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NASH EQUILIBRIUM STRUCTURE OF COX PROCESS HOTELLING GAMES

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Abstract

We study an N-player game where a pure action of each player is to select a nonnegative function on a Polish space supporting a finite diffuse measure, subject to a finite constraint on the integral of the function. This function is used to define the intensity of a Poisson point process on the Polish space. The processes are independent over the players, and the value to a player is the measure of the union of her open Voronoi cells in the superposition point process. Under randomized strategies, the process of points of a player is thus a Cox process, and the nature of competition between the players is akin to that in Hotelling competition games. We characterize when such a game admits Nash equilibria and prove that when a Nash equilibrium exists, it is unique and consists of pure strategies that are proportional in the same proportions as the total intensities. We give examples of such games where Nash equilibria do not exist. A better understanding of the criterion for the existence of Nash equilibria remains an intriguing open problem.

Keywords: Constant-sum game; Cox process; game theory; Hotelling competition; Nash equilibrium; Poisson process

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1. Introduction

1.1. Informal problem formulation

We study *games of spatial competition of the Hotelling type*, where N players compete for space. Here space is modeled as a complete separable metric space (i.e., a *Polish space*) D supporting a *diffuse* (i.e., non-atomic) finite positive measure η on its Borel σ-field, which we denote by $\mathcal{B}$. Throughout the paper we think of D as being endowed with a fixed metric d generating its topology. The game in question is a one-shot game. Each player's action consists in selecting a nonnegative measure on D which is absolutely continuous with respect to η, among the set of all measures with a fixed finite and positive total mass. Since we will always consider D as being metrized by the metric d and endowed with its Borel σ-algebra $\mathcal{B}$, we will not mention d and $\mathcal{B}$ where these can be inferred from the context. Thus, for instance, by a nonnegative measure on D what we actually mean is a nonnegative measure on $(D, \mathcal{B})$.

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Similarly, when we talk about the open Voronoi cell of a point in a configuration of points from D, we implicitly mean that the Voronoi cells are based on the metric d. This mass constraint depends on the player and represents the total available intensity that the player can deploy over the space. The measure chosen by each player results in a Poisson process of points on D with this measure as its intensity measure, these processes being independent. Thus, if a player uses a randomized strategy, the point process on D generated by each player is a *Cox point process*, namely a point process which is conditionally Poisson given its random intensity. The payoff of each player is the total η-measure of the union of the *open Voronoi cells* of the points of the point process of this player, where the open Voronoi cells are evaluated with respect to the superposition of the point processes of all players (if there are no points, which can happen with positive probability, each player gets zero value; further, we need to make a technical assumption on the metric structure of D which ensures that the union of the open Voronoi cells is of full measure, conditioned on there being at least one point in the superposition point process). Our goal in this paper is to study the *Nash equilibria* of this game.

In the two-player case, one motivation for such a formulation comes from a model for defense against threats. The underlying space may be thought of as a model for the set of possible attack modalities, with the metric indicating how similar attacks are to each other. The defender and the attacker are respectively interested in defending against or deploying the different kinds of attacks, and the total η-measure of the union of the open Voronoi cells of the points of each player indicates how well she is doing with regard to her individual objective of getting the upper hand over the other. The stochastic nature of the placement of the points of a player is meant to capture the idea that the deployment of effort only results in success in a stochastic way. The finite total intensity that each player can deploy represents individual budget constraints. The study of Nash equilibria is then motivated by the goal of getting some insights into how the individual players (i.e. the attacker and defender) might play when faced with such an environment. Apart from its possible intrinsic interest, it turns out that it is no more difficult to study the N-player version of the game than the two-player version, for reasons that we will soon see, so we have formulated the problem we study at this apparently broader level of generality.

1.2. Literature survey

The origin of the study of spatial competition models is generally attributed to a paper by Hotelling [17]. Hotelling's model has some additional features, such as prices set by the sellers, which will not play a role in our formulation. Building on the *Cournot duopoly model* ([7], [31, Section 27.5]), as refined by Bertrand ([2], [31, Section 27.9]) and Edgeworth [12], the key innovation of [17] is to introduce spatial aspects to the modeling of the competition between sellers for buyers. Specifically, Hotelling considers the problem faced by two sellers as to where to position themselves along an interval of fixed length, which models a market along which consumers are uniformly distributed, and how to individually set prices for the one identical good that they sell so that each seller maximizes its profit, given the strategy of the other seller. In the model of Hotelling, each consumer incurs a transportation cost proportional to its distance from the seller from which it buys, which then determines which seller it prefers, given the sellers' locations.

Notice that for fixed locations of the two sellers, if price discrimination is not possible (i.e. prices are identical at the two sellers), then consumers go to the closer seller, so the problem of each seller becomes that of how to position itself, in reaction to position of the other seller,

FIGURE 1. Hotelling competition on the unit interval with three sellers, with locations indicated. The individual open Voronoi cells are indicated in color. In the absence of price discrimination, consumers residing at a location in the open Voronoi cell of a seller will go to that seller.

so as to maximize the length of its Voronoi cell in the *Voronoi decomposition* associated to the locations of the two sellers. See [17, Figure 1, p. 45]. The problem considered in this paper is of this purely Voronoi-cell-decomposition type. It is worth noting that the solution concept in [17] or, for that matter, in [7] is already of the type that one would today call a pure-strategy Nash equilibrium, though of course these works predate by several decades the work of von Neumann and Morgenstern [32] and Nash [24, 25], which formalized the notion of Nash equilibrium.

There is by now a vast literature on the Hotelling competition model, and many different variants have been developed and studied. Rather than attempt to survey this work, particularly since the precise problem formulation we consider does not appear in the prior literature in this area, we refer the reader to the surveys in [16, 14, 13, 27], and to the papers in the recent edited volume [22].

Another growing body of work that is related to the themes of this paper is that of Voronoi games; see the seminal papers [1] and [6]. Here also the players are competing to capture regions of space according to the Voronoi decomposition of the underlying space based on the choice of the locations of their points, but, in contrast to the model we consider, the players are assumed to have control over exactly where they can place their points, with the constraint, in some of the literature on Voronoi games, that these locations should lie in a given finite set of potential locations in the ambient space. Further, in contrast to the model we consider, much of this literature assumes that the players place their points one at a time, alternating between the players. There is also a particular interest in this literature in the study of Voronoi games on graphs; see e.g. [30, 11]. Note that this is incompatible with the non-atomic nature of the underlying measure η required for our formulation. Nevertheless, our formulation was partly inspired by a recent work in this area, on so-called *Voronoi choice games*, by Boppana *et al.* [4].

There are also some similarities between the game we analyze and the study of the so-called *Colonel Blotto games* (see [28]), on variants of which also there is a rapidly growing literature. In the game we study, the ability of a player to control exactly where to place her points, as in the literature on Hotelling games or Voronoi games, can be thought of as being replaced by a softer ability to control the intensity or, in effect, the local mean number of points, in the same way as the formulation of the so-called *General Lotto game* (see [20]), softens the ability of individual players to fix the number of soldiers to be placed on each battlefield of the Colonel Blotto game. However, the reward structure of the players in a Colonel Blotto game is completely different from that in the Hotelling games or Voronoi games.

1.3. Notational conventions

We let $\mathbb{N}$ denote the set of natural numbers, and $\mathbb{R}_+$ the set of nonnegative real numbers. We use $:=$ and $=:$ for equality by definition. The indicator of a set or an event E is written as 1_E or $1(E)$.

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1.4. Structure of the paper

Section 2 sets up the structure of two-player Cox process Hotelling games. Section 3 introduces the N-player Cox process Hotelling games and develops several general properties of the value function in these games; these are used later to prove the main results. Section 4 is focused on the case where the underlying Polish space on which the game is played is compact and admits a transitive group of metric-preserving automorphisms, with the base measure of the game being invariant under this group. Section 5 studies the general case of the N-player game and determines the structure of the Nash equilibria when they exist. A necessary and sufficient condition for the existence of Nash equilibria is provided, and also examples are given where no Nash equilibrium exists. Although Nash equilibrium, when it exists, is unique and consists of pure strategies, it is established that Cox process Hotelling games are not ordinal potential games in general. A family of so-called restricted Cox process Hotelling games is defined to provide a vehicle to better understand the meaning of the criterion for the existence of Nash equilibria.

2. Cox process Hotelling games

2.1. Diffuse non-conflicting finite positive measures

Let D be a Polish space. We write $\mathcal{B}$ for the Borel σ-field of D. A finite positive measure η on D is called *diffuse* if for every Borel set $B \in \mathcal{B}$ with $\eta(B) > 0$ there is a Borel subset $C \subset B$ with $0 < \eta(C) < \eta(B)$. One can define a Poisson process on D based on such a diffuse positive measure η on D; see e.g. [8, Chapter 9]. A point process on D is a random counting measure on D. The Poisson process on D with intensity measure η, where η is any finite positive diffuse measure on D, is the point process obtained by first selecting the total number of points according to the Poisson distribution on $\mathbb{N}$ with mean $\eta(D)$, and then sampling independently the location of the points, if any, according to the measure $\eta(.)/\eta(D)$ on $\mathcal{B}$.

Recall that d denotes the fixed metric on D generating its topology. The *open Voronoi cells* of any point process Φ with respect to d can be defined in the usual way. Namely, for x in the support of the point process Φ, the open Voronoi cell $W_\Phi(x)$ consists of those $z \in D$ such that $d(x, z) < d(x', z)$ for all points $x' \neq x$ in the support of Φ. For our purposes, we need to impose on η the condition that the union of the open Voronoi cells of the Poisson process on D with intensity measure η has full measure $\eta(D)$ with probability 1, conditioned on there being at least one point in this Poisson process. If η satisfies this condition, we call it *non-conflicting*. A sufficient condition for this to hold is given in Appendix A.

The importance of imposing a non-conflicting condition can be understood by considering the following example.

Example 1. Suppose D is the disjoint union of two intervals I and J, each of length 1. Assume that every point in I is at distance 2 from every point in J, as depicted in Figure 2. It is straightforward to check that the resulting metric makes D a Polish space and that the measure η on the Borel σ-field of D which corresponds to Lebesgue measure on the two unit intervals comprising D is a finite positive diffuse measure. However, η is not non-conflicting. This is because, for instance, on the event of positive probability that the Poisson process on D with intensity measure η has two points in I and no points in J, each point of J will be at distance 2 from each of the two points of the Poisson process in I, and so will not belong to the open Voronoi cell of either point.

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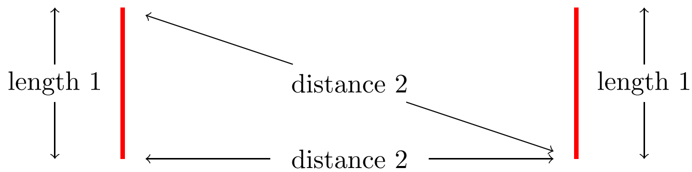

FIGURE 2. A metric space comprising two intervals, each of length 1. Every point in the interval on the left is at distance 2 from every point in the interval on the right. The metric on each interval is the usual one.

Throughout the paper, η is a finite positive diffuse non-conflicting measure on D which will be referred to as the *base measure*. It is straightforward to check that if ν is any finite positive measure on D that is absolutely continuous with respect to η, then ν is also diffuse and non-conflicting. We write $\mathcal{M}(D)$ for the set of nonnegative finite measures on D.

2.2. Radon–Nikodym derivatives

In the Cox process Hotelling games considered in this paper, we think of each individual player as choosing a positive measure on D of fixed total mass that is absolutely continuous with respect to the base measure η. The measure chosen by the player then serves as the intensity measure for a Poisson process of points on D, which we think of as the points belonging to that player. By the Radon–Nikodym theorem, we may identify the measure chosen by the player with its likelihood function with respect to the base measure.

With this viewpoint in mind, for any $\rho > 0$, let

$$\mathcal{C}(\rho) := \left\{ f : D \to \mathbb{R}_+ \text{ s.t. } \int_D f(x)\eta(dx) = \rho \right\}. \tag{1}$$

The set $\mathcal{C}(\rho)$ can be thought of as a subset of $L^1(\eta)$, i.e. the Lebesgue space of $\mathcal{B}$-measurable functions on D that are absolutely integrable with respect to the base measure η, and also as a set of measures on D by identifying $f \in \mathcal{C}(\rho)$ with the measure $f\eta$. With the latter viewpoint in mind, we think of $\mathcal{C}(\rho)$ as endowed with the topology of weak convergence of measures [26], which it inherits as a subset of $\mathcal{M}(D)$. Note that, for all $\rho > 0$ and all $f \in \mathcal{C}(\rho)$, the measure $f\eta$ is diffuse and non-conflicting. Also note that, for all $\rho > 0$, the set $\mathcal{C}(\rho)$ is a convex subset of $\mathcal{M}(D)$. However, since we have endowed $\mathcal{C}(\rho)$ with the topology of weak convergence of measures on D, this set is not closed.

For any $\rho > 0$, the subset of $\mathcal{M}(D)$ comprising measures of total mass ρ, with the topology of weak convergence, can be metrized so as to make it a complete separable metric space [26, Theorem 6.2 and Theorem 6.5]. As a metric space it is first-countable, so it suffices to discuss convergence of sequences rather than of nets [19, Theorem 8]. To discuss randomized strategies of the individual players we need to be able to discuss probability distributions on subsets of the type $\mathcal{C}(\rho)$ of $\mathcal{M}(D)$. This is made possible by the following result.

Lemma 1. *For each $\rho > 0$, the set $\mathcal{C}(\rho)$ is a Borel subset of $\mathcal{M}(D)$ when $\mathcal{M}(D)$ is endowed with the topology of weak convergence.*

Proof. See [21, Theorem 3.5]. □

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2.3. Two-player games

In the two-player version of the game, a pure action of Alice consists of choosing an intensity measure which is absolutely continuous with respect to the measure η. Namely, Alice chooses a nonnegative measurable function $f_A : D \to \mathbb{R}_+$ as the Radon–Nikodym derivative with respect to η of the intensity measure of its Poisson point process. We denote the set of functions from which this choice must be made by $\mathcal{C}_A$, i.e.,

$$\mathcal{C}_A := \mathcal{C}(\rho_A) = \left\{ f_A : D \to \mathbb{R}_+ \text{ s.t. } \int_D f_A(x)\eta(dx) = \rho_A \right\}. \tag{2}$$

Thus ρ_A can be thought of as a constraint on the total intensity that Alice can deploy for her point process.

Similarly, Bob chooses a nonnegative measurable function $f_B : D \to \mathbb{R}_+$ within the class of functions $\mathcal{C}_B$, where

$$\mathcal{C}_B := \mathcal{C}(\rho_B) = \left\{ f_B : D \to \mathbb{R}_+ \text{ s.t. } \int_D f_B(x)\eta(dx) = \rho_B \right\}. \tag{3}$$

Here ρ_A and ρ_B are fixed positive constants. Note that $\mathcal{C}_A$ and $\mathcal{C}_B$ are convex subsets of $\mathcal{M}(D)$, but neither of them is closed in the topology of weak convergence on $\mathcal{M}(D)$.

We denote such a two-player Cox process Hotelling game by $(D, \eta, \rho_A, \rho_B)$.

Assume that Alice plays f_A and Bob plays f_B. The resulting value for Alice, denoted by $V_A\big(f_A, f_B\big)$, is defined as follows. Let Φ_A and Φ_B be independent Poisson point processes on D with intensity measures $f_A\eta$ and $f_B\eta$, respectively. We think of the points of Φ_A as Alice's points, since these result from the choice of f_A by Alice. Similarly we think of the points of Φ_B as Bob's points. Let $\Phi := \Phi_A + \Phi_B$. For all $x \in \Phi$, let $W_\Phi(x)$ denote the open Voronoi cell of x with respect to Φ. Then

$$V_A\big(f_A, f_B\big) := \mathbb{E}\left[\sum_{x\in\Phi_A} \eta(W_\Phi(x)) \right],$$

where the expectation is with respect to the joint law of Φ_A and Φ_B, which are assumed to be independent. Here, by definition, a sum over an empty set is 0. The value for Bob resulting from this pair of actions, denoted by $V_B\big(f_A, f_B\big)$, is determined similarly; i.e., it is

$$V_B\big(f_A, f_B\big) := \mathbb{E}\left[\sum_{x\in\Phi_B} \eta(W_\Phi(x)) \right].$$

In words, the value of Alice is the mean value of the sum of the η-measures of the open Φ-Voronoi cells centered at her points if she has points, and is 0 otherwise, and likewise for the value of Bob. Intuitively, one thinks of the points of Alice and those of Bob as competing to capture the ambient space, with a point in the ambient space D belonging to Alice if the closest point to it is one of Alice, and to Bob otherwise. Under our non-conflicting condition on the base measure η, there is no need to worry about how to break ties.

Note that

$$V_A\big(f_A, f_B\big) + V_B\big(f_A, f_B\big) = \eta(D)\big(1 - e^{-\rho}\big) \tag{4}$$

for all $\big(f_A, f_B\big) \in \mathcal{C}_A \times \mathcal{C}_B$, where $\rho := \rho_A + \rho_B$. This is because, by virtue of our assumption that η is non-conflicting, the sum of the values of Alice and Bob is $\eta(D)$, except on the event where neither Φ_A nor Φ_B has any points, which is an event of probability $1 - e^{-\rho}$. The formula in Equation (4) is formally established in Lemma 2 below.

2.4. Nash equilibrium in two-player games

In view of Lemma 1, any mixed-strategy pair in the two-player game $(D, \eta, \rho_A, \rho_B)$ between Alice and Bob can be written as $(f_A(M_A), f_B(M_B))$, where $M_A \in \mathcal{M}_A$ and $M_B \in \mathcal{M}_B$ are independent random variables representing the randomizations used by Alice and Bob respectively in implementing randomized strategies, and $f_A(m_A) \in \mathcal{C}_A$ (respectively, $f_B(m_B) \in \mathcal{C}_B$) is the choice of Alice (respectively, Bob) in case the realization of her random variable M_A is m_A (respectively, the realization of his random variable M_B is m_B). The value of Alice in such a mixed strategy is $\mathbb{E}\big[V_A\big(f_A\big(M_A\big), f_B\big(M_B\big)\big)\big]$, while that of Bob is $\mathbb{E}\big[V_B\big(f_A\big(M_A\big), f_B\big(M_B\big)\big)\big]$, the expectations being taken with respect to the joint distribution of M_A and M_B, which are independent. As a direct consequence of Equation (4), we have

$$\mathbb{E}\big[V_A\big(f_A\big(M_A\big), f_B\big(M_B\big)\big)\big] + \mathbb{E}\big[V_B\big(f_A\big(M_A\big), f_B\big(M_B\big)\big)\big] = \eta(D)\big(1 - e^{-\rho}\big), \tag{5}$$

where $\rho := \rho_A + \rho_B$.

Definition 1. The pair of independently randomized strategies $(f_A(M_A), f_B(M_B)) \in \mathcal{C}_A \times \mathcal{C}_B$ is called a *Nash equilibrium* of the game between Alice and Bob if, for all $g_A \in \mathcal{C}_A$ and $g_B \in \mathcal{C}_B$, we have

$$\mathbb{E}\big[V_A\big(f_A\big(M_A\big), f_B\big(M_B\big)\big)\big] \geq \mathbb{E}\big[V_A\big(g_A, f_B\big(M_B\big)\big)\big] \tag{6}$$

and

$$\mathbb{E}\big[V_B\big(f_A\big(M_A\big), f_B\big(M_B\big)\big)\big] \geq \mathbb{E}\big[V_B\big(f_A\big(M_A\big), g_B\big)\big]. \tag{7}$$

In words, Alice sees no advantage in playing the strategy g_A instead of her randomized strategy $f_A(M_A)$, given that Bob is playing his randomized strategy $f_B(M_B)$, and similarly for Bob.

3. Supporting lemmas

3.1. A formula for the value

Consider a two-player Cox process Hotelling game $(D, \eta, \rho_A, \rho_B)$ between Alice and Bob, and suppose that the players play the action pair (f_A, f_B). Let Φ_A and Φ_B denote the Poisson processes of points of Alice and Bob respectively on D. Recall that these are independent point processes. Let $\mathbb{P}_A^x$ (respectively, $\mathbb{P}_B^x$) denote the Palm probability [18, Chapter 6] with respect to Φ_A (respectively, Φ_B) at x. By Slivnyak's theorem [18, Lemma 6.14], under the Palm probability with respect to Φ_A at x, Φ_A consists of a point at x and a Poisson process on D of intensity $f_A\eta$, and Φ_B consists of an independent Poisson process on D of intensity $f_B\eta$. The Palm probability with respect to Φ_B at x has a symmetrical description.

From Campbell's formula [18, Section 6.1] we have

$$V_A\big(f_A, f_B\big) = \int_D f_A(x)\mathbb{E}_A^x\big[\eta(W_\Phi(x))\big]\eta(dx).$$

Here the expectation is with respect to the law of Φ under the Palm distribution $\mathbb{P}_A^x$. From the description of this Palm distribution, we have

$$\begin{aligned}
\mathbb{E}_A^x\big[\eta(W_\Phi(x))\big] &= \mathbb{E}_A^x\left[\int_{y\in D} 1_{\{y\in W_\Phi(x)\}}\eta(dy)\right] \\
&\overset{(a)}{=} \int_{y\in D} \mathbb{P}_A^x\big(y \in W_\Phi(x)\big)\eta(dy) \\
&\overset{(b)}{=} \int_{y\in D} e^{-\int_{B(y\to x)}(f_A(u)+f_B(u))\eta(du)}\eta(dy),
\end{aligned}$$

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where

$$B(y \to x) := \{z \in D \text{ s.t. } d(z, y) \le d(x, y)\}.$$

The notation $B(y \to x)$ is supposed to bring to mind a closed ball centered at y having x at its boundary.

In the chain of equations above, Step (a) comes from an application of Fubini's theorem and Step (b) from Slivnyak's theorem. Hence

$$V_A(f_A, f_B) = \int_{x \in D} f_A(x) \int_{y \in D} e^{-\int_{B(y \to x)} (f_A(u)+f_B(u))\eta(du)} \eta(dy)\eta(dx). \tag{8}$$

3.2. N-player games

The point process analysis above and the resulting formula in Equation (8) in the two-player case also leads to a formula in an N-player Cox process Hotelling game for the value $V_i(f_1, \ldots, f_N)$ seen by player i when the pure strategies deployed by the individual players are $f_1, \ldots, f_N$ respectively. For this, let $f_j : D \to \mathbb{R}_+$ be a nonnegative measurable function on D belonging to

$$\mathcal{C}_j := \mathcal{C}(\rho_j) = \left\{ f_j : D \to \mathbb{R}_+ \text{ s.t. } \int_D f_j(x)\eta(dx) = \rho_j \right\}, \tag{9}$$

where the $\rho_j > 0$ are fixed for $1 \le j \le N$. Note that each $\mathcal{C}_j$ is a convex subset of $\mathcal{M}(D)$, but is not closed in the topology of weak convergence. Assume that the individual players play $f_i \in \mathcal{C}_i$ for $1 \le i \le N$. Let Φ_i be a Poisson point process on D with intensity measure $f_i\eta$, with these processes being mutually independent for $1 \le i \le N$, and let $\Phi := \sum_{i=1}^N \Phi_i$. We think of the points of Φ_i as the points of player i, since these result from the choice of f_i, which was made by that player. Then the value of player i, denoted by $V_i(f_1, \ldots, f_N)$, is given by

$$V_i(f_1, \ldots, f_N) := \mathbb{E}\left[\sum_{x \in \Phi_i} \eta(W_\Phi(x)) \right].$$

The expectation is with respect to the joint law of $(\Phi_j, 1 \le j \le n)$, which are independent. Here, by definition, a sum over an empty set is 0.

We denote such an N-player Cox process Hotelling game by $(D, \eta, \rho_1, \ldots, \rho_N)$.

We can write a formula for $V_i(f_1, \ldots, f_N)$, based on Equation (8), by thinking of the points of player i as competing for space in D with the union of the points of the other players. Thus we have

$$V_i(f_1, \ldots, f_N) = \int_{x \in D} f_i(x) \int_{y \in D} e^{-\int_{u \in B(y \to x)} f(u)\eta(du)} \eta(dy)\eta(dx), \tag{10}$$

where $f := \sum_{i=1}^N f_i$.

Note that we must have

$$\sum_{i=1}^N V_i(f_1, \ldots, f_N) = \eta(D)\left(1 - e^{-\rho}\right), \tag{11}$$

where $\rho := \sum_{j=1}^N \rho_j$. This is because we have $\int_{x \in D} f(x)\eta(dx) = \rho$, so the law of the total number of points is Poisson with mean ρ, and thus, by virtue of our assumption that η is non-conflicting, the total value of all the players is $\eta(D)$ except on the event that there are no points in Φ, in which case the total value is 0. The formula in Equation (11) is established in Lemma 2 below.

3.3. Nash equilibrium in N-player games

In view of Lemma 1, any mixed-strategy pair in an N-player Cox process Hotelling game $(D, \eta, \rho_1, \ldots, \rho_N)$ can be written as $(f_1(M_1), \ldots, f_N(M_N))$, where $(M_j \in \mathcal{M}_j, 1 \le j \le N)$ are independent random variables representing the randomizations used by the individual players in implementing their randomized strategies, and $f_j(m_j) \in \mathcal{C}_j$ is the choice of action of player j in case the realization of her random variable M_j is m_j. The value of player i in such a mixed strategy is $\mathbb{E}\big[V_i\big(f_1(M_1), \ldots, f_N(M_N)\big)\big]$, the expectation being taken with respect to the joint distribution of $(M_j, 1 \le j \le N)$, which are independent. As a direct consequence of Equation (11), we have

$$\sum_{i=1}^{N} \mathbb{E}[V_i(f_1(M_1), \ldots, f_N(M_N))] = \eta(D)|\big(1 - e^{-\rho}\big), \tag{12}$$

where $\rho := \sum_{i=1}^{N} \rho_i$.

Definition 2. The vector of independently randomized strategies

$$(f_1(M_1), \ldots, f_N(M_N)) \in \mathcal{C}_1 \times \ldots \times \mathcal{C}_N$$

is called a *Nash equilibrium* of the N-player game if, for all $g_j \in \mathcal{C}_j$, $1 \le j \le N$, we have, for all $1 \le i \le N$,

$$\mathbb{E}\big[V_i(f_1(M_1), \ldots, f_N(M_N))\big] \ge \mathbb{E}\big[V_i\big(g_i, \big(f_j\big(M_j\big), j \ne i\big)\big)\big]. \tag{13}$$

In words, the player i perceives no advantage in playing the strategy g_i instead of the randomized strategy $f_i(M_i)$, given that the other players, i.e. the players $j \ne i$, are playing the individually randomized strategies $\big(f_j\big(M_j\big), j \ne i\big)$.

3.4. A conservation law

To establish the conservation law in Equation (11) and the special case of it in Equation (4), it suffices to demonstrate that for all $\rho > 0$ and all $f \in \mathcal{C}(\rho)$, we have Equation (14) below. This is because we would then immediately obtain the desired conservation laws from Equation (10) and the special case in Equation (8), respectively, by summing these formulas over the individual players. We establish (14) formally in the following lemma.

Lemma 2. *For any $\rho > 0$ and $f \in \mathcal{C}(\rho)$, we have*

$$\int_{x \in D} f(x) \int_{y \in D} e^{-\int_{B(y \to x)} f(u)\eta(du)} \eta(dy)\eta(dx) = \eta(D)\big(1 - e^{-\rho}\big)\,. \tag{14}$$

Proof. We give two proofs of this formula. The first one is probabilistic. Let Φ denote a Poisson process on D with intensity measure $f\eta$. By Campbell's formula and Slivnyak's theorem [18], the integral on the left-hand side of (14) is just

$$\mathbb{E}\left[\sum_{x \in \Phi} \eta(W_\Phi(x))\right],$$

where $W_\Phi(x)$ denotes the open Voronoi cell of $x \in \Phi$ with respect to the Poisson process Φ. If $\Phi(D) = 0$, which happens with probability $e^{-\rho}$, the sum is 0. On the complementary event the sum is $\rho(D)$, since, by virtue of the assumption that η is non-conflicting, the collection of sets $(W_\Phi(x), x \in \Phi)$ form a partition of D up to a set of η-measure 0.

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The second proof is analytical. We have, for each $y \in D$,

$$\int_{x\in D} f(x) e^{-\int_{u\in B(y\to x)} f(u)\eta(du)} \eta(dx) = 1 - e^{-\rho},$$

since, when η is non-conflicting, the integral on the left-hand side of the preceding equation is just the probability that a nonhomogeneous Poisson process with intensity function $f(x)$ with respect to η has at least one point in D (to see this, think of moving out from y by balls of increasing radius till one covers all of D). Thus we have

$$\int_{y\in D}\int_{x\in D} f(x) e^{-\int_{u\in B(y\to x)} f(u)\eta(du)} \eta(dy)\eta(dx) = \eta(D)\big(1 - e^{-\rho}\big).$$

But the left-hand side of the preceding equation equals the left-hand side of Equation (14), by an application of Fubini's theorem. □

3.5. Constant strategies

Consider a two-player Cox process Hotelling game $(D, \eta, \rho_A, \rho_B)$. Write $\bar{\rho}_A$ for the strategy of Alice where she chooses f_A to be the constant $\frac{\rho_A}{\eta(D)}$, and similarly write $\bar{\rho}_B$ for the strategy of Bob where he chooses f_B to be the constant $\frac{\rho_B}{\eta(D)}$. Note that $\bar{\rho}_A \in \mathcal{C}_A$ and $\bar{\rho}_B \in \mathcal{C}_B$.

The following useful lemma gives an explicit formula for the value obtained by Alice and that obtained by Bob in the two-player game $(D, \eta, \rho_A, \rho_B)$ when the players play the constant strategies $\bar{\rho}_A$ and $\bar{\rho}_B$ respectively. Note that, while the strategies are constant, the resulting intensities are $\frac{\rho_A}{\eta(D)}\eta$ and $\frac{\rho_B}{\eta(d)}\eta$ for Alice and Bob respectively.

Lemma 3. *Consider a two-player Cox process Hotelling game* $(D, \eta, \rho_A, \rho_B)$. *Then* $V_A(\bar{\rho}_A, \bar{\rho}_B) = \frac{\rho_A}{\rho}\eta(D)(1 - e^{-\rho})$ *and* $V_B\big(\bar{\rho}_A, \bar{\rho}_B\big) = \frac{\rho_B}{\rho}\eta(D)(1 - e^{-\rho})$, *where* $\rho := \rho_A + \rho_B$.

Proof. We have

$$\begin{aligned}
V_A\big(\bar{\rho}_A, \bar{\rho}_B\big) &\overset{(a)}{=} \int_{x\in D} \frac{\rho_A}{\eta(D)} \int_{y\in D} e^{-\frac{\rho}{\eta(D)}\eta(B(y\to x))}\eta(dy)\eta(dx) \\
&= \frac{\rho_A}{\rho}\int_{x\in D}\int_{y\in D} \frac{\rho}{\eta(D)} e^{-\frac{\rho}{\eta(D)}\eta(B(y\to x))}\eta(dy)\eta(dx) \\
&\overset{(b)}{=} \frac{\rho_A}{\rho}\int_{y\in D}\int_{x\in D} \frac{\rho}{\eta(D)} e^{-\frac{\rho}{\eta(D)}\eta(B(y\to x))}\eta(dx)\eta(dy) \\
&\overset{(c)}{=} \frac{\rho_A}{\rho}\eta(D)(1 - e^{-\rho}).
\end{aligned}$$

Here, Step (a) is from Equation (8), Step (b) is an application of Fubini's theorem, and Step (c) comes from Lemma 2. The formula for $V_B\big(\bar{\rho}_A, \bar{\rho}_B\big)$ results from interchanging the roles of Alice and Bob in this calculation. □

In an N-player Cox process Hotelling game $(D, \eta, \rho_1, \ldots, \rho_M)$, write $\bar{\rho}_i$ for the strategy of player i, $1 \le i \le N$, where she chooses f_i to be the constant $\frac{\rho_i}{\eta(D)}$. We then have the following analogue of Lemma 3.

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Lemma 4. *Consider an N-player Cox process Hotelling game* $(D, \eta, \rho_1, \dots, \rho_N)$. *Then, for all* $1 \le i \le N$, *we have*

$$V_i\big(\bar{\rho}_1, \dots, \bar{\rho}_N\big) = \frac{\rho_i}{\rho}\eta(D)(1 - e^{-\rho}),$$

where $\rho := \sum_{i=1}^N \rho_i$.

Proof. The proof is similar to that of Lemma 3, when one starts with Equation (10) instead of Equation (8) and makes the obvious modifications. □

3.6. Concavity of the value

To close this section, we record a convexity property that will play a key role in establishing the main claims of this paper.

Lemma 5. *Consider a two-player Cox process Hotelling game* $(D, \eta, \rho_A, \rho_B)$. *Fix* $f_B \in \mathcal{C}_B$ *and* $x \in D$. *Then the mapping*

$$f_A \mapsto \int_{y\in D} e^{-\int_{B(y\to x)}(f_A(u)+f_B(u))\eta(du)}\eta(dy)$$

is strictly convex on $\mathcal{C}_A$, *which, we recall, is a convex set.*

As a consequence, $f_A \mapsto V_B(f_A, f_B)$, *for fixed* $f_B \in \mathcal{C}_B$, *is strictly convex on* $\mathcal{C}_A$, *and hence* $f_A \mapsto V_A(f_A, f_B)$, *for fixed* $f_B \in \mathcal{C}_B$, *is strictly concave on* $\mathcal{C}_A$.

Proof. Let $f_A, f'_A \in \mathcal{C}_A$ and $\theta \in [0, 1]$. Then we have

$$\begin{aligned}
&\int_{y\in D} e^{-\int_{B(y\to x)}(\theta f_A(u)+(1-\theta)f'_A(u)+f_B(u))\eta(du)}\eta(dy)\\
&\le \theta \int_{y\in D} e^{-\int_{B(y\to x)}(f_A(u)+f_B(u))\eta(du)}\eta(dy)\\
&+ (1-\theta)\int_{y\in D} e^{-\int_{B(y\to x)}\left(f'_A(u)+f_B(u)\right)\eta(du)}\eta(dy),
\end{aligned}$$

from the convexity of the exponential function, and this inequality is strict if $\theta \notin \{0, 1\}$ and $f_A \ne f'_A$. This proves the first claim of the lemma.

From Equation (8), we have

$$V_B(f_A, f_B) = \int_{x\in D} f_B(x)\int_{y\in D} e^{-\int_{B(y\to x)}(f_A(u)+f_B(u))\eta(du)}\eta(dy)\eta(dx). \tag{15}$$

Since, for fixed $f_B \in \mathcal{C}_B$, the inner integral is strictly convex on $\mathcal{C}_A$ for each $x \in D$, the overall integral is also strictly convex on $\mathcal{C}_A$, which establishes the second claim of the lemma.

Finally, from the conservation rule in Equation (4), we have $V_A(f_A, f_B) = \eta(D)\big(1 - e^{-\rho}\big) - V_B\big(f_A, f_B\big)$, where $\rho := \rho_A + \rho_B$. From the second claim of the lemma, the third claim now follows immediately. □

The main claim of Lemma 5 is the third one about the strict concavity of the value function. This also holds for N-player Cox process Hotelling games. We state this claim formally in the following lemma.

Lemma 6. *Consider an N-player Cox process Hotelling game* $(D, \eta, \rho_1, \dots, \rho_N)$. *Recall that the sets of pure actions* $\mathcal{C}_i$ *for player i,* $1 \le i \le N$, *as defined in Equation* (9), *are convex subsets of* $\mathcal{M}(D)$.

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For each $1 \le i \le N$, the map $f_i \mapsto V_i(f_1, \ldots, f_N)$, for fixed $f_j \in \mathcal{C}_j$, $j \neq i$, is strictly concave on $\mathcal{C}_i$.

Proof. This is a direct consequence of the third claim of Lemma 5, once one observes that the value of player i, when she plays f_i in response to the actions f_j of the players $j \neq i$ in the N-player game, is that same as her value when she plays f_i in response to the pure action $\sum_{j \neq i} f_j$ of the opposing player in a two-player game where the intensity constraint of the opposing player is $\sum_{j \neq i} \rho_j$. □

4. Invariance under a transitive group action

In this section, we consider the case where D is compact and admits a transitive group of metric-preserving automorphisms, and where η is invariant under this group of automorphisms. This scenario covers several interesting concrete cases, such as the metric tori derived from lattice fundamental regions in $\mathbb{R}^d$, with the metric of $\mathbb{R}^d$ and Lebesgue measure; spheres of a fixed radius with the associated uniform measure, which is invariant under rigid rotations; etc.

4.1. Exploiting concavity of the value

The following lemma, which it suffices to state in the two-player case, is the key technical result driving the game-theoretic results in this section.

Lemma 7. *Consider a two-player Cox process Hotelling game $(D, \eta, \rho_A, \rho_B)$ where D is compact and admits a transitive group of metric-preserving automorphisms, and η is invariant under this group of automorphisms. Then, for any strategy $f_A \in \mathcal{C}_A$ of Alice, we have*

$$V_A\big(f_A, \bar{\rho}_B\big) \le V_A\big(\bar{\rho}_A, \bar{\rho}_B\big) = \frac{\rho_A}{\rho}\eta(D)\big(1 - e^{-\rho}\big),$$

where $\rho := \rho_A + \rho_B$. Further, we have the strict inequality

$$V_A\big(f_A, \bar{\rho}_B\big) < V_A\big(\bar{\rho}_A, \bar{\rho}_B\big),$$

except in the case $f_A = \bar{\rho}_A$.

Proof. From Lemma 3, we have $V_A\big(\bar{\rho}_A, \bar{\rho}_B\big) = \frac{\rho_A}{\rho}\eta(D)(1 - e^{-\rho})$, which is one of the claims of this lemma. From Lemma 5 for the choice $f_B = \bar{\rho}_B$, we conclude that $V_A\big(f_A, \bar{\rho}_B\big)$ is strictly concave on $\mathcal{C}_A$. We now use this to conclude that $V_A\big(f_A, \bar{\rho}_B\big)$ is uniquely maximized over $\mathcal{C}_A$ by the choice $f_A = \bar{\rho}_A$. Indeed, if $f_A \in \mathcal{C}_A$, $f_A \neq \bar{\rho}_A$, then we can find a translate f'_A of f_A such that $f_A \neq f'_A$. We have $V_A\big(f_A, \bar{\rho}_B\big) = V_A\big(f'_A, \bar{\rho}_B\big)$, because f_A and f'_A are translates of each other. However, since $f_A \neq f'_A$, we have

$$V_A\left(\frac{1}{2}\big(f_A + f'_A\big), \bar{\rho}_B\right) > \frac{1}{2}V_A\big(f_A, \bar{\rho}_B\big) + \frac{1}{2}V_A\big(f'_A, \bar{\rho}_B\big) = V_A\big(f_A, \bar{\rho}_B\big).$$

Hence, f_A cannot be the maximizer of $V_A\big(f_A, \bar{\rho}_B\big)$ over $\mathcal{C}_A$ unless it is translation-invariant, i.e. unless it equals $\bar{\rho}_A$. This concludes the proof of the lemma. □

4.2. Nash equilibrium structure for two-player games

We are now in a position to determine the Nash equilibrium structure of two-player Cox process Hotelling games $(D, \eta, \rho_A, \rho_B)$ in the context of this section.

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Theorem 1. *Consider a two-player Cox process Hotelling game* $(D, \eta, \rho_A, \rho_B)$ *between two players, Alice and Bob, where D is compact and admits a transitive group of metric-preserving automorphisms, and* η *is invariant under this group of automorphisms. Then* $(\bar{\rho}_A, \bar{\rho}_B)$ *is the unique Nash equilibrium for the game.*

Proof. We need to show that if $(f_A(M_A), f_B(M_B))$ is a Nash equilibrium for the game, then $f_A(M_A) = \bar{\rho}_A$ and $f_B(M_B) = \bar{\rho}_B$ with probability 1.

Suppose first that $f_B(M_B) = \bar{\rho}_B$ with probability 1. If $\mathbb{P}\big(f_A(M_A) \neq \bar{\rho}_A\big) > 0$, then by Lemma 7 we have

$$\mathbb{E}\big[V_A\big(f_A(M_A), f_B(M_B)\big)\big] = \mathbb{E}\big[V_A\big(f_A(M_A), \bar{\rho}_B\big)\big] < V_A\big(\bar{\rho}_A, \bar{\rho}_B\big).$$

On the other hand, since $f_A(M_A)$ is a best response by Alice to the strategy $f_B(M_B)$ of Bob, we have

$$\mathbb{E}\big[V_A\big(f_A(M_A), f_B(M_B)\big)\big] \geq \mathbb{E}\big[V_A\big(\bar{\rho}_A, f_B(M_B)\big)\big] = V_A\big(\bar{\rho}_A, \bar{\rho}_B\big).$$

This contradiction establishes that if $f_B(M_B) = \bar{\rho}_B$ with probability 1, then we must have $f_A(M_A) = \bar{\rho}_A$ with probability 1. A similar argument works to show that if $f_A(M_A) = \bar{\rho}_A$ with probability 1 then we must have $f_B(M_B) = \bar{\rho}_B$ with probability 1.

Thus, it remains to handle the case where we have both $\mathbb{P}\big(f_A(M_A) \neq \bar{\rho}_A\big) > 0$ and $\mathbb{P}\big(f_B(M_B) \neq \bar{\rho}_B\big) > 0$. In this case, since $f_A(M_A)$ is a best response by Alice to the strategy $f_B(M_B)$ of Bob, we have

$$\mathbb{E}\big[V_A\big(f_A(M_A), f_B(M_B)\big)\big] \geq \mathbb{E}\big[V_A\big(\bar{\rho}_A, f_B(M_B)\big)\big], \tag{16}$$

and, since $f_B(M_B)$ is a best response by Bob to the strategy $f_A(M_A)$ of Alice, we have

$$\mathbb{E}\big[V_B\big(f_A(M_A), f_B(M_B)\big)\big] \geq \mathbb{E}\big[V_B\big(f_A(M_A), \bar{\rho}_B\big)\big]. \tag{17}$$

Now, since $\mathbb{P}\big(f_B(M_B) \neq \bar{\rho}_B\big) > 0$, by Lemma 7 we have

$$\mathbb{E}\big[V_B\big(\bar{\rho}_A, f_B(M_B)\big)\big] < V_B\big(\bar{\rho}_A, \bar{\rho}_B\big),$$

so that, by (5), we have

$$\mathbb{E}\big[V_A\big(\bar{\rho}_A, f_B(M_B)\big)\big] > V_A\big(\bar{\rho}_A, \bar{\rho}_B\big).$$

Combining this with (16), we get

$$\mathbb{E}\big[V_A\big(f_A(M_A), f_B(M_B)\big)\big] > V_A\big(\bar{\rho}_A, \bar{\rho}_B\big).$$

Similar reasoning, based on (17), gives

$$\mathbb{E}\big[V_B\big(f_A(M_A), f_B(M_B)\big)\big] > V_B\big(\bar{\rho}_A, \bar{\rho}_B\big).$$

However, putting these inequalities together contradicts the conservation law in (5). This completes the proof of the theorem. □

4.3. Nash equilibrium structure for *N*-player games

Theorem 1 is actually a special case of a uniqueness theorem for Nash equilibria in the general N-player case. The proof in the N-player case also depends on a peculiar feature of Cox process Hotelling games, which is that a player faced with the strategies of the other players, i.e., their individual choices of likelihood functions with respect to the underlying measure η

which result in their individual intensities, receives the same value as she would in a two-player game where she is faced with a single player playing a likelihood with respect to the underlying measure η that results in an intensity equal to the sum of the intensities corresponding to the strategies of the other players. With this in mind, we turn now to the N-player case.

Theorem 2. *Consider an N-player Cox process Hotelling game* $(D, \eta, \rho_1, \ldots, \rho_N)$ *where D is compact and admits a transitive group of metric-preserving automorphisms, and* η *is invariant under this group of automorphisms. Let* $\bar{\rho}_j \in \mathcal{C}_j$ *denote the constant function* $\frac{\rho_j}{\eta(D)}$*, which results in the constant intensity* $\frac{\rho_j}{\eta(D)}\eta$ *for player j.*

Then $(\bar{\rho}_1, \ldots, \bar{\rho}_N)$ *is the unique Nash equilibrium for this game.*

Proof. We need to show that if $(f_1(M_1), \ldots, f_N(M_N))$ is a Nash equilibrium, then $\mathbb{P}(f_j(M_j) = \bar{\rho}_j) = 1$ for all $1 \le j \le N$. To do this, suppose first, after reindexing if needed, that we have $\mathbb{P}(f_1(M_1) \neq \bar{\rho}_1) > 0$ and

$$\mathbb{P}\left(\sum_{j=2}^{N} f_j(M_j) = \sum_{j=2}^{N} \bar{\rho}_j\right) = 1.$$

Then, because $f_1(M_1)$ is a best reaction of player 1 to the individually randomized strategies $(f_j(M_j), 2 \le j \le N)$ of the other players, we have

$$\mathbb{E}\big[V_1\big(f_1(M_1), f_2(M_2), \ldots, f_N(M_N)\big)\big] \ge \mathbb{E}\big[V_1\big(\bar{\rho}_1, f_2(M_2), \ldots, f_N(M_N)\big)\big]. \tag{18}$$

On the other hand, by Lemma 7, we have

$$\mathbb{E}\big[V_1\big(f_1(M_1), f_2(M_2), \ldots, f_N(M_N)\big)\big] < \mathbb{E}\big[V_1\big(\bar{\rho}_1, f_2(M_2), \ldots, f_N(M_N)\big)\big]. \tag{19}$$

To see this, observe that $V_1(f_1(M_1), f_2(M_2), \ldots, f_N(M_N))$ is the same as the value of player 1 in the two-player game in which she plays the randomized strategy $f_1(M_1)$ against a single opponent playing the strategy $\sum_{j=2}^{N} f_j(M_j)$, which we have assumed equals the constant $\sum_{j=2}^{N} \bar{\rho}_j$ with probability 1, and also $V_1(\bar{\rho}_1, f_2(M_2), \ldots, f_N(M_N))$ is the same as the value of player 1 in the two-player game in which she plays the constant strategy $\bar{\rho}_1$ against a single opponent playing the strategy $\sum_{j=2}^{N} f_j(M_j)$, which we have assumed equals the constant $\sum_{j=2}^{N} \bar{\rho}_j$ with probability 1, and so the scenario of Lemma 7 applies to allow us to compare these two values. Note that the inequalities (18) and (19) contradict each other. Thus, we can conclude that if $(f_1(M_1), \ldots, f_N(M_N))$ is a Nash equilibrium, then for every player $1 \le i \le N$ for which $\mathbb{P}(f_i(M_i) \neq \bar{\rho}_i) > 0$, we must also have $\mathbb{P}\Big(\sum_{j \neq i} f_j(M_j) \neq \sum_{j \neq i} \bar{\rho}_j\Big) > 0$.

Suppose now, after reindexing if necessary, that $\mathbb{P}(f_1(M_1) \neq \bar{\rho}_1) > 0$. We have established that we must also have

$$\mathbb{P}\left(\sum_{j=2}^{N} f_j(M_j) \neq \sum_{j=2}^{N} \bar{\rho}_j\right) > 0.$$

Since $f_1(M_1)$ is a best reaction of player 1 to the individually randomized strategies $(f_j(M_j), 2 \le j \le N)$ of the other players, the inequality in (18) holds. Because $\mathbb{P}\big(\sum_{j=2}^{N} f_j(M_j) \neq \sum_{j=2}^{N} \bar{\rho}_j\big) > 0$, from Lemma 7 we also have

$$\sum_{j=2}^{N} \mathbb{E}\big[V_j\big(\bar{\rho}_1, f_2(M_2), \ldots, f_N(M_N)\big)\big] < \sum_{j=2}^{N} V_j\big(\bar{\rho}_1, \bar{\rho}_2, \ldots, \bar{\rho}_N\big). \tag{20}$$

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To see this, note that the sum of the values of the other players $2 \le j \le N$, when player 1 plays the constant strategy $\bar{\rho}_1$, is the same as the value of a single player playing the strategy $\sum_{j=2}^N f_j(M_j)$ against player 1 playing the constant strategy $\bar{\rho}_1$ in a two-player game between player 1 and this single player, where the overall intensity constraint of player 1 continues to be ρ_1 and that of this single player is $\sum_{j=2}^N \rho_j$. Since $\mathbb{P}\Big(\sum_{j=2}^N f_j(M_j) \neq \sum_{j=2}^N \bar{\rho}_j\Big) > 0$, Lemma 7 allows us to conclude that this value is strictly less than the value this single player would get by playing the constant strategy $\sum_{j=2}^N \bar{\rho}_j$ against player 1, who is playing the constant strategy $\bar{\rho}_1$. But this is equal to the sum of the values of the individual players $2 \le j \le N$ in the given N-player game when they individually play the constant strategies $(\bar{\rho}_j, 2 \le j \le N)$ respectively, and player 1 is playing the constant strategy $\bar{\rho}_1$.

Now, in view of the conservation law in Equation (12), we can conclude from the inequality (20) that

$$\mathbb{E}\big[V_1\big(\rho_1, f_2(M_2), \ldots, f_N(M_N)\big)\big] > V_1\big(\bar{\rho}_1, \bar{\rho}_2, \ldots, \bar{\rho}_N\big).$$

Thus, so far, what we have concluded is that if $(f_1(M_1), \ldots, f_N(M_N))$ is a Nash equilibrium, then, for every $1 \le i \le N$ such that $\mathbb{P}\big(f_i(M_i) \neq \bar{\rho}_i\big) > 0$, we must have

$$\mathbb{E}\big[V_i\big(\bar{\rho}_i, \big(f_j(M_j), j \neq i\big)\big)\big] > V_i\big(\bar{\rho}_1, \bar{\rho}_2, \ldots, \bar{\rho}_N\big). \tag{21}$$

Finally, suppose that $\mathbb{P}(f_i(M_i) = \bar{\rho}_i) = 1$ for some $1 \le i \le N$. Then we must have

$$\mathbb{E}\big[V_i\big(\bar{\rho}_i, \big(f_j(M_j), j \neq i\big)\big)\big] \ge V_i\big(\bar{\rho}_1, \bar{\rho}_2, \ldots, \bar{\rho}_N\big). \tag{22}$$

To see this, note that the quantity $\sum_{k \neq i} \mathbb{E}\big[V_k\big(\bar{\rho}_i, \big(f_j(M_j), j \neq i\big)\big)\big]$ is the same as the value of a single player who plays the strategy $\sum_{j \neq i} f_j(M_j)$ in the two-player game against player i playing the constant strategy $\bar{\rho}_i$. By Lemma 7, this is no bigger that the value this single player would get if she played the constant strategy $\sum_{j \neq i} \bar{\rho}_j$, but this value is the same as $\sum_{j \neq i} V_j\big(\bar{\rho}_1, \bar{\rho}_2, \ldots, \bar{\rho}_N\big)$. To deduce (22) from this logic, we apply the conservation rule in Equation (12).

The inequalities in (21) and (22) together result in a contradiction of Equation (12) unless we have $\mathbb{P}(f_i(M_i) = \bar{\rho}_i) = 1$ for all $1 \le i \le N$. This concludes the proof of the theorem. □

5. General results

In this section we discuss the structure of Nash equilibria in a general N-player Cox process Hotelling game $(D, \eta, \rho_1, \ldots, \rho_N)$ without the group-theoretic assumptions of Section 4.

5.1. Structure of the Nash equilibria, assuming one exists

Leaving aside for the moment the question of existence of Nash equilibria, the strict concavity of the value function of a player for fixed choices of the pure actions of the other players, which was established in Lemmas 5 and 6, ensures that any Nash equilibrium that exists must be pure. We discuss this first for the two-player case.

Theorem 3. *Consider a two-player Cox process Hotelling game $(D, \eta, \rho_A, \rho_B)$ between the players Alice and Bob. Suppose $\big(f_A(M_A), f_B(M_B)\big) \in \mathcal{C}_A \times \mathcal{C}_B$ is a Nash equilibrium of the game, as defined in Definition 1. Then there exist $g_A \in \mathcal{C}_A$ and $g_B \in \mathcal{C}_B$ such that*

$$\mathbb{P}((f_A(M_A), f_B(M_B) = (g_A, g_B)) = 1.$$

Proof. We write

$$
\begin{aligned}
\mathbb{E}\big[V_A\big(f_A(M_A), f_B(M_B)\big)\big] &= \mathbb{E}\big[\mathbb{E}\big[V_A(f_A(M_A), f_B(M_B))|M_B\big]\big] \\
&\overset{(a)}{\le} \mathbb{E}\big[\mathbb{E}\big[V_A\big(\mathbb{E}\big[f_A(M_A)|M_B\big], f_B(M_B)\big)|M_B\big]\big] \\
&\overset{(b)}{=} \mathbb{E}\big[\mathbb{E}\big[V_A\big(\mathbb{E}\big[f_A(M_A)\big], f_B(M_B)\big)|M_B\big]\big] \\
&= \mathbb{E}\big[V_A\big(\mathbb{E}\big[f_A(M_A)\big], f_B(M_B)\big].
\end{aligned}
$$

Here, Step (a) comes from the concavity property of the value function established in Lemma 5, and Step (b) comes from independence of M_A and M_B. Since $(f_A(M_A), f_B(M_B))$ is a Nash equilibrium pair, we see from Definition 1 that the inequality in the chain of equations above must be an equality. But then, by the strict concavity property of the value function established in Lemma 5, it follows that $\mathbb{P}\big(f_A(M_A) = \mathbb{E}\big[f_A(M_A)\big]\big) = 1$. A similar argument interchanging the roles of Alice and Bob completes the proof, with g_A being $\mathbb{E}\big[f_A(M_A)\big]$ and g_B being $\mathbb{E}\big[f_B(M_B)\big]$ in the notation of the statement of the lemma. □

The analogue of Theorem 3 also holds in the N-player case.

Theorem 4. *Consider an N-player Cox process Hotelling game* $(D, \eta, \rho_1, \ldots, \rho_N)$. *Suppose* $(f_1(M_1), \ldots, f_N(M_N)) \in \mathcal{C}_1 \times \ldots \times \mathcal{C}_N$ *is a Nash equilibrium of the game, as defined in Definition 2. Then there exist* $g_i \in \mathcal{C}_i$, $1 \le i \le N$, *such that*

$$\mathbb{P}((f_1(M_1), \ldots, f_N(M_N) = (g_1, \ldots, g_N)) = 1.$$

Proof. The proof is similar to that of Theorem 3, with the obvious modifications. The key observation, if one wants to base the proof on Lemma 5, is that the value of player i when she plays the randomized strategy $f_i(M_i)$ in response to the randomized strategies $f_j\big(M_j\big)$, $j \ne i$ of the other players in the N-player game is the same as her value when she plays the randomized strategy $f_i(M_i)$ in response to the randomized strategy $\sum_{j \ne i} f_j\big(M_j\big)$ of the opposing player in a two-player game where the opposing player has the intensity constraint $\sum_{j \ne i} \rho_j$. This observation then leads to the conclusion that $P\big(f_i(M_i) = E\big[f_i(M_i)\big]\big) = 1$, by following the lines of the proof of Theorem 3, and since this holds for all $1 \le i \le N$, this completes the proof.

Alternatively, one can write out the obvious analogue of the sequence of equations in the proof of Theorem 3 by conditioning on $\big(M_j, j \ne i\big)$, for each $1 \le i \le N$, and base the proof on the strict concavity property for the N-player game proved in Lemma 6. □

In fact, the strict concavity property of the value function of a player, as established in Lemmas 5 and 6, together with the constant-sum nature of the game, ensures that if a Nash equilibrium exists, then it is not only pure, but also unique. We state this first in the two-player case.

Theorem 5. *Consider a two-player Cox process Hotelling game* $(D, \eta, \rho_A, \rho_B)$ *between the players Alice and Bob. Suppose* $(f_A, f_B) \in \mathcal{C}_A \times \mathcal{C}_B$ *and* $(g_A, g_B) \in \mathcal{C}_A \times \mathcal{C}_B$ *are pure Nash equilibria of the game. Then* $f_A = g_A$ *and* $f_B = g_B$.

Proof. We will first show that

$$V_A(g_A, g_B) = V_A\big(f_A, f_B\big). \tag{23}$$

The procedure to do this is standard in the theory of constant-sum games, but is reproduced here for convenience. To verify Equation (23), note that $V_A(g_A, f_B) \le V_A(f_A, f_B)$, because (f_A, f_B) is

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a Nash equilibrium; see (6). But then, by the constant-sum nature of the game (see (5)), we have $V_B(g_A, f_B) \geq V_B(f_A, f_B)$. However, since (g_A, g_B) is a Nash equilibrium, we have $V_B(g_A, g_B) \geq V_B(g_A, f_B)$, so we conclude that $V_B(g_A, g_B) \geq V_B(f_A, f_B)$. Interchanging the roles of (f_A, f_B) and (g_A, g_B) then gives $V_B(f_A, f_B) \geq V_B(g_A, g_B)$, which establishes (23).

We next show that

$$V_A(g_A, f_B) = V_A(f_A, f_B). \tag{24}$$

We have $V_B(g_A, g_B) \geq V_B(g_A, f_B)$ because (g_A, g_B) is a Nash equilibrium. Hence, by the constant-sum nature of the game (see Equation (5)), we have $V_A(g_A, g_B) \leq V_A(g_A, f_B)$. In view of Equation (23), this gives $V_A(f_A, f_B) \leq V_A(g_A, f_B)$, but since (f_A, f_B) is a Nash equilibrium, this can only hold with equality; i.e. Equation (24) holds.

Now, since (f_A, f_B) is a Nash equilibrium, we know that f_A is a best response of Alice to the pure strategy f_B of Bob. Thus, Equation (24) tells us that g_A is also a best response of Alice in response to f_B. The strict concavity property of the value function of Alice, proved in Lemma 5, shows that this is possible only if $g_A = f_A$. Interchanging the roles of Alice and Bob, we conclude that we must also have $g_B = f_B$. This concludes the proof of the lemma. □

Assume Nash equilibria exist. Beyond there being a unique pure Nash equilibrium, the special structure of the game allows us to say more about the form of this unique Nash equilibrium. We state the result first in the two-player case.

Theorem 6. *Consider a two-player Cox process Hotelling game $(D, \eta, \rho_A, \rho_B)$ between the players Alice and Bob. Suppose $(f_A, f_B) \in \mathcal{C}_A \times \mathcal{C}_B$ is a pure Nash equilibrium of the game. Then $f_A = \frac{\rho_A}{\rho} f$ and $f_B = \frac{\rho_B}{\rho} f$, where $f := f_A + f_B$.*

Proof. Suppose Alice reacts to f_B by playing $\frac{\rho_A}{\rho_B} f_B$. Let $\rho := \rho_A + \rho_B$. We have

$$\begin{aligned}
V_A\left(\frac{\rho_A}{\rho_B} f_B, f_B\right) &\overset{(a)}{=} \frac{\rho_A}{\rho_B} \int_{x\in D} f_B(x) \int_{y\in D} e^{-\int_{B(y\to x)} \frac{\rho}{\rho_B} f_B(u)\eta(du)} \eta(dy)\eta(dx) \\
&= \frac{\rho_A}{\rho} \int_{x\in D} \frac{\rho}{\rho_B} f_B(x) \int_{y\in D} e^{-\int_{B(y\to x)} \frac{\rho}{\rho_B} f_B(u)\eta(du)} \eta(dy)\eta(dx) \\
&\overset{(b)}{=} \frac{\rho_A}{\rho} \eta(D)\left(1 - e^{-\rho}\right),
\end{aligned}$$

where Step (a) is from Equation (8) and Step (b) is from Equation (14). Since (f_A, f_B) is a Nash equilibrium, it follows that

$$V_A\left(f_A, f_B\right) \geq \frac{\rho_A}{\rho} \eta(D)\left(1 - e^{-\rho}\right).$$

Interchanging the roles of Alice and Bob gives

$$V_B\left(f_A, f_B\right) \geq \frac{\rho_B}{\rho} \eta(D)\left(1 - e^{-\rho}\right).$$

In view of the constant-sum property in Equation (5), it then follows that we have equality in both these inequalities, i.e.

$$V_A\left(f_A, f_B\right) = \frac{\rho_A}{\rho} \eta(D)\left(1 - e^{-\rho}\right),$$

and

$$V_B\left(f_A, f_B\right) = \frac{\rho_B}{\rho} \eta(D)\left(1 - e^{-\rho}\right).$$

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But then we have $V_A(f_A, f_B) = V_A(\frac{\rho_A}{\rho_B} f_B, f_B)$, so the strict concavity property of the value of Alice proved in Lemma 5, together with the assumption that (f_A, f_B) is a Nash equilibrium, implies that $f_A = \frac{\rho_A}{\rho_B} f_B$, which suffices to establish the claim. (One can also go through this argument again interchanging the roles of Alice and Bob, but it is not necessary to do so.) □

The analogue of Theorem 5 and the stronger statement in Theorem 6 also hold in the N-player case. We state these claims together.

Theorem 7. *Consider an N-player Cox process Hotelling game* $(D, \eta, \rho_1, \dots, \rho_N)$. *Suppose* $(f_1, \dots, f_N) \in \mathcal{C}_1 \times \dots \times \mathcal{C}_N$ *and* $(g_1, \dots, g_N) \in \mathcal{C}_1 \times \dots \times \mathcal{C}_N$ *are pure Nash equilibria of the game. Then* $f_i = g_i$ *for* $1 \le i \le N$. *Indeed, if* $(f_1, \dots, f_N) \in \mathcal{C}_1 \times \dots \times \mathcal{C}_N$ *is a pure Nash equilibrium of the game, then* $f_i = \rho_i f$ *for* $1 \le i \le N$, *where* $f := \sum_{i=1}^N f_i$.

Proof. We prove the stronger statement. The proof is similar to that of Theorem 6. Suppose player i reacts to the strategy profile $(f_j, j \neq i)$ by playing

$$\frac{\rho_i}{\sum_{j\neq i} \rho_j} \sum_{j\neq i} f_j.$$

Let $\rho := \sum_{i=1}^N \rho_i$. We have

$$\begin{aligned}
&V_i\left(\frac{\rho_i}{\sum_{j\neq i} \rho_j} \sum_{j\neq i} f_j, (f_j, j\neq i)\right) \\
&\overset{(a)}{=} \frac{\rho_i}{\sum_{j\neq i} \rho_j} \int_{x\in D} \sum_{j\neq i} f_j(x) \int_{y\in D} e^{-\int_{u\in B(y\to x)} \frac{\rho}{\sum_{j\neq i}\rho_j} \sum_{j\neq i} f_j(u)\eta(du)} \eta(dy)\eta(dx) \\
&= \frac{\rho_i}{\rho} \int_{x\in D} \frac{\rho}{\sum_{j\neq i} \rho_j} \sum_{j\neq i} f_j(x) \int_{y\in D} e^{-\int_{u\in B(y\to x)} \frac{\rho}{\sum_{j\neq i}\rho_j} \sum_{j\neq i} f_j(u)\eta(du)} \eta(dy)\eta(dx) \\
&\overset{(b)}{=} \frac{\rho_i}{\rho} \eta(D)\left(1 - e^{-\rho}\right),
\end{aligned}$$

where Step (a) is from Equation (10) and Step (b) is from Equation (14). Since $(f_1, \dots, f_N)$ is a Nash equilibrium, it follows that

$$V_i(f_1, \dots, f_N) \ge \frac{\rho_i}{\rho} \eta(D)\left(1 - e^{-\rho}\right).$$

Since this holds for all $1 \le i \le N$, it follows from Equation (11) that this inequality must hold with equality for all $1 \le i \le N$, i.e. that we have

$$V_i(f_1, \dots, f_N) = \frac{\rho_i}{\rho} \eta(D)\left(1 - e^{-\rho}\right)$$

for all $1 \le i \le N$. But then, for each $1 \le i \le N$ we have

$$V_i\left(f_i = \frac{\rho_i}{\sum_{j\neq i} \rho_j} \sum_{j\neq i} f_j, (f_j, j\neq i)\right) = V_i(f_1, \dots, f_N),$$

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so from the strict concavity property of the value function of player i proved in Lemma 6 and the fact that f_i is a best response of player i to the strategy profile $(f_j, j \neq i)$ of the other players, we must have

$$f_i = \frac{\rho_i}{\sum_{j\neq i} \rho_j} \sum_{j\neq i} f_j,$$

which is the same as $f_i = \frac{\rho_i}{\rho} f$, where $f := \sum_{i=1}^N f_i$. This completes the proof of the theorem. □

The properties of Nash equilibria of Cox process Hotelling games established so far, assuming a Nash equilibrium exists, can be gathered into the following statement.

Theorem 8. *Consider an N-player Cox process Hotelling game* $(D, \eta, \rho_1, \ldots, \rho_N)$. *Let* $\rho := \sum_{i=1}^N \rho_i$. *The game admits a Nash equilibrium if and only if there is a function* $f \in \mathcal{C}(\rho)$ *such that*

$$\int_{x\in D} f(x) \int_{y\in D} e^{-\int_{u\in B(y\to x)} f(u)\eta(du)} \eta(dy)\eta(dx) \\ \geq \int_{x\in D} g(x) \int_{y\in D} e^{-\int_{u\in B(y\to x)} f(u)\eta(du)} \eta(dy)\eta(dx) \tag{25}$$

for all $g \in \mathcal{C}(\rho)$. *If such a function exists, the game has a unique Nash equilibrium, given by the pure strategy profile* $\left(\frac{\rho_1}{\rho} f, \ldots, \frac{\rho_N}{\rho} f\right)$.

Proof. Suppose first that the game admits a Nash equilibrium. By Theorem 7, we know that this Nash equilibrium is unique and is a pure Nash equilibrium of the form $\left(\frac{\rho_1}{\rho} f, \ldots, \frac{\rho_N}{\rho} f\right)$, for some $f \in \mathcal{C}(\rho)$. Since $\frac{\rho_i}{\rho} f$ is a best response of player i to the strategy profile $\left(f_j = \frac{\rho_j}{\rho} f, j \neq i\right)$ of the other players, by the definition of Nash equilibrium (see (13)), we must have

$$\int_{x\in D} \frac{\rho_i}{\rho} f \int_{y\in D} e^{-\int_{u\in B(y\to x)} f(u)\eta(du)} \eta(dy)\eta(dx) \\ \geq \int_{x\in D} g_i(x) \int_{y\in D} e^{-\int_{u\in B(y\to x)} f(u)\eta(du)} \eta(dy)\eta(dx)$$

for all $g_i \in \mathcal{C}(\rho_i)$, which is the same as the condition in (25).

Conversely, suppose the condition in (25) holds for some function $f \in \mathcal{C}(\rho)$. Then, by the definition of Nash equilibrium in (13), we see that the strategy profile $\left(\frac{\rho_1}{\rho} f, \ldots, \frac{\rho_N}{\rho} f\right)$ is a Nash equilibrium of the game. The existence of such a Nash equilibrium then guarantees, by Theorem 7, that it is the unique Nash equilibrium of the game. □

5.2. Relationship with ordinal potential games

Recall that an n-player game, with player i having action set $\mathcal{Y}_i$, is called an *ordinal potential game* [23] if there is a function $P\colon \prod_{i=1}^n \mathcal{Y}_i \to \mathbb{R}$ such that, for all $1 \leq i \leq n$, $y_i, z_i \in Y_i$, and $\left(y_j, j \neq i\right) \in \prod_{j\neq i} Y_j$, we have $V_i\left(y_i, \left(y_j, j \neq i\right)\right) > V_i\left(z_i, \left(y_j, j \neq i\right)\right)$ if and only if $P\left(y_i, \left(y_j, j \neq i\right)\right) > P\left(z_i, \left(y_j, j \neq i\right)\right)$.

We have established that when a Cox process Hotelling game admits a Nash equilibrium it admits a unique pure Nash equilibrium. Since ordinal potential games are a well-known class of games that admit pure-strategy Nash equilibria, this leads naturally to the question of whether Cox process Hotelling games are ordinal potential games. However, it is possible to argue that in general this is not true.

To see this, consider the two-player Cox process Hotelling game between Alice and Bob on D, taken to be a circle of radius 1 centered at the origin in $\mathbb{R}^2$, the base measure η being the Lebesgue measure on D. Thus $\eta(D) = 2\pi$. Suppose that $\rho_A = \rho_B = \frac{\rho}{2}$, where ρ should be thought of as being sufficiently large in a sense that we will make precise shortly. Let $\epsilon > 0$ be sufficiently small $\big($to be precise, we require that $\epsilon < \frac{2\pi}{9}\big)$.

We consider four pure strategies, i.e. elements of $\mathcal{C}\big(\frac{\rho}{2}\big)$, denoted by σ^R, β^R, σ^L, and β^L respectively, defined as follows:

- The strategy σ^R is constant over the arc of the circle of length ϵ centered at $(1, 0)$ and is zero elsewhere.
- Consider the arc of the circle of length $\frac{\epsilon}{2}$ centered at $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and the arc of the circle of length $\frac{\epsilon}{2}$ centered at $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. The strategy β^R is constant over the union of these two arcs and is zero elsewhere.
- The strategy σ^L is constant over the arc of the circle of length ϵ centered at $(-1, 0)$ and is zero elsewhere. It can be considered to be the 'left' version of σ^R, which is the 'right' version of it.
- The strategy β^L is the 'left' version of β^R, which is the 'right' version of it. Namely, β^L is uniform over the union of the two arcs of the circle of length $\frac{\epsilon}{2}$ centered at $\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\left(\frac{-\sqrt{3}}{2}, -\frac{1}{2}\right)$ respectively, and is zero elsewhere.

To be consistent with the convention in the rest of the document, we will use a subscript to indicate the identity of the player playing the strategy. Thus, for instance, the strategy pair $\big(\sigma_A^L, \sigma_B^R\big)$ indicates that Alice is playing the strategy σ^L and Bob is playing the strategy σ^R.

It is straightforward to check that

$$\lim_{\rho\to\infty} V_A\big(\sigma_A^R, \beta_B^R\big) = \frac{\pi}{6} + \frac{\epsilon}{4}$$

and

$$\lim_{\rho\to\infty} V_A\big(\sigma_A^L, \beta_B^R\big) = \frac{5\pi}{6} + \frac{\epsilon}{4}.$$

We will use these facts and their obvious consequences in the following argument.

Suppose there were an ordinal potential function $P : \mathcal{C}\big(\frac{\rho}{2}\big) \times \mathcal{C}\big(\frac{\rho}{2}\big) \to \mathbb{R}$ for this two-player Cox process Hotelling game.

For sufficiently large ρ, we have $V_A\big(\sigma_A^R, \beta_B^R\big) < V_A\big(\sigma_A^L, \beta_B^R\big)$, and so $P\big(\sigma^R, \beta^R\big) < P\big(\sigma^L, \beta^R\big)$.

For sufficiently large ρ we also have $V_B\big(\sigma_A^L, \beta_B^R\big) < V_B\big(\sigma_A^L, \beta_B^L\big)$, and so $P\big(\sigma^L, \beta^R\big) < P\big(\sigma^L, \beta^L\big)$.

But for sufficiently large ρ we also have $V_A\big(\sigma_A^L, \beta_B^L\big) < V_A\big(\sigma_A^R, \beta_B^L\big)$, and so $P\big(\sigma^L, \beta^L\big) < P\big(\sigma^R, \beta^L\big)$.

Finally, for sufficiently large ρ we also have $V_B\big(\sigma_A^R, \beta_B^L\big) < V_B\big(\sigma_A^R, \beta_B^R\big)$, and so $P\big(\sigma^R, \beta^L\big) < P\big(\sigma^R, \beta^R\big)$.

Putting these together leads to a contradiction. Hence this two-player Cox process Hotelling game is not an ordinal potential game.

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5.3. Nash equilibria may not exist

Consider a two-player Cox process Hotelling game $(D, \eta, \rho_A, \rho_B)$. The results of Theorems 3, 5, and 6 are consistent with that of Theorem 1 in the case where D is compact and admits a transitive group of metric-preserving automorphisms, and where η is an invariant measure under the action of this group. This might lead one to expect that the constant intensity pair $\left(\bar{\rho}_A, \bar{\rho}_B\right)$ is a Nash equilibrium for a general two-player Cox process Hotelling game $(D, \eta, \rho_A, \rho_B)$. The following simple example shows that this is not the case.

Example 2. Let D be the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ of the real line, with η being the Lebesgue measure restricted to D. Then, for every two-player Cox process Hotelling game $(D, \eta, \rho_A, \rho_B)$, the constant intensity pair $\left(\bar{\rho}_A, \bar{\rho}_B\right)$ is not a Nash equilibrium of the game.

To see this, first note that $\eta(D) = 1$, so $\bar{\rho}_A = \rho_A$ and $\bar{\rho}_B = \rho_B$. Recall that $\rho := \rho_A + \rho_B$. From Theorem 8, it suffices to find $g \in \mathcal{C}(\rho)$, i.e., $g : \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R}_+$ with $\int_{u=-\frac{1}{2}}^{\frac{1}{2}} g(u)du = \rho$, such that

$$\int_{x=-\frac{1}{2}}^{\frac{1}{2}} g(x) \int_{y=-\frac{1}{2}}^{\frac{1}{2}} e^{-\rho\eta(B(y\to x))} dydx > \int_{x=-\frac{1}{2}}^{\frac{1}{2}} \rho \int_{y=-\frac{1}{2}}^{\frac{1}{2}} e^{-\rho\eta(B(y\to x))} dydx, \tag{26}$$

where we have written dx and dy in the integrals, instead of $\eta(dx)$ and $\eta(dy)$, respectively, because η is the Lebesgue measure. By Equation (14), the integral on the right-hand side of (26) is $1 - e^{-\rho}$. For the integral on the left-hand side of (26), let us first replace $g(x)\, dx$ by $\rho\delta_0(dx)$, where δ_0 is the measure on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ giving mass 1 to the point at the origin; i.e. let us consider the integral

$$\begin{aligned}
\rho \int_{x} = -\frac{1}{2}^{\frac{1}{2}} \int_{y=-\frac{1}{2}}^{\frac{1}{2}} & e^{-\rho\eta(B(y\to x))} dy\delta_0(dx) \\
&= \rho \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\rho\eta(B(y\to 0))} dy \\
&= 2\rho \int_0^{\frac{1}{4}} e^{-2\rho y} dy + 2\rho \int_{\frac{1}{4}}^{\frac{1}{2}} e^{-\frac{\rho}{2}} dy \\
&= 1 - e^{-\frac{\rho}{2}} + \frac{\rho}{2} e^{-\frac{\rho}{2}}.
\end{aligned}$$

For all $\rho > 0$ this integral is strictly bigger than $1 - e^{-\rho}$. It follows that we can find $g \in \mathcal{C}(\rho)$ to get the strict inequality in (26), as desired.

In the two-player game considered in Example 2, one can in fact conclude that there is no Nash equilibrium when the sum of the intensities of the two players is sufficiently small. We state this formally.

Theorem 9. *Let D be the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ of the real line, with η being the Lebesgue measure restricted to D. Then the two-player Cox process Hotelling game $(D, \eta, \rho_A, \rho_B)$ does not admit a Nash equilibrium when $\rho := \rho_A + \rho_B < \log_e 4$.*

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Proof. Theorem 8 tells us that, to prove that a Nash equilibrium does not exist for this two-player game, it suffices to show that for every $f \in \mathcal{C}(\rho)$ there is some $g \in \mathcal{C}(\rho)$ such that

$$\int_{x=-\frac{1}{2}}^{\frac{1}{2}} g(x) \int_{y=-\frac{1}{2}}^{\frac{1}{2}} e^{-\int_{u\in B(y\to x)} f(u)du} dydx > \int_{x=-\frac{1}{2}}^{\frac{1}{2}} f(x) \int_{y=-\frac{1}{2}}^{\frac{1}{2}} e^{-\int_{u\in B(y\to x)} f(u)du} dydx. \tag{27}$$

Let $f \in \mathcal{C}(\rho)$. For $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, define $\psi_x : \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R}_+$ via

$$\psi_x(y) := e^{-\int_{u\in B(y\to x)} f(u)du}.$$

Suppose $\int_{y=-\frac{1}{2}}^{\frac{1}{2}} \psi_x(y)dy$ is not constant in x. Let $x^* \in \arg\max_x \int_{y=-\frac{1}{2}}^{\frac{1}{2}} \psi_x(y)dy$, which exists because $\int_{y=-\frac{1}{2}}^{\frac{1}{2}} \psi_x(y)dy$ is continuous in x over $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Since

$$\int_{y=-\frac{1}{2}}^{\frac{1}{2}} \psi_x(y)dy$$

is not constant in x over $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, and since $\int_{x=-\frac{1}{2}}^{\frac{1}{2}} f(x)dx = \rho$, this implies that

$$\rho \int_{y=-\frac{1}{2}}^{\frac{1}{2}} e^{-\int_{u\in B(y\to x^*)} f(u)du} dy > \int_{x=-\frac{1}{2}}^{\frac{1}{2}} f(x) \int_{y=-\frac{1}{2}}^{\frac{1}{2}} e^{-\int_{u\in B(y\to x)} f(u)du} dydx,$$

from which we can conclude the existence of $g \in \mathcal{C}(\rho)$ satisfying the strict inequality in (27).

From Theorem 8, we also know that if a Nash equilibrium exists it must be pure and of the form $\left(\frac{\rho_A}{\rho} f, \frac{\rho_B}{\rho} f\right)$ for some $f \in \mathcal{C}(\rho)$. We claim that it must further be the case that $f(u) = f(-u)$ for all $u \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, i.e. that f is an *even* function. This is because, by symmetry, if $\left(\frac{\rho_A}{\rho} f, \frac{\rho_B}{\rho} f\right)$ is a Nash equilibrium, then so is $\left(\frac{\rho_A}{\rho} \tilde{f}, \frac{\rho_B}{\rho} \tilde{f}\right)$, where $\tilde{f}(u) := f(-u)$ (so we also have $\tilde{f} \in \mathcal{C}(\rho)$), and then, because the Nash equilibrium is unique, it must be the case that $\tilde{f} = f$.

Thus it suffices to show that when $\rho < \log_e 4$ it is impossible to find an even function $f \in \mathcal{C}(\rho)$ such that $\int_{y=-\frac{1}{2}}^{\frac{1}{2}} \psi_x(y)dy$ is constant in x. We will do this by establishing that for every even function $f \in \mathcal{C}(\rho)$ we have

$$\int_{y=-\frac{1}{2}}^{\frac{1}{2}} \psi_{-\frac{1}{2}}(y)dy < \int_{y=-\frac{1}{2}}^{\frac{1}{2}} \psi_0(y)dy. \tag{28}$$

Observe that $\psi_0(y)$ is an even function of $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Further, we have $\psi_0(y) \geq e^{-\frac{\rho}{2}}$ for all $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, and we have $\psi_0(y) = e^{-\frac{\rho}{2}}$ for $-\frac{1}{2} \leq y \leq -\frac{1}{4}$.

Observe also that $\psi_{-\frac{1}{2}}(y)$ is nonincreasing over $y \in [-\frac{1}{2}, \frac{1}{2}]$, with $\psi_{-\frac{1}{2}}\left(-\frac{1}{2}\right) = 1$, $\psi_{-\frac{1}{2}}\left(-\frac{1}{4}\right) = e^{-\frac{\rho}{2}}$, and $\psi_{-\frac{1}{2}}(y) = e^{-\rho}$ for $0 \leq y \leq \frac{1}{2}$.

From these two sets of observations, we have $\psi_{-\frac{1}{2}}(y) \geq \psi_0(y)$ for $y \in \left[-\frac{1}{2}, -\frac{1}{4}\right]$ and $\psi_{-\frac{1}{2}}(y) \leq \psi_0(y)$ for $y \in \left[-\frac{1}{4}, \frac{1}{2}\right]$. Further, we have

$$\int_{y=-\frac{1}{2}}^{-\frac{1}{4}} \left(\psi_{-\frac{1}{2}}(y) - \psi_0(y)\right) dy \leq \frac{1}{4}\left(1 - e^{-\frac{\rho}{2}}\right)$$

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and

$$\int_{y=0}^{\frac{1}{2}} \Big(\psi_0(y) - \psi_{-\frac{1}{2}}(y)\Big)\, dy \geq \frac{1}{2}\Big(e^{-\frac{\rho}{2}} - e^{-\rho}\Big),$$

while we also have

$$\int_{y=-\frac{1}{4}}^{0} \Big(\psi_0(y) - \psi_{-\frac{1}{2}}(y)\Big)\, dy \geq 0.$$

From this we conclude that

$$\int_{y=-\frac{1}{2}}^{\frac{1}{2}} \Big(\psi_0(y) - \psi_{-\frac{1}{2}}(y)\Big)\, dy \geq \frac{1}{2}\Big(e^{-\frac{\rho}{2}} - e^{-\rho}\Big) - \frac{1}{4}\Big(1 - e^{-\frac{\rho}{2}}\Big) > 0$$

if $0 < \rho < \log_e 4$, which establishes the strict inequality in (28) and completes the proof. A more careful analysis will increase the range of ρ for which one can prove that the game does not admit a Nash equilibrium. □

5.4. Restricted Cox process Hotelling games

A more insightful characterization than the one in Theorem 8 of when Nash equilibria exist in Cox process Hotelling games remains an interesting open problem. The main technical difficulty in proving the existence of Nash equilibria in such games is that the action spaces of the individual players, which are of the form $\mathcal{C}(\rho_i)$, where $\rho_i > 0$ is the intensity budget of player i, are not compact when endowed with the topology of weak convergence. Indeed, we have seen in Section 5.3 that Nash equilibria may not exist in some cases.

To better understand the question of when Nash equilibria exist in such games, we therefore propose to study a family of *restricted Cox process Hotelling games*, where the action space of each individual player is now a compact set. The restrictions can be imposed in such a way that a unique Nash equilibrium will be guaranteed to exist in each such restricted game, it will be pure, and, if the restrictions imposed on the individual players are proportional in a sense made precise below, this unique pure Nash equilibrium will be of proportional form. Furthermore, by varying the restriction, we can vary the compact action space of each player in such a way that the union over all such choices is the full space of allowed actions for that player, namely $\mathcal{C}(\rho_i)$ for player i having a total intensity budget of ρ_i. If a Nash equilibrium did exist for the original Cox process Hotelling game, then this guarantees that it would be discovered as the Nash equilibrium for some profile of restrictions on the action spaces of the individual players, i.e., in one of the restricted Cox process Hotelling games that we consider. Pursuing this direction, which we leave as a topic for future research, may give more insight into what the characterization in Theorem 8 is actually saying.

To carry out this program, we first exhibit, for each $\rho > 0$, a family of compact subsets of $\mathcal{C}(\rho)$ whose union is $\mathcal{C}(\rho)$. Recall that $\mathcal{C}(\rho)$ is a subset of $L^1(\eta)$, where $f \in \mathcal{C}(\rho)$ is identified with the measure $f\eta$ on D, and $\mathcal{C}(\rho)$ endowed with the topology of weak convergence of measures which it inherits as a subset of $\mathcal{M}(D)$. We will now consider $L^1(\eta)$ with its weak topology defined by considering it to be a Banach space with Banach dual $L^\infty(\eta)$; see [29] or [9, p. 44]. To avoid confusion, recall that the topology of weak convergence on $\mathcal{C}(\rho)$ is called the narrow topology in the theory of Banach spaces (see e.g. [3, Vol. 1, p. 176]), and is weaker than the weak topology.

From the theorem of Dunford and Pettis ([9, Theorem 3], [10]), the closure in the weak topology of a subset of $L^1(\eta)$ is compact in the weak topology if and only if it is uniformly

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integrable. Here we recall (see e.g. [3, Vol. 1, Definition 4.5.1], [9, p. 41]) that a subset $S \subseteq L^1(\eta)$ is called uniformly integrable if

$$\lim_{c\to\infty} \sup_{f\in S} \int_D |f(x)| 1(|f(x)| \geq c)\eta(dx) = 0.$$

Furthermore, by the theorem of de la Vallée Poussin ([5], [9, Theorem 2]), for $S \subseteq L^1(\eta)$ to be uniformly integrable it is necessary and sufficient that there be a nondecreasing convex function $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$, with $\Theta(0) = 0$ and $\lim_{x\to\infty} \frac{\Theta(x)}{x} = \infty$, such that

$$\sup_{f\in S} \int_D \Theta(|f(x)|)\eta(dx) < \infty.$$

For $\rho > 0$, $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ a nondecreasing convex function with $\Theta(0) = 0$ and $\lim_{x\to\infty} \frac{\Theta(x)}{x} = \infty$, and $0 < K < \infty$, we propose to consider the subset of $\mathcal{C}(\rho)$ defined by

$$\mathcal{C}(\rho, \Theta, K) := \left\{ f : D \to \mathbb{R}_+ \text{ s.t. } \int_D f(x)\eta(dx) = \rho, \int_D \Theta(f(x))\eta(dx) \leq K \right\}. \tag{29}$$

It can be checked that $\mathcal{C}(\rho, \Theta, K)$ is a closed subset of $L^1(\eta)$ in the weak topology. By the theorem of de la Vallée Poussin, $\mathcal{C}(\rho, \Theta, K)$ is uniformly integrable, so by the Dunford–Pettis theorem it is a compact subset of $L^1(\eta)$ in the weak topology. Since the weak topology on $L^1(\eta)$ is stronger than the topology of weak convergence (i.e., the narrow topology) on $L^1(\eta)$, $\mathcal{C}(\rho, \Theta, K)$ is compact in the topology of weak convergence on $L^1(\eta)$.

For every $f \in L^1(\eta)$ it can be checked that there is some nondecreasing convex function $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$, with $\Theta(0) = 0$ and $\lim_{x\to\infty} \frac{\Theta(x)}{x} = \infty$, such that $\int_D \Theta(|f(x)|)\eta(dx) < \infty$. It follows that the union of $\mathcal{C}(\rho, \Theta, K)$ over all choices of Θ and K equals $\mathcal{C}(\rho)$. We thus have a family of compact subsets of $\mathcal{C}(\rho)$ whose union is $\mathcal{C}(\rho)$.

From the convexity of Θ it is also straightforward to show that each $\mathcal{C}(\rho, \Theta, K)$ is a convex subset of $L^1(\eta)$. As a closed subset of $\mathcal{M}(D)$ in the topology of weak convergence, $\mathcal{C}(\rho, \Theta, K)$ is a Borel subset of $\mathcal{M}(D)$, so we are able discuss probability measures on $\mathcal{C}(\rho, \Theta, K)$.

By an N-player *restricted Cox process Hotelling game* we mean a game with player i, for $1 \leq i \leq N$, having the intensity budget $\rho_i > 0$ and the space of pure actions some $\mathcal{C}(\rho_i, \Theta_i, K_i)$, which, as we have seen, is a compact subset of $\mathcal{C}(\rho_i)$ in the topology of weak convergence. Here, for each $1 \leq i \leq N$, $\Theta_i : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing convex function with $\Theta_i(0) = 0$ and $\lim_{x\to\infty} \frac{\Theta_i(x)}{x} = \infty$, and $0 < K_i < \infty$. Suppose the individual players play the pure actions $f_i \in \mathcal{C}(\rho_i, \Theta_i, K_i)$ for $1 \leq i \leq N$. Let Φ_i be a Poisson point process on D with intensity measure $f_i\eta$, with these processes being mutually independent for $1 \leq i \leq N$, and let $\Phi := \sum_{i=1}^N \Phi_i$. We think of the points of Φ_i as the points of player i, since these result from the choice of f_i, which was made by that player. Then, as before, the value of player i, denoted by $V_i(f_1, \ldots, f_N)$, is given by

$$V_i(f_1, \ldots, f_N) := \mathbb{E}\left[\sum_{x\in\Phi_i} \eta(W_\Phi(x)) \right].$$

The expectation is with respect to the joint law of $(\Phi_j, 1 \leq j \leq n)$, which are independent. Here, by definition, a sum over an empty set is 0.

Consider an N-player restricted Cox process Hotelling game where player i has the space of actions $\mathcal{C}(\rho_i, \Theta_i, K_i)$. Any mixed-strategy N-tuple in this game can be written as

$$(f_1(M_1), \ldots, f_N(M_N)),$$

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where $\big(M_i \in \mathcal{M}_j,\ 1 \le i \le N\big)$ are independent random variables representing the randomizations used by the individual players in implementing their randomized strategies, and $f_i(m_i) \in \mathcal{C}\big(\rho_i, \Theta_i, K_i\big)$ is the choice of action of player i in case the realization of her random variable M_i is m_i. The value of player i in such a mixed strategy is $\mathbb{E}\big[V_i\big(f_1(M_1), \ldots, f_N(M_N)\big)\big]$, the expectation being taken with respect to the joint distribution of $\big(M_j,\ 1 \le j \le N\big)$, which are independent. The vector of independently randomized strategies

$$\big(f_1(M_1), \ldots, f_N(M_N)\big) \in \mathcal{C}(\rho_1, \Theta_1, K_1) \times \ldots \times \mathcal{C}(\rho_N, \Theta_N, K_N)$$

is called a *Nash equilibrium* of the game if, for all $g_j \in \mathcal{C}\big(\rho_j, \Theta_j, K_j\big)$, $1 \le j \le N$, we have, for all $1 \le i \le N$,

$$\mathbb{E}\big[V_i\big(f_1(M_1), \ldots, f_N(M_N)\big)\big] \ge \mathbb{E}\big[V_i\big(g_i, \big(f_j(M_j), j \ne i\big)\big)\big]. \tag{30}$$

Since each $\mathcal{C}\big(\rho_i, \Theta_i, K_i\big)$ is compact and each $V_i : \prod_{j=1}^{N} \mathcal{C}\big(\rho_j, \Theta_j, K_j\big) \to \mathbb{R}_+$ is continuous, the existence of a mixed-strategy Nash equilibrium for every N-player restricted Cox process Hotelling game is guaranteed [15].

We now formally state and prove that every restricted Cox process Hotelling game has unique Nash equilibrium, and that this consists of pure strategies. The following theorem is a combination of the analogue of Theorem 4 and the N-player version of Theorem 5 for restricted Cox process Hotelling games.

Theorem 10. *Consider an N-player restricted Cox process Hotelling game on the Polish space D with base measure η where player i, for $1 \le i \le N$, has the intensity budget $\rho_i > 0$ and the space of pure actions $\mathcal{C}(\rho_i, \Theta_i, K_i)$, where $\Theta_i : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing convex function with $\Theta_i(0) = 0$ and $\lim_{x\to\infty} \frac{\Theta_i(x)}{x} = \infty$, and $0 < K_i < \infty$. Suppose $(f_1(M_1), \ldots, f_N(M_N))$ is a Nash equilibrium of the game, which we know exists. Then there exist $g_i \in \mathcal{C}_i$, $1 \le i \le N$, such that*

$$\mathbb{P}\big(\big(f_1(M_1), \ldots, f_N(M_N) = (g_1, \ldots, g_N)\big) = 1;$$

i.e., the Nash equilibrium is a pure-strategy Nash equilibrium. Furthermore, if $(g_1, \ldots, g_N)$ and $(h_1, \ldots, h_N)$ are two pure-strategy Nash equilibria for the game, then $g_i = h_i$ for all $1 \le i \le N$; i.e. the Nash equilibrium is unique.

Proof. The proof is similar to the proofs of the N-player versions of Theorem 3 and Theorem 5, with the obvious modifications. All that is being used in those proofs is the convexity of the set of allowed actions of each player and the strict concavity of the value function of each player in her own action when the actions of her opponents are fixed. These properties continue to hold in the restricted Cox process Hotelling games that we are now considering. □

When the restrictions on the individual players in an N-player restricted Cox process Hotelling game are proportional to their allowed intensities, we can characterize the Nash equilibrium of the game, which we known by Theorem 10 is unique and consists of pure strategies, in a manner analogous to what was done in Theorem 7. To define what we mean by proportional restrictions, given $\alpha > 0$ and a nondecreasing convex function $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\Theta(0) = 0$ and $\lim_{x\to\infty} \frac{\Theta(x)}{x} = \infty$, we define the function $\Theta^{(\alpha)} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\Theta^{(\alpha)}(x) := \Theta\Big(\frac{x}{\alpha}\Big), x \in \mathbb{R}_+.$$

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Note that $\Theta^{(\alpha)}:\mathbb{R}_+\to\mathbb{R}_+$ is a nondecreasing convex function with $\Theta^{(\alpha)}(0)=0$ and $\lim_{x\to\infty}\frac{\Theta^{(\alpha)}(x)}{x}=\infty$. We can then make the following simple observation.

Lemma 8. *Given $\rho>0$, a nondecreasing convex function $\Theta:\mathbb{R}_+\to\mathbb{R}_+$ with $\Theta(0)=0$ and $\lim_{x\to\infty}\frac{\Theta(x)}{x}=\infty$, and $0<K<\infty$, we have $f\in\mathcal{C}\big(\rho,\Theta^{(\rho)},K\big)$ if and only if $\frac{f}{\rho}\in\mathcal{C}(1,\Theta,K)$.*

Proof. To check whether $\frac{f}{\rho}\in\mathcal{C}(1,\Theta,K)$, we need to check whether $\int_D\frac{f(x)}{\rho}\eta(dx)=1$ and $\int_D\Theta\big(\frac{f(x)}{\rho}\big)\eta(dx)\le K$. Equivalently, we need to check whether $\int_D f(x)\eta(dx)=\rho$ and $\int_D\Theta^{(\rho)}(f(x))\eta(dx)\le K$, i.e. whether $f\in\mathcal{C}\big(\rho,\Theta^{(\rho)},K\big)$. □

The following result characterizes the Nash equilibria of N-player restricted Cox process Hotelling games when the restrictions on the individual players are in proportion to their allowed intensities.

Theorem 11. *Consider an N-player restricted Cox process Hotelling game on the Polish space D with base measure η where player i, for $1\le i\le N$, has the intensity budget $\rho_i>0$ and the space of pure actions $\mathcal{C}\big(\rho_i,\Theta^{(\rho_i)},K\big)$, where $\Theta:\mathbb{R}_+\to\mathbb{R}_+$ is a nondecreasing convex function with $\Theta(0)=0$ and $\lim_{x\to\infty}\frac{\Theta(x)}{x}=\infty$, and $0<K<\infty$. Then the game has a unique Nash equilibrium, which is of the form $(f_1,\ldots,f_N)$, where $f_i=\frac{\rho_i}{\rho}f$ for some $f\in\mathcal{C}\big(\rho,\Theta^{(\rho)}\big),K)$, with $\rho:=\sum_{i=1}^N\rho_i$.*

Proof. The proof is similar to that of Theorem 7, with the obvious modifications. The only thing that needs to be observed is that for any choice of $f_i\in\mathcal{C}\big(\rho_i,\Theta^{(\rho_i)},K\big)$ for $1\le i\le N$, we also have the following:

(i)
$$\frac{\rho_i}{\sum_{j\neq i}\rho_j}\sum_{j\neq i}f_j\in\mathcal{C}\big(\rho_i,\Theta^{(\rho_i)},K\big)\quad\text{for all } 1\le i\le N;$$

(ii)
$$\sum_{i=1}^N f_i=:f\in\mathcal{C}\big(\rho,\Theta^{(\rho)},K\big);$$

(iii)
$$\frac{\rho_i}{\rho}f\in\mathcal{C}\big(\rho_i,\Theta^{(\rho_i)},K\big)\quad\text{for all } 1\le i\le N.$$

To prove (i), by Lemma 8, what we need to show, for all $1\le i\le N$, is that

$$\frac{1}{\sum_{j\neq i}\rho_j}\sum_{j\neq i}f_j\in\mathcal{C}(1,\Theta,K).$$

By Lemma 8 again, we have $\frac{f_j}{\rho_j}\in\mathcal{C}(1,\Theta,K)$ for all $j\neq i$. The desired claim follows from the convexity of $\mathcal{C}(1,\Theta,K)$.

To prove (ii), by Lemma 8, what we need to show is that $\frac{f}{\rho}\in\mathcal{C}(1,\Theta,K)$. By Lemma 8 we have $\frac{f_i}{\rho_i}\in\mathcal{C}(1,\Theta,K)$ for all $1\le i\le N$. The desired claim follows from the convexity of $\mathcal{C}(1,\Theta,K)$.

As for (iii), it is an immediate consequence of Lemma 8 since we have established the claim in (ii). □

Finally, with a view to using restricted Cox process Hotelling games as a vehicle for better understanding the characterization in Theorem 8, we can prove the following analogue of that result for restricted Cox process Hotelling games.

Theorem 12. *Consider an N-player restricted Cox process Hotelling game on the Polish space D with base measure η where player i, for $1 \le i \le N$, has the intensity budget $\rho_i > 0$ and the space of pure actions $\mathcal{C}\big(\rho_i, \Theta^{(\rho_i)}, K\big)$, where $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing convex function with $\Theta(0) = 0$ and $\lim_{x\to\infty} \frac{\Theta(x)}{x} = \infty$, and $0 < K < \infty$. The game admits a unique Nash equilibrium, consisting of pure strategies, defined in terms of a function $f \in \mathcal{C}\big(\rho, \Theta^{(\rho)}, K\big)$, where $\rho := \sum_{i=1}^{N} \rho_i$, such that*

$$\int_{x\in D} f(x) \int_{y\in D} e^{-\int_{u\in B(y\to x)} f(u)\eta(du)}\eta(dy)\eta(dx) \\ \ge \int_{x\in D} g(x) \int_{y\in D} e^{-\int_{u\in B(y\to x)} f(u)\eta(du)}\eta(dy)\eta(dx) \tag{31}$$

for all $g \in \mathcal{C}\big(\rho, \Theta^{(\rho)}, K\big)$. The corresponding unique Nash equilibrium of the game is given by the pure strategy profile $\big(\frac{\rho_1}{\rho} f, \ldots, \frac{\rho_N}{\rho} f\big)$.

Proof. The proof is similar to that of Theorem 7, with the obvious modifications. The key point we need to observe is that, for all $1 \le i \le N$, we have $g \in \mathcal{C}\big(\rho, \Theta^{(\rho)}, K\big)$ if and only if $\frac{\rho_i}{\rho} g \in \mathcal{C}\big(\rho_i, \Theta^{(\rho_i)}, K\big)$. Also, as we have observed earlier, in view of the compactness of the action spaces of the individual players and the continuity of the payoff of an individual player in her action when the actions of her opponents are fixed, the existence of a mixed-strategy Nash equilibrium is guaranteed [15]. □

Remark 1. For the N-player restricted Cox process Hotelling games with proportional restrictions of the kind considered in Theorem 12, that theorem tells us that for each choice of $\rho_i > 0$ for $1 \le i \le N$, nondecreasing convex function $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\Theta(0) = 0$ and $\lim_{x\to\infty} \frac{\Theta(x)}{x} = \infty$, and $0 < K < \infty$, there is a unique function $f \in \mathcal{C}\big(\rho, \Theta^{(\rho)}, K\big)$ which satisfies the inequality in (31) for all $g \in \mathcal{C}\big(\rho, \Theta^{(\rho)}, K\big)$, where $\rho := \sum_{i=1}^{N} \rho_i$. We also know that if, for the given choices of $\rho_i > 0$ for $1 \le i \le N$, the original N-player Cox player Hotelling game considered in Theorem 8 admits a Nash equilibrium, then there will be a function $f \in \mathcal{C}(\rho)$ satisfying the inequality in (25) for all $g \in \mathcal{C}(\rho)$, and, most importantly, this f will be the one satisfying (31) for some choices of nondecreasing convex function $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\Theta(0) = 0$ and $\lim_{x\to\infty} \frac{\Theta(x)}{x} = \infty$, and of $0 < K < \infty$. It is in this sense that the discussion of N-player restricted Cox process Hotelling games gives a vehicle, in principle, for better understanding of the meaning of the criterion in (25) for the existence of Nash equilibria in N-player Cox process Hotelling games.

Appendix A. A sufficient condition for $f\eta$ to be non-conflicting

We give here a sufficient condition for the η-measure of the boundary of the Voronoi tessellation of a Poisson point process Φ of density f with respect to η on D to be zero almost surely. The setting is that of Subsection 2.1, with $\int_D f(x)\eta(dx) = \rho < \infty$, so that Φ has a finite number of points almost surely.

A sufficient condition for the desired property to hold is that

$$\eta\{z \in D \text{ s.t. } \exists X \ne Y \in \Phi \text{ with } d(z, X) = d(z, Y)\} = 0 \quad \text{almost surely,}$$

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which holds if

$$\mathbb{E}[\eta\{z \in D \text{ s.t. } \exists X \neq Y \in \Phi \text{ with } d(z, X) = d(z, Y)\}] = 0.$$

The latter can be written as

$$\sum_{n\geq 2} \frac{\rho^n}{n!} e^{-\rho} n(n-1) \int_{z\in D} \int_{x\in D} \int_{y\in D} 1_{d(z,x)=d(z,y)} \eta(dz) f(x) \eta(dx) f(y) \eta(dy) = 0,$$

so that a sufficient condition for the desired property to hold is that

$$\int_{z\in D} \int_{x\in D} \int_{y\in D} 1_{d(z,x)=d(z,y)} \eta(dz) f(x) \eta(dx) f(y) \eta(dy) = 0. \tag{32}$$

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