \in [0,1]$, and the last equality is obtained using n times equation (12).

For the second term of the sum in the right-hand side of equation (18), we use integration by parts with respect to the functions $-\frac{1}{2}(\log r)^n r^2$ and $-2re^{-r^2}$ to obtain

$$\begin{aligned} \int_{s=1}^{\infty} (\log r)^n e^{-r^2} r^3 \mathrm{d}r &= \left[-\frac{1}{2} (\log r)^n r^2 e^{-r^2} \right]_{r=1}^{\infty} \\ &\quad + \frac{n}{2} \int_{r=1}^{\infty} (\log r)^{n-1} r e^{-r^2} \mathrm{d}r + \int_{r=1}^{\infty} (\log r)^n r e^{-r^2} \mathrm{d}r. \end{aligned} \tag{20}$$

As $[-\frac{1}{2}(\log r)^n r^2 e^{-r^2}]_{r=1}^{\infty} = 0$, we obtain, by using expression (16) in equation (20),

$$\int_{s=1}^{\infty} (\log r)^n e^{-r^2} r^3 \mathrm{d}r \leq \frac{3\sqrt{2}}{4} n!. \tag{21}$$

Putting inequalities (19) and (21) in equation (18), we get

$$\int_{r\in\mathbb{R}_+} |\log r|^n e^{-r^2} r^3 \mathrm{d}s \leq \frac{4\sqrt{6} + 9\sqrt{2}}{12} n!. \tag{22}$$

Now we use inequalities (17) and (22) and we obtain the following estimate:

$$\begin{aligned} &\int_{\substack{r\in\mathbb{R}_+ \\ \rho\in\mathbb{R}_+}} \sum_{k=0}^{m} \binom{m}{k} |\log \rho|^k |\log r|^{m-k} e^{-r^2 - s^2} r^3 \rho \mathrm{d}r \mathrm{d}\rho \\ &\leq \frac{4\sqrt{3}+9}{6} \sum_{k=0}^{m} \binom{m}{k} k!(m-k)! \leq \frac{4\sqrt{3}+9}{6} (m+1)!. \end{aligned} \tag{23}$$

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Putting expression (23) in inequality (11) and using equations (8), (9) and (10), we obtain the desired estimate for expression (7), hence the result. □

4. Proof of Theorem 1.1

In this section, we prove our main result. We follow the notations of Sections 2 and 3.

Proposition 4.1. *Let X be a real algebraic curve equipped with a compatible volume form ω of total volume 1, and let $F \in Pic^1_{\mathbb{R}}(X)$. Fix a sequence of positive real numbers $(a_d)_d$. Then for any $B > 0$ there exist $d_B \in \mathbb{N}$ and a constant c_B such that for any $E \in Pic^0_{\mathbb{R}}(X)$, any $d \geq d_B$ and any sequence of smooth functions $(\varphi_d)_d$ with $dist(supp(\partial\bar{\partial}\varphi_d), \mathbb{R}X) \geq B\frac{\log d}{\sqrt{d}}$, the following holds:*

$$\gamma_{F^d\otimes E}\left\{(\alpha,\beta)\in \mathbb{R}H^0(X,F^d\otimes E)^2, \left|\int_X \log\left(\frac{\pi}{d^{3/2}}\|W_{\alpha\beta}(x)\|\right)\partial\bar{\partial}\varphi_d\right| \geq a_d\right\}$$
$$\leq c_B \exp\left(-\frac{a_d}{2\|\partial\bar{\partial}\varphi_d\|_\infty Vol(Supp(\partial\bar{\partial}\varphi_d))}\right).$$

Here, $dist(\cdot,\cdot)$ is the distance in X induced by ω, $\gamma_{F^d\otimes E}$ is the Gaussian measure on $\mathbb{R}H^0(X,F^d\otimes E)^2$ constructed in Section 3.1 and $\|\cdot\|$ denotes the point-wise norm induced by the Hermitian metrics on F and E given by Proposition 2.1.

Proof. For any $t_d > 0$, let us denote

$$\exp\left(t_d\left|\int_X \log\left(\frac{\pi}{d^{3/2}}\|W_{\alpha\beta}(x)\|\right)\partial\bar{\partial}\varphi_d\right|\right) = \sum_{m=0}^{\infty}\frac{t_d^m}{m!}\left|\int_X \log\left(\frac{\pi}{d^{3/2}}\|W_{\alpha\beta}(x)\|\right)\partial\bar{\partial}\varphi_d\right|^m. \quad (24)$$

Remark that

$$\left|\int_X \log\left(\frac{\pi}{d^{3/2}}\|W_{\alpha\beta}(x)\|\right)\partial\bar{\partial}\varphi_d\right| \geq a_d \Leftrightarrow \exp\left(t_d\left|\int_X \log\left(\frac{\pi}{d^{3/2}}\|W_{\alpha\beta}(x)\|\right)\partial\bar{\partial}\varphi_d\right|\right) \geq e^{t_d a_d}, \quad (25)$$

so that by the Markov inequality we have

$$\gamma_{F^d\otimes E}\left\{(\alpha,\beta)\in \mathbb{R}H^0(X,F^d\otimes E)^2, \left|\int_X \log\frac{\pi}{d^{3/2}}\|W_{\alpha\beta}(x)\|\partial\bar{\partial}\varphi_d\right| \geq a_d\right\}$$
$$\leq e^{-t_d a_d}\int_{\mathbb{R}H^0(X,F^d\otimes E)^2}\exp\left(t_d\left|\int_X \log\left(\frac{\pi}{d^{3/2}}\|W_{\alpha\beta}(x)\|\right)\partial\bar{\partial}\varphi_d\right|\right)d\gamma_{F^d\otimes E}. \quad (26)$$

Now we have

$$\left|\int_X \log\left(\frac{\pi}{d^{3/2}}\|W_{\alpha\beta}(x)\|\right)\partial\bar{\partial}\varphi_d\right|^m \leq \|\partial\bar{\partial}\varphi_d\|_\infty^m\left|\int_{\mathrm{Supp}(\partial\bar{\partial}\varphi_d)}\log\left(\frac{\pi}{d^{3/2}}\|W_{\alpha\beta}(x)\|\right)\omega\right|^m. \quad (27)$$

We then apply the Hölder inequality with m and $m/(m-1)$ for the functions $\log\left(\frac{\pi}{d^{3/2}}\|W_{\alpha\beta}(x)\|\right)$ and 1, so that

$$(27) \leq \|\partial\bar{\partial}\varphi_d\|_\infty^m \mathrm{Vol}(\mathrm{Supp}(\partial\bar{\partial}\varphi_d))^{m-1}\int_{\mathrm{Supp}(\partial\bar{\partial}\varphi_d)}\left|\log\left(\frac{\pi}{d^{3/2}}\|W_{\alpha\beta}(x)\|\right)\right|^m\omega. \quad (28)$$

By Proposition 3.15, there exist $d_B \in \mathbb{N}$ and a positive constant c_B such that for any $d \geq d_B$ we get

$$\text{right-hand side of (28)} \leq \|\partial\bar{\partial}\varphi_d\|_\infty^m \text{Vol}(\text{Supp}(\partial\bar{\partial}\varphi_d))^m c_B(m+1)!. \tag{29}$$

Then by expressions (24), (26) and (29) we have

$$\gamma_{F^d\otimes E}\left\{(\alpha,\beta) \in \mathbb{R}H^0(X, F^d \otimes E)^2, \left|\int_X \log \frac{\|W_{\alpha\beta}(x)\|}{d^{3/2}} \partial\bar{\partial}\varphi_d\right| \geq a_d\right\}$$
$$\leq e^{-t_d a_d} c_B \sum_{m=0}^{\infty} (m+1)\Big(\|\partial\bar{\partial}\varphi_d\|_\infty \cdot \text{Vol}(\text{Supp}(\partial\bar{\partial}\varphi_d))\Big)^m t_d^m. \tag{30}$$

Now we have the identity $\sum_{m=0}^{\infty}(m+1)x^m = \frac{\mathrm{d}}{\mathrm{d}x}\sum_{m=1}^{\infty} x^m = \frac{\mathrm{d}}{\mathrm{d}x}\big(\frac{1}{(1-x)} - 1\big) = \frac{1}{(1-x)^2}$, so that the right-hand side of inequality (30) equals

$$\frac{c_B \exp(-t_d a_d)}{\big(1 - t_d\|\partial\bar{\partial}\varphi_d\|_\infty \cdot \text{Vol}(\text{Supp}(\partial\bar{\partial}\varphi_d))\big)^2}. \tag{31}$$

Putting $t_d = \big(2\|\partial\bar{\partial}\varphi_d\|_\infty \cdot \text{Vol}(\text{Supp}(\partial\bar{\partial}\varphi_d))\big)^{-1}$, we get the result. □

Lemma 4.2 ([5, Lemma 2]). *There exist positive constants C_i, $i \in \{1, \ldots, 4\}$ and a family of cutoff functions $\chi_t : X \to [0,1]$, defined for $t \in (0, t_0]$, for some $t_0 > 0$, such that*

1. *Vol(supp$(\partial\bar{\partial}\chi_t)$) $\leq C_1 t$,*
2. *Vol$(X \setminus \chi_t^{-1}(1)) \leq C_2 t$,*
3. $\|\partial\bar{\partial}\chi_t\|_{L^\infty} \leq C_3 t^{-2}$ *and*
4. *dist(supp$(\chi_t), \mathbb{R}X)) \geq C_4 t$.*

We now prove the following fibre-wise version of Theorem 1.1:

Theorem 4.3. *Let $\ell(d)$ be a sequence of positive real numbers such that $\ell(d) \geq B(\log d)$ for some $B > 0$. Then there exist positive constants c_1 and c_2 such that*

$$\mu_{F^d\otimes E}\big\{u \in \mathcal{M}_d^{\mathbb{R}}(X, F^d \otimes E), \#(\mathit{Crit}(u) \cap \mathbb{R}X) \geq \ell(d)\sqrt{d}\big\} \leq c_1 e^{-c_2\ell(d)^2}.$$

Here, $\mu_{F^d\otimes E}$ is the probability measure defined in Definition 2.6 and $\mathcal{M}_d^{\mathbb{R}}(X, F^d \otimes E)$ is defined in Definition 2.9.

Proof. For any pair of real global sections $(\alpha,\beta) \in \mathbb{R}H^0(X, F^d \otimes E)^2$ without common zeros, let $u_{\alpha\beta}$ be the real branched covering defined by $x \mapsto [\alpha(x) : \beta(x)]$. Consider the set

$$\mathcal{C}_{\ell(d)} := \{(\alpha,\beta) \in \mathbb{R}H^0(X, F^d \otimes E)^2, \#(\text{Crit}(u_{\alpha\beta}) \cap \mathbb{R}X) \geq \ell(d)\sqrt{d}\big\}. \tag{32}$$

Note that this set is a cone in $\mathbb{R}H^0(X, F^d \otimes E)^2$. By Proposition 3.3, this implies that the Gaussian measure of $\mathcal{C}_{\ell(d)}$ equals the Fubini–Study measure of its projectivisation, which is exactly the measure we want to estimate. In order to obtain the result, we will then compute the Gaussian measure of cone (32). Moreover, by Proposition 3.14, we have that $x \in \text{Crit}(u_{\alpha\beta})$ if and only if $W_{\alpha\beta}(x) = 0$, so that in order to compute $\#\text{Crit}(u_{\alpha\beta})$, we can compute the number of zeros of $W_{\alpha\beta}$. To do this, we will use the Poincaré–Lelong formula – that is, the following equality between currents:

$$\omega_d - \sum_{x\in\{W_{\alpha\beta}=0\}} \delta_x = \frac{1}{2\pi i}\partial\bar{\partial}\log\|W_{\alpha\beta}\|, \tag{33}$$

where $\|\cdot\|$ is the (induced) metric on $F^{2d}\otimes E^2\otimes T_X^*$ given by Proposition 2.1 and ω_d is the corresponding curvature form. Remark that ω_d equals $2d\cdot\omega + O(1)$ (the term $2d\cdot\omega$ comes from the curvature form of $F^{2d}\otimes E^2$ and the term $O(1)$ from the curvature form of T_X^*). Moreover, remark that the Hermitian metric $\frac{\pi}{d^{3/2}}\|\cdot\|$ has the same curvature as the Hermitian metric $\|\cdot\|$, because the curvature form is not affected by a multiplicative constant. Then equation (33) can also be read as

$$2d\cdot\omega + O(1) - \sum_{x\in\{W_{\alpha\beta}=0\}} \delta_x = \frac{1}{2\pi i}\partial\bar{\partial}\log\left(\frac{\pi}{d^{3/2}}\|W_{\alpha\beta}\|\right), \tag{34}$$

where the equality is in the sense of currents. We will apply equation (34) for the functions χ_{t_d}, given by Lemma 4.2, for $t_d = \frac{\ell(d)}{4C_2\sqrt{d}}$, where C_2 is the constant appearing in the lemma. By equation (34), we then get

$$\frac{1}{2\pi}\left|\int_X \log\left(\frac{\pi}{d^{3/2}}\|W_{\alpha\beta}\|\right)\partial\bar{\partial}\chi_{t_d}\right| \geq \left|2d\left(1-\frac{\ell(d)}{4\sqrt{d}}\right) + O(1) - \sum_{x\in\{W_{\alpha\beta}=0\}} \chi_{\frac{\ell(d)}{\sqrt{d}}}(x)\right|. \tag{35}$$

Note that for any pair of real global sections (α,β) in the cone $\mathcal{C}_{\ell(d)}$ defined in expression (32), we have

$$\sum_{x\in\{W_{\alpha\beta}=0\}} \chi_{\frac{\ell(d)}{\sqrt{d}}}(x) \leq 2d + 2g - 2 - \ell(d)\sqrt{d}, \tag{36}$$

where g is the genus of X. Then, putting inequality (36) into inequality (35), we get

$$\frac{1}{2\pi}\left|\int_X \log\left(\frac{\pi}{d^{3/2}}\|W_{\alpha\beta}\|\right)\partial\bar{\partial}\chi_{\frac{\ell(d)}{\sqrt{d}}}\right| \geq \frac{1}{2}\ell(d)\sqrt{d} + O(1),$$

for any $(\alpha,\beta)\in\mathcal{C}_{\ell(d)}$. Then for d large enough, cone (32) is included in the set

$$\left\{(\alpha,\beta)\in\mathbb{R}H^0(X,F^d\otimes E)^2, \left|\int_X \log\left(\frac{\pi}{d^{3/2}}\|W_{\alpha\beta}\|\right)\partial\bar{\partial}\chi_{\frac{\ell(d)}{\sqrt{d}}}\right| \geq \ell(d)\sqrt{d}\right\}.$$

The result then follows from Proposition 4.1 and Lemma 4.2. □

Proof of Theorem 1.1. We fix a degree 1 real holomorphic line bundle F over X, so that for any $L\in \mathrm{Pic}^d_{\mathbb{R}}(X)$ there exists an unique degree 0 real holomorphic line bundle $E\in\mathrm{Pic}^0_{\mathbb{R}}(X)$ such that $L = F^d\otimes E$. The result then follows by integrating the inequality appearing in Theorem 4.3 along the compact base $\mathrm{Pic}^0_{\mathbb{R}}(X)\simeq \mathrm{Pic}^d_{\mathbb{R}}(X)$ (the last isomorphism is given by the choice of the degree 1 real line bundle F). □

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