Forum of Mathematics, Sigma (2024), **Vol. 12**:e93 1–37
doi:10.1017/fms.2024.89

RESEARCH ARTICLE

Two-sided permutation statistics via symmetric functions

Ira M. Gessel[1] and Yan Zhuang[2]

[1]Department of Mathematics, Brandeis University, 415 South St., Waltham, MA 02453, USA; E-mail: gessel@brandeis.edu.
[2]Department of Mathematics and Computer Science, Davidson College, 405 N. Main St., Davidson, NC 28035, USA;
E-mail: yazhuang@davidson.edu (corresponding author).

Received: 11 November 2023; **Revised:** 10 July 2024; **Accepted:** 15 August 2024

2020 Mathematics Subject Classification: *Primary* – 05A15; *Secondary* – 05A05, 05E05

Abstract
Given a permutation statistic st, define its inverse statistic ist by $\mathrm{ist}(\pi) := \mathrm{st}(\pi^{-1})$. We give a general approach, based on the theory of symmetric functions, for finding the joint distribution of st_1 and ist_2 whenever st_1 and st_2 are descent statistics: permutation statistics that depend only on the descent composition. We apply this method to a number of descent statistics, including the descent number, the peak number, the left peak number, the number of up-down runs and the major index. Perhaps surprisingly, in many cases the polynomial giving the joint distribution of st_1 and ist_2 can be expressed as a simple sum involving products of the polynomials giving the (individual) distributions of st_1 and st_2. Our work leads to a rederivation of Stanley's generating function for doubly alternating permutations, as well as several conjectures concerning real-rootedness and γ-positivity.

Contents

1. Introduction

Let $\mathfrak{S}_n$ denote the symmetric group of permutations of the set $[n] := \{1, 2, \ldots, n\}$. We call $i \in [n-1]$ a *descent* of $\pi \in \mathfrak{S}_n$ if $\pi(i) > \pi(i+1)$. Let

$$\operatorname{des}(\pi) := \sum_{\pi(i)>\pi(i+1)} 1 \quad \text{and} \quad \operatorname{maj}(\pi) := \sum_{\pi(i)>\pi(i+1)} i$$

be the number of descents and the sum of descents of π, respectively. The *descent number* des and *major index* maj are classical permutation statistics dating back to MacMahon [17].

Given a permutation statistic st, let us define its *inverse statistic* ist by $\operatorname{ist}(\pi) := \operatorname{st}(\pi^{-1})$. This paper is concerned with the general problem of finding the joint distribution of a permutation statistic st and its inverse statistic ist over the symmetric group $\mathfrak{S}_n$. This was first done by Carlitz, Roselle and Scoville [5] for des and by Roselle [20] for maj. Among the results of Carlitz, Roselle and Scoville was the elegant generating function formula

$$\sum_{n=0}^{\infty} \frac{A_n(s,t)}{(1-s)^{n+1}(1-t)^{n+1}} x^n = \sum_{i,j=0}^{\infty} \frac{s^i t^j}{(1-x)^{ij}} \tag{1.1}$$

for the *two-sided Eulerian polynomials* $A_n(s,t)$ defined by

$$A_n(s,t) := \sum_{\pi \in \mathfrak{S}_n} s^{\operatorname{des}(\pi)+1} t^{\operatorname{ides}(\pi)+1}$$

for $n \geq 1$ and $A_0(s,t) := 1$.[1] (Throughout this paper, all polynomials encoding distributions of permutation statistics will be defined to be 1 for $n = 0$.) Roselle gave a similar formula for the joint distribution of maj and imaj over $\mathfrak{S}_n$. Note that extracting coefficients of x^n from both sides of Equation (1.1) yields the formula

$$\frac{A_n(s,t)}{(1-s)^{n+1}(1-t)^{n+1}} = \sum_{i,j=0}^{\infty} \binom{ij+n-1}{n} s^i t^j, \tag{1.2}$$

which Petersen [19] later proved using the technology of putting balls in boxes. Equation (1.2) may be compared with the formula

[1]Carlitz, Roselle and Scoville actually considered the joint distribution of ascents (elements of $[n-1]$ which are not descents) and inverse ascents, but this is the same as the joint distribution of descents and inverse descents by symmetry.

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$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{k=0}^{\infty} k^n t^k \tag{1.3}$$

for the (ordinary) *Eulerian polynomials*

$$A_n(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\operatorname{des}(\pi)+1}.$$

Several years after the work of Carlitz–Roselle–Scoville and Roselle, Garsia and Gessel [11] used the theory of P-partitions to derive the formula

$$\sum_{n=0}^{\infty} \frac{A_n(s,t,q,r)}{(1-s)(1-qs)\cdots(1-q^n s)(1-t)(1-rt)\cdots(1-r^n t)} x^n = \sum_{i,j=0}^{\infty} s^i t^j \prod_{k=0}^{i} \prod_{l=0}^{j} \frac{1}{1-xq^k r^l} \tag{1.4}$$

for the quadrivariate polynomials

$$A_n(s,t,q,r) := \sum_{\pi \in \mathfrak{S}_n} s^{\operatorname{des}(\pi)} t^{\operatorname{ides}(\pi)} q^{\operatorname{maj}(\pi)} r^{\operatorname{imaj}(\pi)}.$$

The Garsia–Gessel formula (1.4) specializes to both the Carlitz–Roselle–Scoville formula (1.1) for $(\operatorname{des}, \operatorname{ides})$ as well as Roselle's formula for $(\operatorname{maj}, \operatorname{imaj})$.

The two-sided Eulerian polynomials have since received renewed attention due to a refined γ-positivity conjecture of Gessel, which was later proven by Lin [16]. The joint statistic $(\operatorname{des}, \operatorname{maj}, \operatorname{ides}, \operatorname{imaj})$ and the sum $\operatorname{des} + \operatorname{ides}$ have also been studied in the probability literature; see, for example, [4, 6, 7, 27].

Throughout this paper, let us call a pair of the form $(\operatorname{st}, \operatorname{ist})$ a *two-sided permutation statistic* and the distribution of this statistic over $\mathfrak{S}_n$ the *two-sided distribution* of st. If we are taking st to be a pair $(\operatorname{st}_1, \operatorname{st}_2)$ such as $(\operatorname{des}, \operatorname{maj})$, then we consider $(\operatorname{st}_1, \operatorname{ist}_1, \operatorname{st}_2, \operatorname{ist}_2)$ a two-sided statistic as well.

1.1. Descent statistics

We use the notation $L \vDash n$ to indicate that L is a composition of n. Every permutation can be uniquely decomposed into a sequence of maximal increasing consecutive subsequences—or equivalently, maximal consecutive subsequences with no descents—which we call *increasing runs*. The descent composition of π, denoted $\operatorname{Comp}(\pi)$, is the composition whose parts are the lengths of the increasing runs of π in the order that they appear. For example, the increasing runs of $\pi = 72485316$ are 7, 248, 5, 3, and 16, so $\operatorname{Comp}(\pi) = (1, 3, 1, 1, 2)$.

A permutation statistic st is called a *descent statistic* if $\operatorname{Comp}(\pi) = \operatorname{Comp}(\sigma)$ implies $\operatorname{st}(\pi) = \operatorname{st}(\sigma)$—that is, if st depends only on the descent composition. Whenever st is a descent statistic, we may write $\operatorname{st}(L)$ for the value of st on any permutation with descent composition L. Both des and maj are descent statistics, and in this paper, we will also consider the following descent statistics:

- The *peak number* pk. We call i (where $2 \le i \le n-1$) a *peak* of $\pi \in \mathfrak{S}_n$ if $\pi(i-1) < \pi(i) > \pi(i+1)$. Then $\operatorname{pk}(\pi)$ is the number of peaks of π.
- The *left peak number* lpk. We call $i \in [n-1]$ a *left peak* of $\pi \in \mathfrak{S}_n$ if either i is a peak of π, or if $i = 1$ and $\pi(1) > \pi(2)$. Then $\operatorname{lpk}(\pi)$ is the number of left peaks of π.
- The *number of biruns* br. A *birun* of a permutation π is a maximal monotone consecutive subsequence. Then $\operatorname{br}(\pi)$ is the number of biruns of π.
- The *number of up-down runs* udr. An *up-down run* of π is either a birun of π, or $\pi(1)$ if $\pi(1) > \pi(2)$. Then $\operatorname{udr}(\pi)$ is the number of up-down runs of π.[2]

[2]Equivalently, $\operatorname{udr}(\pi)$ is equal to the length of the longest alternating subsequence of π, which was studied in depth by Stanley [22].

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For example, if $\pi = 624731598$, then the peaks of π are 4 and 8; the left peaks of π are 1, 4 and 8; the biruns of π are 62, 247, 731, 159 and 98; and the up-down runs of π are 6, 62, 247, 731, 159 and 98. Therefore, we have $\mathrm{pk}(\pi) = 2$, $\mathrm{lpk}(\pi) = 3$, $\mathrm{br}(\pi) = 5$ and $\mathrm{udr}(\pi) = 6$. Other examples of descent statistics include the number of valleys, double ascents, double descents and alternating descents (see [29] for definitions).

In this paper, we give a general approach to the problem of finding the two-sided distribution of st whenever st is a descent statistic. In fact, our approach can be used to find distributions of *mixed two-sided statistics*: The joint distribution of st_1 and ist_2 when st_1 and st_2 are (possibly different) descent statistics. Our approach utilizes a theorem of Foulkes [9] at the intersection of permutation enumeration and symmetric function theory: The number of permutations π with prescribed descent composition L whose inverse π^{-1} has descent composition M is equal to the scalar product $\langle r_L, r_M\rangle$ of two ribbon skew Schur functions (defined in Section 2.1). Thus, if f is a generating function for ribbon functions that keeps track of a descent statistic st_1 and g is a similar generating function for a descent statistic st_2, then the scalar product $\langle f, g\rangle$ should give a generating function for $(\mathrm{st}_1, \mathrm{ist}_2)$.

To illustrate this idea, let us sketch how our approach can be used to rederive the formula (1.1) of Carlitz, Roselle and Scoville for the two-sided Eulerian polynomials. We have

$$\frac{1}{1-tH(x)} = \frac{1}{1-t} + \sum_{n=1}^{\infty}\sum_{L\vDash n} \frac{t^{\mathrm{des}(L)+1}}{(1-t)^{n+1}} r_L x^n,$$

where $H(z) \coloneqq \sum_{n=0}^{\infty} h_n z^n$ is the ordinary generating function for the complete symmetric functions h_n. Then, upon applying Foulkes's theorem, we get

$$\begin{aligned}\left\langle \frac{1}{1-sH(x)}, \frac{1}{1-tH(1)}\right\rangle &= \frac{1}{(1-s)(1-t)} + \sum_{n=1}^{\infty}\sum_{L,M\vDash n} \frac{s^{\mathrm{des}(L)+1}t^{\mathrm{des}(M)+1}}{(1-s)^{n+1}(1-t)^{n+1}}\langle r_L, r_M\rangle x^n \\ &= \sum_{n=0}^{\infty} \frac{A_n(s,t)}{(1-s)^{n+1}(1-t)^{n+1}} x^n.\end{aligned}$$

However, calculating the same scalar product but in a different way yields

$$\left\langle \frac{1}{1-sH(x)}, \frac{1}{1-tH(1)}\right\rangle = \sum_{i,j=0}^{\infty} \frac{s^i t^j}{(1-x)^{ij}},$$

and equating these two expressions recovers (1.1).

In fact, a third way of calculating the same scalar product leads to a new formula expressing the two-sided Eulerian polynomials $A_n(s,t)$ in terms of the Eulerian polynomials $A_n(t)$. Let us use the notations $\lambda \vdash n$ and $|\lambda| = n$ to indicate that λ is a partition of n, and let $l(\lambda)$ denote the number of parts of λ. We write $\lambda = (1^{m_1}2^{m_2}\cdots)$ to mean that λ has m_1 parts of size 1, m_2 parts of size 2 and so on, and define $z_\lambda \coloneqq 1^{m_1}m_1!\,2^{m_2}m_2!\,\cdots$. Then it can be shown that

$$\left\langle \frac{1}{1-sH(x)}, \frac{1}{1-tH(1)}\right\rangle = \sum_{\lambda} \frac{1}{z_\lambda} \frac{A_{l(\lambda)}(s)A_{l(\lambda)}(t)}{(1-s)^{l(\lambda)+1}(1-t)^{l(\lambda)+1}} x^{|\lambda|},$$

leading to

$$A_n(s,t) = \sum_{\lambda\vdash n} \frac{1}{z_\lambda}((1-s)(1-t))^{n-l(\lambda)} A_{l(\lambda)}(s)A_{l(\lambda)}(t).$$

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Collecting terms with $l(\lambda) = k$ then gives the formula

$$A_n(s,t) = \frac{1}{n!}\sum_{k=0}^{n} c(n,k)((1-s)(1-t))^{n-k} A_k(s)A_k(t),$$

(see Theorem 3.9), where the $c(n,k)$ are the unsigned Stirling numbers of the first kind [23, p. 26]. Perhaps surprisingly, many of the 'two-sided polynomials' that we study in this paper can be similarly expressed as a simple sum involving products of the univariate polynomials encoding the distributions of the individual statistics.

Foulkes's theorem was also used by Stanley [21] in his study of alternating permutations; some of our results, notably Theorems 3.5 and 4.5, generalize results of his.

1.2. Outline

We organize this paper as follows. Section 2 is devoted to background material on symmetric functions. While we assume familiarity with basic symmetric function theory at the level of Stanley [24, Chapter 7], we shall use this section to establish notation, recall some elementary facts that will be important for our work and to give an exposition of various topics and results needed to develop our approach to two-sided statistics; these include Foulkes's theorem, plethysm and symmetric function generating functions associated with descent statistics.

Our main results are given in Sections 3–6. We begin in Section 3 by using our symmetric function approach to prove formulas for the two-sided statistic $(\mathrm{pk}, \mathrm{ipk}, \mathrm{des}, \mathrm{ides})$, which we then specialize to formulas for $(\mathrm{pk}, \mathrm{ipk})$, $(\mathrm{pk}, \mathrm{ides})$ and $(\mathrm{des}, \mathrm{ides})$. We continue in Section 4 by deriving analogous formulas for $(\mathrm{lpk}, \mathrm{ilpk}, \mathrm{des}, \mathrm{ides})$ and $(\mathrm{lpk}, \mathrm{ipk}, \mathrm{des}, \mathrm{ides})$, and their specializations. Notably, our results for the two-sided distributions of pk and of lpk lead to a rederivation of Stanley's [21] generating function formula for 'doubly alternating permutations': alternating permutations whose inverses are alternating.

Section 5 considers (mixed) two-sided distributions involving the number of up-down runs, as well as a couple involving the number of biruns. In Section 6, we give a rederivation of the Garsia–Gessel formula for $(\mathrm{maj}, \mathrm{imaj}, \mathrm{des}, \mathrm{ides})$ using our approach and give formulas for several mixed two-sided distributions involving the major index.

We conclude in Section 7 with several conjectures concerning real-rootedness and γ-positivity of some of the polynomials appearing in our work.

2. Symmetric functions background

Let Λ denote the $\mathbb{Q}$-algebra of symmetric functions in the variables $x_1, x_2, \ldots$. We recall the important bases for Λ: the monomial symmetric functions m_λ, the complete symmetric functions h_λ, the elementary symmetric functions e_λ, the power sum symmetric functions p_λ and the Schur functions s_λ. As usual, we write $h_{(n)}$ as h_n, $e_{(n)}$ as e_n and $p_{(n)}$ as p_n.

We will also work with symmetric functions with coefficients involving additional variables such as s, t, y, z and α, as well as symmetric functions of unbounded degree like

$$H(z) = \sum_{n=0}^{\infty} h_n z^n \quad \text{and} \quad E(z) \coloneqq \sum_{n=0}^{\infty} e_n z^n.$$

We adopt the notation

$$H \coloneqq H(1) = \sum_{n=0}^{\infty} h_n \quad \text{and} \quad E \coloneqq E(1) = \sum_{n=0}^{\infty} e_n.$$

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2.1. The scalar product and Foulkes's theorem

Let $\langle\cdot,\cdot\rangle\colon \Lambda\times\Lambda\to\mathbb{Q}$ denote the usual scalar product on symmetric functions defined by

$$\langle m_\lambda, h_\mu\rangle = \begin{cases} 1, & \text{if } \lambda=\mu \\ 0, & \text{otherwise,} \end{cases}$$

for all partitions λ and μ and extending bilinearly, that is, by requiring that $\{m_\lambda\}$ and $\{h_\mu\}$ be dual bases. Then we have

$$\langle p_\lambda, p_\mu\rangle = \begin{cases} z_\lambda, & \text{if } \lambda=\mu \\ 0, & \text{otherwise,} \end{cases}$$

for all λ and μ [23, Proposition 7.9.3]. We extend the scalar product in the obvious way to symmetric functions involving other variables as well as symmetric functions of unbounded degree. Note that the scalar product is not always defined for the latter; for example, we have $\langle H(z),H\rangle=\sum_{n=0}^\infty z^n$ but $\langle H,H\rangle$ is undefined.

Given a composition L, let r_L denote the skew Schur function of ribbon shape L. That is, for $L=(L_1,L_2,\dots,L_k)$, we have

$$r_L=\sum_{i_1,\dots,i_n} x_{i_1}x_{i_2}\cdots x_{i_n},$$

where the sum is over all $i_1,\dots,i_n$ satisfying

$$\underbrace{i_1\le\cdots\le i_{L_1}}_{L_1} > \underbrace{i_{L_1+1}\le\cdots\le i_{L_1+L_2}}_{L_2} > \cdots > \underbrace{i_{L_1+\cdots+L_{k-1}+1}\le\cdots\le i_n}_{L_k}.$$

The next theorem, due to Foulkes [9] (see also [12, Theorem 5] and [24, Corollary 7.23.8]), will play a pivotal role in our approach to two-sided descent statistics.

Theorem 2.1. *Let L and M be compositions. Then $\langle r_L, r_M\rangle$ is the number of permutations π with descent composition L such that π^{-1} has descent composition M.*

Foulkes's theorem is a special case of a more general theorem of Gessel on quasisymmetric generating functions, which we briefly describe below. For a composition $L=(L_1,L_2,\dots,L_k)$, let $\operatorname{Des}(L) := \{L_1, L_1+L_2,\dots,L_1+\cdots+L_{k-1}\}$, and recall that the fundamental quasisymmetric function F_L is defined by

$$F_L := \sum_{\substack{i_1\le i_2\le\cdots\le i_n \\ i_j<i_{j+1} \text{ if } j\in\operatorname{Des}(L)}} x_{i_1}x_{i_2}\cdots x_{i_n}.$$

Moreover, given a set Π of permutations, its *quasisymmetric generating function* $Q(\Pi)$ is defined by

$$Q(\Pi) := \sum_{\pi\in\Pi} F_{\operatorname{Comp}(\pi)}.$$

The following is Corollary 4 of Gessel [12].

Theorem 2.2. *Suppose that $Q(\Pi)$ is a symmetric function. Then the number of permutations in Π with descent composition L is equal to $\langle r_L, Q(\Pi)\rangle$.*

Because r_M is the quasisymmetric generating function for permutations whose inverse has descent composition M, Foulkes's theorem follows from Theorem 2.2.

Gessel and Reutenauer [13] showed that, for any partition λ, the quasisymmetric generating function for permutations with cycle type λ is a symmetric function, and they used this fact along with Theorem 2.2 to study the joint distribution of maj and des over sets of permutations with restricted cycle structure, including cyclic permutations, involutions and derangements. The present authors later studied the distributions of (pk, des), (lpk, des) and udr over permutations with restricted cycle structure [15]. Our current work is a continuation of this line of research but for inverse descent classes.

2.2. Plethysm

Let A be a $\mathbb{Q}$-algebra of formal power series (possibly containing Λ). Consider the operation $\Lambda \times A \to A$, where the image of $(f, a) \in \Lambda \times A$ is denoted $f[a]$, defined by the following two properties:

1. For any $i \geq 1$, $p_i[a]$ is the result of replacing each variable in a with its ith power.
2. For any fixed $a \in A$, the map $f \mapsto f[a]$ is a $\mathbb{Q}$-algebra homomorphism from Λ to A.

In other words, we have $p_i[f(x_1, x_2, \ldots)] = f(x_1^i, x_2^i, \ldots)$ for any $f \in \Lambda$. If f contains variables other than the x_i, then they are all raised to the ith power as well. For example, if q and t are variables, then $p_i[q^2 t p_m] = q^{2i} t^i p_{im}$. The map $(f, a) \mapsto f[a]$ is called *plethysm*. As with the scalar product, we extend plethysm in the obvious way to symmetric functions of unbounded degree with coefficients involving other variables; whenever we do so, we implicitly assume that any infinite sums involved converge as formal power series so that the plethysms are defined.

We will need several technical lemmas involving plethysm in order to evaluate the scalar products needed in our work. All of these lemmas are from [15] by the present authors (or are easy consequences of results from [15]).

A *monic term* is any monomial with coefficient 1.

Lemma 2.3. *Let $\mathsf{m} \in A$ be a monic term not containing any of the variables $x_1, x_2, \ldots$. Then the maps $f \mapsto \langle f, H(\mathsf{m})\rangle$ and $f \mapsto \langle H(\mathsf{m}), f\rangle$ are $\mathbb{Q}$-algebra homomorphisms on A.*

Proof. By a special case of [15, Lemma 2.5], we have

$$f[\mathsf{m}] = \langle f, H(\mathsf{m})\rangle = \langle H(\mathsf{m}), f\rangle$$

and the result follows from the fact that $f \mapsto f[\mathsf{m}]$ is a $\mathbb{Q}$-algebra homomorphism. □

Lemma 2.4. *Let $y \in A$ be a variable and $k \in \mathbb{Z}$. Then the map $f \mapsto \left\langle f, E(y)^k H^k\right\rangle$ is a $\mathbb{Q}$-algebra homomorphism on A.*

Proof. Given $f \in A$ and a variable $\alpha \in A$, it is known [15, Lemma 3.2] that

$$f[k(1-\alpha)] = \left\langle f, E(-\alpha)^k H^k\right\rangle.$$

Then the map $f \mapsto \left\langle f, E(y)^k H^k\right\rangle$ is obtained by composing the map $f \mapsto f[k(1-\alpha)]$ with evaluation at $\alpha = -y$, both of which are $\mathbb{Q}$-algebra homomorphisms. □

Lemma 2.5. *Let $\alpha, \beta \in A$ be variables, and let k be an integer. Then:*

(a) $H(\beta)[k(1-\alpha)] = \left\langle H(\beta), E(-\alpha)^k H^k\right\rangle = (1-\alpha\beta)^k/(1-\beta)^k$.
(b) $E(\beta)[k(1-\alpha)] = \left\langle E(\beta), E(-\alpha)^k H^k\right\rangle = (1+\beta)^k/(1+\alpha\beta)^k$.

Proof. Part (a) is a special case of Lemma 2.4 (d) of [15]. Part (b) follows immediately from part (a) and the well-known identity $E(\beta) = (H(-\beta))^{-1}$. □

The two lemmas below are Lemmas 3.5 and 6.6 of [15], respectively.

Lemma 2.6. *Let $f, g \in A$, and let $\mathsf{m} \in A$ be a monic term. Then $\langle f[X+\mathsf{m}], g\rangle = \langle f, H[\mathsf{m}X]g\rangle$.*

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Lemma 2.7. *Let $\alpha \in A$ be a variable and $\mathsf{m} \in A$ a monic term. Then:*

(a) $H(\alpha)[X + \mathsf{m}] = H(\alpha)/(1 - \alpha\mathsf{m})$
(b) $E(\alpha)[X + \mathsf{m}] = (1 + \alpha\mathsf{m})E(\alpha)$.

2.3. Symmetric function generating functions for descent statistics

As mentioned in the introduction, we will need generating functions for the ribbon skew Schur functions r_L which keep track of the descent statistics that we are studying. Such generating functions were produced in previous work by the present authors and are stated in the next lemma. Part (a) is essentially a commutative version of [14, Lemma 17], whereas (b)–(d) are given in [15, Lemma 2.8].

Lemma 2.8. *We have*

$$\sum_{n=0}^{\infty} t^n \prod_{k=0}^{n} H(q^k x) = \sum_{n=0}^{\infty} \frac{\sum_{L\vDash n} q^{\operatorname{maj}(L)} t^{\operatorname{des}(L)} r_L}{(1-t)(1-qt)\cdots(1-q^n t)} x^n, \tag{a}$$

$$\frac{1}{1 - tE(yx)H(x)} = \frac{1}{1-t} + \frac{1}{1+y} \sum_{n=1}^{\infty} \sum_{L\vDash n} \left(\frac{1+yt}{1-t}\right)^{n+1} \left(\frac{(1+y)^2 t}{(y+t)(1+yt)}\right)^{\operatorname{pk}(L)+1} \left(\frac{y+t}{1+yt}\right)^{\operatorname{des}(L)+1} r_L x^n, \tag{b}$$

$$\frac{H(x)}{1 - tE(yx)H(x)} = \frac{1}{1-t} + \sum_{n=1}^{\infty} \sum_{L\vDash n} \frac{(1+yt)^n}{(1-t)^{n+1}} \left(\frac{(1+y)^2 t}{(y+t)(1+yt)}\right)^{\operatorname{lpk}(L)} \left(\frac{y+t}{1+yt}\right)^{\operatorname{des}(L)} r_L x^n, \tag{c}$$

and

$$\frac{1 + tH(x)}{1 - t^2 E(x)H(x)} = \frac{1}{1-t} + \frac{1}{2(1-t)^2} \sum_{n=1}^{\infty} \sum_{L\vDash n} \frac{(1+t^2)^n}{(1-t^2)^{n-1}} \left(\frac{2t}{1+t^2}\right)^{\operatorname{udr}(L)} r_L x^n. \tag{d}$$

Some of these generating functions have nice power sum expansions that are expressible in terms of Eulerian polynomials and type B Eulerian polynomials. The nth *type B Eulerian polynomial* $B_n(t)$ encodes the distribution of the type B descent number over the nth hyperoctahedral group (see [30, Section 2.3] for definitions) but can also be defined by the formula

$$\frac{B_n(t)}{(1-t)^{n+1}} = \sum_{k=0}^{\infty} (2k+1)^n t^k$$

analogous to Equation (1.3). Recall that $l(\lambda)$ is the number of parts of the partition λ. We also define $o(\lambda)$ to be the number of odd parts of λ, and use the notation $\sum_{\lambda \text{ odd}}$ for a sum over partitions in which every part is odd.

The next lemma is [15, Lemma 2.9].

Lemma 2.9. *We have*

$$\frac{1}{1 - tE(yx)H(x)} = \sum_{\lambda} \frac{p_\lambda}{z_\lambda} \frac{A_{l(\lambda)}(t)}{(1-t)^{l(\lambda)+1}} x^{|\lambda|} \prod_{k=1}^{l(\lambda)} (1 - (-y)^{\lambda_k}) \tag{a}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}$ are the parts of λ,

$$\frac{H(x)}{1 - tE(x)H(x)} = \sum_{\lambda} \frac{p_\lambda}{z_\lambda} \frac{B_{o(\lambda)}(t)}{(1-t)^{o(\lambda)+1}} x^{|\lambda|}, \tag{b}$$

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and

$$\frac{1+tH(x)}{1-tE(x)H(x)} = \sum_{\lambda \text{ odd}} \frac{p_\lambda}{z_\lambda} 2^{l(\lambda)} \frac{A_{l(\lambda)}(t^2)}{(1-t^2)^{l(\lambda)+1}} x^{|\lambda|} + t \sum_{\lambda} \frac{p_\lambda}{z_\lambda} \frac{B_{o(\lambda)}(t^2)}{(1-t^2)^{o(\lambda)+1}} x^{|\lambda|}. \tag{c}$$

2.4. *Sums involving z_λ and Stanley's formula for doubly alternating permutations*

The remainder of this section is not strictly about symmetric functions but relates to the constants z_λ which appear in symmetric function theory, and a connection to an enumeration formula of Stanley obtained via symmetric function techniques.

To derive some of our formulas later on, we will need to evaluate several sums like

$$\sum_{\substack{\lambda \vdash n \\ l(\lambda)=k}} \frac{1}{z_\lambda}. \tag{2.1}$$

The key to evaluating these sums is the well-known fact that $n!/z_\lambda$ is the number of permutations in $\mathfrak{S}_n$ of cycle type λ; see, for example, [23, Proposition 1.3.2] and [24, pp. 298–299]. Thus, Equation (2.1) is equal to $c(n,k)/n!$, where as before, $c(n,k)$ is the unsigned Stirling number of the first kind, which counts permutations in $\mathfrak{S}_n$ with k cycles.

Now, let $d(n,k)$ be the number of permutations in $\mathfrak{S}_n$ with k cycles, all of odd length; let $e(n,k)$ be the number of permutations in $\mathfrak{S}_n$ with k odd cycles (and any number of even cycles); and let $f(n,k,m)$ be the number of permutations in $\mathfrak{S}_n$ with k odd cycles and m cycles in total. Note that $d(n,k)$, $e(n,k)$ and $f(n,k,m)$ are 0 if $n-k$ is odd. Then by the same reasoning as above, we have the additional sum evaluations in the next lemma.

Lemma 2.10. *We have*

$$\sum_{\substack{\lambda \vdash n \\ l(\lambda)=k}} \frac{1}{z_\lambda} = \frac{c(n,k)}{n!}, \tag{a}$$

$$\sum_{\substack{\lambda \vdash n \\ \text{odd} \\ l(\lambda)=k}} \frac{1}{z_\lambda} = \frac{d(n,k)}{n!}, \tag{b}$$

$$\sum_{\substack{\lambda \vdash n \\ o(\lambda)=k}} \frac{1}{z_\lambda} = \frac{e(n,k)}{n!}, \tag{c}$$

and

$$\sum_{\substack{\lambda \vdash n \\ o(\lambda)=k \\ l(\lambda)=m}} \frac{1}{z_\lambda} = \frac{f(n,k,m)}{n!}. \tag{d}$$

The numbers $c(n,k)$, $d(n,k)$, $e(n,k)$ and $f(n,k,m)$ all have simple exponential generating functions. Let

$$L(u) \coloneqq \frac{1}{2} \log \frac{1+u}{1-u} = \sum_{m=1}^{\infty} \frac{u^{2m-1}}{2m-1}.$$

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Proposition 2.11. *We have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} c(n,k)v^k \frac{u^n}{n!} = e^{-v\log(1-u)} = \frac{1}{(1-u)^v}, \tag{a}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} d(n,k)v^k \frac{u^n}{n!} = e^{vL(u)} = \left(\frac{1+u}{1-u}\right)^{v/2}, \tag{b}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} e(n,k)v^k \frac{u^n}{n!} = \frac{e^{vL(u)}}{\sqrt{1-u^2}} = \frac{(1+u)^{(v-1)/2}}{(1-u)^{(v+1)/2}}, \tag{c}$$

and

$$\sum_{n=0}^{\infty}\sum_{k,m=0}^{n} f(n,k,m)v^k w^m \frac{u^n}{n!} = \frac{e^{vwL(u)}}{(1-u^2)^{w/2}} = \frac{(1+u)^{(v-1)w/2}}{(1-u)^{(v+1)w/2}}. \tag{d}$$

Proof. We prove only (c); the proofs of the other formulas are similar (and (a) is well known). For (c) we want to count permutations in which odd cycles are weighted v and even cycles are weighted 1. So by the exponential formula for permutations [24, Corollary 5.1.9], we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{n} e(n,k)\frac{u^n}{n!}v^k &= \exp\left(v\sum_{m=1}^{\infty}\frac{u^{2m-1}}{2m-1} + \sum_{m=1}^{\infty}\frac{u^{2m}}{2m}\right)\\ &= \exp\left(\frac{v}{2}\big(-\log(1-u)+\log(1+u)\big) + \frac{1}{2}\big(-\log(1-u)-\log(1+u)\big)\right)\\ &= \frac{(1+u)^{(v-1)/2}}{(1-u)^{(v+1)/2}}.\end{aligned}$$

□

A permutation π is called *alternating* if $\pi(1) > \pi(2) < \pi(3) > \pi(4) < \cdots$, and it is well known that alternating permutations in $\mathfrak{S}_n$ are counted by the nth *Euler number* E_n, whose exponential generating function is $\sum_{n=0}^{\infty} E_n x^n/n! = \sec x + \tan x$. The series $L(u)$ defined above makes an appearance in Stanley's work on *doubly alternating permutations*: alternating permutations whose inverses are alternating. Let $\tilde{E}_n$ denote the number of doubly alternating permutations in $\mathfrak{S}_n$. Stanley showed that the ordinary generating function for doubly alternating permutations is

$$\sum_{n=0}^{\infty}\tilde{E}_n u^n = \frac{1}{\sqrt{1-u^2}}\sum_{r=0}^{\infty}E_{2r}^2\frac{L(u)^{2r}}{(2r)!} + \sum_{r=0}^{\infty}E_{2r+1}^2\frac{L(u)^{2r+1}}{(2r+1)!} \tag{2.2}$$

[21, Theorem 3.1]. Stanley's proof of Equation (2.2) uses Foulkes's theorem, but Equation (2.2) can also be recovered from our results, as shall be demonstrated at the end of Sections 3.2 and 4.3.

We note that Stanley's formula (2.2) can be expressed in terms of the numbers $d(n,k)$ and $e(n,k)$. By Proposition 2.11, we have

$$\frac{1}{\sqrt{1-u^2}}\frac{L(u)^{2r}}{(2r)!} = \sum_{n=0}^{\infty} e(n,2r)\frac{u^n}{n!} \quad\text{and}\quad \frac{L(u)^{2r+1}}{(2r+1)!} = \sum_{d=0}^{\infty} d(n,2r+1)\frac{u^n}{n!}.$$

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Thus, Equation (2.2) is equivalent to the statement that

$$\tilde{E}_n = \begin{cases} \dfrac{1}{n!}\displaystyle\sum_r e(n,2r)E_{2r}^2 = \dfrac{1}{n!}\sum_{k=0}^n e(n,k)E_k^2, & \text{if } n \text{ is even,} \\ \dfrac{1}{n!}\displaystyle\sum_r d(n,2r+1)E_{2r+1}^2 = \dfrac{1}{n!}\sum_{k=0}^n d(n,k)E_k^2, & \text{if } n \text{ is odd.} \end{cases}$$

3. Two-sided peak and descent statistics

We are now ready to proceed to the main body of our work. Our first task will be to derive formulas for the two-sided distribution of (pk, des)—that is, the joint distribution of pk, ipk, des and ides over $\mathfrak{S}_n$—and then we shall specialize our results to the (mixed) two-sided statistics (pk, ipk), (pk, ides) and (des, ides).

3.1. Peaks, descents and their inverses

Define the polynomials $P_n^{(\mathrm{pk},\mathrm{ipk},\mathrm{des},\mathrm{ides})}(y,z,s,t)$ by

$$P_n^{(\mathrm{pk},\mathrm{ipk},\mathrm{des},\mathrm{ides})}(y,z,s,t) \coloneqq \sum_{\pi\in\mathfrak{S}_n} y^{\mathrm{pk}(\pi)+1} z^{\mathrm{ipk}(\pi)+1} s^{\mathrm{des}(\pi)+1} t^{\mathrm{ides}(\pi)+1},$$

which encodes the desired two-sided distribution.

Theorem 3.1. *We have*

$$\frac{1}{(1-s)(1-t)} + \sum_{n=1}^{\infty} \frac{\left(\frac{(1+ys)(1+zt)}{(1-s)(1-t)}\right)^{n+1} P_n^{(\mathrm{pk},\mathrm{ipk},\mathrm{des},\mathrm{ides})}\left(\frac{(1+y)^2 s}{(y+s)(1+ys)}, \frac{(1+z)^2 t}{(z+t)(1+zt)}, \frac{y+s}{1+ys}, \frac{z+t}{1+zt}\right)}{(1+y)(1+z)} x^n \tag{a}$$
$$= \sum_{i,j=0}^{\infty} \left(\frac{(1+yx)(1+zx)}{(1-yzx)(1-x)}\right)^{ij} s^i t^j$$

and, for all $n \geq 1$, we have

$$\frac{\left(\frac{(1+ys)(1+zt)}{(1-s)(1-t)}\right)^{n+1} P_n^{(\mathrm{pk},\mathrm{ipk},\mathrm{des},\mathrm{ides})}\left(\frac{(1+y)^2 s}{(y+s)(1+ys)}, \frac{(1+z)^2 t}{(z+t)(1+zt)}, \frac{y+s}{1+ys}, \frac{z+t}{1+zt}\right)}{(1+y)(1+z)} \tag{b}$$
$$= \sum_{\lambda\vdash n} \frac{\prod_{i=1}^{l(\lambda)}(1-(-y)^{\lambda_i})(1-(-z)^{\lambda_i})}{z_\lambda} \frac{A_{l(\lambda)}(s)A_{l(\lambda)}(t)}{(1-s)^{l(\lambda)+1}(1-t)^{l(\lambda)+1}}.$$

Proof. To prove this theorem, we shall compute $\langle (1-sE(yx)H(x))^{-1}, (1-tE(z)H)^{-1}\rangle$ in three different ways. First, from Lemma 2.8 (b) we have

$$\left\langle \frac{1}{1-sE(yx)H(x)}, \frac{1}{1-tE(z)H} \right\rangle$$
$$= \frac{1}{(1-s)(1-t)} + \sum_{m,n=1}^{\infty} \frac{\left(\frac{1+ys}{1-s}\right)^{m+1}\left(\frac{1+zt}{1-t}\right)^{n+1} \sum_{L\vDash m,\, M\vDash n} N_{L,M}\langle r_L, r_M\rangle}{(1+y)(1+z)} x^m, \tag{3.1}$$

where

$$N_{L,M} := \left(\frac{(1+y)^2 s}{(y+s)(1+ys)}\right)^{\mathrm{pk}(L)+1} \left(\frac{(1+z)^2 t}{(z+t)(1+zt)}\right)^{\mathrm{pk}(M)+1} \left(\frac{y+s}{1+ys}\right)^{\mathrm{des}(L)+1} \left(\frac{z+t}{1+zt}\right)^{\mathrm{des}(M)+1}.$$

Upon applying Foulkes's theorem (Theorem 2.1), Equation (3.1) simplifies to

$$\begin{aligned}
&\left\langle \frac{1}{1-sE(yx)H(x)}, \frac{1}{1-tE(z)H} \right\rangle \\
&= \frac{1}{(1-s)(1-t)} + \sum_{n=1}^{\infty} \frac{\left(\frac{(1+ys)(1+zt)}{(1-s)(1-t)}\right)^{n+1} \sum_{\pi\in\mathfrak{S}_n} N_{\mathrm{Comp}(\pi),\mathrm{Comp}(\pi^{-1})}}{(1+y)(1+z)} x^n \\
&= \frac{1}{(1-s)(1-t)} + \sum_{n=1}^{\infty} \frac{\left(\frac{(1+ys)(1+zt)}{(1-s)(1-t)}\right)^{n+1} P_n^{(\mathrm{pk},\mathrm{ipk},\mathrm{des},\mathrm{ides})}\left(\frac{(1+y)^2 s}{(y+s)(1+ys)}, \frac{(1+z)^2 t}{(z+t)(1+zt)}, \frac{y+s}{1+ys}, \frac{z+t}{1+zt}\right)}{(1+y)(1+z)} x^n.
\end{aligned} \tag{3.2}$$

Second, we have

$$\begin{aligned}
\left\langle \frac{1}{1-sE(yx)H(x)}, \frac{1}{1-tE(z)H} \right\rangle &= \left\langle \sum_{i=0}^{\infty} E(yx)^i H(x)^i s^i, \sum_{j=0}^{\infty} E(z)^j H^j t^j \right\rangle \\
&= \sum_{i,j=0}^{\infty} \left\langle E(yx)^i H(x)^i, E(z)^j H^j \right\rangle s^i t^j \\
&= \sum_{i,j=0}^{\infty} \left\langle E(yx), E(z)^j H^j \right\rangle^i \left\langle H(x), E(z)^j H^j \right\rangle^i s^i t^j \\
&= \sum_{i,j=0}^{\infty} \frac{(1+yx)^{ij}}{(1-yzx)^{ij}} \frac{(1+zx)^{ij}}{(1-x)^{ij}} s^i t^j,
\end{aligned} \tag{3.3}$$

where the last two steps are obtained using Lemmas 2.4 and 2.5, respectively. Finally, from Lemma 2.9 (a) we have

$$\begin{aligned}
&\left\langle \frac{1}{1-sE(yx)H(x)}, \frac{1}{1-tE(z)H} \right\rangle \\
&= \sum_{\lambda,\mu} \frac{\prod_{i=1}^{l(\lambda)}(1-(-y)^{\lambda_i}) \prod_{j=1}^{l(\mu)}(1-(-z)^{\mu_i})}{z_\lambda z_\mu} \frac{A_{l(\lambda)}(s) A_{l(\mu)}(t)}{(1-s)^{l(\lambda)+1}(1-t)^{l(\mu)+1}} \langle p_\lambda, p_\mu \rangle x^{|\lambda|} \\
&= \sum_{\lambda} \frac{\prod_{i=1}^{l(\lambda)}(1-(-y)^{\lambda_i})(1-(-z)^{\lambda_i})}{z_\lambda} \frac{A_{l(\lambda)}(s) A_{l(\lambda)}(t)}{(1-s)^{l(\lambda)+1}(1-t)^{l(\lambda)+1}} x^{|\lambda|}.
\end{aligned} \tag{3.4}$$

Combining Equation (3.2) with Equation (3.3) yields part (a), whereas combining Equation (3.2) with Equation (3.4) and then extracting coefficients of x^n yields part (b). □

The formulas in Theorem 3.1 and others appearing later in this paper can be 'inverted' upon making appropriate substitutions. For example, Theorem 3.1 (a) can be rewritten as

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$$\frac{1}{(1-\alpha)(1-\beta)}+\frac{1}{(1+u)(1+v)}\sum_{n=1}^{\infty}\left(\frac{(1+u\alpha)(1+v\beta)}{(1-\alpha)(1-\beta)}\right)^{n+1}P_n^{(\mathrm{pk},\mathrm{ipk},\mathrm{des},\mathrm{ides})}(y,z,s,t)x^n$$
$$=\sum_{i,j=0}^{\infty}\left(\frac{(1+ux)(1+vx)}{(1-uvx)(1-x)}\right)^{ij}\alpha^i\beta^j,$$

where

$$u=\frac{1+s^2-2ys-(1-s)\sqrt{(1+s)^2-4ys}}{2(1-y)s},\qquad \alpha=\frac{(1+s)^2-2ys-(1+s)\sqrt{(1+s)^2-4ys}}{2ys},$$
$$v=\frac{1+t^2-2zt-(1-t)\sqrt{(1+t)^2-4zt}}{2(1-z)t},\quad\text{and}\quad \beta=\frac{(1+t)^2-2zt-(1+t)\sqrt{(1+t)^2-4zt}}{2zt}.$$

3.2. Peaks and inverse peaks

We will now consider specializations of the polynomials $P_n^{(\mathrm{pk},\mathrm{ipk},\mathrm{des},\mathrm{ides})}(y,z,s,t)$ that give the distributions of the (mixed) two-sided statistics $(\mathrm{pk},\mathrm{ipk})$, $(\mathrm{pk},\mathrm{ides})$ and $(\mathrm{des},\mathrm{ides})$. Let us begin with the two-sided distribution of pk, which is encoded by

$$P_n^{(\mathrm{pk},\mathrm{ipk})}(s,t)\coloneqq P_n^{(\mathrm{pk},\mathrm{ipk},\mathrm{des},\mathrm{ides})}(s,t,1,1)=\sum_{\pi\in\mathfrak{S}_n}s^{\mathrm{pk}(\pi)+1}t^{\mathrm{ipk}(\pi)+1}.$$

Theorem 3.2. *We have*

$$\frac{1}{(1-s)(1-t)}+\frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{(1+s)(1+t)}{(1-s)(1-t)}\right)^{n+1}P_n^{(\mathrm{pk},\mathrm{ipk})}\left(\frac{4s}{(1+s)^2},\frac{4t}{(1+t)^2}\right)x^n=\sum_{i,j=0}^{\infty}\left(\frac{1+x}{1-x}\right)^{2ij}s^it^j \tag{a}$$

and, for all $n\geq 1$, we have

$$\left(\frac{(1+s)(1+t)}{(1-s)(1-t)}\right)^{n+1}P_n^{(\mathrm{pk},\mathrm{ipk})}\left(\frac{4s}{(1+s)^2},\frac{4t}{(1+t)^2}\right)=4\sum_{i,j=0}^{\infty}\sum_{k=0}^{n}\binom{2ij}{k}\binom{2ij+n-k-1}{n-k}s^it^j \tag{b}$$

and

$$\left(\frac{(1+s)(1+t)}{(1-s)(1-t)}\right)^{n+1}P_n^{(\mathrm{pk},\mathrm{ipk})}\left(\frac{4s}{(1+s)^2},\frac{4t}{(1+t)^2}\right)=\frac{1}{n!}\sum_{k=0}^{n}4^{k+1}d(n,k)\frac{A_k(s)A_k(t)}{(1-s)^{k+1}(1-t)^{k+1}} \tag{c}$$

with $d(n,k)$ as defined in Section 2.4.

Proof. Parts (a) and (c) are obtained from evaluating Theorem 3.1 (a) and (b), respectively, at $y=z=1$, with the sum on λ in (c) evaluated by Lemma 2.10 (b). From the identities

$$(1+x)^k=\sum_{n=0}^{k}\binom{k}{n}x^n\quad\text{and}\quad(1-x)^{-k}=\sum_{n=0}^{\infty}\binom{k+n-1}{n}x^n,$$

we obtain

$$\left(\frac{1+x}{1-x}\right)^{2ij}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{2ij}{k}\binom{2ij+n-k-1}{n-k}x^n. \tag{3.5}$$

Substituting Equation (3.5) into (a) and extracting coefficients of x^n yields (b). □

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Table 1. *Joint distribution of* pk *and* ipk *over* $\mathfrak{S}_n$

n	$P_n^{(\mathrm{pk},\mathrm{ipk})}(s,t)$
1	st
2	$2st$
3	$(3s+s^2)t+(s+s^2)t^2$
4	$(4s+4s^2)t+(4s+12s^2)t^2$
5	$(5s+10s^2+s^3)t+(10s+66s^2+12s^3)t^2+(s+12s^2+3s^3)t^3$
6	$(6s+20s^2+6s^3)t+(20s+248s^2+148s^3)t^2+(6s+148s^2+118s^3)t^3$
7	$(7s+35s^2+21s^3+s^4)t+(35s+739s^2+969s^3+81s^4)t^2+(21s+969s^2+1719s^3+141s^4)t^3+(s+81s^2+171s^3+19s^4)t^4$

The first several polynomials $P_n^{(\mathrm{pk},\mathrm{ipk})}(s,t)$ are displayed in Table 1. The coefficients of $s^k t$—equivalently, the coefficients of st^k by symmetry—are characterized by the following proposition.

Proposition 3.3. *For any $n \geq 1$ and $k \geq 0$, the number of permutations $\pi \in \mathfrak{S}_n$ with* $\mathrm{pk}(\pi) = k$ *and* $\mathrm{ipk}(\pi) = 0$ *is equal to* $\binom{n}{2k+1}$.

Proposition 3.3 was first stated as a corollary of a more general result of Troyka and Zhuang [26], namely, that for any composition L of $n \geq 1$, there is exactly one permutation in $\pi \in \mathfrak{S}_n$ with descent composition L such that $\mathrm{ipk}(\pi) = 0$. However, it is also possible to prove Proposition 3.3 directly from Theorem 3.2.

Next, we obtain a surprisingly simple formula expressing the two-sided peak polynomials $P_n^{(\mathrm{pk},\mathrm{ipk})}(s,t)$ in terms of products of the ordinary peak polynomials

$$P_n^{\mathrm{pk}}(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{pk}(\pi)+1}$$

giving the distribution of the peak number over $\mathfrak{S}_n$.

Theorem 3.4. *For all $n \geq 1$, we have*

$$P_n^{(\mathrm{pk},\mathrm{ipk})}(s,t) = \frac{1}{n!}\sum_{k=0}^{n} d(n,k)((1-s)(1-t))^{\frac{n-k}{2}} P_k^{\mathrm{pk}}(s) P_k^{\mathrm{pk}}(t).$$

Since $d(n,k) = 0$ when $n-k$ is odd, the above formula does not involve any square roots.

Proof. It is known [25] that

$$A_n(t) = \left(\frac{1+t}{2}\right)^{n+1} P_n^{\mathrm{pk}}\left(\frac{4t}{(1+t)^2}\right) \tag{3.6}$$

for $n \geq 1$. Combining Equation (3.6) with Theorem 3.2 (c) and then replacing the variables s and t with u and v, respectively, yields

$$P_n^{(\mathrm{pk},\mathrm{ipk})}\left(\frac{4u}{(1+u)^2}, \frac{4v}{(1+v)^2}\right) = \frac{1}{n!}\sum_{k=0}^{n} d(n,k)\left(\frac{(1-u)(1-v)}{(1+u)(1+v)}\right)^{n-k} P_k^{\mathrm{pk}}\left(\frac{4u}{(1+u)^2}\right) P_k^{\mathrm{pk}}\left(\frac{4v}{(1+v)^2}\right).$$

Setting $s = 4u/(1+u)^2$ and $t = 4v/(1+v)^2$, we obtain

$$P_n^{(\mathrm{pk},\mathrm{ipk})}(s,t) = \frac{1}{n!}\sum_{k=0}^{n} d(n,k)\left(\frac{(1-u)(1-v)}{(1+u)(1+v)}\right)^{n-k} P_k^{\mathrm{pk}}(s) P_k^{\mathrm{pk}}(t). \tag{3.7}$$

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It can be verified that $u = 2s^{-1}(1-\sqrt{1-s})-1$ and $v = 2t^{-1}(1-\sqrt{1-t})-1$, which lead to

$$\frac{(1-u)(1-v)}{(1+u)(1+v)} = \sqrt{(1-s)(1-t)}. \tag{3.8}$$

Substituting Equation (3.8) into Equation (3.7) completes the proof. □

The substitution used in the proof of Theorem 3.4 allows us to invert the formulas in Theorem 3.2 and others involving the same quadratic transformation. For example, Theorem 3.2 (b) is equivalent to

$$\frac{P_n^{(\mathrm{pk},\mathrm{ipk})}(s,t)}{(1-s)^{\frac{n+1}{2}}(1-t)^{\frac{n+1}{2}}} = 4\sum_{i,j=0}^{\infty}\sum_{k=0}^{n}\binom{2ij}{k}\binom{2ij+n-k-1}{n-k}u^i v^j,$$

where, as before, $u = 2s^{-1}(1-\sqrt{1-s})-1$ and $v = 2t^{-1}(1-\sqrt{1-t})-1$. In fact, we can express u and v in terms of the Catalan generating function

$$C(x) := \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n$$

as $u = (s/4)C(s/4)^2$ and $v = (t/4)C(t/4)^2$.

If we multiply the formula of Theorem 3.4 by u^n and sum over n using Proposition 2.11 (b), we obtain the following generating function for the two-sided peak polynomials $P_n^{(\mathrm{pk},\mathrm{ipk})}(s,t)$.

Theorem 3.5. *We have*

$$\sum_{n=0}^{\infty}P_n^{(\mathrm{pk},\mathrm{ipk})}(s,t)u^n = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{L(\sqrt{(1-s)(1-t)}u)}{\sqrt{(1-s)(1-t)}}\right)^k P_k^{\mathrm{pk}}(s)P_k^{\mathrm{pk}}(t),$$

with $L(u)$ as defined in Section 2.4.

We can derive the odd part of Stanley's generating function (2.2) for doubly alternating permutations from Theorem 3.5. While an alternating permutation π satisfies $\pi(1) > \pi(2) < \pi(3) > \pi(4) < \cdots$, we say that π is *reverse alternating* if $\pi(1) < \pi(2) > \pi(3) < \pi(4) > \cdots$. It is evident by symmetry that reverse alternating permutations are also counted by the Euler numbers E_n. As shown by Stanley [21], the number of doubly alternating permutations in $\mathfrak{S}_n$ is equal to the number $\tilde{E}_n$ of reverse alternating permutations in $\mathfrak{S}_n$ whose inverses are reverse alternating.

It is readily verified that a permutation π in $\mathfrak{S}_n$ has at most $(n-1)/2$ peaks and has exactly $(n-1)/2$ peaks if and only if n is odd and π is reverse alternating. Thus, we have

$$\lim_{s\to 0}P_k^{\mathrm{pk}}(s^{-2})s^{k+1} = \lim_{s\to 0}\sum_{\pi\in\mathfrak{S}_k}s^{k-2\,\mathrm{pk}(\pi)-1} = \begin{cases}0, & \text{if } k \text{ is even,}\\ E_k, & \text{if } k \text{ is odd.}\end{cases}$$

Similarly, we have

$$\lim_{s\to 0}P_n^{(\mathrm{pk},\mathrm{ipk})}(s^{-2},t^{-2})s^{n+1}t^{n+1} = \begin{cases}0, & \text{if } n \text{ is even,}\\ \tilde{E}_n, & \text{if } n \text{ is odd.}\end{cases}$$

So, to obtain Stanley's formula from Theorem 3.5, we first replace s with s^{-2}, t with t^{-2} and u with stu; then we multiply by st and take the limit as $s,t\to 0$. The substitution takes

$$\left(\frac{L(\sqrt{(1-s)(1-t)}u)}{\sqrt{(1-s)(1-t)}}\right)^k P_k^{\mathrm{pk}}(s)P_k^{\mathrm{pk}}(t)$$

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to

$$\left(\frac{L(\sqrt{(s^2-1)(t^2-1)}u)}{\sqrt{(1-s^{-2})(1-t^{-2})}}\right)^k P_k^{\mathrm{pk}}(s^{-2})P_k^{\mathrm{pk}}(t^{-2}) = \left(\frac{L(\sqrt{(s^2-1)(t^2-1)}u)}{\sqrt{(s^2-1)(t^2-1)}}\right)^k s^k P_k^{\mathrm{pk}}(s^{-2})t^k P_k^{\mathrm{pk}}(t^{-2});$$

multiplying by st and taking $s, t \to 0$ gives $L(u)^k E_k^2$ for k odd and 0 for k even, as desired.

We will later get the even part of Stanley's generating function (2.2) in a similar way from Theorem 4.4, which is the analogue of Theorem 3.5 for left peaks.

Before proceeding, we note that every formula like Theorem 3.4 has an analogous generating function like Theorem 3.5. We will only write out these generating functions for (pk, ipk) and (lpk, ilpk) – which are used in our rederivation of Stanley's formula – as well as for (des, ides).

3.3. *Peaks and inverse descents*

Next, define the polynomials $P_n^{(\mathrm{pk},\mathrm{ides})}(s,t)$ by

$$P_n^{(\mathrm{pk},\mathrm{ides})}(s,t) \coloneqq P_n^{(\mathrm{pk},\mathrm{ipk},\mathrm{des},\mathrm{ides})}(s,1,1,t) = \sum_{\pi\in\mathfrak{S}_n} s^{\mathrm{pk}(\pi)+1}t^{\mathrm{ides}(\pi)+1}.$$

We omit the proof of the next theorem as it is similar to that of Theorem 3.2, with the main difference being that we specialize at $y = 1$ and $z = 0$ (as opposed to $y = z = 1$).

Theorem 3.6. *We have*

$$\frac{1}{(1-s)(1-t)} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1+s}{(1-s)(1-t)}\right)^{n+1} P_n^{(\mathrm{pk},\mathrm{ides})}\left(\frac{4s}{(1+s)^2},t\right) = \sum_{i,j=0}^{\infty}\left(\frac{1+x}{1-x}\right)^{ij} s^i t^j \tag{a}$$

and, for all $n \geq 1$, we have

$$\left(\frac{1+s}{(1-s)(1-t)}\right)^{n+1} P_n^{(\mathrm{pk},\mathrm{ides})}\left(\frac{4s}{(1+s)^2},t\right) = 2\sum_{i,j=0}^{\infty}\sum_{k=0}^{n}\binom{ij}{k}\binom{ij+n-k-1}{n-k}s^i t^j \tag{b}$$

and

$$\left(\frac{1+t}{(1-s)(1-t)}\right)^{n+1} P_n^{(\mathrm{pk},\mathrm{ides})}\left(\frac{4s}{(1+s)^2},t\right) = \frac{1}{n!}\sum_{k=0}^{n}2^{k+1}d(n,k)\frac{A_k(s)A_k(t)}{(1-s)^{k+1}(1-t)^{k+1}}. \tag{c}$$

We display the first several polynomials $P_n^{(\mathrm{pk},\mathrm{ides})}(s,t)$ in Table 2, collecting terms with the same power of s in order to display the symmetry in their coefficients. This symmetry follows from the fact that the *reverse* $\pi^r \coloneqq \pi(n)\cdots\pi(2)\pi(1)$ of a permutation $\pi \in \mathfrak{S}_n$ satisfies $\mathrm{ides}(\pi^r) = n-1-\mathrm{ides}(\pi)$ [31, Proposition 2.6 (c)] but has the same number of peaks as π.

Furthermore, observe that the coefficients of $t^k s$ in $P_n^{(\mathrm{pk},\mathrm{ides})}(s,t)$ are binomial coefficients; this is a consequence of the following proposition, which is Corollary 9 of [26].

Proposition 3.7. *For any $n \geq 1$ and $k \geq 0$, the number of permutations $\pi \in \mathfrak{S}_n$ with $\mathrm{des}(\pi) = k$ and $\mathrm{ipk}(\pi) = 0$ is equal to $\binom{n-1}{k}$.*

In addition to being symmetric, the coefficients of s^k in $P_n^{(\mathrm{pk},\mathrm{ides})}(s,t)$ seem to be unimodal polynomials in t, which we know holds for $k = 0$ in light of Proposition 3.7. In the last section of this paper, we will state the conjecture that the polynomials $[s^k]\,P_n^{(\mathrm{pk},\mathrm{ides})}(s,t)$ are in fact γ-positive, a property which implies unimodality and symmetry.

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Table 2. *Joint distribution of* pk *and* ides *over* $\mathfrak{S}_n$

n	$P_n^{(\mathrm{pk},\mathrm{ides})}(s,t)$
1	ts
2	$(t+t^2)s$
3	$(t+2t^2+t^3)s+2t^2s^2$
4	$(t+3t^2+3t^3+t^4)s+(8t^2+8t^3)s^2$
5	$(t+4t^2+6t^3+4t^4+t^5)s+(20t^2+48t^3+20t^4)s^2+(2t^2+12t^3+12t^4)s^3$
6	$(t+5t^2+10t^3+10t^4+5t^5+t^6)s+(40t^2+168t^3+168t^4+40t^5)s^2+(12t^2+124t^3+124t^4+12t^5)s^3$
7	$(t+6t^2+15t^3+20t^4+15t^5+6t^6+t^7)s+(70t^2+448t^3+788t^4+448t^5+70t^6)s^2+(42t^2+672t^3+1452t^4+672t^5$ $+42t^6)s^3+(2t^2+56t^3+156t^4+56t^5+2t^6)s^4$

The next formula expresses $P_n^{(\mathrm{pk},\mathrm{ides})}(s,t)$ in terms of the products $P_k^{\mathrm{pk}}(s)A_k(t)$. The proof is very similar to that of Theorem 3.4 and so it is omitted.

Theorem 3.8. *For all $n \geq 1$, we have*

$$P_n^{(\mathrm{pk},\mathrm{ides})}(s,t) = \frac{1}{n!}\sum_{k=0}^{n} d(n,k)\Big((1-s)^{1/2}(1-t)\Big)^{n-k} P_k^{\mathrm{pk}}(s)A_k(t).$$

3.4. Descents and inverse descents

To conclude this section, we state the analogous results for the two-sided Eulerian polynomials

$$A_n(s,t) = P_n^{(\mathrm{pk},\mathrm{ipk},\mathrm{des},\mathrm{ides})}(1,1,s,t) = \sum_{\pi\in\mathfrak{S}_n} s^{\mathrm{des}(\pi)+1}t^{\mathrm{ides}(\pi)+1}.$$

Recall that parts (a) and (b) were originally due to Carlitz, Roselle and Scoville [5], whereas (c) and (d) are new.

Theorem 3.9. *We have*

$$\sum_{n=0}^{\infty} \frac{A_n(s,t)}{(1-s)^{n+1}(1-t)^{n+1}}x^n = \sum_{i,j=0}^{\infty} \frac{s^i t^j}{(1-x)^{ij}} \tag{a}$$

and, for all $n \geq 1$, we have

$$\frac{A_n(s,t)}{(1-s)^{n+1}(1-t)^{n+1}} = \sum_{i,j=0}^{\infty} \binom{ij+n-1}{n} s^i t^j \tag{b}$$

and

$$A_n(s,t) = \frac{1}{n!}\sum_{k=0}^{n} c(n,k)((1-s)(1-t))^{n-k}A_k(s)A_k(t). \tag{c}$$

Moreover, we have

$$\sum_{n=0}^{\infty} A_n(s,t)u^n = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\log\frac{1}{1-(1-s)(1-t)u}\right)^k \frac{A_k(s)A_k(t)}{(1-s)^k(1-t)^k}. \tag{d}$$

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Parts (a)–(c) are proven similarly to Theorems 3.2, 3.4 and 3.6 except that we evaluate Theorem 3.1 at $y = z = 0$. Part (c) can also be derived directly from (b):

$$\begin{aligned}\frac{A_n(s,t)}{(1-s)^{n+1}(1-t)^{n+1}} &= \sum_{i,j=0}^{\infty} \binom{ij+n-1}{n} s^i t^j \\ &= \sum_{i,j=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} c(n,k)(ij)^k s^i t^j \\ &= \frac{1}{n!} \sum_{k=0}^{n} c(n,k) \sum_{i=0}^{\infty} i^k s^i \sum_{j=0}^{\infty} j^k t^j \\ &= \frac{1}{n!} \sum_{k=0}^{n} c(n,k) \frac{A_k(s)}{(1-t)^{k+1}} \frac{A_k(t)}{(1-t)^{k+1}}.\end{aligned}$$

Part (d) is obtained from (c) using Proposition 2.11 (a).

4. Two-sided left peak statistics

4.1. Left peaks, descents and their inverses

In this section, we will examine (mixed) two-sided distributions involving the left peak number lpk. Let us first derive a generating function formula for the polynomials

$$P_n^{(\mathrm{lpk},\mathrm{ilpk},\mathrm{des},\mathrm{ides})}(y,z,s,t) := \sum_{\pi\in\mathfrak{S}_n} y^{\mathrm{lpk}(\pi)} z^{\mathrm{ilpk}(\pi)} s^{\mathrm{des}(\pi)} t^{\mathrm{ides}(\pi)}$$

giving the joint distribution of lpk, ilpk, des and ides over $\mathfrak{S}_n$.

Theorem 4.1. *We have*

$$\begin{aligned}\frac{1}{(1-s)(1-t)} + \sum_{n=1}^{\infty} \frac{(1+ys)^n(1+zt)^n P_n^{(\mathrm{lpk},\mathrm{ilpk},\mathrm{des},\mathrm{ides})}\left(\frac{(1+y)^2 s}{(y+s)(1+ys)}, \frac{(1+z)^2 t}{(z+t)(1+zt)}, \frac{y+s}{1+ys}, \frac{z+t}{1+zt}\right)}{(1-s)^{n+1}(1-t)^{n+1}} x^n \\ = \sum_{i,j=0}^{\infty} \frac{(1+yx)^{i(j+1)}(1+zx)^{(i+1)j}}{(1-yzx)^{ij}(1-x)^{(i+1)(j+1)}} s^i t^j.\end{aligned}$$

Proof. From Lemma 2.8 (c), we have

$$\begin{aligned}&\left\langle \frac{H(x)}{1-sE(yx)H(x)}, \frac{H}{1-tE(z)H} \right\rangle \\ &\qquad = \frac{1}{(1-s)(1-t)} + \sum_{m,n=1}^{\infty} \sum_{\substack{L\vDash m\\ M\vDash n}} \frac{(1+ys)^m(1+zt)^n}{(1-s)^{m+1}(1-t)^{n+1}} \acute{N}_{L,M} \langle r_L, r_M \rangle x^m, \qquad (4.1)\end{aligned}$$

where

$$\acute{N}_{L,M} := \left(\frac{(1+y)^2 s}{(y+s)(1+ys)}\right)^{\mathrm{lpk}(L)} \left(\frac{(1+z)^2 t}{(z+t)(1+zt)}\right)^{\mathrm{ilpk}(M)} \left(\frac{y+s}{1+ys}\right)^{\mathrm{des}(L)} \left(\frac{z+t}{1+zt}\right)^{\mathrm{des}(M)}.$$

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By Foulkes's theorem (Theorem 2.1), Equation (4.1) simplifies to

$$\begin{aligned}&\left\langle \frac{1}{1-sE(yx)H(x)}, \frac{1}{1-tE(z)H}\right\rangle\\&= \frac{1}{(1-s)(1-t)} + \sum_{n=1}^{\infty} \frac{(1+ys)^n(1+zt)^n \sum_{\pi\in\mathfrak{S}_n} \acute{N}_{\mathrm{Comp}(\pi),\mathrm{Comp}(\pi^{-1})}}{(1-s)^{n+1}(1-t)^{n+1}} x^n\\&= \frac{1}{(1-s)(1-t)} + \sum_{n=1}^{\infty} \frac{(1+ys)^n(1+zt)^n P_n^{(\mathrm{lpk},\mathrm{ilpk},\mathrm{des},\mathrm{ides})}\left(\frac{(1+y)^2 s}{(y+s)(1+ys)}, \frac{(1+z)^2 t}{(z+t)(1+zt)}, \frac{y+s}{1+ys}, \frac{z+t}{1+zt}\right)}{(1-s)^{n+1}(1-t)^{n+1}} x^n.\end{aligned} \tag{4.2}$$

We now compute the same scalar product in a different way. We have

$$\begin{aligned}\left\langle \frac{H(x)}{1-sE(yx)H(x)}, \frac{H}{1-tE(z)H}\right\rangle &= \left\langle \sum_{i=0}^{\infty} E(yx)^i H(x)^{i+1} s^i, \sum_{j=0}^{\infty} E(z)^j H^{j+1} t^j \right\rangle\\&= \sum_{i,j=0}^{\infty} \langle E(yx)^i H(x)^{i+1}, E(z)^j H^{j+1}\rangle s^i t^j\\&= \sum_{i,j=0}^{\infty} \langle (E(yx)^i H(x)^{i+1})[X+1], E(z)^j H^j\rangle s^i t^j\\&= \sum_{i,j=0}^{\infty} \langle (E(yx)[X+1])^i (H(x)[X+1])^{i+1}, E(z)^j H^j\rangle s^i t^j\\&= \sum_{i,j=0}^{\infty} \left\langle \frac{(1+yx)^i}{(1-x)^{i+1}} E(yx)^i H(x)^{i+1}, E(z)^j H^j \right\rangle s^i t^j\\&= \sum_{i,j=0}^{\infty} \frac{(1+yx)^i}{(1-x)^{i+1}} \langle E(yx), E(z)^j H^j\rangle^i \langle H(x), E(z)^j H^j\rangle^{i+1} s^i t^j\\&= \sum_{i,j=0}^{\infty} \frac{(1+yx)^{i(j+1)}}{(1-yzx)^{ij}} \frac{(1+zx)^{(i+1)j}}{(1-x)^{(i+1)(j+1)}} s^i t^j;\end{aligned} \tag{4.3}$$

here, we are using Lemma 2.6 for the third equality, and in the last three steps, we apply Lemmas 2.7, 2.4 and 2.5, respectively. Combining (4.2) and (4.3) completes the proof. □

4.2. *Left peaks, inverse peaks, descents and inverse descents*

Next, let us consider the joint distribution of lpk, ipk, des and ides over $\mathfrak{S}_n$. Define

$$P_n^{(\mathrm{lpk},\mathrm{ipk},\mathrm{des},\mathrm{ides})}(y,z,s,t) \coloneqq \sum_{\pi\in\mathfrak{S}_n} y^{\mathrm{lpk}(\pi)} z^{\mathrm{ipk}(\pi)+1} s^{\mathrm{des}(\pi)} t^{\mathrm{ides}(\pi)+1}.$$

Theorem 4.2. *We have*

$$\begin{aligned}\frac{1}{(1-s)(1-t)} + \sum_{n=1}^{\infty} \frac{(1+ys)^n(1+zt)^{n+1} P_n^{(\mathrm{lpk},\mathrm{ipk},\mathrm{des},\mathrm{ides})}\left(\frac{(1+y)^2 s}{(y+s)(1+ys)}, \frac{(1+z)^2 t}{(z+t)(1+zt)}, \frac{y+s}{1+ys}, \frac{z+t}{1+zt}\right)}{(1+z)(1-s)^{n+1}(1-t)^{n+1}} x^n\\= \sum_{i,j=0}^{\infty} \left(\frac{1+yx}{1-yzx}\right)^{ij} \left(\frac{1+zx}{1-x}\right)^{(i+1)j} s^i t^j.\end{aligned}$$

Proof. The two sides of the above equation are obtained from evaluating the scalar product

$$\left\langle \frac{H}{1 - sE(yx)H(x)}, \frac{1}{1 - tE(z)H} \right\rangle$$

in two different ways; the proof is similar to that of Theorem 3.1 and so we omit the details. □

4.3. Left peaks and inverse left peaks

We now consider specializations of $P_n^{(\mathrm{lpk},\mathrm{ilpk},\mathrm{des},\mathrm{ides})}(y,z,s,t)$ and $P_n^{(\mathrm{lpk},\mathrm{ipk},\mathrm{des},\mathrm{ides})}(y,z,s,t)$, beginning with the two-sided left peak polynomials

$$P_n^{(\mathrm{lpk},\mathrm{ilpk})}(s,t) \coloneqq P_n^{(\mathrm{lpk},\mathrm{ilpk},\mathrm{des},\mathrm{ides})}(s,t,1,1) = \sum_{\pi \in \mathfrak{S}_n} s^{\mathrm{lpk}(\pi)} t^{\mathrm{ilpk}(\pi)}.$$

Theorem 4.3. *We have*

$$\frac{1}{(1-s)(1-t)} + \sum_{n=1}^{\infty} \frac{((1+s)(1+t))^n}{((1-s)(1-t))^{n+1}} P_n^{(\mathrm{lpk},\mathrm{ilpk})}\left(\frac{4s}{(1+s)^2}, \frac{4t}{(1+t)^2}\right) x^n \\ = \sum_{i,j=0}^{\infty} \frac{(1+x)^{2ij+i+j}}{(1-x)^{2ij+i+j+1}} s^i t^j \tag{a}$$

and, for all $n \geq 1$, we have

$$\frac{((1+s)(1+t))^n}{((1-s)(1-t))^{n+1}} P_n^{(\mathrm{lpk},\mathrm{ilpk})}\left(\frac{4s}{(1+s)^2}, \frac{4t}{(1+t)^2}\right) = \sum_{i,j=0}^{\infty} \sum_{k=0}^{n} \binom{2ij+i+j}{k} \binom{2ij+i+j+n-k}{n-k} s^i t^j \tag{b}$$

and

$$\frac{((1+s)(1+t))^n}{((1-s)(1-t))^{n+1}} P_n^{(\mathrm{lpk},\mathrm{ilpk})}\left(\frac{4s}{(1+s)^2}, \frac{4t}{(1+t)^2}\right) = \sum_{k=0}^{n} e(n,k) \frac{B_k(s)B_k(t)}{(1-s)^{k+1}(1-t)^{k+1}} \tag{c}$$

with $e(n,k)$ as defined in Section 2.4.

Proof. Evaluating Theorem 4.1 at $y = z = 1$ yields part (a), whereas (b) follows from (a) and

$$\frac{(1+x)^{2ij+i+j}}{(1-x)^{2ij+i+j+1}} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2ij+i+j}{k} \binom{2ij+i+j+n-k}{n-k} x^n.$$

To prove (c), first observe that Lemma 2.9 (b) implies

$$\left\langle \frac{H(x)}{1 - sE(x)H(x)}, \frac{H}{1 - tEH} \right\rangle = \sum_{\lambda,\mu} \frac{1}{z_\lambda z_\mu} \frac{B_{o(\lambda)}(s) B_{o(\mu)}(t)}{(1-s)^{o(\lambda)+1}(1-t)^{o(\mu)+1}} \langle p_\lambda, p_\mu \rangle x^{|\lambda|} \\ = \sum_{\lambda} \frac{1}{z_\lambda} \frac{B_{o(\lambda)}(s) B_{o(\lambda)}(t)}{(1-s)^{o(\lambda)+1}(1-t)^{o(\lambda)+1}} x^{|\lambda|},$$

but this same scalar product is also given by

$$\left\langle \frac{H(x)}{1 - sE(x)H(x)}, \frac{H}{1 - tEH} \right\rangle = \frac{1}{(1-s)(1-t)} \sum_{n=1}^{\infty} \frac{((1+s)(1+t))^n}{((1-s)(1-t))^{n+1}} P_n^{(\mathrm{lpk},\mathrm{ilpk})}\left(\frac{4s}{(1+s)^2}, \frac{4t}{(1+t)^2}\right) x^n,$$

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Table 3. *Joint distribution of* lpk *and* ilpk *over* $\mathfrak{S}_n$

n	$P_n^{(\mathrm{lpk},\mathrm{ilpk})}(s,t)$
1	1
2	$1+st$
3	$1+5st$
4	$1+(15s+3s^2)t+(3s+2s^2)t^2$
5	$1+(35s+23s^2)t+(23s+38s^2)t^2$
6	$1+(70s+100s^2+9s^3)t+(100s+335s^2+44s^3)t^2+(9s+44s^2+8s^3)t^3$
7	$1+(126s+324s^2+93s^3)t+(324s+1951s^2+836s^3)t^2+(93s+836s^2+456s^3)t^3$

which is obtained from evaluating Equation (4.2) at $y=z=1$. Equating these two expressions, extracting coefficients of x^n, and applying Lemma 2.10 (c) completes the proof. □

See Table 3 for the first several polynomials $P_n^{(\mathrm{lpk},\mathrm{ilpk})}(s,t)$.

Next, we express these two-sided left peak polynomials as a sum involving products of the ordinary left peak polynomials

$$P_n^{\mathrm{lpk}}(t) := \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{lpk}(\pi)}.$$

Theorem 4.4. *For all $n\geq 1$, we have*

$$P_n^{(\mathrm{lpk},\mathrm{ilpk})}(s,t) = \frac{1}{n!}\sum_{k=0}^{n} e(n,k)((1-s)(1-t))^{\frac{n-k}{2}} P_k^{\mathrm{lpk}}(s)P_k^{\mathrm{lpk}}(t).$$

Proof. The proof follows in the same way as the proof of Theorem 3.4 except that we use the formula

$$B_n(t) = (1+t)^n P_n^{\mathrm{lpk}}\left(\frac{4t}{(1+t)^2}\right) \tag{4.4}$$

[18, Proposition 4.15] in place of (3.6). The details are omitted. □

From Theorem 4.4 and Proposition 2.11 (c), we can obtain a generating function for the two-sided left peak polynomials, analogous to Theorem 3.5 for the two-sided peak polynomials.

Theorem 4.5. *We have*

$$\sum_{n=0}^{\infty} P_n^{(\mathrm{lpk},\mathrm{ilpk})}(s,t)u^n = \frac{1}{\sqrt{1-(1-s)(1-t)u^2}}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{L(\sqrt{(1-s)(1-t)}u)}{\sqrt{(1-s)(1-t)}}\right)^k P_k^{\mathrm{lpk}}(s)P_k^{\mathrm{lpk}}(t).$$

Just as we derived from Theorem 3.5 the odd part of Stanley's formula (2.2) for doubly alternating permutations, we can derive the even part of (2.2) from Theorem 4.5. First, we note that a permutation π in $\mathfrak{S}_n$ has at most $n/2$ left peaks and has exactly $n/2$ left peaks if and only if n is even and π is alternating. Then we have

$$\lim_{s\to 0} P_k^{\mathrm{lpk}}(s^{-2})s^k = \lim_{s\to 0}\sum_{\pi\in\mathfrak{S}_k} s^{k-2\,\mathrm{lpk}(\pi)} = \begin{cases} 0, & \text{if } n \text{ is odd,}\\ E_k, & \text{if } n \text{ is even,}\end{cases}$$

and similarly

$$\lim_{s\to 0} P_n^{(\mathrm{lpk},\mathrm{ilpk})}(s^{-2},t^{-2})s^n t^n = \begin{cases} 0, & \text{if } n \text{ is odd,}\\ \tilde{E}_n, & \text{if } n \text{ is even.}\end{cases}$$

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Therefore, to obtain the even part of Stanley's formula, we take Theorem 4.5 and replace s with s^{-2}, t with t^{-2} and u with stu; then we take the limit as $s, t \to 0$, similarly to the computation for the odd part.

4.4. Left peaks and inverse peaks

We proceed to give analogous formulas for the polynomials

$$P_n^{(\mathrm{lpk},\mathrm{ipk})}(s,t) := P_n^{(\mathrm{lpk},\mathrm{ipk})}(s,t,1,1) = \sum_{\pi\in\mathfrak{S}_n} s^{\mathrm{lpk}(\pi)}t^{\mathrm{ipk}(\pi)+1}$$

encoding the joint distribution of lpk and ipk over $\mathfrak{S}_n$.

Theorem 4.6. *We have*

$$\frac{1}{(1-s)(1-t)} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(1+s)^n(1+t)^{n+1}}{((1-s)(1-t))^{n+1}}P_n^{(\mathrm{lpk},\mathrm{ipk})}\left(\frac{4s}{(1+s)^2},\frac{4t}{(1+t)^2}\right)x^n = \sum_{i,j=0}^{\infty}\left(\frac{1+x}{1-x}\right)^{2ij+j}s^it^j \tag{a}$$

and, for all $n \geq 1$, we have

$$\frac{(1+s)^n(1+t)^{n+1}}{((1-s)(1-t))^{n+1}}P_n^{(\mathrm{lpk},\mathrm{ipk})}\left(\frac{4s}{(1+s)^2},\frac{4t}{(1+t)^2}\right) = 2\sum_{i,j=0}^{\infty}\sum_{k=0}^{n}\binom{2ij+j}{k}\binom{2ij+j+n-k-1}{n-k}s^it^j \tag{b}$$

and

$$\frac{(1+s)^n(1+t)^{n+1}}{((1-s)(1-t))^{n+1}}P_n^{(\mathrm{lpk},\mathrm{ipk})}\left(\frac{4s}{(1+s)^2},\frac{4t}{(1+t)^2}\right) = \frac{1}{n!}\sum_{k=0}^{n}2^{k+1}d(n,k)\frac{B_k(s)A_k(t)}{(1-s)^{k+1}(1-t)^{k+1}}. \tag{c}$$

Proof. The proof of parts (a) and (b) is the same as that of Theorem 4.3 (a) and (b), except that we specialize Theorem 4.2 as opposed to Theorem 4.1. To prove (c), notice that from Lemma 2.9 (a)–(b) we have

$$\begin{aligned}\left\langle\frac{1}{1-sE(x)H(x)},\frac{H}{1-tEH}\right\rangle &= \sum_{\lambda\text{ odd}}\sum_{\mu}\frac{2^{l(\lambda)}}{z_\lambda z_\mu}\frac{A_{l(\lambda)}(s)B_{o(\mu)}(t)}{(1-s)^{l(\lambda)+1}(1-t)^{o(\mu)+1}}\langle p_\lambda,p_\mu\rangle x^{|\mu|}\\ &= \sum_{\lambda\text{ odd}}\frac{2^{l(\lambda)}}{z_\lambda}\frac{A_{l(\lambda)}(s)B_{l(\lambda)}(t)}{(1-s)^{l(\lambda)+1}(1-t)^{l(\lambda)+1}}x^{|\lambda|},\end{aligned} \tag{4.5}$$

and the same scalar product can be shown to be equal to

$$\begin{aligned}\left\langle\frac{1}{1-sE(x)H(x)},\frac{H}{1-tEH}\right\rangle &= \frac{1}{(1-s)(1-t)}\\ &\quad+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(1+s)^{n+1}(1+t)^n}{((1-s)(1-t))^{n+1}}P_n^{(\mathrm{lpk},\mathrm{ipk})}\left(\frac{4s}{(1+s)^2},\frac{4t}{(1+t)^2}\right)x^n\end{aligned} \tag{4.6}$$

by using Lemma 2.8 (b)–(c). Equating Equations (4.5) and (4.6), extracting coefficients of x^n, and then applying Lemma 2.10 (b) completes the proof of (c). □

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Table 4. *Joint distribution of* lpk *and* ipk *over* $\mathfrak{S}_n$

n	$P_n^{(\mathrm{lpk},\mathrm{ipk})}(s,t)$
1	t
2	$(1+s)t$
3	$(1+3s)t+2st^2$
4	$(1+6s+s^2)t+(12s+4s^2)t^2$
5	$(1+10s+5s^2)t+(42s+46s^2)t^2+(6s+10s^2)t^3$
6	$(1+15s+15s^2+s^3)t+(112s+272s^2+32s^3)t^2+(52s+192s^2+28s^3)t^3$
7	$(1+21s+35s^2+7s^3)t+(252s+1136s^2+436s^3)t^2+(252s+1776s^2+852s^3)t^3+(18s+164s^2+90s^3)t^4$

Table 4 displays the first several polynomials $P_n^{(\mathrm{lpk},\mathrm{ipk})}(s,t)$. The following proposition, which is Corollary 11 of [26], characterizes the coefficients of $s^k t$ in $P_n^{(\mathrm{lpk},\mathrm{ipk})}(s,t)$.

Proposition 4.7. *For any $n \geq 1$ and $k \geq 0$, the number of permutations $\pi \in \mathfrak{S}_n$ with* $\mathrm{lpk}(\pi) = k$ *and* $\mathrm{ipk}(\pi) = 0$ *is equal to* $\binom{n}{2k}$.

The next formula expresses the polynomial $P_n^{(\mathrm{pk},\mathrm{ilpk})}(s,t)$ as a sum involving products of the peak and left peak polynomials. The proof follows in the same way as that of Theorem 3.4, except that we begin by substituting both Equations (3.6) and (4.4) into Theorem 4.6 (c).

Theorem 4.8. *For all $n \geq 1$, we have*

$$P_n^{(\mathrm{lpk},\mathrm{ipk})}(s,t) = \frac{1}{n!}\sum_{k=0}^{n} d(n,k)((1-s)(1-t))^{\frac{n-k}{2}} P_k^{\mathrm{lpk}}(s)P_k^{\mathrm{pk}}(t).$$

4.5. Left peaks and inverse descents

Finally, define

$$P_n^{(\mathrm{lpk},\mathrm{ides})}(s,t) \coloneqq P_n^{(\mathrm{lpk},\mathrm{ipk},\mathrm{des},\mathrm{ides})}(s,1,t,1) = \sum_{\pi\in\mathfrak{S}_n} s^{\mathrm{lpk}(\pi)} t^{\mathrm{ides}(\pi)+1}.$$

We omit the proofs of the following theorems as they are very similar to the proofs of analogous results presented earlier.

Theorem 4.9. *We have*

$$\frac{1}{(1-s)(1-t)} + \sum_{n=1}^{\infty} \frac{(1+s)^n}{((1-s)(1-t))^{n+1}} P_n^{(\mathrm{lpk},\mathrm{ides})}\left(\frac{4s}{(1+s)^2}, t\right)x^n = \sum_{i,j=0}^{\infty} \frac{(1+x)^{i(j+1)}}{(1-x)^{(i+1)(j+1)}} s^i t^j, \quad \text{(a)}$$

and, for all $n \geq 1$, we have

$$\frac{(1+s)^n}{((1-s)(1-t))^{n+1}} P_n^{(\mathrm{lpk},\mathrm{ides})}\left(\frac{4s}{(1+s)^2}, t\right) = \sum_{i,j=0}^{\infty}\sum_{k=0}^{n} \binom{i(j+1)}{k}\binom{ij+i+j+n-k}{n-k} s^i t^j \quad \text{(b)}$$

and

$$\frac{(1+s)^n}{((1-s)(1-t))^{n+1}} P_n^{(\mathrm{lpk},\mathrm{ides})}\left(\frac{4s}{(1+s)^2}, t\right) = \frac{1}{n!}\sum_{k,m=0}^{n} f(n,k,m) \frac{B_k(s)A_m(t)}{(1-s)^{k+1}(1-t)^{m+1}} \quad \text{(c)}$$

with $f(n,k,m)$ as defined in Section 2.4.

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Table 5. *Joint distribution of* lpk *and* ides *over* $\mathfrak{S}_n$

n	$P_n^{(\mathrm{lpk},\mathrm{ides})}(s,t)$
1	t
2	$t+st^2$
3	$t+4st^2+st^3$
4	$t+(10s+s^2)t^2+(7s+4s^2)t^3+st^4$
5	$t+(20s+6s^2)t^2+(27s+39s^2)t^3+(10s+16s^2)t^4+st^5$
6	$t+(35s+21s^2+s^3)t^2+(77s+205s^2+20s^3)t^3+(53s+213s^2+36s^3)t^4+(13s+40s^2+4s^3)t^5+st^6$
7	$t+(56s+56s^2+8s^3)t^2+(182s+776s^2+233s^3)t^3+(200s+1480s^2+736s^3)t^4+(88s+719s^2+384s^3)t^5+(16s+80s^2+24s^3)t^6+st^7$

Theorem 4.10. *For all $n \geq 1$, we have*

$$P_n^{(\mathrm{lpk},\mathrm{ides})}(s,t) = \frac{1}{n!}\sum_{k,m=0}^{n} f(n,k,m)(1-s)^{\frac{n-k}{2}}(1-t)^{n-m}P_k^{\mathrm{lpk}}(s)A_m(t).$$

See Table 5 for the first several polynomials $P_n^{(\mathrm{lpk},\mathrm{ides})}(s,t)$.

5. Up-down runs and biruns

We will now give analogous formulas for (mixed) two-sided distributions involving the number of up-down runs, as well as a couple involving the number of biruns.

5.1. Up-down runs and inverse up-down runs

Consider the two-sided up-down run polynomials

$$P_n^{(\mathrm{udr},\mathrm{iudr})}(s,t) := \sum_{\pi\in\mathfrak{S}_n} s^{\mathrm{udr}(\pi)}t^{\mathrm{iudr}(\pi)}.$$

Theorem 5.1. *We have*

$$\frac{1}{(1-s)(1-t)} + \frac{1}{4(1-s)^2(1-t)^2}\sum_{n=1}^{\infty}\frac{(1+s^2)^n(1+t^2)^n}{(1-s^2)^{n-1}(1-t^2)^{n-1}}P_n^{(\mathrm{udr},\mathrm{iudr})}\left(\frac{2s}{1+s^2},\frac{2t}{1+t^2}\right)x^n$$
$$= \sum_{i,j=0}^{\infty}\left(\frac{1+x}{1-x}\right)^{2ij}\left(1+s\left(\frac{1+x}{1-x}\right)^{j}+t\left(\frac{1+x}{1-x}\right)^{i}+st\frac{(1+x)^{i+j}}{(1-x)^{i+j+1}}\right)s^{2i}t^{2j}.$$

The proof of Theorem 5.1 follows the same structure as our proofs from Sections 3–4: We compute an appropriate scalar product in multiple ways and set them equal to each other. We shall provide the full proof here but will omit the proofs for all remaining results in this section due to their similarity with what has been presented earlier.

Proof. First, Lemma 2.8 (d) along with Foulkes's theorem (Theorem 2.1) leads to

$$\left\langle \frac{1+sH(x)}{1-s^2E(x)H(x)}, \frac{1+tH}{1-t^2EH}\right\rangle = \frac{1}{(1-s)(1-t)} + \frac{1}{4(1-s)^2(1-t)^2}\sum_{n=1}^{\infty}\frac{(1+s^2)^n(1+t^2)^n}{(1-s^2)^{n-1}(1-t^2)^{n-1}}P_n^{(\mathrm{udr},\mathrm{iudr})}\left(\frac{2s}{1+s^2},\frac{2t}{1+t^2}\right)x^n. \qquad (5.1)$$

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Next, using Lemmas 2.4 and 2.5, we obtain

$$\begin{aligned}\left\langle \frac{1+sH(x)}{1-s^2E(x)H(x)}, \frac{1}{1-t^2EH}\right\rangle &= \sum_{i,j=0}^{\infty}\langle (1+sH(x))E(x)^iH(x)^i, E^jH^j\rangle s^{2i}t^{2j}\\ &= \sum_{i,j=0}^{\infty}\Big(\langle E(x)^iH(x)^i, E^jH^j\rangle + s\langle E(x)^iH(x)^{i+1}, E^jH^j\rangle\Big)s^{2i}t^{2j}\\ &= \sum_{i,j=0}^{\infty}\Big(\langle E(x), E^jH^j\rangle^i\langle H(x), E^jH^j\rangle^i + s\langle E(x), E^jH^j\rangle^i\langle H(x), E^jH^j\rangle^{i+1}\Big)s^{2i}t^{2j}\\ &= \sum_{i,j=0}^{\infty}\left(\left(\frac{1+x}{1-x}\right)^{2ij} + s\left(\frac{1+x}{1-x}\right)^{2ij+j}\right)s^{2i}t^{2j}\\ &= \sum_{i,j=0}^{\infty}\left(\frac{1+x}{1-x}\right)^{2ij}\left(1+s\left(\frac{1+x}{1-x}\right)^j\right)s^{2i}t^{2j}. \end{aligned} \tag{5.2}$$

Similarly, using Lemmas 2.4–2.7, we obtain

$$\begin{aligned}\left\langle \frac{1+sH(x)}{1-s^2E(x)H(x)}, \frac{tH}{1-t^2EH}\right\rangle &= \sum_{i,j=0}^{\infty}\langle (1+sH(x))E(x)^iH(x)^i, E^jH^{j+1}\rangle s^{2i}t^{2j+1}\\ &= \sum_{i,j=0}^{\infty}\langle ((1+sH(x))E(x)^iH(x)^i)[X+1], E^jH^j\rangle s^{2i}t^{2j+1}\\ &= \sum_{i,j=0}^{\infty}\langle (E(x)^iH(x)^i + sE(x)^iH(x)^{i+1})[X+1], E^jH^j\rangle s^{2i}t^{2j+1}\\ &= \sum_{i,j=0}^{\infty}\langle (E(x)^iH(x)^i + sE(x)^iH(x)^{i+1})[X+1], E^jH^j\rangle s^{2i}t^{2j+1}\\ &= \sum_{i,j=0}^{\infty}\left\langle \left(\frac{1+x}{1-x}\right)^i E(x)^iH(x)^i + s\frac{(1+x)^i}{(1-x)^{i+1}}E(x)^iH(x)^{i+1}, E^jH^j\right\rangle s^{2i}t^{2j+1}\\ &= \sum_{i,j=0}^{\infty}\left(\left(\frac{1+x}{1-x}\right)^i\langle E(x), E^jH^j\rangle^i\langle H(x), E^jH^j\rangle^i\right.\\ &\qquad \left. + s\frac{(1+x)^i}{(1-x)^{i+1}}\langle E(x), E^jH^j\rangle^i\langle H(x), E^jH^j\rangle^{i+1}\right)s^{2i}t^{2j+1}\\ &= \sum_{i,j=0}^{\infty}\left(\left(\frac{1+x}{1-x}\right)^{2ij+i} + s\frac{(1+x)^{2ij+i+j}}{(1-x)^{2ij+i+j+1}}\right)s^{2i}t^{2j+1}\\ &= \sum_{i,j=0}^{\infty}\left(\frac{1+x}{1-x}\right)^{2ij}\left(t\left(\frac{1+x}{1-x}\right)^i + st\frac{(1+x)^{i+j}}{(1-x)^{i+j+1}}\right)s^{2i}t^{2j}. \end{aligned} \tag{5.3}$$

Summing Equations (5.2) and (5.3) yields

$$\left\langle \frac{1+sH(x)}{1-s^2E(x)H(x)}, \frac{1+tH}{1-t^2EH}\right\rangle = \sum_{i,j=0}^{\infty}\left(\frac{1+x}{1-x}\right)^{2ij}\left(1+s\left(\frac{1+x}{1-x}\right)^j + t\left(\frac{1+x}{1-x}\right)^i + st\frac{(1+x)^{i+j}}{(1-x)^{i+j+1}}\right)s^{2i}t^{2j};$$

comparing this with Equation (5.1) completes the proof. □

Table 6. *Joint distribution of* udr *and* iudr *over* $\mathfrak{S}_n$

n	$P_n^{(\mathrm{udr},\mathrm{iudr})}(s,t)$
1	st
2	$st + s^2t^2$
3	$st + (2s^2 + s^3)t^2 + (s^2 + s^3)t^3$
4	$st + (3s^2 + 3s^3 + s^4)t^2 + (3s^2 + 6s^3 + 2s^4)t^3 + (s^2 + 2s^3 + 2s^4)t^4$
5	$st + (4s^2 + 6s^3 + 4s^4 + s^5)t^2 + (6s^2 + 19s^3 + 13s^4 + 5s^5)t^3 + (4s^2 + 13s^3 + 21s^4 + 7s^5)t^4 + (s^2 + 5s^3 + 7s^4 + 3s^5)t^5$
6	$st + (5s^2 + 10s^3 + 10s^4 + 5s^5 + s^6)t^2 + (10s^2 + 45s^3 + 47s^4 + 38s^5 + 8s^6)t^3 + (10s^2 + 47s^3 + 109s^4 + 78s^5 + 24s^6)t^4$ $+ (5s^2 + 38s^3 + 78s^4 + 70s^5 + 20s^6)t^5 + (s^2 + 8s^3 + 24s^4 + 20s^5 + 8s^6)t^6$

The first several polynomials $P_n^{(\mathrm{udr},\mathrm{iudr})}(s,t)$ are given in Table 6. We note that the permutations in $\mathfrak{S}_n$ with n up-down runs are precisely the alternating permutations in $\mathfrak{S}_n$, so the coefficient of s^nt^n of $P_n^{(\mathrm{udr},\mathrm{iudr})}(s,t)$ is the number of doubly alternating permutations in $\mathfrak{S}_n$. In fact, Stanley's formula (2.2) for doubly alternating permutations can be derived from Theorem 5.1, although it is simpler to work with our formulas for (pk, ipk) and (lpk, ilpk).

To invert Theorem 5.1 and other formulas in this section, we will need to solve $s = 2u/(1+u^2)$ for u, and the solution is given by $u = s^{-1}(1-\sqrt{1-s^2}) = (s/2)C(s^2/4)$, where $C(x)$ is the Catalan number generating function. Hence, Theorem 5.1 can be written as

$$\frac{1}{(1-u)(1-v)} + \frac{1}{4(1-u)^2(1-v)^2}\sum_{n=1}^{\infty}\frac{(1+u^2)^n(1+v^2)^n}{(1-u^2)^{n-1}(1-v^2)^{n-1}}P_n^{(\mathrm{udr},\mathrm{iudr})}(s,t)x^n$$
$$= \sum_{i,j=0}^{\infty}\left(\frac{1+x}{1-x}\right)^{2ij}\left(1 + u\left(\frac{1+x}{1-x}\right)^j + v\left(\frac{1+x}{1-x}\right)^i + uv\frac{(1+x)^{i+j}}{(1-x)^{i+j+1}}\right)u^{2i}v^{2j},$$

where $u = s^{-1}(1-\sqrt{1-s^2}) = (s/2)C(s^2/4)$ and $v = t^{-1}(1-\sqrt{1-t^2}) = (t/2)C(t^2/4)$.

5.2. Up-down runs, inverse peaks and inverse descents

Next, let us study the statistic (udr, ipk, ides) and its specializations (udr, ipk) and (udr, ides). Define

$$P_n^{(\mathrm{udr},\mathrm{ipk},\mathrm{ides})}(s,y,t) := \sum_{\pi\in\mathfrak{S}_n} s^{\mathrm{udr}(\pi)}y^{\mathrm{ipk}(\pi)+1}t^{\mathrm{ides}(\pi)+1},$$
$$P_n^{(\mathrm{udr},\mathrm{ipk})}(s,t) := P_n^{(\mathrm{udr},\mathrm{ipk},\mathrm{ides})}(s,t,1) = \sum_{\pi\in\mathfrak{S}_n} s^{\mathrm{udr}(\pi)}t^{\mathrm{ipk}(\pi)+1} \text{ and}$$
$$P_n^{(\mathrm{udr},\mathrm{ides})}(s,t) := P_n^{(\mathrm{udr},\mathrm{ipk},\mathrm{ides})}(s,1,t) = \sum_{\pi\in\mathfrak{S}_n} s^{\mathrm{udr}(\pi)}t^{\mathrm{ides}(\pi)+1}.$$

Theorem 5.2. *We have*

$$\frac{1}{(1-s)(1-t)} + \sum_{n=1}^{\infty}\frac{(1+s^2)^n}{(1-s^2)^{n-1}}\left(\frac{1+yt}{1-t}\right)^{n+1}\frac{P_n^{(\mathrm{udr},\mathrm{ipk},\mathrm{ides})}\left(\frac{2s}{1+s^2},\frac{(1+y)^2t}{(y+t)(1+yt)},\frac{y+t}{1+yt}\right)}{2(1-s)^2(1+y)}x^n \tag{a}$$
$$= \sum_{i,j=0}^{\infty}\left(\frac{(1+x)(1+yx)}{(1-yx)(1-x)}\right)^{ij}\left(1 + s\left(\frac{1+yx}{1-x}\right)^j\right)s^{2i}t^j$$

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and, for all $n \geq 1$, we have

$$\frac{(1+s^2)^n}{(1-s^2)^{n-1}}\left(\frac{1+yt}{1-t}\right)^{n+1}\frac{P_n^{(\mathrm{udr,ipk,ides})}\left(\frac{2s}{1+s^2},\frac{(1+y)^2t}{(y+t)(1+yt)},\frac{y+t}{1+yt}\right)}{2(1-s)^2(1+y)}$$
$$=\sum_{\substack{\lambda\vdash n\\ \text{odd}}}\frac{2^{l(\lambda)}}{z_\lambda}\frac{A_{l(\lambda)}(s^2)A_{l(\lambda)}(t)}{(1-s^2)^{l(\lambda)+1}(1-t)^{l(\lambda)+1}}\prod_{k=1}^{l(\lambda)}(1-(-y)^{\lambda_k})$$
$$+s\sum_{\lambda\vdash n}\frac{1}{z_\lambda}\frac{B_{o(\lambda)}(s^2)A_{l(\lambda)}(t)}{(1-s^2)^{o(\lambda)+1}(1-t)^{l(\lambda)+1}}\prod_{k=1}^{l(\lambda)}(1-(-y)^{\lambda_k}). \tag{b}$$

As before, we set $y = 1$ to specialize to (udr, ipk), immediately arriving at parts (a) and (b) of the following theorem. Part (c) is proven similarly to Theorem 3.4, except that we also use the formula

$$2^nA_n(t^2)+tB_n(t^2)=\frac{(1+t)^2(1+t^2)^n}{2}P_n^{\mathrm{udr}}\left(\frac{2t}{1+t^2}\right)$$

[15, Section 6.3], where

$$P_n^{\mathrm{udr}}(t) := \sum_{\pi\in\mathfrak{S}_n}t^{\mathrm{udr}(\pi)}.$$

Theorem 5.3. *We have*

$$\frac{1}{(1-s)(1-t)}+\frac{1}{4(1-s)^2}\sum_{n=1}^{\infty}\frac{(1+s^2)^n}{(1-s^2)^{n-1}}\left(\frac{1+t}{1-t}\right)^{n+1}P_n^{(\mathrm{udr,ipk})}\left(\frac{2s}{1+s^2},\frac{4t}{(1+t)^2}\right)x^n$$
$$=\sum_{i,j=0}^{\infty}\left(\frac{1+x}{1-x}\right)^{2ij}\left(1+s\left(\frac{1+x}{1-x}\right)^j\right)s^{2i}t^j \tag{a}$$

and, for all $n \geq 1$, we have

$$\frac{1}{4(1-s)^2}\frac{(1+s^2)^n}{(1-s^2)^{n-1}}\left(\frac{1+t}{1-t}\right)^{n+1}P_n^{(\mathrm{udr,ipk})}\left(\frac{2s}{1+s^2},\frac{4t}{(1+t)^2}\right)$$
$$=\frac{1}{n!}\sum_{k=0}^{n}2^kd(n,k)\frac{(2^kA_k(s^2)+sb:k(s^2))A_k(t)}{(1-s^2)^{k+1}(1-t)^{k+1}} \tag{b}$$

and

$$P_n^{(\mathrm{udr,ipk})}(s,t)=\frac{1}{n!}\sum_{k=0}^{n}d(n,k)\Big((1-s^2)(1-t)\Big)^{\frac{n-k}{2}}P_k^{\mathrm{udr}}(s)P_k^{\mathrm{pk}}(t). \tag{c}$$

The first several polynomials $P_n^{(\mathrm{udr,ipk})}(s,t)$ are displayed in Table 7. The coefficients of s^kt in $P_n^{(\mathrm{udr,ipk})}(s,t)$ are binomial coefficients, just like the coefficients of st^k in $P_n^{(\mathrm{pk,ides})}(s,t)$ from Table 2.

Proposition 5.4. *For any $n \geq 1$ and $k \geq 0$, the number of permutations $\pi \in \mathfrak{S}_n$ with $\mathrm{udr}(\pi) = k$ and $\mathrm{ipk}(\pi) = 0$ is equal to $\binom{n-1}{k-1}$.*

Unlike the analogous results given earlier, Proposition 5.4 was not proven in [26], so we shall supply a proof here.

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Table 7. *Joint distribution of* udr *and* ipk *over* $\mathfrak{S}_n$

n	$P_n^{(\mathrm{udr},\mathrm{ipk})}(s,t)$
1	st
2	$(s+s^2)t$
3	$(s+2s^2+s^3)t+(s^2+s^3)t^2$
4	$(s+3s^2+3s^3+s^4)t+(4s^2+8s^3+4s^4)t^2$
5	$(s+4s^2+6s^3+4s^4+s^5)t+(10s^2+32s^3+34s^4+12s^5)t^2+(s^2+5s^3+7s^4+3s^5)t^3$
6	$(s+5s^2+10s^3+10s^4+5s^5+s^6)t+(20s^2+92s^3+156s^4+116s^5+32s^6)t^2+(6s^2+46s^3+102s^4+90s^5+28s^6)t^3$
7	$(s+6s^2+15s^2+20s^4+15s^5+6s^6+s^7)t+(35s^2+217s^3+522s^4+614s^5+355s^6+81s^7)t^2+(21s^2+231s^3+738s^4$ $+1038s^5+681s^6+171s^7)t^3+(s^2+17s^3+64s^4+100s^5+71s^6+19s^7)t^4$

Proof. Let us define the *up-down composition* of a permutation π, denoted $\mathrm{udComp}(\pi)$, in the following way: If $u_1, u_2, \ldots, u_k$ are the lengths of the up-down runs of π in the order that they appear, then $\mathrm{udComp}(\pi) := (u_1, u_2 - 1, \ldots, u_k - 1)$. For example, if $\pi = 312872569$, then $\mathrm{udComp}(\pi) = (1, 1, 2, 2, 3)$. Note that if $\pi \in \mathfrak{S}_n$, then $\mathrm{udComp}(\pi)$ is a composition of n. It is not hard to verify that the descent composition determines the up-down composition and vice versa.

According to [26, Theorem 5], for any composition $L \vDash n$, there exists exactly one permutation $\pi \in \mathfrak{S}_n$ with descent composition L such that $\mathrm{ipk}(\pi) = 0$. In light of this fact and Proposition 3.7, it suffices to construct a bijection between compositions $L \vDash n$ with k parts and compositions $K \vDash n$ satisfying $\mathrm{udr}(K) = k$. This bijection is obtained by mapping L to the descent composition K of any permutation with up-down composition L. For example, the composition $L = (1, 1, 2, 2, 3)$ is mapped to $K = (1, 3, 1, 4)$, which is the descent composition of the permutation π from above. □

Setting $y = 0$ in Theorem 5.2 yields the analogous result for (udr, ides).

Theorem 5.5. *We have*

$$\frac{1}{(1-s)(1-t)} + \frac{1}{2(1-s)^2}\sum_{n=1}^{\infty}\frac{(1+s^2)^n}{(1-s^2)^{n-1}(1-t)^{n+1}}P_n^{(\mathrm{udr},\mathrm{ides})}\left(\frac{2s}{1+s^2},t\right)x^n \qquad \text{(a)}$$
$$= \sum_{i,j=0}^{\infty}\left(\frac{1+x}{1-x}\right)^{ij}\left(1+\frac{s}{(1-x)^j}\right)s^{2i}t^j$$

and, for all $n \geq 1$, we have

$$\frac{(1+s^2)^n}{2(1-s)^2(1-s^2)^{n-1}(1-t)^{n+1}}P_n^{(\mathrm{udr},\mathrm{ides})}\left(\frac{2s}{1+s^2},t\right)x^n \qquad \text{(b)}$$
$$= \frac{1}{n!}\sum_{k=0}^{n}2^k d(n,k)\frac{A_k(s^2)A_k(t)}{(1-s^2)^{k+1}(1-t)^{k+1}} + \frac{s}{n!}\sum_{k,m=0}^{n} f(n,k,m)\frac{B_k(s^2)A_m(t)}{(1-s^2)^{k+1}(1-t)^{m+1}}.$$

Table 8 contains the first several polynomials $P_n^{(\mathrm{udr},\mathrm{ides})}(s,t)$.

5.3. Up-down runs, inverse left peaks and inverse descents

Define

$$P_n^{(\mathrm{udr},\mathrm{ilpk},\mathrm{ides})}(s,y,t) := \sum_{\pi\in\mathfrak{S}_n} s^{\mathrm{udr}(\pi)}y^{\mathrm{ilpk}(\pi)}t^{\mathrm{ides}(\pi)} \quad \text{and}$$
$$P_n^{(\mathrm{udr},\mathrm{ilpk})}(s,t) := P_n^{(\mathrm{udr},\mathrm{ilpk},\mathrm{ides})}(s,t,1) = \sum_{\pi\in\mathfrak{S}_n} s^{\mathrm{udr}(\pi)}t^{\mathrm{ilpk}(\pi)}.$$

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Table 8. *Joint distribution of* udr *and* ides *over* $\mathfrak{S}_n$

n	$P_n^{(\mathrm{udr},\mathrm{ides})}(s,t)$
1	st
2	$st + s^2t^2$
3	$st + (2s^2 + 2s^3)t^2 + s^2t^3$
4	$st + (3s^2 + 7s^3 + s^4)t^2 + (3s^2 + 4s^3 + 4s^4)t^3 + s^2t^4$
5	$st + (4s^2 + 16s^3 + 4s^4 + 2s^5)t^2 + (6s^2 + 21s^3 + 27s^4 + 12s^5)t^3 + (4s^2 + 6s^3 + 14s^4 + 2s^5)t^4 + s^2t^5$
6	$st + (5s^2 + 30s^3 + 10s^4 + 11s^5 + s^6)t^2 + (10s^2 + 67s^3 + 101s^4 + 104s^5 + 20s^6)t^3 + (10s^2 + 43s^3 + 125s^4 + 88s^5 + 36s^6)t^4 + (5s^2 + 8s^3 + 32s^4 + 8s^5 + 4s^6)t^5 + s^2t^6$

Theorem 5.6. *We have*

$$\frac{1}{(1-s)(1-t)} + \frac{1}{2(1-s)^2}\sum_{n=1}^{\infty}\frac{(1+s^2)^n(1+yt)^nP_n^{(\mathrm{udr},\mathrm{ilpk},\mathrm{ides})}\left(\frac{2s}{1+s^2},\frac{(1+y)^2t}{(y+t)(1+yt)},\frac{y+t}{1+yt}\right)}{(1-s^2)^{n-1}(1-t)^{n+1}}x^n$$
$$= \sum_{i,j=0}^{\infty}\left(\frac{1+x}{1-x}\right)^{i(j+1)}\left(\frac{1+yx}{1-yx}\right)^{ij}\left(1+s\frac{(1+yx)^j}{(1-x)^{j+1}}\right)s^{2i}t^j.$$

Setting $y = 1$ in Theorem 5.6 yields part (a) of the following theorem. Part (b) is obtained by computing the scalar product

$$\left\langle \frac{1+sH(x)}{1-sE(x)H(x)}, \frac{H}{1-tEH}\right\rangle$$

using Lemmas 2.8 (c)–(d) and 2.9 (b)–(c). Part (c) is obtained from (b) in a way similar to the proof of Theorem 4.4, making use of Equation (4.4).

Theorem 5.7. *We have*

$$\frac{1}{(1-s)(1-t)} + \frac{1}{2(1-s)^2}\sum_{n=1}^{\infty}\frac{(1+s^2)^n(1+t)^n}{(1-s^2)^{n-1}(1-t)^{n+1}}P_n^{(\mathrm{udr},\mathrm{ilpk})}\left(\frac{2s}{1+s^2},\frac{4t}{(1+t)^2}\right)x^n$$
$$= \sum_{i,j=0}^{\infty}\left(\frac{1+x}{1-x}\right)^{i(2j+1)}\left(1+s\frac{(1+x)^j}{(1-x)^{j+1}}\right)s^{2i}t^j \qquad \text{(a)}$$

and, for all $n \geq 1$, we have

$$\frac{(1+s^2)^n(1+t)^n}{2(1-s)^2(1-s^2)^{n-1}(1-t)^{n+1}}P_n^{(\mathrm{udr},\mathrm{ilpk})}\left(\frac{2s}{1+s^2},\frac{4t}{(1+t)^2}\right) \qquad \text{(b)}$$
$$= \frac{1}{n!}\sum_{k=0}^{n}2^kd(n,k)\frac{A_k(s^2)B_k(t)}{(1-s^2)^{k+1}(1-t)^{k+1}} + \frac{s}{n!}\sum_{k=0}^{n}e(n,k)\frac{B_k(s^2)B_k(t)}{(1-s^2)^{k+1}(1-t)^{k+1}}$$

Table 9. *Joint distribution of* udr *and* ilpk *over* $\mathfrak{S}_n$

n	$P_n^{(\mathrm{udr},\mathrm{ilpk})}(s,t)$
1	s
2	$s+s^2t$
3	$s+(3s^2+2s^3)t$
4	$s+(6s^2+9s^3+3s^4)t+(s^2+2s^3+2s^4)t^2$
5	$s+(10s^2+25s^3+17s^4+6s^5)t+(5s^2+18s^3+28s^4+10s^5)t^2$
6	$s+(15s^2+55s^3+57s^4+43s^5+9s^6)t+(15s^2+85s^3+187s^4+148s^5+44s^6)t^2+(s^2+8s^3+24s^4+20s^5+8s^6)t^3$
7	$s+(21s^2+105s^3+147s^4+177s^5+75s^6+18s^7)t+(35s^2+289s^3+847s^4+1104s^5+672s^6+164s^7)t^2+(7s^2+86s^3+350s^4+486s^5+366s^6+90s^7)t^3$

and

$$(1+s)^2(1+s^2)^n P_n^{(\mathrm{udr},\mathrm{ilpk})}\left(\frac{2s}{1+s^2},t\right) \tag{c}$$
$$=\frac{1}{n!}\sum_{k=0}^{n}2^{k+1}d(n,k)\Big((1-s^2)(1-t)^{1/2}\Big)^{n-k}A_k(s^2)P_k^{\mathrm{lpk}}(t)$$
$$+\frac{2s}{n!}\sum_{k=0}^{n}e(n,k)\Big((1-s^2)(1-t)^{1/2}\Big)^{n-k}B_k(s^2)P_k^{\mathrm{lpk}}(t).$$

See Table 9 for the first several polynomials $P_n^{(\mathrm{udr},\mathrm{ilpk})}(s,t)$.

5.4. Biruns

Recall that a birun is a maximal monotone consecutive subsequence, whereas an up-down run is either a birun or an initial descent. In [15], the authors gave the formula

$$\frac{2+tH(x)+tE(x)}{1-t^2E(x)H(x)}=\frac{2}{1-t}+\frac{2t}{(1-t)^2}xh_1+\frac{(1+t)^3}{2(1-t)}\sum_{n=2}^{\infty}\sum_{L\vDash n}\frac{(1+t^2)^{n-1}}{(1-t^2)^n}\left(\frac{2t}{1+t^2}\right)^{\mathrm{br}(L)}x^n r_L$$

(cf. Lemma 2.8), which allows us to produce formulas for (mixed) two-sided distributions involving the number of biruns. For example, computing the scalar product

$$\left\langle \frac{2+sH(x)+sE(x)}{1-s^2E(x)H(x)},\frac{2+tH+tE}{1-t^2EH}\right\rangle \tag{5.4}$$

would yield a formula for the two-sided distribution of br, but we chose not to derive this formula as it would be complicated to write down. Looking back at the formula in Theorem 5.1 for the two-sided distribution of udr, we see that the right-hand side is a summation whose summands are each a sum involving four terms, which is because the numerators in the scalar product

$$\left\langle \frac{1+sH(x)}{1-s^2E(x)H(x)},\frac{1+tH}{1-t^2EH}\right\rangle$$

that we sought to compute has two terms each. The numerators in the scalar product (5.4) contain three terms each, which will lead to nine terms in the formula for $(\mathrm{br},\mathrm{ibr})$ as opposed to four.

On the other hand, formulas for the polynomials

$$P_n^{(\mathrm{br},\mathrm{ipk})}(s,t):=\sum_{\pi\in\mathfrak{S}_n}s^{\mathrm{br}(\pi)}t^{\mathrm{ipk}(\pi)+1}\quad\text{and}\quad P_n^{(\mathrm{br},\mathrm{ides})}(s,t):=\sum_{\pi\in\mathfrak{S}_n}s^{\mathrm{br}(\pi)}t^{\mathrm{ides}(\pi)+1}$$

have fewer such terms, and so we present them below.

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Table 10. *Joint distribution of* br *and* ipk *over* $\mathfrak{S}_n$

n	$P_n^{(\mathrm{br},\mathrm{ipk})}(s,t)$
1	st
2	$2st$
3	$(2s+2s^2)t+2s^2t^2$
4	$(2s+4s^2+2s^3)t+(8s^2+8s^3)t^2$
5	$(2s+6s^2+6s^3+2s^4)t+(20s^2+44s^3+24s^4)t^2+(2s^2+8s^3+6s^4)t^3$
6	$(2s+8s^2+12s^3+8s^4+2s^5)t+(40s^2+144s^3+168s^4+64s^5)t^2+(12s^2+80s^3+124s^4+56s^5)t^3$
7	$(2s+10s^2+20s^3+20s^4+10s^5+2s^6)t+(70s^2+364s^3+680s^4+548s^5+162s^6)t^2+(42s^2+420s^3+1056s^4$ $+1020s^5+342s^6)t^3+(2s^2+32s^3+96s^4+104s^5+38s^6)t^4$

Table 11. *Joint distribution of* br *and* ides *over* $\mathfrak{S}_n$

n	$P_n^{(\mathrm{br},\mathrm{ides})}(s,t)$
1	ts
2	$(t+t^2)s$
3	$(t+t^3)s+4t^2s^2$
4	$(t+t^4)s+(6t^2+6t^3)s^2+(5t^2+5t^3)s^3$
5	$(t+t^5)s+(8t^2+12t^3+8t^4)s^2+(14t^2+30t^3+14t^4)s^3+(4t^2+24t^3+4t^4)s^4$
6	$(t+t^6)s+(10t^2+20t^3+20t^4+10t^5)s^2+(28t^2+90t^3+90t^4+28t^5)s^3+(14t^2+136t^3+136t^4+14t^5)s^4+(5t^2$ $+56t^3+56t^4+5t^5)s^5$

Theorem 5.8. *We have*

$$\frac{1}{(1-s)(1-t)}+\frac{2stx}{(1-s)^2(1-t)^2}+\frac{(1+s)^3}{8(1-s)}\sum_{n=2}^{\infty}\frac{(1+s^2)^{n-1}(1+t)^{n+1}P_n^{(\mathrm{br},\mathrm{ipk})}\left(\frac{2s}{1+s^2},\frac{4t}{(1+t)^2}\right)}{(1-s^2)^n(1-t)^{n+1}}x^n \quad \text{(a)}$$
$$=\sum_{i,j=0}^{\infty}\left(\frac{1+x}{1-x}\right)^{2ij}\left(1+s\left(\frac{1+x}{1-x}\right)^j\right)s^{2i}t^j$$

and

$$\frac{1}{(1-s)(1-t)}+\frac{stx}{(1-s)^2(1-t)^2}+\frac{(1+s)^3}{4(1-s)}\sum_{n=2}^{\infty}\frac{(1+s^2)^{n-1}P_n^{(\mathrm{br},\mathrm{ides})}\left(\frac{2s}{1+s^2},t\right)}{(1-s^2)^n(1-t)^{n+1}}x^n \quad \text{(b)}$$
$$=\frac{1}{2}\sum_{i,j=0}^{\infty}\left(\frac{1+x}{1-x}\right)^{ij}\left(2+\frac{s}{(1-x)^j}+s(1+x)^j\right)s^{2i}t^j.$$

The first several polynomials $P_n^{(\mathrm{br},\mathrm{ipk})}(s,t)$ and $P_n^{(\mathrm{br},\mathrm{ides})}(s,t)$ are displayed in Tables 10–11. Notice that the coefficients of s^kt in $P_n^{(\mathrm{br},\mathrm{ipk})}(t)$ are twice the coefficients of s^kt in $P_{n-1}^{(\mathrm{udr},\mathrm{ipk})}(t)$; we shall give a simple proof of this fact.

Proposition 5.9. *For any $n\geq 2$ and $k\geq 0$, the number of permutations in $\mathfrak{S}_n$ with $\mathrm{br}(\pi)=k$ and $\mathrm{ipk}(\pi)=0$ is equal to $2\binom{n-2}{k-1}$, twice the number of permutations in $\mathfrak{S}_{n-1}$ with $\mathrm{udr}(\pi)=k$ and $\mathrm{ipk}(\pi)=0$.*

Proof. In light of Proposition 5.4 and its proof, it suffices to construct a one-to-two map from compositions $L\vDash n-1$ with k parts to compositions $K\vDash n$ satisfying $\mathrm{br}(K)=k$. We claim that such a map is given by sending $L=(L_1,L_2,\dots,L_k)$ to the descent compositions corresponding to the up-down compositions $J=(L_1+1,L_2,\dots,L_k)$ and $J'=(1,L_1,L_2,\dots,L_k)$. It is easy to see that $\mathrm{br}(\pi)=k$ if and only if $\mathrm{udComp}(\pi)$ has k parts with initial part greater than 1 or if $\mathrm{udComp}(\pi)$ has $k+1$ parts with initial part 1, so this map is well defined. For example, let $L=(4,1,1,2)$. Then permutations with

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up-down compositions $J = (5, 1, 1, 2)$ and $J' = (1, 4, 1, 1, 2)$ have descent compositions $K = (5, 2, 1, 1)$ and $K' = (1, 1, 1, 1, 2, 3)$, respectively, so our map sends L to K and K'. The reverse procedure is given by first taking the up-down composition corresponding to the descent composition input and then removing the first part if it is equal to 1 and subtracting 1 from the first part otherwise. □

In Table 11, we first collect powers of s in displaying the $P_n^{(\mathrm{br},\mathrm{ides})}(s,t)$ to showcase the symmetry present in the coefficients of s^k. This symmetry can be explained in the same way as for the polynomials $P_n^{(\mathrm{pk},\mathrm{ides})}(s,t)$: Upon taking reverses, we have that $\mathrm{ides}(\pi^r) = n - 1 - \mathrm{ides}(\pi)$ but $\mathrm{br}(\pi^r) = \mathrm{br}(\pi)$. Note that the coefficients of s^k for $k \geq 2$ also seem to be unimodal, but in general they are not γ-positive.

6. Major index

6.1. A rederivation of the Garsia–Gessel formula

We now return full circle by showing how our approach can be used to rederive Garsia and Gessel's formula (1.4) for the polynomials $A_n(s,t,q,r) = \sum_{\pi\in\mathfrak{S}_n} s^{\mathrm{des}(\pi)} t^{\mathrm{ides}(\pi)} q^{\mathrm{maj}(\pi)} r^{\mathrm{imaj}(\pi)}$.

Proof of the Garsia–Gessel formula. We seek to compute the the scalar product

$$\left\langle \sum_{i=0}^{\infty} s^i \prod_{k=0}^{i} H(q^k x), \sum_{j=0}^{\infty} t^j \prod_{l=0}^{j} H(r^l) \right\rangle$$

in two different ways. First, from Lemma 2.8 (a) we have

$$\begin{aligned}
&\left\langle \sum_{i=0}^{\infty} s^i \prod_{k=0}^{i} H(q^k x), \sum_{j=0}^{\infty} t^j \prod_{l=0}^{j} H(r^l) \right\rangle \\
&\qquad = \sum_{m,n=0}^{\infty} \frac{\sum_{L\vDash m, M\vDash n} s^{\mathrm{des}(L)} t^{\mathrm{des}(M)} q^{\mathrm{maj}(L)} r^{\mathrm{maj}(M)} \langle r_L, r_M\rangle}{(1-s)(1-qs)\cdots(1-q^m s)(1-t)(1-rt)\cdots(1-r^n t)} x^m \\
&\qquad = \sum_{n=0}^{\infty} \frac{\sum_{\pi\in\mathfrak{S}_n} s^{\mathrm{des}(\pi)} t^{\mathrm{ides}(\pi)} q^{\mathrm{maj}(\pi)} r^{\mathrm{imaj}(\pi)}}{(1-s)(1-qs)\cdots(1-q^n s)(1-t)(1-rt)\cdots(1-r^n t)} x^n \\
&\qquad = \sum_{n=0}^{\infty} \frac{A_n(s,t,q,r)}{(1-s)(1-qs)\cdots(1-q^n s)(1-t)(1-rt)\cdots(1-r^n t)} x^n,
\end{aligned} \tag{6.1}$$

where, as usual, we apply Foulkes's theorem (Theorem 2.1) to calculate the scalar product $\langle r_L, r_M\rangle$. Second, observe that

$$\begin{aligned}
\left\langle \sum_{i=0}^{\infty} s^i \prod_{k=0}^{i} H(q^k x), \sum_{j=0}^{\infty} t^j \prod_{l=0}^{j} H(r^l) \right\rangle &= \sum_{i,j=0}^{\infty} s^i t^i \left\langle \prod_{k=0}^{i} H(q^k x), \prod_{l=0}^{j} H(r^l) \right\rangle \\
&= \sum_{i,j=0}^{\infty} s^i t^i \prod_{k=0}^{i} \prod_{l=0}^{j} \left\langle H(q^k x), H(r^l) \right\rangle
\end{aligned}$$

by Lemma 2.3. To evaluate $\left\langle H(q^k x), H(r^l)\right\rangle$, let us recall that $h_n = \sum_{\lambda\vdash n} m_\lambda$. Then

$$\begin{aligned}
\left\langle H(q^k x), H(r^l)\right\rangle &= \left\langle \sum_{i=0}^{\infty} h_i q^{ki} x^i, \sum_{j=0}^{\infty} h_j r^{lj} \right\rangle \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\lambda\vdash i} \left\langle m_\lambda, h_j\right\rangle q^{ki} x^i r^{lj}
\end{aligned}$$

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$$= \sum_{j=0}^{\infty} q^{kj} x^j r^{lj}$$
$$= \frac{1}{1 - xq^k r^l},$$

whence

$$\left\langle \sum_{i=0}^{\infty} s^i \prod_{k=0}^{i} H(q^k x), \sum_{j=0}^{\infty} t^j \prod_{l=0}^{j} H(r^l) \right\rangle = \sum_{i,j=0}^{\infty} s^i t^i \prod_{k=0}^{i} \prod_{l=0}^{j} \frac{1}{1 - xq^k r^l}. \tag{6.2}$$

Combining Equations (6.1) and (6.2) yields Garsia and Gessel's formula (1.4). □

6.2. Major index and other statistics

Lemma 2.8 (a) also enables us to derive formulas for mixed two-sided distributions involving the major index. Define

$$P_n^{(\mathrm{maj},\mathrm{ipk},\mathrm{des},\mathrm{ides})}(q,y,s,t) := \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{maj}(\pi)} y^{\mathrm{ipk}(\pi)+1} s^{\mathrm{des}(\pi)} t^{\mathrm{ides}(\pi)+1},$$
$$P_n^{(\mathrm{maj},\mathrm{ilpk},\mathrm{des},\mathrm{ides})}(q,y,s,t) := \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{maj}(\pi)} y^{\mathrm{ilpk}(\pi)} s^{\mathrm{des}(\pi)} t^{\mathrm{ides}(\pi)}, \quad \text{and}$$
$$P_n^{(\mathrm{maj},\mathrm{iudr},\mathrm{des})}(q,s,t) := \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{maj}(\pi)} s^{\mathrm{iudr}(\pi)} t^{\mathrm{des}(\pi)}.$$

We leave the proofs of the following formulas to the interested reader.

Theorem 6.1. *We have*

$$\frac{1}{(1-s)(1-t)} + \frac{1}{1+y} \sum_{n=1}^{\infty} \left(\frac{1+yt}{1-t}\right)^{n+1} \frac{P_n^{(\mathrm{maj},\mathrm{ipk},\mathrm{des},\mathrm{ides})}\left(q, \frac{(1+y)^2 t}{(y+t)(1+yt)}, s, \frac{y+t}{1+yt}\right)}{(1-s)(1-qs)\cdots(1-q^n s)} x^n$$
$$= \sum_{i,j=0}^{\infty} s^i t^j \prod_{k=0}^{i} \left(\frac{1+q^k yx}{1-q^k x}\right)^j, \tag{a}$$

$$\frac{1}{(1-s)(1-t)} + \sum_{n=1}^{\infty} \frac{(1+t)^n}{(1-t)^{n+1}} \frac{P_n^{(\mathrm{maj},\mathrm{ilpk},\mathrm{des},\mathrm{ides})}\left(q, \frac{(1+y)^2 t}{(y+t)(1+yt)}, s, \frac{y+t}{1+yt}\right)}{(1-s)(1-qs)\cdots(1-q^n s)} x^n$$
$$= \sum_{i,j=0}^{\infty} s^i t^j \prod_{k=0}^{i} \frac{(1+q^k yx)^j}{(1-q^k x)^{j+1}} \tag{b}$$

and

$$\frac{1}{(1-s)(1-t)} + \frac{1}{2(1-s)^2} \sum_{n=1}^{\infty} \frac{(1+s^2)^n}{(1-s^2)^{n-1}} \frac{P_n^{(\mathrm{maj},\mathrm{iudr},\mathrm{des})}\left(q, \frac{2s}{1+s^2}, t\right)}{(1-t)(1-qt)\cdots(1-q^n t)}$$
$$= \sum_{i,j=0}^{\infty} s^{2i} t^j \left(1 + s \prod_{k=0}^{j} \frac{1}{1-q^k x}\right) \prod_{k=0}^{j} \left(\frac{1+q^k x}{1-q^k x}\right)^i. \tag{c}$$

One may obtain formulas for (maj, ipk), (maj, ilpk) and (maj, iudr) by specializing Theorem 6.1 appropriately. For example, setting $y = 1$ in Theorem 6.1 (a), multiplying both sides by $1 - s$, and then taking the limit of both sides as $s \to 1$ yields

$$\frac{1}{1-t} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1+t}{1-t}\right)^{n+1} \frac{P_n^{(\mathrm{maj},\mathrm{ipk})}\left(q, \frac{4t}{(1+t)^2}\right)}{(1-q)(1-q^2)\cdots(1-q^n)} x^n = \sum_{j=0}^{\infty} t^j \prod_{k=0}^{\infty}\left(\frac{1+q^k x}{1-q^k x}\right)^j,$$

where $P_n^{(\mathrm{maj},\mathrm{ipk})}(q,t) := P_n^{(\mathrm{maj},\mathrm{ipk},\mathrm{des},\mathrm{ides})}(q,t,1,1) = \sum_{\pi\in\mathfrak{S}_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{ipk}(\pi)+1}$.

7. Conjectures

We conclude with a discussion of some conjectures concerning some of the permutation statistic distributions studied in this paper.

7.1. Real-rootedness

A univariate polynomial is called *real-rooted* if it has only real roots. We conjecture that the distributions of ipk and ilpk over permutations in $\mathfrak{S}_n$ with any fixed value of pk, lpk, and udr—as well as that of ipk upon fixing br—are all encoded by real-rooted polynomials. This conjecture has been empirically verified for all $n \le 50$; our formulas from Sections 3–6 played a crucial role in formulating and gathering supporting evidence for this conjecture as they have allowed us to efficiently compute the polynomials in question.

Conjecture 7.1. *The following polynomials are real-rooted for all $n \ge 1$:*

(a) $[s^{k+1}]\, P_n^{(\mathrm{pk},\mathrm{ipk})}(s,t) = \sum_{\substack{\pi\in\mathfrak{S}_n \\ \mathrm{pk}(\pi)=k}} t^{\mathrm{ipk}(\pi)+1}$ *for* $0 \le k \le \lfloor (n-1)/2 \rfloor$,

(b) $[s^{k}]\, P_n^{(\mathrm{lpk},\mathrm{ilpk})}(s,t) = \sum_{\substack{\pi\in\mathfrak{S}_n \\ \mathrm{lpk}(\pi)=k}} t^{\mathrm{ilpk}(\pi)}$ *for* $0 \le k \le \lfloor n/2 \rfloor$,

(c) $[s^{k}]\, P_n^{(\mathrm{pk},\mathrm{ilpk})}(t,s) = \sum_{\substack{\pi\in\mathfrak{S}_n \\ \mathrm{lpk}(\pi)=k}} t^{\mathrm{ipk}(\pi)+1}$ *for* $0 \le k \le \lfloor n/2 \rfloor$,

(d) $[s^{k+1}]\, P_n^{(\mathrm{pk},\mathrm{ilpk})}(s,t) = \sum_{\substack{\pi\in\mathfrak{S}_n \\ \mathrm{pk}(\pi)=k}} t^{\mathrm{ilpk}(\pi)}$ *for* $0 \le k \le \lfloor (n-1)/2 \rfloor$,

(e) $[s^{k}]\, P_n^{(\mathrm{udr},\mathrm{ipk})}(s,t) = \sum_{\substack{\pi\in\mathfrak{S}_n \\ \mathrm{udr}(\pi)=k}} t^{\mathrm{ipk}(\pi)+1}$ *for* $1 \le k \le n$,

(f) $[s^{k}]\, P_n^{(\mathrm{udr},\mathrm{ilpk})}(s,t) = \sum_{\substack{\pi\in\mathfrak{S}_n \\ \mathrm{udr}(\pi)=k}} t^{\mathrm{ilpk}(\pi)}$ *for* $1 \le k \le n$ *and*

(g) $[s^{k}]\, P_n^{(\mathrm{br},\mathrm{ipk})}(s,t) = \sum_{\substack{\pi\in\mathfrak{S}_n \\ \mathrm{br}(\pi)=k}} t^{\mathrm{ipk}(\pi)+1}$ *for* $1 \le k \le n$.

Conjecture 7.1 would imply that all of these polynomials are unimodal and log-concave and can be used to show that the distributions of ipk and ilpk over permutations in $\mathfrak{S}_n$ with a fixed value k for the relevant statistics each converge to a normal distribution as $n \to \infty$.

It is worth noting that the peak and left peak polynomials

$$P_n^{\mathrm{pk}}(t) = \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{pk}(\pi)+1} = \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{ipk}(\pi)+1} \quad \text{and} \quad P_n^{\mathrm{lpk}}(t) = \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{lpk}(\pi)} = \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{ilpk}(\pi)}$$

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are known to be real-rooted (see, e.g., [18, 25]), so Conjecture 7.1 would also have the consequence that both of these 'real-rooted distributions' can be partitioned into real-rooted distributions based on the value of another statistic.

7.2. Gamma-positivity

Real-rootedness provides a powerful method of proving unimodality results in combinatorics, but when the polynomials in question are known to be symmetric,[3] an alternate avenue to unimodality is γ-positivity. Any symmetric polynomial $f(t) \in \mathbb{R}[t]$ with center of symmetry $n/2$ can be written uniquely as a linear combination of the polynomials $\{t^j(1+t)^{n-2j}\}_{0\leq j\leq \lfloor n/2\rfloor}$—referred to as the *gamma basis*—and $f(t)$ is called *γ-positive* if its coefficients in the gamma basis are nonnegative. The γ-positivity of a polynomial directly implies its unimodality, and γ-positivity has appeared in many contexts within combinatorics and geometry; see [1] for a detailed survey.

The prototypical example of a family of γ-positive polynomials are the Eulerian polynomials $A_n(t)$, as established by Foata and Schützenberger [8], and there is also a sizable literature on the γ-positivity of 'Eulerian distributions' (polynomials encoding the distribution of the descent number des) over various restricted subsets of $\mathfrak{S}_n$. For example, the Eulerian distribution over linear extensions of sign-graded posets [2], r-stack-sortable permutations [3], separable permutations [10] and involutions [28] are all known to be γ-positive. The two-sided Eulerian distribution (des, ides) is also known to satisfy a refined γ-positivity property which was conjectured by Gessel and later proved by Lin [16].

Define

$$\hat{A}_{n,k}(t) := \sum_{\substack{\pi\in\mathfrak{S}_n \\ \mathrm{pk}(\pi)=k}} t^{\mathrm{ides}(\pi)+1}$$

to be the polynomial encoding the distribution of ides over permutations in $\mathfrak{S}_n$ with k peaks, or equivalently, the distribution of des (i.e., the Eulerian distribution) over permutations in $\mathfrak{S}_n$ whose inverses have k peaks. We conjecture that these polynomials are γ-positive as well.

Conjecture 7.2. *For all $n \geq 1$ and $0 \leq k \leq \lfloor (n-1)/2 \rfloor$, the polynomials $\hat{A}_{n,k}(t)$ are γ-positive with center of symmetry $(n+1)/2$.*

In fact, we shall give a stronger conjecture which refines by 'pinnacle sets'. Given a permutation $\pi \in \mathfrak{S}_n$, the *pinnacle set* $\mathrm{Pin}(\pi)$ of π is defined by

$$\mathrm{Pin}(\pi) := \{\, \pi(k) : \pi(k-1) < \pi(k) > \pi(k+1) \text{ and } 2 \leq k \leq n-1 \,\}.$$

In other words, $\mathrm{Pin}(\pi)$ contains all of the values $\pi(k)$ at which k is a peak of π. Given $n \geq 1$ and $S \subseteq [n]$, define the polynomial $\hat{A}^{\mathrm{ides}}_{n,S}(t)$ by

$$\hat{A}^{\mathrm{ides}}_{n,S}(t) := \sum_{\substack{\pi\in\mathfrak{S}_n \\ \mathrm{Pin}(\pi)=S}} t^{\mathrm{ides}(\pi)+1};$$

this gives the distribution of the inverse descent number over permutations in $\mathfrak{S}_n$ with a fixed pinnacle set S or, equivalently, the Eulerian distribution over permutations in $\mathfrak{S}_n$ with 'inverse pinnacle set' S.

Conjecture 7.3. *For all $n \geq 1$ and $S \subseteq [n]$, the polynomials $\hat{A}^{\mathrm{ides}}_{n,S}(t)$ are γ-positive with center of symmetry $(n+1)/2$.*

[3]The use of the term 'symmetric polynomial' in this context is different from that in symmetric function theory. Here, a symmetric polynomial refers to a univariate polynomial whose coefficients form a symmetric sequence.

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Note that

$$\hat{A}^{\mathrm{ides}}_{n,k}(t) = \sum_{\substack{S\subseteq[n]\\|S|=k}} \hat{A}^{\mathrm{ides}}_{n,S}(t).$$

Because the sum of γ-positive polynomials with the same center of symmetry is again γ-positive, a positive resolution to Conjecture 7.3 would imply Conjecture 7.2. However, we do not have nearly as much empirical evidence to support Conjecture 7.3. Since we do not have a formula for the polynomials $\hat{A}^{\mathrm{ides}}_{n,k}(t)$, we were only able to verify Conjecture 7.3 up to $n = 10$, whereas Conjecture 7.2 has been verified for all $n \leq 80$ with the assistance of Theorem 3.6 (a).

Acknowledgements. We thank Kyle Petersen for insightful discussions concerning the work in this paper, and in particular for his suggestion to look at refining Conjecture 7.2 by pinnacle sets. We also thank two anonymous referees for their careful reading of an earlier version of this paper and providing thoughtful comments.

Competing interests. The authors have no competing interest to declare.

Financial support. YZ was partially supported by an AMS-Simons Travel Grant and NSF grant DMS-2316181.

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https://doi.org/10.1017/fms.2024.89 Published online by Cambridge University Press

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