Macroeconomic Dynamics, **23**, 2019, 2941–2958. Printed in the United States of America.
doi:10.1017/S1365100517001006

THE DEMAND FOR LIQUID ASSETS: EVIDENCE FROM THE MINFLEX LAURENT DEMAND SYSTEM WITH CONDITIONALLY HETEROSKEDASTIC ERRORS

DONGFENG CHANG
Shandong University

APOSTOLOS SERLETIS
University of Calgary

We investigate the demand for money and the degree of substitutability among monetary assets in the United States using the generalized Leontief and the Minflex Laurent (ML) models as suggested by Serletis and Shahmoradi (2007). In doing so, we merge the demand systems literature with the recent financial econometrics literature, relaxing the homoskedasticity assumption and instead assuming that the covariance matrix of the errors of flexible demand systems is time-varying. We also pay explicit attention to theoretical regularity, treating the curvature property as a maintained hypothesis. Our findings indicate that only the curvature constrained ML model with a Baba, Engle, Kraft, and Kroner (BEKK) specification for the conditional covariance matrix is able to generate inference consistent with theoretical regularity.

Keywords: Flexible Functional Forms, Volatility, Theoretical Regularity, Divisia Monetary Aggregates.

1. INTRODUCTION

This paper focuses on the demand for money in the United States and investigates the degree of substitutability among monetary assets using the flexible functional forms approach. This approach, introduced by Diewert (1971), has been widely used to investigate the interrelated problems of estimation of monetary asset demand functions and monetary aggregation issues. See, for example, Barnett (1983), Ewiss and Fisher (1984, 1985), Serletis and Robb (1986), Fisher and

This paper is based on Chapter 4 of Dongfeng Chang's Ph.D. dissertation at the University of Calgary. We would like to thank Bill Barnett and the following members of Dongfeng's dissertation committee: Herbert Emery, Daniel Gordon, Ron Kneebone, David Walls, and Philip Chang. Address correspondence to: Apostolos Serletis, Department of Economics, University of Calgary, Calgary, Alberta T2N 1N4, Canada; e-mail: Serletis@ucalgary.ca; Website: http://econ.ucalgary.ca/serletis.htm

 1365-1005/18

TABLE 1. A summary of flexible functional forms estimation of monetary assets demand

Author(s)	Model used	Curvature imposed	Homoscedasticity assumed
Barnett (1983)	Minflex Laurent	✓	✓
Ewis and Fisher (1984)	Translog		✓
Ewis and Fisher (1985)	Fourier		✓
Serletis and Robb (1986)	Translog		✓
Serletis (1987, 1988)	Translog		✓
Fisher and Fleissig (1994, 1997)	Fourier		✓
Fleissig (1997)	Minflex, GL, Translog		✓
Fleissig and Swoflord (1996)	AIM		✓
Drake, Fleissig, and Mullineux (1999)	AIM		✓
Fleissig and Serletis (2002)	Fourier		✓
Drake, Fleissig, and Swoflord (2003)	AIM		✓
Drake and Fleissig (2004)	Fourier		✓
Serletis and Shahmoradi (2005)	AIM and Fourier	✓	✓
Serletis and Shahmoradi (2007)	GL, BTL, AIDS, Minflex, NQ	✓	✓
Drake and Fleissig (2010)	Fourier		✓
Serletis and Feng (2010)	AIM	✓	✓

Notes: GL, generalized Leontief; AIM, asymptotically ideal model; BTL, basic translog; AIDS, almost ideal demand system; NQ, normalized quadratic.

Fleissig (1994, 1997), Fleissig and Serletis (2002), and Serletis and Shahmoradi (2005, 2007), among others.

As noted by Barnett (2002), the usefulness of flexible functional forms depends on whether they satisfy the theoretical regularity conditions of positivity, monotonicity, and curvature, and in the empirical monetary demand literature there has been a tendency to ignore theoretical regularity, as can be seen, for example, in Table 1. In this regard, as Barnett (2002, pp. 199) put it, without satisfaction of all three theoretical regularity conditions, "... the second-order conditions for optimizing behavior fail, and duality theory fails. The resulting first-order conditions, demand functions, and supply functions become invalid."

Motivated by these considerations, in this paper we follow Serletis and Shahmoradi (2005, 2007) and estimate the degree of substitutability among monetary assets paying explicit attention to theoretical regularity, treating the curvature property as a maintained hypothesis.

Furthermore, in the empirical demand systems literature, existing studies typically assume that the covariance matrix of the error terms associated with the demand equations is homoskedastic, as summarized in Table 1. In this paper, we merge the demand systems literature with the recent financial econometrics literature by following Serletis and Isakin (2017) to incorporate heteroskedastic variance in the demand system estimation subject to full satisfaction of regularity conditions. In particular, we relax the homoskedasticity assumption and instead assume that the covariance matrix of the errors of flexible demand systems is time-varying. By doing so, we achieve superior modeling using parametric nonlinear demand systems that capture certain important features of the data.

To obtain our estimates of the demand for money in the United States, we use the monthly time series data on monetary asset quantities and their user costs recently produced by Barnett et al. (2013) and maintained within the Center of Financial Stability (CFS) program Advances in Monetary and Financial Measurement (AMFM). Our investigation is in the context of the best performed flexible functional forms in Serletis and Shahmoradi (2007): The generalized Leontief (GL) model of Diewert (1973, 1974) and the Minflex Laurent (ML) model introduced by Barnett (1983) and Barnett and Lee (1985). Our findings indicate that only the curvature constrained ML model with a Baba, Engle, Kraft, and Kroner (BEKK) specification [see Engle and Kroner (1995)] for the conditional covariance matrix is able to generate results consistent with theoretical regularity.

The rest of the paper is organized as follows. Section 2 provides a discussion of the representative agent's problem and Section 3 presents the two locally flexible functional forms, paying explicit attention to the imposition of curvature. Section 4 discusses the data, whereas Section 5 focuses on related econometric issues and on the way to incorporate the BEKK specification for the conditional covariance matrix. Section 6 presents the empirical results and the final section briefly concludes the paper.

2. THE REPRESENTATIVE AGENT'S PROBLEM

We assume that the representative consumer has the following utility function:

$$u = u(\boldsymbol{c}, l, \boldsymbol{x}), \tag{1}$$

where $\boldsymbol{c}$ is a vector of consumption goods, l is leisure, and $\boldsymbol{x}$ is a vector of monetary asset quantities. The consumer maximizes (1) subject to the budget constraint:

$$\boldsymbol{q}'\boldsymbol{c} + wl + \boldsymbol{p}'\boldsymbol{x} = m,$$

where $\boldsymbol{q}$ is a vector of prices of the consumption goods, $\boldsymbol{c}$, w is the wage rate, $\boldsymbol{p}$ is the corresponding vector of monetary asset user costs, and m is total income.

We assume that monetary assets are as a group separable from consumption goods, $\boldsymbol{c}$, and leisure, l. That is, it is possible to write (1) as

$$u = u\left[\boldsymbol{c}, l,\ f(\boldsymbol{x})\right], \tag{2}$$

where $f(\boldsymbol{x})$ is the aggregator function over monetary assets, $\boldsymbol{x}$. The requirement of (direct) weak separability in $\boldsymbol{x}$ is that the marginal rate of substitution between any two components of $\boldsymbol{x}$ does not depend upon the values of $\boldsymbol{c}$ and l, meaning that the demand for monetary assets is independent of relative prices outside

https://doi.org/10.1017/S1365100517001006 Published online by Cambridge University Press

TABLE 2. Monetary assets/components

A	
1	Currency
2	Travelers checks
3	Demand deposits
4	Other checkable deposits at banks including Super Now accounts
5	Other checkable deposits at thrifts including Super Now accounts
B	
6	Savings deposits at banks including money market deposit accounts
7	Savings deposits at thrifts including money market deposit accounts
C	
8	Small denomination time deposits at commercial banks
9	Small denomination time deposits at thrift institutions

the monetary sector—see Leontief (1947) and Sono (1961). Under the weak separability assumption, we will focus on the representative agent facing the following problem:

$$\max_{\boldsymbol{x}} f(\boldsymbol{x}) \quad \text{subject to} \quad \boldsymbol{p}'\boldsymbol{x} = y, \tag{3}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_9)$ is the vector of monetary asset quantities described in Table 2, $\boldsymbol{p} = (p_1, p_2, \ldots, p_9)$ is the corresponding vector of user costs, and y is the total expenditure on the services of monetary assets. For details regarding the theory of multistage optimization in the context of consumer theory, see Strotz (1957, 1959), Gorman (1959), and Blackorby et al. (1978).

Because the functional forms that we use in this paper are parameter intensive, we face the problem of having a large number of parameters in estimation. To reduce the number of parameters, we follow Serletis and Shahmoradi (2007) and separate the group of assets into three collections based on empirical pretesting. Thus, the monetary utility function in (3) can be written as

$$f(\boldsymbol{x}) = f\left[f_A(x_1, x_2, x_3, x_4, x_5),\ f_B(x_6, x_7),\ f_C(x_8, x_9)\right],$$

where the subaggregate functions f_i $(i = A, B, C)$ provide subaggregate measures of monetary services.

Instead of using the simple-sum index, currently in use by the Federal Reserve and most central banks around the world, to construct the monetary subaggregates, f_i $(i = A, B, C)$, we follow Barnett (1980) and use the Divisia quantity index to allow for less than perfect substitutability among the relevant monetary components. The Divisia index (in discrete time) is defined as

$$\log M_t^D - \log M_{t-1}^D = \sum_{j=1}^{n} s_{jt}^{*}(\log x_{jt} - \log x_{jt-1}),$$

https://doi.org/10.1017/S1365100517001006 Published online by Cambridge University Press

according to which the growth rate of the aggregate is the weighted average of the growth rates of the component quantities, with the Divisia weights being defined as the expenditure shares averaged over the two periods of the change, $s_{jt}^* = (1/2)(s_{jt} + s_{j,t-1})$ for $j = 1, \ldots, n$, where $s_{jt} = \pi_{jt}x_{jt}/\pi_{kt}x_{kt}$ is the expenditure share of asset j during period t , and π_{jt} is the nominal user cost of asset j, derived in Barnett (1978),

$$\pi_{jt} = p_t^* \frac{R_t - r_{jt}}{1 + R_t},$$

which is just the opportunity cost of holding a dollar's worth of the jth asset. Above, p_t^* is the true-cost-of-living index, r_{jt} is the market yield on the jth asset, and R_t is the yield available on a benchmark asset that is held only to carry wealth between multiperiods.

3. FLEXIBLE DEMAND SYSTEMS

In this section, we briefly discuss the GL and the ML models that we use to approximate the unknown underlying indirect utility function of the representative economic agent as well as the procedure of imposing the curvature conditions in each model. Both models are locally flexible and capable of approximating any unknown function up to the second order.

3.1. The Generalized Leontief

According to Diewert (1974), the GL flexible functional form has the following reciprocal indirect utility function:

$$h(\boldsymbol{v}) = a_0 + \sum_{i=1}^{n} a_i v_i^{1/2} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} v_i^{1/2} v_j^{1/2}, \tag{4}$$

where $\boldsymbol{v} = [v_1, v_2, \ldots, v_n]$ is a vector of income normalized user costs with $v_i = p_i/y$, where p_i is the user cost of asset i and y is the total expenditure on the n assets. $\boldsymbol{B} = \left[\beta_{ij}\right]$ is an $n \times n$ symmetric matrix of parameters and a_0 and a_i are other parameters, for a total of $(n^2 + 3n + 2)/2$ parameters. The GL share equations, derived using the logarithmic form of Roy's identity are

$$s_i = \frac{a_i v_i^{1/2} + \sum_{j=1}^{n} \beta_{ij} v_i^{1/2} v_j^{1/2}}{\sum_{j=1}^{n} a_j v_j^{1/2} + \sum_{k=1}^{n} \sum_{m=1}^{n} \beta_{km} v_k^{1/2} v_m^{1/2}}, \qquad i = 1, \ldots, n. \tag{5}$$

Since the share equations are homogeneous of degree zero in the parameters, we follow Barnett and Lee (1985) and impose the following normalization in estimation:

$$2\sum_{i=1}^{n} a_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij} = 1. \tag{6}$$

Curvature condition of the GL reciprocal indirect utility function requires the Hessian matrix to be negative semidefinite. Local curvature can be imposed using the Serletis and Shahmoradi (2007) procedure by evaluating the Hessian terms of (4) at $\mathbf{v}^* = \mathbf{1}$, as follows:

$$H_{ij} = -\delta_{ij}\left(a_i + \sum_{j=1,\, j\neq i}^{n} \beta_{ij}\right) + \left(1-\delta_{ij}\right)\beta_{ij},$$

where δ_{ij} is the Kronecker delta (that is, $\delta_{ij} = 1$ when $i = j$ and 0 otherwise).

Replacing $\boldsymbol{H}$ with $-\boldsymbol{K}\boldsymbol{K}'$, where $\boldsymbol{K}$ is an $n \times n$ lower triangular matrix so that $-\boldsymbol{K}\boldsymbol{K}'$ is by construction a negative semidefinite matrix. The above can be written as

$$-\left(KK'\right)_{ij} = -\delta_{ij}\left(a_i + \sum_{j=1,\, j\neq i}^{n} \beta_{ij}\right) + \left(1-\delta_{ij}\right)\beta_{ij}. \tag{7}$$

Solving for the a_i and β_{ij} terms as a function of the $\left(KK'\right)_{ij}$, we can get the restrictions that ensure the negative semidefiniteness of the Hessian matrix.

In particular, when $i \neq j$, equation (7) implies that

$$\beta_{ij} = -\left(KK'\right)_{ij}, \tag{8}$$

and when $i = j$, it implies that

$$\left(KK'\right)_{ii} = a_i + \sum_{j=1,\, j\neq i}^{n} \beta_{ij}.$$

Substituting β_{ij} from (8) into the above equation, we get

$$a_i = \sum_{j=1}^{n} \left(KK'\right)_{ij}. \tag{9}$$

for $i, j = 1, \ldots, n$. See Serletis and Shahmoradi (2007) for an example with $n = 3$.

3.2. The Minflex Laurent Model

The ML model, introduced by Barnett (1983) and Barnett and Lee (1985), is a special case of the Full Laurent model also introduced by Barnett (1983). Following

https://doi.org/10.1017/S1365100517001006 Published online by Cambridge University Press

Barnett (1983), the Full Laurent reciprocal indirect utility function is

$$h(\boldsymbol{v}) = a_0 + 2\sum_{i=1}^{n} a_i v_i^{1/2} + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} v_i^{1/2} v_j^{1/2} - 2\sum_{i=1}^{n} b_i v_i^{-1/2} - \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} v_i^{-1/2} v_j^{-1/2}, \quad \textbf{(10)}$$

where a_0, a_i, a_{ij}, b_i, and b_{ij} are unknown parameters and v_i denotes the income normalized price, p_i/y.

By assuming that $b_i = 0, b_{ii} = 0\,\forall i, a_{ij}b_{ij} = 0\,\forall i,\ j$, and forcing the off diagonal elements of the symmetric matrices $\boldsymbol{A} \equiv [a_{ij}]$ and $\boldsymbol{B} \equiv [b_{ij}]$ to be nonnegative, (10) reduces to the ML reciprocal indirect utility function:

$$h(\boldsymbol{v}) = a_0 + 2\sum_{i=1}^{n} a_i v_i^{1/2} + \sum_{i=1}^{n} a_{ii} v_i + \sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n} a_{ij}^2 v_i^{1/2} v_j^{1/2} - \sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n} b_{ij}^2 v_i^{-1/2} v_j^{-1/2}. \quad \textbf{(11)}$$

Note that the off diagonal elements of $\boldsymbol{A}$ and $\boldsymbol{B}$ are nonnegative as they are raised to the power of two.

By applying Roy's identity to (11), the share equations of the ML demand system are

$$s_i = \frac{a_i v_i^{1/2} + a_{ii} v_i + \sum_{\substack{j=1\\ i\neq j}}^{n} a_{ij}^2 v_i^{1/2} v_j^{1/2} + \sum_{\substack{j=1\\ i\neq j}}^{n} b_{ij}^2 v_i^{-1/2} v_j^{-1/2}}{\sum_{i=1}^{n} a_i v_i^{1/2} + \sum_{i=1}^{n} a_{ii} v_i + \sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n} a_{ij}^2 v_i^{1/2} v_j^{1/2} + \sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n} b_{ij}^2 v_i^{-1/2} v_j^{-1/2}}. \quad \textbf{(12)}$$

Since the share equations are homogeneous of degree zero in the parameters, we follow Barnett and Lee (1985) and impose the following normalization in the estimation of (12):

$$\sum_{i=1}^{n} a_{ii} + 2\sum_{i=1}^{n} a_i + \sum_{\substack{j=1\\ i\neq j}}^{n} a_{ij}^2 - \sum_{\substack{j=1\\ i\neq j}}^{n} b_{ij}^2 = 1. \quad \textbf{(13)}$$

Hence, there are

$$1 + n + \frac{n(n+1)}{2} + \frac{n(n-1)}{2}$$

parameters in (11), but the $n\,(n-1)\,/2$ equality restrictions, $a_{ij}b_{ij} = 0\ \forall i,\ j$, and the normalization (13) reduce the number of parameters in equation (12) to $\left(n^2 + 3n\right)/2$.

As shown by Barnett (1983, Theorem A.3), (11) is globally concave for every $\boldsymbol{v} \geq \mathbf{0}$, if all parameters are nonnegative, as in that case (11) would be a sum of concave functions. If the initially estimated parameters of the vector $\boldsymbol{a}$ and matrix $\boldsymbol{A}$ are not nonnegative, curvature can be imposed globally by replacing each unsquared parameter by a squared parameter, as in Barnett (1983).

4. DATA

We use monthly data on monetary asset quantities and their user costs for the nine items listed in Table 2, recently produced by Barnett et al. (2013) and maintained within the CFS program AMFM. The sample period is from 1967:1 to 2015:3 (a total of 579 observations). For a detailed discussion of the data and the methodology of the calculation of user costs, see Barnett et al. (2013) and http://www.centerforfinancialstability.org. As we require real per capita asset quantities for our empirical work, we have divided each measure of monetary services by the United States Consumer Price Index (US CPI) (all items) and total US population in each period.

Because demand system estimation requires heavy dimension reduction (as already noted in Section 2), we use the Divisia index to reduce the dimension of each model by constructing the three subaggregates shown in Table 2. In particular, subaggregate A (M1) is composed of currency, traveler's checks, and other checkable deposits, including Super NOW accounts issued by commercial banks and thrifts (series 1 to 5 in Table 2). Subaggregate B (savings deposits) is composed of savings deposits issued by commercial banks and thrifts (series 6 and 7), and subaggregate C (Time deposits) is composed of small time deposits issued by commercial banks and thrifts (series 8 and 9). Divisia user cost indices for each of these subaggregates are calculated by applying Fisher's (1922) weak factor reversal test.

5. ECONOMETRIC ISSUES

In order to estimate share equation systems such as (5) and (12), a stochastic version must be specified. Also, since only exogenous variables appear on the right-hand side, it seems reasonable to assume that the observed share in the ith equation deviates from the true share by an additive disturbance term u_i. Thus, the share equation system for each model at time t can be written in matrix form as

$$\boldsymbol{s}_t = \boldsymbol{g}(\boldsymbol{p}_t, y_t, \vartheta) + \boldsymbol{u}_t, \tag{14}$$

where $\boldsymbol{s} = (s_1, \ldots, s_n)'$, $\boldsymbol{g}\big(\boldsymbol{p}_t, y_t, \vartheta\big) = \big(\boldsymbol{g}_1\left(\boldsymbol{p}_t, y_t, \vartheta\right), \ldots, \boldsymbol{g}_n\left(\boldsymbol{p}_t, y_t, \vartheta\right)\big)'$, ϑ is the parameter vector to be estimated, and $\boldsymbol{g}_i\left(\boldsymbol{p}_t, y_t, \vartheta\right)$ is given by the right-hand side of each of (5) and (12).

In this literature, it has been typically assumed that

$$\boldsymbol{u}_t \sim N\left(\mathbf{0}, \Omega\right), \tag{15}$$

https://doi.org/10.1017/S1365100517001006 Published online by Cambridge University Press

where $\boldsymbol{u} = (u_{1t}, \ldots, u_{nt})'$, $\mathbf{0}$ is a null matrix, and Ω is the $n \times n$ symmetric positive definite error covariance matrix.

Since the budget shares sum to 1, the disturbance covariance matrix Ω is singular. To address this issue, Barten (1969) showed that maximum likelihood estimates can be obtained by arbitrarily dropping any equation in the system. We follow Barten (1969) and drop the last equation in each model.

In this paper, we follow Serletis and Isakin (2017) and relax the homoskedasticity assumption in (15) by assuming that the n-dimensional error vector is normally distributed with zero-mean and time-varying covariance matrix Ω_t with respect to information set I_{t-1}

$$\boldsymbol{u}_t \mid I_{t-1} \sim N(\mathbf{0}, \Omega_t). \tag{16}$$

As before, the error terms of the demand system sum to zero and we drop the last equation to avoid singularity and consider the corresponding $(n-1) \times (n-1)$ covariance matrix Φ_t. We also assume the BEKK GARCH(1,1) with $K = 1$ representation for the conditional variance matrix

$$\Phi_t = \boldsymbol{C}'\boldsymbol{C} + \boldsymbol{B}'\Phi_{t-1}\boldsymbol{B} + \boldsymbol{A}'\epsilon_{t-1}\epsilon'_{t-1}\boldsymbol{A},$$

where $\epsilon_t = \left(u_{1t}, u_{2t}, \ldots, u_{n-1,t}\right)$. In our case of 3 goods, each of the demand systems with a BEKK specification for the covariance matrix Φ_t has the following conditional variance and covariance equations

$$\begin{aligned}
h_{11,t} &= c_{11}^2 + b_{11}^2 h_{11,t-1} + 2b_{11}b_{21}h_{12,t-1} + b_{21}^2 h_{22,t-1} \\
&\quad + a_{11}^2 u_{1,t-1}^2 + 2a_{11}a_{21}u_{1,t-1}u_{2,t-1} + a_{21}^2 u_{2,t-1}^2 \\
h_{12,t} &= c_{11}c_{12} + b_{11}b_{21}h_{11,t-1} + (b_{11}b_{22} + b_{12}b_{21})h_{12,t-1} + b_{21}b_{22}h_{22,t-1} \\
&\quad + a_{11}a_{12}u_{1,t-1}^2 + (a_{11}a_{22} + a_{12}a_{21})u_{1,t-1}u_{2,t-1} + a_{21}a_{22}u_{2,t-1}^2 \\
h_{22,t} &= c_{12}^2 + c_{22}^2 + b_{12}^2 h_{11,t-1} + 2b_{12}b_{22}h_{12,t-1} + b_{22}^2 h_{22,t-1} \\
&\quad + a_{12}^2 u_{1,t-1}^2 + 2a_{12}a_{22}u_{1,t-1}u_{2,t-1} + a_{22}^2 u_{2,t-1}^2.
\end{aligned} \tag{17}$$

See Serletis and Isakin (2017) for a detailed discussion. All estimations are performed in Estima RATS.

6. EMPIRICAL EVIDENCE

In Tables 3 and 4, we report a summary of results from the GL and ML models in terms of parameter estimates (with p-values in parentheses) for the mean equations (5) and (12), and the variance equations (17). We also report positivity, monotonicity, and curvature violations, Lagrange Multiplier test results for autoregressive conditional heteroscedasticity (ARCH) effects, as well as log likelihood values, when the models are estimated without the curvature conditions imposed (in the first column), with the curvature conditions imposed (in the second column), and with both BEKK errors and curvature conditions imposed (in the last column).

https://doi.org/10.1017/S1365100517001006 Published online by Cambridge University Press

TABLE 3. Generalized Leontief parameter estimates

Parameter	Unrestricted	Curvature imposed	BEKK errors and curvature imposed
		Mean equations	
a_1	0.245 (0.000)	0.186 (0.000)	−0.038 (0.000)
a_2	0.360 (0.000)	0.154 (0.000)	0.029 (0.000)
β_{11}	0.494 (0.000)	−0.159 (0.000)	0.031 (0.007)
β_{12}	−0.655 (0.000)	0.044 (0.000)	0.247 (0.000)
β_{13}	0.181 (0.000)	0.194 (0.000)	0.226 (0.000)
β_{22}	0.421 (0.000)	0.159 (0.000)	0.303 (0.000)
β_{23}	−0.091 (0.000)	−0.061 (0.000)	−0.131 (0.000)
β_{33}	0.028 (0.000)	0.030 (0.000)	−0.047 (0.000)
	Variance equations (with the 3rd mean equation deleted)		
c_{11}			0.002 (0.000)
c_{12}			0.002 (0.015)
c_{22}			0.001 (0.039)
a_{11}			−0.625 (0.000)
a_{12}			0.044 (0.111)
a_{21}			0.010 (0.610)
a_{22}			−0.639 (0.000)
b_{11}			0.897 (0.000)
b_{12}			0.018 (0.158)
b_{21}			0.003 (0.680)
b_{22}			0.862 (0.000)
Violations			
Positivity	0	0	0
Monotonicity	0	0	0
Curvature	579	268	287
LM test (χ^2)			
$\widehat{u}_1$	530.498 (0.000)	512.215 (0.000)	
$\widehat{u}_2$	486.963 (0.000)	508.604 (0.000)	
Log L	2212.805	2113.510	3011.555

Notes: Sample period, monthly data 1967:1–2015:3 (579 observations).
Numbers in parentheses are p-values.

As can be seen in the first column of Tables 3 and 4, both models satisfy positivity and monotonicity at all sample observations when the curvature conditions are not imposed, but both unrestricted models violate curvature when curvature is not imposed. Because regularity has not been attained (by luck), we follow Barnett (2002) and estimate the models by imposing the curvature conditions, using the methodology discussed in Section 3. The results are disappointing in the case of the GL model. As can be seen in the second column of Table 3, the imposition of local curvature on the GL model reduces the number of curvature violations, but does

TABLE 4. Minflex Laurent parameter estimates

Parameter	Unrestricted	Curvature imposed	BEKK errors and curvature imposed
		Mean equations	
a_1	−0.086 (0.000)	0.062 (0.000)	0.000 (0.000)
a_2	−0.209 (0.000)	0.004 (0.000)	0.030 (0.000)
a_3	−0.174 (0.000)	0.000 (0.000)	0.000 (0.000)
a_{11}	0.469 (0.000)	0.299 (0.000)	0.148 (0.000)
a_{13}	−0.498 (0.000)	−0.278 (0.000)	−0.412 (0.000)
a_{33}	0.138 (0.000)	0.016 (0.000)	0.061 (0.000)
a_{22}	0.846 (0.000)	0.405 (0.000)	0.395 (0.000)
b_{12}	0.039 (0.000)	0.044 (0.000)	0.028 (0.000)
	Variance equations (with the 3rd mean equation deleted)		
c_{11}			0.005 (0.000)
c_{12}			0.003 (0.000)
c_{22}			0.001 (0.003)
a_{11}			0.161 (0.008)
a_{12}			0.144 (0.013)
a_{21}			0.042 (0.000)
a_{22}			0.330 (0.046)
b_{11}			−0.995 (0.000)
b_{12}			0.034 (0.959)
b_{21}			−0.008 (0.121)
b_{22}			−0.936 (0.000)
Violations			
Positivity	0	0	0
Monotonicity	0	0	0
Curvature	579	0	0
LM test (χ^2)			
$\widehat{u}_1$	531.524 (0.000)	502.545 (0.000)	
$\widehat{u}_2$	497.551 (0.000)	507.133 (0.000)	
Log L	2273.443	2066.706	2852.363

Notes: Sample period, monthly data 1967:1–2015:3 (579 observations).
Numbers in parentheses are p-values.

not completely eliminate them. Only the curvature restricted ML model satisfies full theoretical regularity (see the second column of Table 4). Moreover, when we relax the homoskedasticity assumption and model the curvature constrained demand systems with the BEKK GARCH(1,1) errors specification, we find that only the curvature constrained ML model with BEKK errors satisfies all three regularity conditions at all data points (see the last column of Tables 3 and 4).

To verify that homoskedasticity is not a good assumption in this literature, we report χ^2 statistics of Lagrange Multiplier tests for the models estimated under the homoskedasticity assumption (see the first and second columns) in Tables 3 and 4. Results show statistically significant evidence of ARCH effects when the models are estimated under the homoskedasticity assumption. This is also supported by the plots of the estimated squared residuals, $\widehat{u}_1^2$ and $\widehat{u}_2^2$, of the unrestricted and curvature restricted ML model in Figures 1 and 2, respectively; similar figures for the GL model are available upon request. Overall, based on our evidence, only the curvature constrained ML model with a BEKK specification for the conditional variance matrix is able to provide inference that is consistent with full theoretical regularity and the time series properties of the data.

In the demand systems approach to the estimation of economic relationships, the primary interest, especially in policy analysis, is in how the arguments of the underlying function affect the quantities demanded. This is conventionally expressed in terms of the income and price elasticities and the Allen and Morishima elasticities of substitution. These elasticities can be calculated from the estimated budget share equations by writing the left-hand side as

$$x_i = \frac{s_i y}{p_i}, \qquad i = 1, \ldots, n.$$

In particular, the income elasticities can be calculated by

$$\eta_{iy} = 1 + \frac{y}{s_i}\frac{\partial s_i}{\partial y}, \qquad i = 1, \ldots, n$$

and the Marshallian (or uncompensated) price elasticities by

$$\eta_{ij} = \frac{p_j}{s_i}\frac{\partial s_i}{\partial p_j} - \delta_{ij}, \qquad i, j = 1, \ldots, n$$

where δ_{ij} is the Kronecker delta (that is, $\delta_{ij} = 1$ when $i = j$ and 0 otherwise). The Allen (1938) elasticities of substitution can be calculated by

$$\sigma_{ij}^a = \eta_{iy} + \frac{\eta_{ji}}{s_i} = \sigma_{ji}^a, \qquad i, j = 1, \ldots, n$$

and the Morishima (1967) elasticities of substitution by

$$\sigma_{ij}^m = s_i(\sigma_{ji}^a - \sigma_{ii}^a), \qquad i, j = 1, \ldots, n.$$

We report the income and the own- and cross-price elasticities evaluated at the mean of the data (with p-values in parentheses) in Table 5, the Allen elasticities of substitution in Table 6, and the Morishima elasticities of substitution in Table 7. We do so for each of the three monetary subaggregates, A, B, and C, and only for the curvature constrained ML model with a BEKK specification for the conditional variance matrix, since this is the only model that satisfies full theoretical regularity and reflects the data generating process. Since the elasticities are functions of the

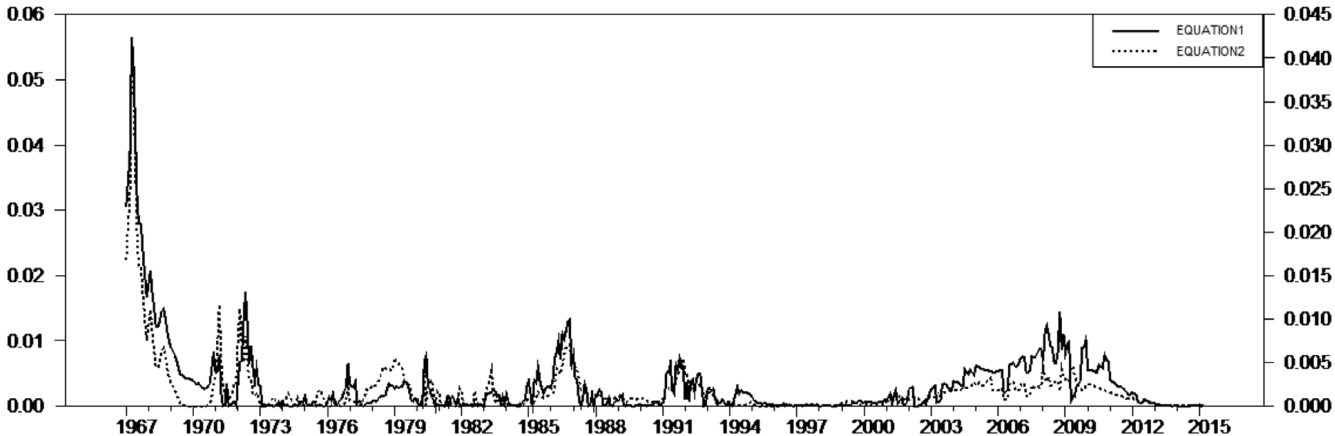

FIGURE 1. Squared residuals of the two estimated share equations of the unrestricted ML model.

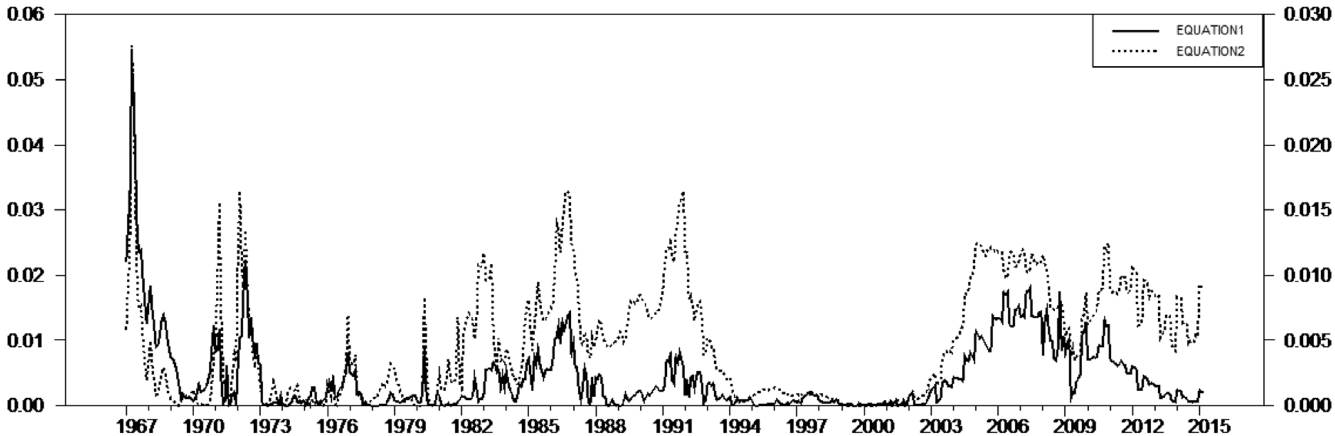

FIGURE 2. Squared residuals of the two estimated share equations of the curvature constrained ML model.

https://doi.org/10.1017/S1365100517001006 Published online by Cambridge University Press

TABLE 5. Curvature Constrained Minflex Laurent with a BEKK Specification: Income and Price Elasticities

	Elasticities			
	Income	Own- andcross-price		
Subaggregate i	η_i	η_{iA}	η_{iB}	η_{iC}
(A)	1.171 (0.000)	−0.988 (0.000)	−0.154 (0.000)	−0.028 (0.000)
(B)	0.864 (0.000)	−0.177 (0.000)	−0.639 (0.000)	−0.020 (0.000)
(C)	2.304 (0.000)	−0.800 (0.000)	−0.488 (0.000)	−1.015 (0.000)

Notes: Sample period, monthly data 1967:1–2015:3 (579 observations).
Numbers in parentheses are p-values.

TABLE 6. Curvature constrained Minflex Laurent with a BEKK specification: Allen elasticities of substitution

	Allen elasticities		
Subaggregate i	σ^a_{iA}	σ^a_{iB}	σ^a_{iC}
(A)	−1.944 (0.000)	−0.409 (0.000)	−0.247 (0.000)
(B)		−1.697 (0.000)	−0.173 (0.000)
(C)			−8.840 (0.000)

Notes: Sample period, monthly data 1967:1–2015:3 (579 observations).
Numbers in parentheses are p-values.

TABLE 7. Curvature constrained Minflex Laurent with a BEKK specification: Morishima elasticities of substitution

	Morishima elasticities		
Subaggregate i	σ^m_{iA}	σ^m_{iB}	σ^m_{iC}
(A)		0.811 (0.000)	0.188 (0.000)
(B)	0.485 (0.000)		0.152 (0.000)
(C)	0.987 (0.000)	0.995 (0.000)	

Notes: Sample period, monthly data 1967:1–2015:3 (579 observations).
Numbers in parentheses are p-values.

parameter estimates, we construct the reported standard errors using the Delta method.

As expected, all the income elasticities reported in Table 5, η_A, η_B, and η_C, are positive and statistically significant ($\eta_A = 1.171$ with a p-value of 0.000, $\eta_B = 0.864$ with a p-value of 0.000, and $\eta_C = 2.304$ with a p-value of 0.000), implying that M1 (A), savings deposits (B), and time deposits (C) are all normal goods, which is consistent with economic theory and the existing literature. The

https://doi.org/10.1017/S1365100517001006 Published online by Cambridge University Press

own-price elasticities, η_{ii}, are all negative, as predicted by the theory, and statistically significant ($\eta_{AA} = -0.988$ with a p-value of 0.000, $\eta_{BB} = -0.639$ with a p-value of 0.000, and $\eta_{CC} = -1.015$ with a p-value of 0.000). For the cross-price elasticities, η_{ij}, economic theory does not predict any signs, but we note that the off-diagonal terms in Table 5 are negative, indicating that the assets taken as a whole are gross complements. This is (qualitatively) consistent with the evidence reported by Serletis and Shahmoradi (2005) using the Fourier and AIM (asymptotically ideal model) globally flexible functional forms and that by Serletis and Shahmoradi (2007) using the ML and GL models.

In addition to the standard Marshallian income and price elasticities, we show estimates of the Allen elasticities of substitution in Table 6, evaluated at the means of the data. As expected, the three diagonal terms of the Allen own elasticities of substitution for the three assets are negative ($\sigma^a_{AA} = -1.944$ with a p-value of 0.000, $\sigma^a_{BB} = -1.697$ with a p-value of 0.000, and $\sigma^a_{CC} = -8.840$ with a p-value of 0.000). However, because the Allen elasticity of substitution produces ambiguous results off diagonal, we use the Morishima elasticity of substitution to investigate the substitutability/complementarity relationship between monetary assets—see Blackorby and Russell (1989) for more details. Based on the Morishima elasticities of substitution shown in Table 7, the assets are Morishima substitutes, with all Morishima elasticities of substitution being less than unity.

7. COMPARISON WITH OTHER STUDIES

It is difficult to provide a comparison between our results and those obtained in previous studies using different flexible functional forms and different monetary assets. Moreover, as we have already mentioned, most of the money demand studies listed in Table 1 do not produce inference consistent with neoclassical microeconomic theory, and all of the studies listed in Table 1 are based on the homoskedasticity assumption.

Our results, however, are generally consistent with those reported by Serletis and Shahmoradi (2007) who investigate the same monetary assets using the ML model as we do in this paper. That is, the monetary assets are Morishima substitutes, with all the Morishima elasticities of substitution being less than unity. This is evidence against what Barnett (2016) refers to as the "Linearity Condition," which requires infinite elasticities of substitution. It is also consistent with much of the earlier literature that was based on different demand systems and different monetary assets. It means that the simple-sum monetary aggregates used by the Federal Reserve (and other central banks around the world) are inconsistent with neoclassical microeconomic theory, and therefore should be abandoned.

8. CONCLUSION

We investigate the demand for money and the degree of substitutability among monetary assets in the United States in the context of four of the most widely used flexible functional forms. We also merge the demand systems literature with the recent financial econometrics literature, relaxing the homoskedasticity assumption. In doing so, we make a valuable and novel contribution to the demand systems literature by showing how one can use standard time series techniques in order to obtain improved estimates of the income and price elasticities and the Allen and Morishima elasticities of substitution. The evidence indicates that the elasticities of substitution among the monetary assets are very low, implying that approximation with a linear index, such as the simple sum index, requiring infinite elasticities of substitution is invalid.

REFERENCES

Allen, R. G. D. (1938) *Mathematical Analysis for Economists*. London: Macmillan.

Barnett, W. A. (1978) The user cost of money. *Economics Letters* 1, 145–149.

Barnett, W. A. (1980) Economic monetary aggregates: An application of aggregation and index number theory. *Journal of Econometrics* 14, 11–48.

Barnett, W. A. (1983) New indices of money supply and the flexible Laurent demand system. *Journal of Business and Economic Statistics* 1, 7–23.

Barnett, W. A. (2002) Tastes and technology: Curvature is not sufficient for regularity. *Journal of Econometrics* 108, 199–202.

Barnett, W. A. (2016) Friedman and Divisia monetary measures. In R. Cord and D. Hammond (eds.), *Milton Friedman: Contributions to Economics and Public Policy*, pp. 265–291. Oxford, UK: Oxford University Press.

Barnett, W. A. and Y. W. Lee (1985) The global properties of the minflex Laurent, generalized Leontief, and translog flexible functional forms. *Econometrica* 53, 1421–1437.

Barnett, W. A., J. Liu, R. S. Mattson, and J. van den Noort (2013) The new CFS Divisia monetary aggregates: Design, construction, and data sources. *Open Economies Review* 24, 101–124.

Barten, A. P. (1969) Maximum likelihood estimation of a complete system of demand equations. *European Economic Review* 1, 7–73.

Blackorby, C., D. Primont, and R. R. Russell (1978) *Duality, Separability, and Functional Structure*. Amsterdam: North-Holland.

Blackorby, C., D. Primont, and R. R. Russell (1989) Will the real elasticity of substitution please stand up? *American Economic Review* 79, 882–888.

Diewert, W. E. (1971) An application of the Shephard duality theorem: A generalized Leontief production function. *Journal of Political Economy* 79, 481–507.

Diewert, W. E. (1973) Functional forms for profit and transformation functions. *Journal of Economic Theory* 6, 284–316.

Diewert, W. E. (1974) Applications of duality theory. In M. Intriligator and D. Kendrick (eds.), *Frontiers in Quantitive Economics*, pp. 106–171. Amsterdam: North-Holland.

Drake, L. and A. R. Fleissig (2004) Semi-nonparametric estimates of currency substitution: The demand for sterling in Europe. *Review of International Economics* 12, 374–394.

Drake, L. and A. R. Fleissig (2010) Substitution between monetary assets and consumer goods: New evidence on the monetary transmission mechanism. *Journal of Banking and Finance* 34, 2811–2821.

Drake, L., A. R. Fleissig, and A. Mullineaux (1999) Are 'risky' assets substitutes for 'monetary assets'? Evidence from an AIM demand system. *Economic Inquiry* 37, 510–526.

Drake, L., A. R. Fleissig, and J. L. Swofford (2003) A semi-nonparametric approach to the demand for U.K. monetary assets. *Economica* 70, 99–120.

Engle, R. F. and K. F. Kroner (1995) Multivariate simultaneous generalized ARCH. *Econometric Theory* 11, 122–150.

Ewiss, N. A. and D. Fisher (1984) The translog utility function and the demand for money in the United States. *Journal of Money, Credit and Banking* 16, 34–52.

Ewiss, N. A. and D. Fisher (1985) Toward a consistent estimate of the substitutability between money and near monies: An application of the Fourier flexible form. *Journal of Macroeconomics* 7, 151–174.

Fisher, D. and A. R. Fleissig (1994) Money demand in a flexible dynamic Fourier expenditure system. *Federal Reserve Bank of St. Louis Review* 76, 117–128.

Fisher, D. and A. R. Fleissig (1997) Monetary aggregation and the demand for assets. *Journal of Money, Credit and Banking* 29, 458–475.

Fisher, I. (1922) *The Making of Index Numbers: A Study of Their Varieties, Tests, and Reliability.* Boston: Houghton Mifflin.

Fleissig, A. R. (1997) The dynamic Laurent flexible form and long-run analysis. *Journal of Applied Econometrics* 12, 687–699.

Fleissig, A. R. and A. Serletis (2002) Seminonparametric estimates of substitution for Canadian monetary assets. *Canadian Journal of Economics* 35, 78–91.

Fleissig, A. R. and J. L. Swofford (1996) A dynamic asymptotically ideal model of money demand. *Journal of Monetary Economics* 37, 371–380.

Gorman, W. M. (1959) Separable utility and aggregation. *Econometrica* 27, 469–481.

Leontief, W. W. (1947) An introduction to a theory of the internal structure of functional relationships. *Econometrica* 15, 361–373.

Morishima, M. (1967) A few suggestions on the theory of elasticity (in Japanese). *Keizai Hyoron (Economic Review)* 16, 144–150.

Serletis, A. (1987) The demand for Divisia M1, M2, and M3 in the United States. *Journal of Macroeconomics* 9, 567–591.

Serletis, A. (1988) Translog flexible functional forms and substitutability of monetary assets. *Journal of Business and Economic Statistics* 6, 59–67.

Serletis, A. and G. Feng (2010) Semi-nonparametric estimates of currency substitution between the Canadian dollar and the U.S. dollar. *Macroeconomic Dynamics* 14, 29–55.

Serletis, A. and M. Isakin (2017) Stochastic volatility demand systems. *Econometric Reviews* 36, 1111–1122.

Serletis, A. and A. L. Robb (1986) Divisia aggregation and substitutability among monetary assets. *Journal of Money, Credit and Banking* 18, 430–446.

Serletis, A. and A. Shahmoradi (2005) Semi-nonparametric estimates of the demand for money in the United States. *Macroeconomic Dynamics* 9, 542–559.

Serletis, A. and A. Shahmoradi (2007) Flexible functional forms, curvature conditions, and the demand for assets. *Macroeconomic Dynamics* 11, 455–486.

Sono, M. (1961) The effect of price changes on the demand and supply of separable goods. *International Economic Review* 2, 239–271.

Strotz, R. H. (1957) The empirical implications of a utility tree. *Econometrica* 25, 169–180.

Strotz, R. H. (1959) The utility tree-A correction and further appraisal. *Econometrica* 27, 482–488.