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ON DETERMINATION OF A GAUGE FIELD ON $\mathbb{R}^d$ FROM ITS NON-ABELIAN RADON TRANSFORM ALONG ORIENTED STRAIGHT LINES

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Abstract We consider the inverse connection problem consisting of determining a gauge field on $\mathbb{R}^d$ from its non-abelian Radon transform along oriented straight lines. The determination is considered modulo gauge transformations. Our results include: global uniqueness theorems for $d \geqslant 3$, new local uniqueness theorems for $d = 2$, constructive proofs (i.e. proofs containing reconstruction procedures), counterexamples to the global uniqueness for $d = 2$, a reduction to the attenuated X-ray transform.

Keywords: gauge field; non-abelian Radon transform; inverse connection problem

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1. Introduction

We consider a collection $a = (a_1, \dots, a_d)$, where

$$\left.\begin{array}{l} a_i,\ i = 1, \dots, d, \text{ are sufficiently regular functions on } \mathbb{R}^d \text{ taking} \\ \text{their values in } M(n, \mathbb{C}), \text{ sufficiently rapidly vanishing at infinity,} \end{array}\right\} \tag{1.1}$$

for example,

$$a_i \in C^{\alpha,1+\varepsilon}(\mathbb{R}^d, M(n, \mathbb{C})), \quad i = 1, \dots, d, \quad \text{for some } \alpha > 0 \text{ and } \varepsilon > 0 \tag{1.1$_a$}$$

(where α is the degree of regularity and $1 + \varepsilon$ is the vanishing rate at infinity in terms of $O(|x|^{-1-\varepsilon})$ (in a sense depending on α) as $|x| \to \infty$), where we use the notation of § 2. We say also that a is a gauge field on $\mathbb{R}^d$.

Let η denote the trivial vector bundle with the base $\mathbb{R}^d$ and the fibre $\mathbb{C}^n$. The gauge field a generates the following $GL(n, \mathbb{C})$-connection on η: for this connection the covariant gradient $\nabla = (\nabla_1, \dots, \nabla_d)$ of the sections of η is given by

$$\nabla_i \psi(x) = \left(\frac{\partial}{\partial x_i} + a_i(x)\right)\psi(x), \quad i = 1, \dots, d, \quad x \in \mathbb{R}^d, \tag{1.2}$$

where ψ is a section of η.

Concerning the definition of a connection on a bundle and related definitions and facts see [**1**].

Let $T\mathbb{S}^{d-1}$ denote the space of all oriented straight lines in $\mathbb{R}^d$. The manifold $T\mathbb{S}^{d-1}$ can be given by the formula (3.17). The aforementioned gauge field a determines the map $S : T\mathbb{S}^{d-1} \to GL(n, \mathbb{C})$, where $S(\gamma)$ for fixed $\gamma \in T\mathbb{S}^{d-1}$ is the operator of the parallel transport of the fibre $\mathbb{C}^n$ along γ (from $-\infty$ to $+\infty$ on γ) according to the connection given by (1.2) (in terms of the covariant gradient). This definition implies that S is the scattering matrix for the equation

$$\theta\nabla\psi(x,\theta) = 0, \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}, \tag{1.3}$$

where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$, θ is a spectral parameter, ∇ is the covariant gradient given by (1.2), $\psi \in L^\infty(\mathbb{R}^d, M(n, \mathbb{C}))$ for fixed θ. That is

$$S(x,\theta) = \lim_{s\to+\infty} \psi^+(x + s\theta, \theta), \quad (x,\theta) \in T\mathbb{S}^{d-1}, \tag{1.4}$$

where $T\mathbb{S}^{d-1}$ is given by (3.17), $\psi^+(\cdot, \theta)$ is the solution of (1.3) specified by

$$\lim_{s\to-\infty} \psi^+(x + s\theta, \theta) = I \quad \text{for } x \in \mathbb{R}^d, \tag{1.5}$$

where I is the $n \times n$ identity matrix.

We say also that S is the non-abelian Radon transform along oriented straight lines (or the non-abelian X-ray transform) of the gauge field a. Note that S is invariant under gauge transformations of the form

$$a \to a' = (a'_1, \dots, a'_d), \tag{1.6}$$

where

$$a'_i = g^{-1}a_i g + g^{-1}\partial_i g, \quad \partial_i g(x) = \frac{\partial g(x)}{\partial x_i}, \quad i = 1, \dots, d, \quad x \in \mathbb{R}^d,$$

$$\left.\begin{array}{l} g \text{ is a sufficiently regular function on } \mathbb{R}^d \text{ taking its values in} \\ GL(n,\mathbb{C}), \text{ sufficiently rapidly approaching } I \text{ at infinity.} \end{array}\right\} \tag{1.7}$$

For example, the transform S of a gauge field a satisfying (1.1_a) and the property (1.1_a) itself are invariant under gauge transformations of the form (1.6), where

$$\left.\begin{array}{l} g \text{ takes its values in } GL(n,\mathbb{C}), \\ g - I \in C^{0,\varepsilon}(\mathbb{R}^d, M(n,\mathbb{C})), \quad \partial_i g \in C^{\alpha,1+\varepsilon}(\mathbb{R}^d, M(n,\mathbb{C})), \quad i = 1,\dots,d. \end{array}\right\} \tag{1.7$_a$}$$

We consider now the following inverse scattering (or inverse connection) problem.

Problem. *Given the transform S of a gauge field a with the property (1.1), find a modulo the gauge transformations (1.6), (1.7) (for example, under the conditions (1.1_a), (1.7_a)).*

This problem was considered recently in [**16**] for the compactly supported case. In addition, related problems were being investigated in many works (see [**5**, **9**, **10**, **12**, **18**–**20**]). Below in the introduction we give, in particular, some comments in this connection.

Let a^+_ω, $\omega \in \mathbb{S}^{d-1}$, denote a' related with a satisfying (1.1) by (1.6) for $g = \psi^+(\cdot, \omega)$.

In the present paper we obtain, in particular, the following results (formulated using the notation of § 2).

(I) For $d \geqslant 3$, we show that a gauge field a satisfying (1.1_a), $\alpha = 2$, is determined modulo the gauge transformations (1.6), (1.7_a) by its transform S. In addition, the proof contains a reconstruction procedure of a^+_ω from S, ε and an upper bound for $\sum_{i=1}^{d} \|a_i\|_{2,1+\varepsilon}$ for any $\omega \in \mathbb{S}^{d-1}$. We give, in a parallel way, similar results for a gauge field a satisfying (3.4 b) for $i = 1, \dots, d$, $\alpha = 2$ and preassigned r. (See Theorem 6.1 and Corollary 6.1.)

(II) For $d = 2$, we show that a gauge field a satisfying (1.1_a) and such that

$$\sum_{i=1}^{2} \|a_i\|_{\alpha,1+\varepsilon,\rho}$$

is small enough with respect to ρ for fixed α, ε and ρ is determined modulo the gauge transformations (1.6), (1.7_a) by its transform S. In addition, we give a reconstruction procedure of a^+_ω from S for any $\omega \in \mathbb{S}^1$. We give, in a parallel way, similar results for a gauge field a satisfying (3.4 b) for $i = 1, 2$, $d = 2$. (See Theorem 5.2 and Corollary 5.3.)

(III) For $d = 2$, for any $\alpha > 0$ and $\varepsilon > 0$ we show by explicit examples that, in general, a gauge field a satisfying (1.1_a) and considered up to (1.6), (1.7_a) is not determined by its transform S. In the examples in question $n = 2$, $a_1 \equiv 0$, a_2 is a rational function on $\mathbb{R}^2$ taking its values in $su(n)$, $S \equiv I$ and a considered up to (1.6), (1.7_a) (as well as up to the gauge transformations (1.6), where $g \in C^1(\mathbb{R}^2, GL(n, \mathbb{C}))$) differs from $a' \equiv (0, 0)$. (See Theorem 7.1.)

Note also that, for $d = 1$, a gauge field a satisfying (1.1_a) is determined up to (1.6), (1.7_a) from its transform S. In addition, $a^+_\omega \equiv 0$ for any initial a.

We emphasize that in the aforementioned global (for $d \geqslant 3$) and local (for $d = 2$) constructive uniqueness results obtained under the assumptions (1.1_a) (with $\alpha = 2$ for $d \geqslant 3$) we deal with the least possible vanishing rate at infinity (by assuming that $\varepsilon > 0$ only) in the sense that under the assumptions (1.1_a), where $\alpha > 0$ and $\varepsilon = 0$, the transform S is not defined, in general (even for $\alpha = +\infty$ and $n = 1$). In addition, the aforementioned examples of item (III) show that the global uniqueness under the assumptions (1.1_a) for $d = 2$ fails (even for $\alpha = +\infty$) for any fixed $\varepsilon \in \,]0, +\infty[$.

The scheme of the proof of the aforementioned results for $d \geqslant 3$ contains the following the most principle components.

(a) The key point is that S determines a^+_ω (in a constructive way if an upper bound for a suitable norm of a is known). The other is a corollary. (See Theorem 6.1, Remark 6.1 and Corollary 6.1.)

(b) We show that S determines a^+_ω, first, for $d=3$ and then, as a corollary, for $d>3$.

(c) For $d=3$ we show, first, that S determines a^+_ω on $\mathbb{R}^3\backslash B_\tau$, where $B_\tau=\{x\in\mathbb{R}^3 \mid |x|<\tau\}$ and τ is large enough in such a way that a restricted to each two-dimensional plane $X\subset\mathbb{R}^3\backslash B_\tau$ is small enough in a suitable sense. We do it using the aforementioned results of item (II) for $d=2$. Further, as an intermediate step, for any $t>0$, we show that S and a^+_ω on $\mathbb{R}^3\backslash B_t$ determine the non-abelian X-ray transform $S^+_{\omega,(t)}$ of the field $a^+_{\omega,(t)}$ coinciding with a^+_ω on B_t and being identically zero on $\mathbb{R}^3\backslash B_t$. Further, using also that $(a^+_{\omega,(t)})^+_\omega=a^+_{\omega,(t)}$, we show that S determines a^+_ω on $\mathbb{R}^3\backslash B_{\tau_i}$, where τ_i, $i=1,\dots,k$, is a suitable finite sequence such that $\tau_k=0$, $\tau_j<\tau_i$ for $j>i$, $\tau_1=\tau$. (See Propositions 6.1–6.3 and Proof of Theorem 6.1 for $d=3$.)

This proof is given in detail in § 6. From the comments of Uhlmann on this paper, we learned that the idea to use local uniqueness results for $d=2$ to get global uniqueness results for $d=3$ is not new in the sense that such an idea was used earlier (see § 3 of [**4**]) in proving that the attenuated X-ray transformation (being linear and involving no gauge invariance considerations) is injective for the regular compactly supported case for $d=3$. In § 8 we obtain the attenuated X-ray transform as a reduction of the non-abelian Radon transform.

To obtain the aforementioned results for $d=2$ of item (II) we proceed from a formal method of [**12**] for solving (by means of Riemann conjugation problems) the inverse scattering problem for the equation

$$(\zeta\partial_{\bar z}-\zeta^{-1}(\partial_z+B)+A)\psi=0, \tag{1.8}$$

where ζ is a spectral parameter, $\zeta\in\mathbb{C}$, $|\zeta|=1$, A, B are sufficiently regular functions on $\mathbb{C}$ taking their values in $M(n,\mathbb{C})$, sufficiently rapidly vanishing at infinity, sufficiently small in a suitable sense, $\psi\in L^\infty(\mathbb{C},M(n,\mathbb{C}))$ for fixed ζ, $\partial_z=\partial/\partial z$, $\partial_{\bar z}=\partial/\partial\bar z$, z, $\bar z$ are the standard coordinates on $\mathbb{C}$. Note that, at least, on a formal level the equation (1.3) for $d=2$ and sufficiently small a_1, a_2 can be taken to the form (1.8) with $A\equiv 0$ using a gauge transformation and a change of variables. We generalize the aforementioned formal method of [**12**] to the case of the equation (1.9) for $d=2$ and sufficiently small a_0, a_1, a_2 and justify the resulting version of the method by a proper analysis. Our results and proofs in this connection are given in detail in § 3 (for $d=2$), § 4 and § 5. In particular, for the equation (1.9) for $d=2$, under our smallness assumptions, the transform S determines the collection a in its gauge setting a^+_ω (denoting a' given by (1.10) for $g=\psi^+_0(\cdot,\omega)$, where ψ^+_0 denotes the function ψ^+ for the equation (1.9) with a_0 replaced by zero) for any $\omega\in\mathbb{S}^1$ by the following scheme:

(a) S (written as $S^\perp$ according to (4.76 a)) determines $Q^\perp_{+,\pm}$, $Q^\perp_{-,\pm}$ via Riemann conjugation problems according to Propositions 5.1 and 5.2;

(b) $Q^\perp_{\pm,-}$, $Q^\perp_{\pm,+}$ determine R by the formulae (4.83), (4.84);

(c) R (written using complex notation) determines $\tilde\psi_\pm$ via a Riemann conjugation problem according to Propositions 5.3 and 5.4;

(d) $\tilde{\psi}_\pm$ determine the collection a in its gauge setting $\tilde{a}$ (given by (5.4 b)–(5.4 e), (5.5) using complex notation) according to Proposition 5.3;

(e) $\tilde{a}$ determines a^+_ω by (5.32 c), (5.32 d).

In addition, the Riemann conjugation problems of inverse scattering arise from the equations (4.50)–(4.53), (4.85), (4.86) of Propositions 4.1 and 4.2 of direct 'scattering' with complex spectral parameter.

Note that results on inverse scattering for (1.8) are given in [**12**] in the framework of the inverse scattering method for solving a $(2+1)$-dimensional system suggested in [**12**] as a reduction of the self-dual Yang–Mills equation in $2+2$ dimensions.

To obtain the aforementioned examples for $d=2$ of item (III) we use results of [**19**] and subsequent results of [**18**] concerning soliton solutions of an integrable $(2+1)$-dimensional system suggested in [**19**] as a reduction (different from the reduction of [**12**]) of the self-dual Yang–Mills equation in $2+2$ dimensions. The results of [**18**, **19**] in question include, actually, some results on inverse spectral problem for the equation (1.3) for $d=2$, $a_1\equiv 0$ and complexified $\theta\in\Sigma=\{\theta\in\mathbb{C}^2 \mid \theta^2=1\}$. In addition, the scattering matrix S defined by (1.4), (1.5) is not considered in [**18**, **19**]. See § 7 for details.

Concerning the most recent results (obtained before [**14**]) on solving the aforementioned system of [**19**] by the inverse spectral method involving the spectral problem (1.3) for $d=2$, $a_1\equiv 0$ see [**5**]. The scattering matrix S defined by (1.4), (1.5) is not considered in [**5**].

In the present paper we consider also the inverse scattering problem for the equation

$$\theta\nabla\psi(x,\theta)+a_0(x)\psi(x,\theta)=0,\quad x\in\mathbb{R}^d,\quad \theta\in\mathbb{S}^{d-1}, \tag{1.9}$$

where θ is a spectral parameter, ∇ is the covariant gradient given by (1.2), a_0 is of the same functional space that a_i, $i=1,\dots,d$, $\psi(\cdot,\theta)\in L^\infty(\mathbb{R}^d,M(n,\mathbb{C}))$. We say that the equation (1.9) (as well as the equation (1.3)) is the X-ray connection equation.

The scattering matrix S for (1.9) is defined by (1.4), where $\psi^+(\cdot,\theta)$ is now the solution of (1.9) specified by (1.5). We say also that S is the non-abelian Radon transform along oriented straight lines (or the non-abelian X-ray transform) of the collection $a=(a_0,a_1,\dots,a_d)$. The scattering matrix S for (1.9) is invariant under gauge transformations of the form

$$a=(a_0,a_1,\dots,a_d)\to a'=(a'_0,a'_1,\dots,a'_d), \tag{1.10}$$

where

$$a'_i=g^{-1}a_ig+g^{-1}\partial_ig,\quad i=1,\dots,d,\quad a'_0=g^{-1}a_0g,$$

and g is a function satisfying (1.7).

We show that aforementioned results of items (I), (II) for the case of the equation (1.3) admit a direct generalization to the case of the equation (1.9).

To explain a geometric sense of the equation (1.9), consider the Minkowski space $\mathbb{R}^{d+1}_{1,d}$ with the coordinates (t,x), $t\in\mathbb{R}$, $x\in\mathbb{R}^d$. For any oriented straight line γ in $\mathbb{R}^d$, $\gamma=(y,\theta)\in T\mathbb{S}^{d-1}$, we consider the family $l(\gamma,p)$, $p\in\mathbb{R}$, of light rays in $\mathbb{R}^{d+1}_{1,d}$:

$$l(\gamma,p)=\{(t,x)\in\mathbb{R}^{d+1}_{1,d} \mid t=2^{-1/2}s+p,\ x=2^{-1/2}s\theta+y,\ s\in\mathbb{R}\} \tag{1.11}$$

(up to orientation) and it is assumed that the orientation of $l(\gamma,p)$ is given by the vector $2^{-1/2}(1,\theta)$, $p\in\mathbb{R}$. The scattering matrix $S(\gamma)$ for the equation (1.9) for fixed $\gamma\in T\mathbb{S}^{d-1}$ is the operator of the parallel transport along $l(\gamma,p)$ (from $-\infty$ to $+\infty$ on $l(\gamma,p)$, for an arbitrary $p\in\mathbb{R}$) of the fibre $\mathbb{C}^n$ of the trivial vector bundle over $\mathbb{R}^{d+1}_{1,d}$ according to the $GL(n,\mathbb{C})$-connection with the covariant gradient $\nabla=(\nabla_0,\nabla_1,\dots,\nabla_d)$ of the sections given by

$$\left.\begin{aligned}\nabla_0\psi(t,x)&=\left(\frac{\partial}{\partial t}+a_0(x)\right)\psi(t,x),\\ \nabla_i\psi(t,x)&=\left(\frac{\partial}{\partial x_i}+a_i(x)\right)\psi(t,x),\end{aligned}\right\}\quad i=1,\dots,d,\quad (t,x)\in\mathbb{R}^{d+1}_{1,d}. \tag{1.12}$$

Consider now the scattering matrix S for (1.9) under the additional condition that a_i, $i=0,1,\dots,d$, take values in $M(n,\mathbb{C})$ for the case $n=1$. In this case the following formula holds:

$$S(\gamma)=\exp(-Pa(\gamma)), \tag{1.13 a}$$

$$Pa(\gamma)\stackrel{\text{def}}{=}\int_\gamma\sum_{i=1}^d a_i\,\mathrm{d}x_i+\int_\gamma a_0\,\mathrm{d}s, \tag{1.13 b}$$

s is a natural parameter on γ, $\gamma\in T\mathbb{S}^{d-1}$. Thus, the scattering matrix S for (1.9) for the case $n=1$ is reduced to the abelian Radon transform along oriented straight lines Pa of a collection a. For the case when $a_i\equiv 0$, $i=1,\dots,d$, and for the case when $a_0\equiv 0$, explicit inversion formulae and a characterization of the image (in terms of differential equations for $d\geqslant 3$) for the transformation P are given in the literature (see, for example, [**6**, **7**, **13**]). One can generalize these results for the case when the both a_0 and $\sum_{i-1}^d a_i\,\mathrm{d}x_i$ are present in (1.13 *b*). (An inversion way for the latter case is given in [**17**].) In addition, concerning the characterization of the image of the non-abelian Radon transformation along oriented straight lines of collections a, for $n\geqslant 2$, one can obtain results in this direction, for $d\geqslant 3$, using methods of [**8**–**10**].

Consider now the scattering matrix S for (1.9) under the additional assumptions that

$$n=2,\quad a_i\equiv 0,\quad i=1,\dots,d,\quad a_0=\begin{pmatrix}\mu & f\\ 0 & 0\end{pmatrix}.$$

In this case the following formula holds

$$S(\gamma)=\begin{pmatrix}\exp(-P\mu(\gamma)) & -P_\mu f(\gamma)\\ 0 & 1\end{pmatrix},\quad \gamma\in TS^{d-1}, \tag{1.14}$$

where P is the classical X-ray transformation of the transmission tomography, P_μ is the attenuated X-ray transformation of the emission tomography (see § 8 for details).

In the present paper, in §§ 3–6, the results are given for the case A, that is under the assumptions (3.4 *a*) (with additional specifications), and, in a parallel way, for the case B, that is under the assumptions (3.4 *b*) (with additional specifications). Our motivation to consider the case B consists of the following.

(1) To prove the results of § 6 for the case A for $d = 3$, we use, in particular, results of §§ 3–6 for the case B for $d \in \{2, 3\}$.

(2) The results for the case B and, say, for $a_0 \equiv 0$ can be interpreted as results on an inverse connection problem for the trivial vector bundle over $\mathcal{D}$ with the fibre $\mathbb{C}^n$, where $\mathcal{D} = \bar{B}_r = \{x \in \mathbb{R}^d \mid |x| \leqslant r\}$. In addition, one can generalize these results for the case when

$$\left.\begin{array}{l}\mathcal{D} \subset \mathbb{R}^d \text{ is a strictly convex (in the strong sense)} \\ \text{compact domain with smooth boundary.}\end{array}\right\} \tag{1.15}$$

Note that our counterexamples to the global uniqueness for $d = 2$ (mentioned above and given in details in § 7) are not compactly supported. The global uniqueness problem of inverse scattering for the equation (1.3), (1.9) with sufficiently regular compactly supported coefficients remains open for $d = 2$.

Earlier, the problem of determining a G-connection on a vector bundle over a simple compact Riemannian manifold $(M, \tilde{g})$, from the known parallel transport operator between every two boundary points of M along the geodesic (of the metric $\tilde{g}$) joining these points was considered in [**16**], at least, for the case when the fibre is $\mathbb{C}^n$ and $G = U(n)$. In addition, the determination is considered up to an automorphism of the bundle which is identical on the boundary ∂M (i.e. up to a gauge transformation). The main result of [**16**] on this problem is the uniqueness theorem under small norms assumptions (i.e. a local uniqueness theorem). In [**16**], the proof of this result contains no reconstruction procedure. As an example of a simple compact Riemannian manifold one can take a domain $\mathcal{D}$ satisfying (1.15) with the Euclidean metric. Therefore, the main result of [**16**] includes a result on the aforementioned inverse connection problem for the trivial vector bundle over $\mathcal{D}$ satisfying (1.15) with the fibre $\mathbb{C}^n$.

The work [**16**] was preceded by [**20**]. The work [**20**] deals with the problem of determining an $n \times n$ matrix function on a compact simply connected planar domain with smooth boundary from its multiplicative integrals along a family of curves (with suitable properties) joining boundary points. The main result of [**20**] on this problem is the uniqueness theorem under small norm assumptions and a related stability estimate. In [**20**], the proof of this result contains no reconstruction procedure. (Although, in principle, proceeding from this result and using considerations with epsilon chains one can propose some reconstruction procedure of a function of a compact subclass.) As an example of a domain with a curves family satisfying the assumptions of [**20**] one can take a domain $\mathcal{D}$ satisfying (1.15) with the family of geodesics of the Euclidean metric in $\mathcal{D}$. Therefore, the main result of [**20**] includes a result on the inverse scattering problem for the equation (1.9) for $d = 2$, $a_1 \equiv 0$, $a_2 \equiv 0$.

The present work was stimulated by [**16**]. In the present work we deal only with the case of parallel transport (or multiplicative integration) along Euclidean geodesics. For this case the progress of the present work in compare with [**16, 20**] includes: global uniqueness theorems for $d \geqslant 3$, new local uniqueness theorems for $d = 2$, constructive proofs (i.e. proofs containing reconstruction procedures), no restrictions by the limits of the compactly supported case, counterexamples to the global uniqueness for $d = 2$ and, for example, a pure geometric interpretation of the inverse scattering problem for the equation (1.9). Presentation of the attenuated X-ray transform as a reduction of the non-abelian X-ray transform (according to the formulae (1.14), (8.5)) is also a new result of the present work.

Note now that non-abelian Radon transformation of gauge fields on $\mathbb{R}^d$ along oriented straight lines discussed in the present paper is a real version of the non-abelian Radon transformation of gauge fields on $\mathbb{C}^d$ (or $\mathbb{R}^d$) along complex straight lines discussed in [**10**], at least, for $d = 4$. In addition, the latter transformation taken along complex light lines, only, is called the Radon–Penrose transformation. The non-abelian Radon transform along complex straight lines of a gauge field $a = (a_1, \dots, a_d)$ on $\mathbb{R}^d$ satisfying (1.1) is actually some $\bar{\partial}$-'scattering' data for the equation (1.3) with complexified $\theta \in \Sigma = \{\theta \in \mathbb{C}^d \mid \theta^2 = 1\}$. The work [**10**] contains, in particular, a global uniqueness theorem (which remains valid for $d \geqslant 2$) and a reconstruction procedure on determination of a gauge field from its non-abelian Radon transform along complex straight lines and a characterization of the image for this transformation. It should be noted, in addition, that by its construction the non-abelian Radon transform of a gauge field along complex straight lines contains the condition of unique solvability of inverse problem equations. In contrast, the non-abelian Radon transform of a gauge field along (real) oriented straight lines has a completely transparent real geometric sense. As a result, in particular, for the case of determination of a gauge field from its non-abelian Radon transform along (real) oriented straight lines there are counterexamples to the global uniqueness for $d = 2$ and the proof of the global uniqueness theorems for $d \geqslant 3$ is rather complicated from technical point of view.

Note, finally, that the spectral problem (1.3) (including the case of complexified $\theta \in \Sigma = \{\theta \in \mathbb{C}^d \mid \theta^2 = 1\}$) arises as a high energy limit of the Schrödinger equation in an external Yang–Mills field (see [**2**, **8**–**10**]). (In addition, the spectral problem (1.9) for $n = 1$ arises as a high energy limit of the wave equation with first order perturbation in [**17**].) Using the global uniqueness theorem (of the present paper) on inverse scattering for (1.3) for $d \geqslant 3$, one can obtain a global uniqueness theorem on inverse scattering for the Schrödinger equation in an external Yang–Mills field at high energies for $d \geqslant 3$.

The present paper (without results of § 8) corresponds to the preprint [**14**].

2. Functional spaces and related notation

We consider

$$C^{\alpha,\sigma}(\mathcal{D}, \mathcal{V}) = \{f \in C^{[\alpha]}(\mathcal{D}, \mathcal{V}) \mid \|f\|_{\alpha,\sigma} < +\infty\}, \tag{2.1}$$

where $\alpha \geqslant 0$, $\sigma \geqslant 0$, $[\alpha]$ is the integer part of α, $\mathcal{D}$ is an open domain or the closure of an open domain in $\mathbb{R}^d$ (for example, $\mathcal{D} = \mathbb{R}^d$), $\mathcal{V}$ is a subset in $\mathcal{M}_{m\times n}$, where $\mathcal{M}_{m\times n}$ is the space of $m \times n$ matrices with complex elements, $C^k(\mathcal{D}, \mathcal{V})$, $k \in \mathbb{N} \cup 0$, is the space of k-times continuously differentiable functions on $\mathcal{D}$ with values in $\mathcal{V}$,

$$\|f\|_{\alpha,\sigma} = \|f\|_{\alpha,\sigma,1}, \tag{2.2}$$

where

$$\|f\|_{0,\sigma,\rho} = \sup_{x\in\mathcal{D}} (\rho + |x|)^{\sigma} |f(x)|, \tag{2.3}$$

$$\left.\begin{aligned} \|f\|_{\alpha,\sigma,\rho} &= \max(\|f\|_{0,\sigma,\rho}, \|f\|'_{\alpha,\sigma,\rho}), && 0<\alpha\leqslant 1,\\ \|f\|'_{\alpha,\sigma,\rho} &= \sup_{\substack{x\in\mathcal{D},\\ x+y\in\mathcal{D},\\ |y|\leqslant 1}} (\rho+|x|)^{\sigma}|y|^{-\alpha}|f(x+y)-f(x)|, && 0<\alpha\leqslant 1, \end{aligned}\right\} \tag{2.4}$$

$$\|f\|_{\alpha,\sigma,\rho} = \max\Big(\|f\|_{\beta,\sigma,\rho}, \max_{|J|=\beta} \|\partial^J f\|_{\alpha-\beta,\sigma,\rho}\Big) \quad \text{for } \beta\leqslant\alpha\leqslant\beta+1,\ \beta\in\mathbb{N} \tag{2.5}$$

(where $\partial^J = \partial^{|J|}/\partial x_1^{J_1}\dots\partial x_d^{J_d}$, $J\in(\mathbb{N}\cup 0)^d$, $|J| = \sum_{i=1}^d J_i$), where $\rho>0$,

$$|c| = \max_{\substack{1\leqslant i\leqslant m,\\ 1\leqslant j\leqslant n}} |c_{ij}| \quad \text{for } c\in\mathcal{M}_{m\times n}. \tag{2.6}$$

Let $\mathcal{D}$ be an open domain or the closure of an open domain in $\mathbb{C}$, $\alpha\geqslant 0$, $\sigma\geqslant 0$, $\mathcal{V}\subseteq\mathcal{M}_{m\times n}$ and f be a function on $\mathcal{D}$ with values in $\mathcal{V}$. Then we write $f\in C^{\alpha,\sigma}(\mathcal{D},\mathcal{V})$ if and only if $f_{\mathbb{R}}\in C^{\alpha,\sigma}(\mathcal{D}_{\mathbb{R}},\mathcal{V})$, where $\mathcal{D}_{\mathbb{R}} = \{x\in\mathbb{R}^2 \mid x_1+\mathrm{i}x_2\in\mathcal{D}\}$, $f_{\mathbb{R}}(x) = f(x_1+\mathrm{i}x_2)$ for $x\in\mathcal{D}_{\mathbb{R}}$ (the notation $f=f(z)$ does not mean that f is a holomorphic function in z), and, by definition, $\|f\|_{\alpha,\sigma,\rho} = \|f_{\mathbb{R}}\|_{\alpha,\sigma,\rho}$, $\rho>0$.

We consider also

$$C^{\alpha,\sigma}(X_Y,\mathcal{V}) = \{f\in C(X,\mathcal{V}) \mid \|f\|_{\alpha,(Y),\sigma} < +\infty\}, \tag{2.7}$$

where $0\leqslant\alpha<1$, $\sigma\geqslant 0$, $X=\mathbb{R}^d$, Y is a non-zero subspace in X, $\mathcal{V}\subseteq\mathcal{M}_{m\times n}$,

$$\|f\|_{\alpha,(Y),\sigma} = \|f\|_{\alpha,(Y),\sigma,1}, \tag{2.8}$$

where

$$\|f\|_{0,(Y),\sigma,\rho} = \|f\|_{0,\sigma,\rho} = \sup_{x\in X} (\rho+|x|)^{\sigma}|f(x)|, \tag{2.9}$$

$$\left.\begin{aligned} \|f\|_{\alpha,(Y),\sigma,\rho} &= \max(\|f\|_{0,\sigma,\rho}, \|f\|'_{\alpha,(Y),\sigma,\rho}), && 0<\alpha<1,\\ \|f\|'_{\alpha,(Y),\sigma,\rho} &= \sup_{\substack{x\in X,\\ y\in Y,\\ |y|\leqslant 1}} (\rho+|x|)^{\sigma}|y|^{-\alpha}|f(x+y)-f(y)|, && 0<\alpha<1, \end{aligned}\right\} \tag{2.10}$$

where $\rho>0$.

Note that

$$C^{\alpha,\sigma}(X,\mathcal{V}) = C^{\alpha,\sigma}(X_X,\mathcal{V}),$$
$$\|f\|_{\alpha,\sigma} = \|f\|_{\alpha,(X),\sigma}, \qquad \|f\|_{\alpha,\sigma,\rho} = \|f\|_{\alpha,(X),\sigma,\rho}, \qquad \|f\|'_{\alpha,\sigma,\rho} = \|f\|'_{\alpha,(X),\sigma,\rho},$$

where $0\leqslant\alpha<1$, $\sigma\geqslant 0$, $X=\mathbb{R}^d$, $f\in C(X,\mathcal{V})$. We use the following abbreviation

$$\|f\|_0 = \|f\|_{0,0} \quad \text{for } f\in C(\mathcal{D},\mathcal{V}).$$

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We consider the following special subsets of $\mathcal{M}_{n\times n}$:

$$GL(n,\mathbb{C}) = \{c \in \mathcal{M}_{n\times n} \mid \det c \neq 0\},$$
$$\left.\begin{aligned} U(n) &= \{c \in \mathcal{M}_{n\times n} \mid cc^* = I\}, \quad u(n) = \{c \in \mathcal{M}_{n\times n} \mid c^* = -c\}, \\ SU(n) &= \{c \in U(n) \mid \det c = 1\}, \quad su(n) = \{c \in u(n) \mid \operatorname{tr} c = 0\}, \end{aligned}\right\}$$

where $c^* = \bar{c}^{\mathrm{T}}$, I is the $n \times n$ identity matrix. The space $\mathcal{M}_{n\times n}$ is denoted also as $M(n,\mathbb{C})$.

3. Direct scattering for the X-ray connection equation

Consider the X-ray connection equation

$$\theta\partial_x\psi + v(x,\theta)\psi = 0, \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}, \tag{3.1}$$

where $\mathbb{S}^{d-1} = \{\theta \in \mathbb{R}^d \mid \theta^2 = 1\}$, θ is a spectral parameter,

$$\partial_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}\right), \qquad \theta\partial_x = \sum_{i=1}^{d} \theta_i \frac{\partial}{\partial x_i}, \tag{3.2}$$

$$v(x,\theta) = a_0(x) + \sum_{i=1}^{d} \theta_i a_i(x), \tag{3.3}$$

a_i, $i = 0, 1, \dots, d$, satisfy (3.4 a) or (3.4 b), ψ satisfies (3.6):

$$a_i \in C^{\alpha,1+\varepsilon}(\mathbb{R}^d, \mathcal{M}_{n\times n}), \quad i = 0, 1, \dots, d, \tag{3.4 a}$$

for some $\alpha > 0$ and some $\varepsilon > 0$ or

$$\left.\begin{aligned} a_i(x) &= \chi_+(r - |x|)b_i(x), \quad x \in \mathbb{R}^d, \\ b_i(x) &\in C^{\alpha,0}(\mathbb{R}^d, \mathcal{M}_{n\times n}), \quad i = 0, 1, \dots, d, \end{aligned}\right\} \tag{3.4 b}$$

for some $\alpha > 0$ and some $r \geqslant 0$, where

$$\chi_+(s) = \begin{cases} 1 & \text{for } s > 0, \\ 0 & \text{for } s \leqslant 0; \end{cases} \tag{3.5}$$

$$\psi \in L^\infty(\mathbb{R}^d, \mathcal{M}_{n\times n}) \quad \text{for fixed } \theta, \tag{3.6}$$

where $d \in \mathbb{N}$, $n \in \mathbb{N}$. For a collection $u = (u_0, u_1, \dots, u_d)$, $u_i \in C^{\alpha,\sigma}(\mathbb{R}^d, \mathcal{M}_{n\times n})$, $i = 0, 1, \dots, d$, we use the notation

$$\|u\|_{\alpha,\sigma,\rho} = \sum_{i=0}^{d} \|u_i\|_{\alpha,\sigma,\rho}, \qquad \|u\|_{\alpha,\sigma,1} = \|u\|_{\alpha,\sigma}, \qquad \|u\|_{\alpha,0} = \|u\|_{\alpha}, \tag{3.7}$$

where $\alpha \geqslant 0$, $\sigma \geqslant 0$, $\rho > 0$ and for $\|u_i\|_{\alpha,\sigma,\rho}$, $i = 0, 1, \dots, d$, we use the definition given in § 2.

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For $\theta \in \mathbb{S}^{d-1}$ we consider the solutions $\psi^{\pm}(\cdot, \theta)$ of (3.1) specified by the conditions

$$\lim_{s \to -\infty} \psi^{+}(x + s\theta, \theta) = I, \qquad (3.8\,a)$$

$$\lim_{s \to +\infty} \psi^{-}(x + s\theta, \theta) = I, \qquad (3.8\,b)$$

where I is the $n \times n$ identity matrix, $x \in \mathbb{R}^d$.

We consider the function

$$S(x, \theta) = (\psi^{-}(x, \theta))^{-1} \psi^{+}(x, \theta), \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}. \qquad (3.9)$$

(The fact that $\det \psi^{-} \neq 0$ follows from (3.40) for $\psi = \psi^{-}$ and (3.8 *b*).)

We use the following terminology (going back to the scattering theory for the Schrödinger equation): the functions $\psi^{\pm}$ are the wave functions and the function S is the scattering matrix for the equation (3.1).

The following formulae hold:

$$\theta \partial_x S(x, \theta) = 0, \qquad (3.10)$$

$$S(x, \theta) = \lim_{s \to +\infty} \psi^{+}(x + s\theta, \theta), \qquad (3.11)$$

$$S(x, \theta) = T \exp \int_{-\infty}^{+\infty} -v(x + t\theta, \theta)\, \mathrm{d}t, \qquad (3.12)$$

$$\psi^{+}(x + s\theta, \theta) = T \exp \int_{-\infty}^{s} -v(x + t\theta, \theta)\, \mathrm{d}t, \qquad (3.13\,a)$$

$$\psi^{-}(x + s\theta, \theta) = \left(T \exp \int_{s}^{+\infty} -v(x + t\theta, \theta)\, \mathrm{d}t \right)^{-1}, \qquad (3.13\,b)$$

where $x \in \mathbb{R}^d$, $\theta \in \mathbb{S}^{d-1}$, T denotes t-ordering (see, for example, Part II, § 25 of [**1**]).

The formula (3.10) follows from (3.1) for $\psi = \psi^{\pm}$ and (3.9). The formula (3.11) follows from (3.8 *b*), (3.9), (3.10). The formulae (3.13) follow from the equation (3.1) written as

$$\left(\frac{\mathrm{d}}{\mathrm{d}s} \right) \psi(x + s\theta, \theta) + v(x + s\theta, \theta) \psi(x + s\theta, \theta) = 0 \qquad (3.14)$$

for $\psi = \psi^{\pm}$ and the conditions (3.8). The formula (3.12) follows from (3.11), (3.13 *a*).

Due to (3.10),

$$S(x, \theta) = S(\pi_\theta x, \theta), \quad \theta \in \mathbb{S}^{d-1}, \quad x \in \mathbb{R}^d, \qquad (3.15)$$

where

π_θ is the orthogonal projector of $\mathbb{R}^d$ on the subspace $X_\theta = \{x \in \mathbb{R}^d \mid x\theta = 0\}$. (3.16)

Due to (3.15), S on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ is uniquely determined by S on $T\mathbb{S}^{d-1}$, where

$$T\mathbb{S}^{d-1} = \{(x, \theta) \mid x \in \mathbb{R}^d,\ \theta \in \mathbb{S}^{d-1},\ x\theta = 0\}. \qquad (3.17)$$

We interpret $T\mathbb{S}^{d-1}$ as the set of all rays in $\mathbb{R}^d$. As a ray γ we understand a straight line with fixed orientation. If $\gamma = (x,\theta) \in T\mathbb{S}^{d-1}$, then $\gamma = \{y \in \mathbb{R}^d \mid y = x + t\theta,\ t \in \mathbb{R}\}$ (up to orientation) and θ gives the orientation of γ. The scattering matrix $S(x,\theta)$ at fixed $(x,\theta) \in T\mathbb{S}^{d-1}$ is uniquely determined by $a_0(x)$ and $\sum_{i=1}^{d} a_i(x)\,\mathrm{d}x_i$ restricted to the ray $\gamma = (x,\theta)$ (as to the straight line) and by the orientation of γ. To obtain this statement we use the equation (3.14) for $\psi = \psi^{\pm}$ and the formulae (3.8), (3.9) at fixed $(x,\theta) \in T\mathbb{S}^{d-1}$.

The functions $\psi^{\pm}(\cdot,\theta)$ satisfy the following integral equations

$$\psi^{+}(\cdot,\theta) = I - D_{-\theta} v_\theta \psi^{+}(\cdot,\theta), \psi^{+}(\cdot,\theta) \in L^{\infty}(\mathbb{R}^d, \mathcal{M}_{n\times n}), \tag{3.18 a}$$

$$\psi^{-}(\cdot,\theta) = I + D_{\theta} v_\theta \psi^{-}(\cdot,\theta), \psi^{-}(\cdot,\theta) \in L^{\infty}(\mathbb{R}^d, \mathcal{M}_{n\times n}), \tag{3.18 b}$$

where

$$D_{\mp\theta} v_\theta f = D_{\mp\theta}(v_\theta f), \tag{3.19}$$

$$v_\theta f(x) = v(x,\theta) f(x), \tag{3.20}$$

$$D_\theta \varphi(x) = \int_0^{+\infty} \varphi(x + t\theta)\,\mathrm{d}t, \tag{3.21}$$

where $\theta \in \mathbb{S}^{d-1}$, $x \in \mathbb{R}^d$. Due to Lemma A.4, the equations (3.18) are uniquely solvable by the method of successive approximations.

The following formula holds:

$$S(\cdot,\theta) = I - P_\theta v_\theta \psi^{+}(\cdot,\theta), \tag{3.22}$$

where

$$P_\theta v_\theta f = P_\theta(v_\theta f), \tag{3.23}$$

v_θ is defined by (3.20),

$$P_\theta \varphi(x) = \int_{-\infty}^{+\infty} \varphi(x + t\theta)\,\mathrm{d}t, \tag{3.24}$$

where $\theta \in \mathbb{S}^{d-1}$, $x \in \mathbb{R}^d$. The formula (3.22) follows from (3.18 *a*), (3.11).

We use the following terminology: the operator D_θ defined by (3.21) is the divergent beam transform at fixed direction $\theta \in \mathbb{S}^{d-1}$; the operator P_θ defined by (3.24) is the X-ray transform at fixed direction $\theta \in \mathbb{S}^{d-1}$.

Proposition 3.1A. *Let a_i, $i = 0, 1, \dots, d$, satisfy (3.4 a). Let $a = (a_0, a_1, \dots, a_d)$. Then we have the following estimates:*

(1) *if $0 < \alpha \leqslant 1$, then*

$$\psi^{\pm}, S \in C(\mathbb{R}^d \times \mathbb{S}^{d-1}, GL(n,\mathbb{C})), \tag{3.25}$$

$$\max(|\psi^{\pm}(x,\theta) - I|, |(\psi^{\pm}(x,\theta))^{-1} - I|) \leqslant \exp(nc_1(\rho,\varepsilon,\pm\theta,x)\|a\|_{0,1+\varepsilon,\rho}) - 1, \tag{3.26 a}$$

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$$\max(|\psi^{\pm}(x+y,\theta)-\psi^{\pm}(x,\theta)|,|(\psi^{\pm}(x+y,\theta))^{-1}-(\psi^{\pm}(x,\theta))^{-1}|)$$
$$\leqslant 2^{1+\varepsilon}(\exp(2^{1+\varepsilon}nc_1(\rho,\varepsilon,\pm\theta,x)\|a\|_{\alpha,1+\varepsilon,\rho})-1)|y|^{\alpha}, \quad (3.26\,b)$$

$$\max(\|\psi^{\pm}(\cdot,\theta)\|_{\alpha,0},\|(\psi^{\pm}(\cdot,\theta))^{-1}\|_{\alpha,0})\leqslant 2^{1+\varepsilon}\exp(2^{3+\varepsilon}n\varepsilon^{-1}\rho^{-\varepsilon}\|a\|_{\alpha,1+\varepsilon,\rho}), \quad (3.26\,c)$$

$$|S(x,\theta)-I|\leqslant \exp(2nc_7(\rho,\varepsilon,|\pi_\theta x|)\|a\|_{0,1+\varepsilon,\rho})-1, \quad (3.27\,a)$$
$$|S(x+y,\theta)-S(x,\theta)|\leqslant 2^{1+\varepsilon}(\exp(2^{2+\varepsilon}nc_7(\rho,\varepsilon,|\pi_\theta x|)\|a\|_{\alpha,1+\varepsilon,\rho})-1)|y|^{\alpha}; \quad (3.27\,b)$$

(2) *if* $1\leqslant\alpha\leqslant 2$, *then*
$$\psi^{\pm}(\cdot,\theta)\in C^1(\mathbb{R}^d,GL(n,\mathbb{C})), \quad (3.28)$$

$$|\partial_j\psi^{\pm}(x,\theta)|\leqslant nc_1(\rho,\varepsilon,\pm\theta,x)\|a\|_{1,1+\varepsilon,\rho}\exp(8n\varepsilon^{-1}\rho^{-\varepsilon}\|a\|_{0,1+\varepsilon,\rho}), \quad (3.29\,a)$$
$$\|\partial_j\psi^{\pm}(\cdot,\theta)\|_{\alpha-1,0}\leqslant 2^{5+3\varepsilon}n\varepsilon^{-1}\rho^{-\varepsilon}\|a\|_{\alpha,1+\varepsilon,\rho}\exp(2^{4+\varepsilon}n\varepsilon^{-1}\rho^{-\varepsilon}\|a\|_{\alpha-1,1+\varepsilon,\rho}); \quad (3.29\,b)$$

(3) *if* $\alpha=2$, *then*
$$\psi^{\pm}(\cdot,\theta)\in C^2(\mathbb{R}^d,GL(n,\mathbb{C})), \quad (3.30)$$

where $c_1(\rho,\varepsilon,\theta,x)$, $c_7(\rho,\varepsilon,s)$ *are given by (A.6), (A.27),* $x,y\in\mathbb{R}^d$, $|y|\leqslant 1$, $\theta\in\mathbb{S}^{d-1}$, $\rho>1$, $\partial_j=\partial/\partial x_j$, $j=1,\dots,d$.

Let

$$\mathcal{D}_{r,\delta,\theta}=\mathcal{D}^1_{r,\delta}\cup\mathcal{D}^2_{r,\delta,\theta}, \quad (3.31\,a)$$
$$\mathcal{D}^1_{r,\delta}=\{x\in\mathbb{R}^d\mid |x|\leqslant r-\delta\}, \quad (3.31\,b)$$
$$\mathcal{D}^2_{r,\delta,\theta}=\{x\in\mathbb{R}^d\mid |x|\geqslant r+\delta,|\pi_\theta x|\leqslant r-\delta\}, \quad (3.31\,c)$$

where $0<\delta<r$, $\theta\in\mathbb{S}^{d-1}$.

Proposition 3.1B. *Let* a_i, $i=0,1,\dots,d$, *satisfy (3.4 b). Let* $b=(b_0,b_1,\dots,b_d)$. *Then we have the following estimates:*

(1) *if* $0<\alpha\leqslant 1$, *then*
$$\psi^{\pm},S\in C(\mathbb{R}^d\times\mathbb{S}^{d-1},GL(n,\mathbb{C})), \quad (3.32)$$
$$\max(|\psi^{\pm}(x,\theta)-I|,|(\psi^{\pm}(x,\theta))^{-1}-I|)\leqslant\exp(nc_3(|\pi_\theta x|,r)\|b\|_0)-1, \quad (3.33\,a)$$
$$\max(\|\psi^{\pm}(\cdot,\theta)-I\|_{\beta,(X_\theta),0},\|(\psi^{\pm}(\cdot,\theta))^{-1}-I\|_{\beta,(X_\theta),0})$$
$$\leqslant\exp(2nr\|b\|_{\beta,0})-1+nc_4(r)\|b\|_{\beta,0}\exp(4nr\|b\|_{\beta,0}),$$
$$0<\beta\leqslant\min(\tfrac12,\alpha), \quad (3.33\,b)$$

$$|S(x,\theta)-I|\leqslant\exp(nc_3(|\pi_\theta x|,r)\|b\|_0)-1, \quad (3.34\,a)$$
$$|S(x+y,\theta)-S(x,\theta)|\leqslant(\exp(2nr\|b\|_{\beta,0})-1+nc_4(r)\|b\|_{\beta,0}\exp(4nr\|b\|_{\beta,0}))|y|^{\beta},$$
$$0<\beta\leqslant\min(\tfrac12,\alpha); \quad (3.34\,b)$$

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(2) *if* $1 \leqslant \alpha \leqslant 2$, *then*

$$\psi^{\pm}(\cdot,\theta) \in C^1(\mathcal{D}_{r,\delta,\theta}, GL(n,\mathbb{C})), \tag{3.35}$$

$$|\partial_j \psi^{\pm}(x,\theta)| \leqslant \chi_+(\sqrt{r^2-(\pi_\theta x)^2} \pm \theta x)\chi_+(r-|\pi_\theta x|) \times 2r(1+(r^2-(\pi_\theta x)^2)^{-1/2})n^2\|b\|_{1,0}\exp(4nr\|b\|_0), \tag{3.36 a}$$

$$\|\partial_j \psi^{\pm}(\cdot,\theta)|_{\mathcal{D}_{r,\delta,\theta}}\|_{\beta,0} \leqslant \text{const.}(n,\beta,\|b\|_{\alpha,0},r,\delta), \quad 0 \leqslant \beta \leqslant \min(\tfrac{1}{2},\alpha-1); \tag{3.36 b}$$

(3) *if* $\alpha = 2$, *then*

$$\psi^{\pm}(\cdot,\theta) \in C^2(\mathcal{D}_{r,\delta,\theta}, GL(n,\mathbb{C})), \tag{3.37}$$

where $c_3(s,r)$, $c_4(r)$ *are given by (A.13), (A.15), the constant of (3.36 b) also can be given explicitly,* $x,y \in \mathbb{R}^d$, $|y| \leqslant 1$, $\theta \in \mathbb{S}^{d-1}$, $0 < \delta < r$, $\partial_j = \partial/\partial x_j$, $j = 1,\dots,d$.

We obtain Propositions 3.1A and 3.1B using (3.8), (3.9), (3.11), (3.18), Lemmas A.4_a, A.4_b and the following general facts about the equation (3.1):

(i) if $\psi \in L^\infty(\mathbb{R}^d, \mathcal{M}_{n\times n})$ satisfies (3.1) for fixed θ, then

$$\theta\partial_x \det\psi + \operatorname{tr} v(x,\theta)\det\psi = 0, \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}; \tag{3.38}$$

(ii) if $\psi \in L^\infty(\mathbb{R}^d, GL(n,\mathbb{C}))$ satisfies (3.1) for fixed θ, then

$$\left.\begin{aligned} \theta\partial_x\psi^{-1} - \psi^{-1}v(x,\theta) = 0, \\ \theta\partial_x(\psi^{-1})^{\mathrm{T}} - (v(x,\theta))^{\mathrm{T}}(\psi^{-1})^{\mathrm{T}} = 0, \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}. \end{aligned}\right\} \tag{3.39}$$

More precisely, the proof of Propositions 3.1A and 3.1B consists of the following.

(1) Using (3.8), (3.38) for $\psi = \psi^{\pm}$ we obtain that

$$\det\psi^{\pm}(x,\theta) = \exp\biggl(\mp\int_0^{+\infty} \operatorname{tr} v(x \mp t\theta,\theta)\,\mathrm{d}t\biggr) \neq 0, \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}. \tag{3.40}$$

(2) The estimates (3.26), (3.33) for $\psi^{\pm}$ (not yet for $(\psi^{\pm})^{-1}$) follow from the equations (3.18) (which we solve by the method of successive approximations) and the estimates (A.43), (A.44), (A.49), (A.50).

(3) To obtain (3.25), (3.32) for $\psi^{\pm}$ we use the proof of (3.26), (3.33) for $\psi^{\pm}$ and the formulae (3.40), (A.47), (A.54). The properties (3.25), (3.32) for S follow from (3.25), (3.32) for $\psi^{\pm}$ and (3.9).

(4) The fact that $\psi^{\pm}$ are specified by (3.8), the properties (3.25), (3.32) for $\psi^{\pm}$, the equation (3.39) for $\psi = \psi^{\pm}$, the equality $\|-f^{\mathrm{T}}\|_{s_1,s_2,s_3} = \|f\|_{s_1,s_2,s_3}$ and the estimates (3.26), (3.33) for $\psi^{\pm}$ imply the estimates (3.26), (3.33) for $(\psi^{\pm})^{-1}$.

(5) The estimates (3.27), (3.34) follow from (3.26), (3.33) for ψ^+ and (3.11).

(6) For the case $1 \leqslant \alpha$, using (3.18) we obtain that $\partial_j \psi^{\pm}(\cdot, \theta)$ are solutions (given by the method of successive approximations) of the following equations

$$\left.\begin{aligned} \partial_j \psi^{+}(\cdot,\theta) &= f_j^{+}(\cdot,\theta) - D_{-\theta} v_\theta \partial_j \psi^{+}(\cdot,\theta),\\ \partial_j \psi^{-}(\cdot,\theta) &= f_j^{-}(\cdot,\theta) + D_{\theta} v_\theta \partial_j \psi^{-}(\cdot,\theta), \end{aligned}\right\} \tag{3.41}$$

where

$$f_j^{\pm}(x,\theta) = \mp \int_0^{+\infty} \frac{\partial v(x \mp t\theta, \theta)}{\partial x_j} \psi^{\pm}(x \mp t\theta, \theta)\,\mathrm{d}t, \tag{3.42}$$

where $x \in \mathbb{R}^d$, $\theta \in \mathbb{S}^{d-1}$, $j \in \{1, \dots, d\}$. In addition, for the case of the conditions (3.4 b), the following is valid:

$$\left.\begin{aligned} &\frac{\partial v(x,\theta)}{\partial x_j} = 2\delta(r^2 - x^2) x_j u(x,\theta) + v_j(x,\theta),\\ &v_j(x,\theta) = \chi_+(r - |x|)\frac{\partial u(x,\theta)}{\partial x_j}, \qquad u(x,\theta) = b_0(x) + \sum_{i=1}^{d} \theta_i b_i(x); \end{aligned}\right\} \tag{3.43}$$

$$f_j^{\pm}(x,\theta) = f_{j,1}^{\pm}(x,\theta) + f_{j,2}^{\pm}(x,\theta), \tag{3.44 a}$$

$$f_{j,1}^{\pm}(x,\theta) = \mp \int_0^{+\infty} v_j(x \mp t\theta, \theta)\psi^{\pm}(x \mp t\theta, \theta)\,\mathrm{d}t, \tag{3.44 b}$$

$$f_{j,2}^{\pm}(x,\theta) = g_1^{\pm}(x,\theta) m_{j,1}^{\pm}(\pi_\theta x, \theta) + g_2^{\pm}(x,\theta) m_{j,2}^{\pm}(\pi_\theta x, \theta), \tag{3.44 c}$$

$$\left.\begin{aligned} m_{j,1}^{\pm}(\pi_\theta x,\theta) &= \mp((\pi_\theta x)_j - \sqrt{r^2 - (\pi_\theta x)^2}\theta_j)\\ &\qquad \times \frac{u(\pi_\theta x - \sqrt{r^2 - (\pi_\theta x)^2}\theta, \theta)\psi^{\pm}(\pi_\theta x - \sqrt{r^2 - (\pi_\theta x)^2}\theta, \theta)}{\sqrt{r^2 - (\pi_\theta x)^2}},\\ m_{j,2}^{\pm}(\pi_\theta x,\theta) &= \mp((\pi_\theta x)_j + \sqrt{r^2 - (\pi_\theta x)^2}\theta_j)\\ &\qquad \times \frac{u(\pi_\theta x + \sqrt{r^2 - (\pi_\theta x)^2}\theta, \theta)\psi^{\pm}(\pi_\theta x + \sqrt{r^2 - (\pi_\theta x)^2}\theta, \theta)}{\sqrt{r^2 - (\pi_\theta x)^2}}, \end{aligned}\right\} \tag{3.44 d}$$

$$\left.\begin{aligned} g_1^{\pm}(x,\theta) &= \chi_+(r - |\pi_\theta x|)\chi_+(\pm(\sqrt{r^2 - (\pi_\theta x)^2} + \theta x))I,\\ g_2^{\pm}(x,\theta) &= \chi_+(r - |\pi_\theta x|)\chi_+(\pm(-\sqrt{r^2 - (\pi_\theta x)^2} + \theta x))I; \end{aligned}\right\} \tag{3.44 e}$$

$$\partial_j \psi^{\pm}(x,\theta) = \varphi_j^{\pm}(x,\theta) + \mu_1^{\pm}(x,\theta) m_{j,1}^{\pm}(\pi_\theta x,\theta) + \mu_2^{\pm}(x,\theta) m_{j,2}^{\pm}(\pi_\theta x,\theta), \tag{3.45}$$

where $\varphi_j^{\pm}(\cdot,\theta)$, $\mu_1^{\pm}(\cdot,\theta)$, $\mu_2^{\pm}(\cdot,\theta)$ are the solutions (given by the method of successive approximations) of the following equations

$$\varphi_j^{\pm}(\cdot,\theta) = f_{j,1}^{\pm}(\cdot,\theta) \mp D_{\mp\theta} v_\theta \varphi_j^{\pm}(\cdot,\theta), \tag{3.46}$$

$$\left.\begin{aligned} \mu_1^{\pm}(\cdot,\theta) &= g_1^{\pm}(\cdot,\theta) \mp D_{\mp\theta} v_\theta \mu_1^{\pm}(\cdot,\theta),\\ \mu_2^{\pm}(\cdot,\theta) &= g_2^{\pm}(\cdot,\theta) \mp D_{\mp\theta} v_\theta \mu_2^{\pm}(\cdot,\theta), \end{aligned}\right\} \tag{3.47}$$

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and (as a corollary of the equations (3.18) and (3.47), the method of successive approximations and the identities $v_\theta g_1^+(\cdot,\theta) = v_\theta I$, $v_\theta g_2^+(\cdot,\theta) \equiv 0$, $v_\theta g_1^-(\cdot,\theta) \equiv 0$, $v_\theta g_2^-(\cdot,\theta) = v_\theta I$)

$$\left.\begin{array}{ll} \mu_1^+(\cdot,\theta) - g_1^+(\cdot,\theta) = \psi^+(\cdot,\theta) - I, & \mu_2^+(\cdot,\theta) = g_2^+(\cdot,\theta), \\ \mu_1^-(\cdot,\theta) = g_1^-(\cdot,\theta), & \mu_2^-(\cdot,\theta) - g_2^-(\cdot,\theta) = \psi^-(\cdot,\theta) - I. \end{array}\right\} \tag{3.48}$$

(7) The estimates (3.28), (3.29) follow from the equations (3.41), the formula (3.42), the estimates (3.26 a), (3.26 c), (A.43), (A.46), Remark A.1$_a$ and the inequality

$$(\exp(nc_1(\rho,\varepsilon,\pm\theta,x)\|a\|_{0,1+\varepsilon,\rho}) - 1)2^{3/2}\varepsilon^{-1}\rho^{-\varepsilon} \leqslant c_1(\rho,\varepsilon,\pm\theta,x)(\exp(n2^{3/2}\varepsilon^{-1}\rho^{-\varepsilon}\|a\|_{0,1+\varepsilon,\rho}) - 1).$$

The estimates (3.35), (3.36) follow from (3.44 d), (3.44 e), (3.45), (3.46), (3.48), (3.1) for $\psi = \psi^\pm$, (3.33), (A.49)–(A.52) and Remark A.1$_b$.

(8) The proof of (3.30), (3.37) is similar to the proof of (3.28), (3.35).

In this paper we also use the following statement.

Statement 3.1. *Let the assumptions (3.4 a) or (3.4 b) be valid and a_i, $i = 0,1,\dots,d$, take values in $u(n)$. Then $\psi^\pm$ take values in $U(n)$.*

Proof of Statement 3.1. If $\psi \in L^\infty(\mathbb{R}^d, GL(n,\mathbb{C}))$ satisfies (3.1) for fixed θ then

$$\theta\nabla_x(\psi^{-1})^* - (v(x,\theta))^*(\psi^{-1})^* = 0, \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}. \tag{3.49}$$

Statement 3.1 follows from (3.49) for $\psi = \psi^\pm$, the fact that $\psi^\pm$ are specified by (3.8), the definition (3.3) and the definitions of $u(n)$ and $U(n)$. The proof is completed. □

4. Direct 'scattering' for the two-dimensional X-ray connection equation with complex spectral parameter

Consider the two-dimensional X-ray connection equation

$$\theta\partial_x\psi + v(x,\theta)\psi = 0, \quad x \in \mathbb{R}^2, \quad \theta \in \Sigma, \tag{4.1}$$

where

$$\Sigma = \{\theta \in \mathbb{C}^2 \mid \theta^2 = \theta_1^2 + \theta_2^2 = 1\}, \tag{4.2}$$

$$\partial_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right), \qquad \theta\partial_x = \theta_1\frac{\partial}{\partial x_1} + \theta_2\frac{\partial}{\partial x_2}, \tag{4.3}$$

$$v(x,\theta) = \theta_1 a_1(x) + \theta_2 a_2(x) + a_0(x), \tag{4.4}$$

a_i, $i = 0,1,2$, satisfy (3.4 a) or (3.4 b) for $d = 2$, ψ satisfies (3.6) for $d = 2$, $\theta \in \Sigma$. Consider

$$\mathbb{S}^1 = \{\theta \in \mathbb{R}^2 \mid \theta^2 = 1\}.$$

In (4.1) the spectral parameter θ is complex, in general, and it is real if and only if it belongs to $\mathbb{S}^1$. The equation (4.1) for $\theta \in \mathbb{S}^1$ is the equation (3.1) for $d = 2$.

For $\theta \in \Sigma\backslash\mathbb{S}^1$ we consider the solution $\psi(\cdot,\theta)$ of (4.1) defined as the solution of the following integral equation (provided that it is uniquely solvable):

$$\psi(\cdot,\theta) = I - G_\theta v_\theta \psi(\cdot,\theta), \quad \psi(\cdot,\theta) \in L^\infty(\mathbb{R}^2, \mathcal{M}_{n\times n}), \tag{4.5}$$

where

$$G_\theta v_\theta f = G_\theta(v_\theta f), \tag{4.6}$$

$$v_\theta f(x) = v(x,\theta) f(x), \tag{4.7}$$

$$G_\theta \varphi(x) = \int_{\mathbb{R}^2} G(x-y,\theta)\varphi(y)\,\mathrm{d}y, \tag{4.8}$$

$$G(x,\theta) = \frac{\operatorname{sgn}(\operatorname{Re}\theta_1 \operatorname{Im}\theta_2 - \operatorname{Re}\theta_2 \operatorname{Im}\theta_1)}{-2\pi\mathrm{i}(\theta_2 x_1 - \theta_1 x_2)}, \tag{4.9}$$

$$\theta\partial_x G(x,\theta) = \delta(x), \tag{4.10}$$

where $\theta \in \Sigma\backslash\mathbb{S}^1$, $x \in \mathbb{R}^2$.

Statement 4.1. *Under assumptions (3.4 a) or (3.4 b) for $d = 2$, for $\theta \in \Sigma\backslash\mathbb{S}^1$, a function ψ with the properties*

$$\psi \in C(\mathbb{R}^2, \mathcal{M}_{n\times n}), \quad \psi(x) \to I \quad \text{as } |x| \to \infty \tag{4.11}$$

is a solution of (4.1) if and only if it is a solution of (4.5).

We obtain this statement using (4.10) and Lemma A.10.

For $\theta \in \mathbb{S}^1$ we consider the solutions $\psi_\pm(\cdot,\theta)$ of (4.1) defined as the solutions of the following integral equations (provided that they are uniquely solvable):

$$\left.\begin{aligned} \psi_+(\cdot,\theta) &= I - G_{+,\theta} v_\theta \psi_+(\cdot,\theta), \quad \psi_+(\cdot,\theta) \in C^{\beta,0}(\mathbb{R}^2, \mathcal{M}_{n\times n}),\\ \psi_-(\cdot,\theta) &= I - G_{-,\theta} v_\theta \psi_-(\cdot,\theta), \quad \psi_-(\cdot,\theta) \in C^{\beta,0}(\mathbb{R}^2, \mathcal{M}_{n\times n}),\end{aligned}\right\} \tag{4.12}$$

where

$$\left.\begin{aligned} &0 < \beta \leqslant \alpha \text{ under assumptions } (3.4\,a),\\ &0 < \beta \leqslant \min(\tfrac{1}{2},\alpha) \text{ under assumptions } (3.4\,b),\end{aligned}\right\} \tag{4.13}$$

$$G_{\pm,\theta} v_\theta f = G_{\pm,\theta}(v_\theta f), \tag{4.14}$$

$$v_\theta f(x) = v(x,\theta) f(x), \tag{4.15}$$

$$G_{\pm,\theta}\varphi(x) = \lim_{0<\varepsilon\to 0} G_{\omega(\pm\varepsilon)}\varphi(x), \tag{4.16}$$

$$\omega(\varepsilon) = \sqrt{1+\varepsilon^2}\,\theta + \mathrm{i}\varepsilon\theta^\perp \in \Sigma\backslash\mathbb{S}^1 \quad \text{for } \varepsilon \neq 0,$$
$$\sqrt{1+\varepsilon^2} > 0 \quad \text{for } \varepsilon \in \mathbb{R}, \theta^\perp = (-\theta_2,\theta_1) \quad \text{for } \theta = (\theta_1,\theta_2),$$

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$$G_{\pm,\theta}\varphi(x) = \int_{\mathbb{R}^2} G_\pm(x-y,\theta)\varphi(y)\,\mathrm{d}y, \tag{4.17}$$

$$G_\pm(x,\theta) = \frac{\pm 1}{2\pi\mathrm{i}(\theta^\perp x \mp \mathrm{i}0\theta x)}, \tag{4.18}$$

$$\theta\partial_x G_\pm(x,\theta) = \delta(x), \tag{4.19}$$

where $\theta \in \mathbb{S}^1$, $x \in \mathbb{R}^2$. Note that the functions $\psi_\pm$ defined by (4.12) differ (in general) from the wave functions $\psi^\pm$ (for $d=2$) defined in § 3.

The following formulae hold:

$$G_\pm(x,\theta) = \mathrm{p.v.}\frac{\pm 1}{2\pi\mathrm{i}\theta^\perp x} + \tfrac{1}{2}\delta(\theta^\perp x)\,\mathrm{sgn}(\theta x), \tag{4.20}$$

$$G_{\pm,\theta}\varphi(x) = \tfrac{1}{2}(D_{-\theta}\varphi(x) - D_\theta\varphi(x)) \pm \frac{1}{2\mathrm{i}} H P_\theta^\perp \varphi(\theta^\perp x), \tag{4.21}$$

$$G_{\pm,\theta}v_\theta f(x) = \tfrac{1}{2}(D_{-\theta}v_\theta f(x) - D_\theta v_\theta f(x)) \pm \frac{1}{2\mathrm{i}} H P_\theta^\perp v_\theta f(\theta^\perp x), \tag{4.22}$$

where D_θ, P_θ are defined by (3.21), (3.24), $d=2$,

$$P_\theta^\perp f(s) = P_\theta f(s\theta^\perp), \quad \theta \in \mathbb{S}^1, \quad s \in \mathbb{R}, \tag{4.23}$$

and H is the Hilbert transform:

$$Hg(s) = \frac{1}{\pi}\mathrm{p.v.}\int_{\mathbb{R}} \frac{g(t)}{s-t}\,\mathrm{d}t, \quad s \in \mathbb{R}. \tag{4.24}$$

We use the formula (4.22) in order to obtain the estimates for $G_{\pm,\theta}v_\theta$ given in Lemma A.8. These estimates are necessary for us in order to deal with the integral equations (4.12), (4.5).

In addition to the variables $x \in \mathbb{R}^2$, $\theta \in \Sigma$, we use also the variables $z \in \mathbb{C}$, $\lambda \in \mathbb{C}$, where

$$\left.\begin{aligned} z = x_1 + \mathrm{i}x_2, \qquad \bar{z} = x_1 - \mathrm{i}x_2, \qquad \lambda = \theta_1 + \mathrm{i}\theta_2,\\ \theta_1 = (\lambda + \lambda^{-1})/2, \qquad \theta_2 = (\lambda - \lambda^{-1})/(2\mathrm{i}). \end{aligned}\right\} \tag{4.25}$$

In the variable λ the surface Σ is $\mathbb{C}\backslash 0$ and the circle $\mathbb{S}^1$ is

$$T = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}. \tag{4.26}$$

We use also the following notation

$$\left.\begin{aligned} D_+ = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}, \quad \bar{D}_+ = D_+ \cup T,\\ D_- = \{\lambda \in \bar{\mathbb{C}} \mid |\lambda| > 1\}, \quad \bar{D}_- = D_- \cup T, \end{aligned}\right\} \tag{4.27}$$

where $\bar{\mathbb{C}} = \mathbb{C} \cup \infty = \mathbb{C}P^1$.

Using the variables z, λ, we write aforementioned $a_i(x)$, $v(x,\theta)$, $\psi(x,\theta)$, $G(x,\theta)$, G_θ, v_θ, where $x \in \mathbb{R}^2$, $\theta \in \Sigma$, and $\psi_\pm(x,\theta)$, $G_\pm(x,\theta)$, $G_{\pm,\theta}$, where $x \in \mathbb{R}^2$, $\theta \in \mathbb{S}^1$, as $a_i(z)$, $v(z,\lambda)$, $\psi(z,\lambda)$, $G(z,\lambda)$, G_λ, v_λ, where $z \in \mathbb{C}$, $\lambda \in \mathbb{C}\backslash 0$, and $\psi_\pm(z,\lambda)$, $G_\pm(z,\lambda)$, $G_{\pm,\lambda}$, where $z \in \mathbb{C}$, $\lambda \in T$.

In the variables z, λ the equations (4.1), (4.5), (4.12) take the form (4.28), (4.31), (4.36), the formulae (4.3), (4.6)–(4.9), (4.14)–(4.16) take the form (4.29), (4.32)–(4.35), (4.37)–(4.39):

$$(\lambda\partial_z + \lambda^{-1}\partial_{\bar z})\psi + v(z,\lambda)\psi = 0, \quad z \in \mathbb{C}, \quad \lambda \in \mathbb{C}\backslash 0, \tag{4.28}$$

where $\partial_z = \partial/\partial z$, $\partial_{\bar z} = \partial/\partial \bar z$,

$$v(z,\lambda) = \lambda a_-(z) + \lambda^{-1} a_+(z) + a_0(z), \tag{4.29}$$

$$a_-(z) = (a_1(z) - \mathrm{i}a_2(z))/2, \qquad a_+(z) = (a_1(z) + \mathrm{i}a_2(z))/2; \tag{4.30}$$

$$\psi(\cdot,\lambda) = I - G_\lambda v_\lambda \psi(\cdot,\lambda), \qquad \psi(\cdot,\lambda) \in L^\infty(\mathbb{C}, \mathcal{M}_{n\times n}), \tag{4.31}$$

where

$$G_\lambda v_\lambda f = G_\lambda(v_\lambda f), \tag{4.32}$$

$$v_\lambda f(z) = v(z,\lambda) f(z), \tag{4.33}$$

$$G_\lambda \varphi(z) = \int_{\mathbb{C}} G(z-\xi,\lambda)\varphi(\xi)\,\mathrm{d}\xi_{\mathrm{R}}\,\mathrm{d}\xi_{\mathrm{I}}, \quad \xi_{\mathrm{R}} = \operatorname{Re}\xi, \quad \xi_{\mathrm{I}} = \operatorname{Im}\xi, \tag{4.34}$$

$$G(z,\lambda) = \frac{\operatorname{sgn}(1-|\lambda|)}{2\pi\mathrm{i}(\mathrm{i}/2)(\lambda\bar z - z/\lambda)}, \tag{4.35}$$

where $z \in \mathbb{C}$, $\lambda \in \mathbb{C}\backslash(0 \cup T)$;

$$\psi_\pm(\cdot,\lambda) = I - G_{\pm,\lambda} v_\lambda \psi_\pm(\cdot,\lambda), \qquad \psi_\pm(\cdot,\lambda) \in C^{\beta,0}(\mathbb{C}, \mathcal{M}_{n\times n}), \tag{4.36}$$

where β satisfies (4.13),

$$G_{\pm,\lambda} v_\lambda f = G_{\pm,\lambda}(v_\lambda f), \tag{4.37}$$

$$v_\lambda f(z) = v(z,\lambda) f(z), \tag{4.38}$$

$$G_{\pm,\lambda}\varphi(z) = \lim_{0<\varepsilon\to 0} G_{(1\mp\varepsilon)\lambda}\varphi(z), \tag{4.39}$$

where $z \in \mathbb{C}$, $\lambda \in T$. The following formulae hold:

$$\frac{\partial}{\partial\bar\lambda} G_\lambda v_\lambda f(z) = 0, \quad \lambda \in \mathbb{C}\backslash(T \cup 0), \tag{4.40}$$

$$\lim_{\lambda\to 0} G_\lambda v_\lambda f(z) = C a_+ f(z), \tag{4.41}$$

$$\lim_{\lambda\to\infty} G_\lambda v_\lambda f(z) = \bar C a_- f(z), \tag{4.42}$$

where

$$C a_+ f = C(a_+ f), \qquad \bar C a_- f = \bar C(a_- f), \tag{4.43}$$

$$a_\pm f(z) = a_\pm(z) f(z), \tag{4.44}$$

$$C\varphi(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\varphi(\xi)}{\xi - z}\,\mathrm{d}\xi_{\mathrm{R}}\,\mathrm{d}\xi_{\mathrm{I}}, \tag{4.45}$$

$$\bar C\varphi(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\varphi(\xi)}{\bar\xi - \bar z}\,\mathrm{d}\xi_{\mathrm{R}}\,\mathrm{d}\xi_{\mathrm{I}}, \tag{4.46}$$

where a_i, $i = 0,1,2$, satisfy (3.4 a) or (3.4 b), $d = 2$, $f \in L^\infty(\mathbb{C}, \mathcal{M}_{n\times n})$, $z \in \mathbb{C}$.

Due to (4.41), (4.42), the equation (4.31) for $\lambda = 0$ takes the form (4.47) and for $\lambda = \infty$ takes the form (4.48):

$$\psi_{+,0} = I - Ca_+\psi_{+,0}, \quad \psi_{+,0} \in L^\infty(\mathbb{C}, \mathcal{M}_{n\times n}), \tag{4.47 a}$$

$$\psi_{-,0} = I - \bar{C}a_-\psi_{-,0}, \quad \psi_{-,0} \in L^\infty(\mathbb{C}, \mathcal{M}_{n\times n}). \tag{4.48 a}$$

Under assumptions (3.4 a) or (3.4 b) for $d = 2$, the following statements are valid:

a function $\psi_{+,0}$ is a solution of (4.47 a), if and only if

$$\partial_{\bar z}\psi_{+,0} + a_+(z)\psi_{+,0} = 0, \quad \psi_{+,0} \in C(\mathbb{C}, \mathcal{M}_{n\times n}), \quad \psi_{+,0}(z) - I \to 0 \quad \text{as } |z| \to \infty; \tag{4.47 b}$$

a function $\psi_{-,0}$ is a solution of (4.48 a), if and only if

$$\partial_z\psi_{-,0} + a_-(z)\psi_{-,0} = 0, \quad \psi_{-,0} \in C(\mathbb{C}, \mathcal{M}_{n\times n}), \quad \psi_{-,0}(z) - I \to 0 \quad \text{as } |z| \to \infty. \tag{4.48 b}$$

These statements are similar to Statement 4.1.

Proposition 4.1A. *Let a_i, $i = 0, 1, 2$, satisfy (3.4 a), $0 < \alpha < 1$, $d = 2$, $a = (a_0, a_1, a_2)$ and*

$$\delta = nc_{18}(\alpha, \varepsilon, 0, \varepsilon'')\rho^{\varepsilon''-\varepsilon}\|a\|_{\alpha,1+\varepsilon,\rho} < 1, \tag{4.49}$$

for some $\varepsilon'' \in\ ]0, \varepsilon[$, where $c_{18}(\alpha, \varepsilon, \varepsilon', \varepsilon'')$ is the constant of Lemma A.8$_a$. Then the equation (4.5) for $\theta \in \Sigma\backslash\mathbb{S}^1$, (4.12) for $\theta \in \mathbb{S}^1$, (4.31) for $\lambda \in \mathbb{C}\backslash(T \cup 0)$, (4.36) for $\lambda \in T$, (4.47), (4.48) are uniquely solvable and the following formulae hold:

$$\frac{\partial}{\partial\bar\lambda}\psi(z, \lambda) = 0 \quad \text{for } \lambda \in \mathbb{C}\backslash(T \cup 0), \quad z \in \mathbb{C}, \tag{4.50}$$

$$\psi_+(z, \cdot) \in C(\bar{D}_+, GL(n, \mathbb{C})), \quad z \in \mathbb{C}, \tag{4.51 a}$$

where

$$\psi_+(z, \lambda) \overset{\text{def}}{=} \psi(z, \lambda) \quad \text{for } \lambda \in D_+\backslash 0, \qquad \psi_+(z, 0) \overset{\text{def}}{=} \psi_{+,0}(z), \tag{4.51 b}$$

$$\psi_-(z, \cdot) \in C(\bar{D}_-, GL(n, \mathbb{C})), \quad z \in \mathbb{C}, \tag{4.52 a}$$

where

$$\psi_-(z, \lambda) \overset{\text{def}}{=} \psi(z, \lambda) \quad \text{for } \lambda \in D_-\backslash\infty, \qquad \psi_-(z, \infty) \overset{\text{def}}{=} \psi_{-,0}(z), \tag{4.52 b}$$

$$R(z, \cdot) \in C(T, GL(n, \mathbb{C})), \tag{4.53 a}$$

where

$$R(z, \lambda) \overset{\text{def}}{=} (\psi_-(z, \lambda))^{-1}\psi_+(z, \lambda), \quad z \in \mathbb{C}, \quad \lambda \in T, \tag{4.53 b}$$

$$(\lambda\partial_z + \lambda^{-1}\partial_{\bar z})R(z, \lambda) = 0, \quad z \in \mathbb{C}, \quad \lambda \in T, \tag{4.54}$$

$$\max(\|\psi_\pm(\cdot, \lambda) - I\|_{\alpha,0}, \|(\psi_\pm(\cdot, \lambda))^{-1} - I\|_{\alpha,0}) < \delta(1 - \delta)^{-1} \quad \text{for } \lambda \in \bar{D}_\pm, \tag{4.55}$$

$$\|R(\cdot, \lambda) - I\|_{\alpha,0} \leqslant 2\delta(1 - \delta)^{-1}(1 + n\delta(1 - \delta)^{-1}), \quad \lambda \in T, \tag{4.56}$$

$$\|R(z, \cdot) - I\|_0 \leqslant \delta(1 - \delta)^{-1}(2 + n\delta(1 - \delta)^{-1}), \quad z \in \mathbb{C}, \tag{4.57}$$

where in (4.50)–(4.57) we use the variables z, λ.

Proposition 4.1B. *Let a_i, $i = 0, 1, 2$, satisfy (3.4b), $0 < \alpha < 1$, $d = 2$, $b = (b_0, b_1, b_2)$ and*

$$\delta = n^2 c_{21}(\beta, \varepsilon, 0)(2+r)^{\varepsilon} c_4(r)(1 + c_{21}(\beta, \varepsilon, 0)(2+r)^{\varepsilon} c_4(r))(\|b\|_{\beta,0})^2 < 1 \tag{4.58}$$

for $\beta = \min(\frac{1}{2}, \alpha)$ and some $\varepsilon > 0$, where $c_4(r) = 2^{3/2} r^{1/2} + 2r$, $c_{21}(\beta, \varepsilon, \varepsilon')$ is the constant of Lemma A.8$_b$. Then the equations (4.5) for $\theta \in \Sigma \backslash \mathbb{S}^1$, (4.12) for $\theta \in \mathbb{S}^1$, (4.31) for $\lambda \in \mathbb{C} \backslash (T \cup 0)$, (4.36) for $\lambda \in T$, (4.47), (4.48) are uniquely solvable, the formulae (4.50)–(4.54) are valid and the following estimates hold:

$$\max(\|\psi_{\pm}(\cdot, \lambda) - I\|_{\beta,0}, \|(\psi_{\pm}(\cdot, \lambda))^{-1} - I\|_{\beta,0}) \leqslant p \quad \text{for } \lambda \in \bar{D}_{\pm}, \tag{4.59}$$

$$\max(\|\psi_{\pm}(\cdot, \theta) - I\|_{\beta,(X_\theta),0}, \|(\psi_{\pm}(\cdot, \theta))^{-1} - I\|_{\beta,(X_\theta),0}) \leqslant q \quad \text{for } \theta \in \mathbb{S}^1, \tag{4.60 a}$$

$$\max(\|\psi_{\pm}(\cdot, \lambda) - I\|_{0}, \|(\psi_{\pm}(\cdot, \lambda))^{-1} - I\|_{0}) \leqslant q \quad \text{for } \lambda \in \bar{D}_{\pm}, \tag{4.60 b}$$

$$\|R(\cdot, \lambda) - I\|_{\beta,0} \leqslant 2p(1 + np), \quad \lambda \in T, \tag{4.61}$$

$$\|R(z, \cdot) - I\|_{0} \leqslant q(2 + nq), \quad z \in \mathbb{C}, \tag{4.62}$$

where

$$p = \mu + \nu + \delta(1-\delta)^{-1}(1 + \mu + \nu), \tag{4.63}$$

$$q = \mu(1-\mu)^{-1}, \tag{4.64}$$

$$\mu = n c_{21}(\beta, \varepsilon, 0)(2+r)^{\varepsilon} c_4(r)\|b\|_{\beta,0} \leqslant \delta^{1/2}, \tag{4.65}$$

$$\nu = n\|b\|_{\beta,0}, \tag{4.66}$$

where in (4.59), (4.60 b), (4.61)–(4.66) we use the variables z, λ, in (4.60 a) we use the variables x, θ; X_θ is defined in (3.16).

We obtain Propositions 4.1A and 4.1B, using (4.31), (4.36), (4.39), (4.40), (4.47), (4.48), Statement 4.1, Lemmas A.8, A.9 and the following general facts about the equation (4.1):

if $\psi \in L^{\infty}(\mathbb{R}^2, \mathcal{M}_{n \times n})$ satisfies (4.1) for fixed θ, then

$$\theta \partial_x \det \psi + \operatorname{tr} v(x, \theta) \det \psi = 0, \quad x \in \mathbb{R}^2, \quad \theta \in \Sigma; \tag{4.67}$$

if $\psi \in L^{\infty}(\mathbb{R}^2, GL(n, \mathbb{C}))$ satisfies (4.1) for fixed θ, then

$$\left.\begin{aligned} \theta \partial_x \psi^{-1} - \psi^{-1} v(x, \theta) = 0, \\ \theta \partial_x (\psi^{-1})^{\mathrm{T}} - (v(x, \theta))^{\mathrm{T}} (\psi^{-1})^{\mathrm{T}} = 0. \end{aligned}\right\} \tag{4.68}$$

More precisely, the proof of Propositions 4.1A and 4.1B consists of the following.

(1) Using Statement 4.1 and the equation (4.67) we obtain that

$$\det \psi(x, \theta) = \exp\left(-\int_{\mathbb{R}^2} G(x-y, \theta) \operatorname{tr} v(y, \theta)\, \mathrm{d}y\right) \neq 0, \tag{4.69}$$

where ψ is a solution of (4.5), $\theta \in \Sigma \backslash \mathbb{S}^1$, G is given by (4.9).

(2) The estimate (4.55) for $\psi_\pm - I$ follows from the equations (4.31), (4.36), (4.47 a), (4.48 a) (which we solve by the method of successive approximations) and the estimates (A.85), (A.96), (A.98), (A.100). To obtain the estimate (4.59) for $\psi_\pm - I$ we use the equations

$$\psi(\cdot,\lambda) = I - G_\lambda v_\lambda I + (G_\lambda v_\lambda)^2\psi(\cdot,\lambda) \quad \text{for } \lambda \in \mathbb{C}\backslash(T\cup 0), \tag{4.70 a}$$

$$\psi_\pm(\cdot,\lambda) = I - G_{\pm,\lambda} v_\lambda I + (G_{\pm,\lambda} v_\lambda)^2\psi_\pm(\cdot,\lambda) \quad \text{for } \lambda \in T, \tag{4.70 b}$$

$$\psi_{+,0} = I - Ca_+ I + (Ca_+)^2\psi_{+,0}, \tag{4.70 c}$$

$$\psi_{-,0} = I - \bar{C}a_- I + (\bar{C}a_-)^2\psi_{-,0}, \tag{4.70 d}$$

(which we solve by the method of successive approximations) and the estimates (A.89), (A.91), (A.92), (A.101), (A.102), (A.104), (A.106).

(3) To obtain the properties (4.50)–(4.52) we use the proof of (4.55), (4.59) for $\psi_\pm - I$ and the formulae (4.39), (4.40), (A.86), (A.94), (4.22), (A.4), (A.12), (A.25), (A.26), (A.34), (A.35), (A.80), (4.69).

(4) To obtain (4.60) for $\psi_\pm - I$ we use (4.50)–(4.52), (4.36), (A.89) and the maximum principle for holomorphic functions.

(5) Statement 4.1, the properties (4.51), (4.52), the equation (4.68) for a solution ψ of (4.5), the equality $\|-f^{\mathrm{T}}\|_{s_1,s_2,s_3} = \|f\|_{s_1,s_2,s_3}$ and the estimates (4.55), (4.59), (4.60) for $\psi_\pm - I$ imply the estimates (4.55), (4.59), (4.60) for $(\psi_\pm)^{-1} - I$.

(6) The properties (4.53 a), (4.54) follow from (4.51 a), (4.52 a), (4.53 b), (4.1), (4.68). The estimates (4.56), (4.57) follow from (4.53 b), (4.55). The estimates (4.61), (4.62) follow from (4.53 b), (4.59), (4.60).

Under assumptions of Proposition 4.1A or Proposition 4.1B we consider

$$R(x,\theta) = (\psi_-(x,\theta))^{-1}\psi_+(x,\theta) \tag{4.71}$$

(the formula (4.53 b) in the coordinates x, θ),

$$\left.\begin{aligned} Q_{+,\pm}(x,\theta) &= \lim_{s\to\pm\infty} \psi_+(x+s\theta,\theta), \\ Q_{-,\pm}(x,\theta) &= \lim_{s\to\pm\infty} \psi_-(x+s\theta,\theta), \end{aligned}\right\} \tag{4.72}$$

where $x \in \mathbb{R}^2$, $\theta \in \mathbb{S}^1$. The following formulae hold:

$$\theta\partial_x R(x,\theta) = 0 \tag{4.73}$$

(the formula (4.54) in the coordinates x, θ),

$$\left.\begin{aligned} \theta\partial_x Q_{+,\pm}(x,\theta) &= 0, \\ \theta\partial_x Q_{-,\pm}(x,\theta) &= 0, \end{aligned}\right\} \tag{4.74}$$

$$\left.\begin{aligned} S(x,\theta) &= Q_{+,+}(x,\theta)(Q_{+,-}(x,\theta))^{-1}, \\ S(x,\theta) &= Q_{-,+}(x,\theta)(Q_{-,-}(x,\theta))^{-1}, \end{aligned}\right\} \tag{4.75}$$

where $x \in \mathbb{R}^2$, $\theta \in \mathbb{S}^1$.

Taking into account (3.15), (4.73), (4.74), we use the following notation

$$S^{\perp}(p,\theta) = S(p\theta^{\perp},\theta), \tag{4.76 a}$$

$$R^{\perp}(p,\theta) = R(p\theta^{\perp},\theta), \tag{4.76 b}$$

$$Q^{\perp}_{+,\pm}(p,\theta) = Q_{+,\pm}(p\theta^{\perp},\theta), Q^{\perp}_{-,\pm}(p,\theta) = Q_{-,\pm}(p\theta^{\perp},\theta), \tag{4.76 c}$$

where $p \in \mathbb{R}$, $\theta \in \mathbb{S}^1$.

The following formulae hold:

$$\left.\begin{aligned} Q^{\perp}_{+,\pm}(\cdot,\theta) &= I - \frac{1}{2\mathrm{i}} H_{\mp} P^{\perp}_{\theta} v_{\theta} \psi_{+}(\cdot,\theta), \\ Q^{\perp}_{-,\pm}(\cdot,\theta) &= I + \frac{1}{2\mathrm{i}} H_{\pm} P^{\perp}_{\theta} v_{\theta} \psi_{-}(\cdot,\theta), \end{aligned}\right\} \tag{4.77}$$

$$\left.\begin{aligned} \det Q^{\perp}_{+,\pm}(\cdot,\theta) &= \exp\left(-\frac{1}{2\mathrm{i}} H_{\mp} P^{\perp}_{\theta} \operatorname{tr} v(\cdot,\theta)\right), \\ \det Q^{\perp}_{-,\pm}(\cdot,\theta) &= \exp\left(\frac{1}{2\mathrm{i}} H_{\pm} P^{\perp}_{\theta} \operatorname{tr} v(\cdot,\theta)\right), \end{aligned}\right\} \tag{4.78}$$

where

$$H_{\pm} g(s) = Hg(s) \mp \mathrm{i} g(s), \quad s \in \mathbb{R}, \tag{4.79}$$

H is defined by (4.24), $P^{\perp}_{\theta}$ is defined by (4.23), $\theta \in \mathbb{S}^1$.

To obtain (4.77) we use (4.72), (4.12), (4.22), (3.20), (3.21), (3.24), (4.79). To obtain (4.78) we use (4.72), (4.69), (4.51 *a*), (4.52 *a*), (4.16), (4.22), (3.20), (3.21), (3.24), (4.79).

There are the formulae

$$H_{\pm} g(s) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{g(t)}{s \pm \mathrm{i}0 - t}\, \mathrm{d}t, \quad s \in \mathbb{R}. \tag{4.80}$$

We use the notation

$$\mathbb{C}_{\pm} = \{ s \in \mathbb{C} \mid \pm \operatorname{Im} s > 0 \}, \quad \bar{\mathbb{C}}_{\pm} = \mathbb{C}_{\pm} \cup \mathbb{R} \cup \infty. \tag{4.81}$$

In addition to $H_{\pm} g(s)$ for $s \in \mathbb{R}$, we consider

$$H_{\pm} g(s) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{g(t)}{s - t}\, \mathrm{d}t \quad \text{for } s \in \mathbb{C}_{\pm}. \tag{4.82}$$

Proposition 4.2A. *Let the assumptions of Proposition 4.1A be valid. Then the following formulae hold:*

$$R(x,\theta) = R^{\perp}(x\theta^{\perp},\theta), \tag{4.83}$$

$$R^{\perp}(s,\theta) = (Q^{\perp}_{-,-}(s,\theta))^{-1} Q^{\perp}_{+,-}(s,\theta), \tag{4.84 a}$$

$$R^{\perp}(s,\theta) = (Q^{\perp}_{-,+}(s,\theta))^{-1} Q^{\perp}_{+,+}(s,\theta), \tag{4.84 b}$$

$$Q^{\perp}_{+,+}(s,\theta)(Q^{\perp}_{+,-}(s,\theta))^{-1} = S^{\perp}(s,\theta), \tag{4.85 a}$$

$$Q^{\perp}_{-,+}(s,\theta)(Q^{\perp}_{-,-}(s,\theta))^{-1} = S^{\perp}(s,\theta), \tag{4.85 b}$$

where $x \in \mathbb{R}^2$, $s \in \mathbb{R}$, $\theta \in \mathbb{S}^1$;

$$Q^{\perp}_{+,\pm}(\cdot,\theta), \quad Q^{\perp}_{-,\mp}(\cdot,\theta) \in C(\bar{\mathbb{C}}_{\mp}, GL(n,\mathbb{C})), \quad \theta \in \mathbb{S}^1, \tag{4.86 a}$$

$$\frac{\partial}{\partial \bar{s}} Q^{\perp}_{+,\pm}(s,\theta) = 0, \quad \frac{\partial}{\partial \bar{s}} Q^{\perp}_{-,\mp}(s,\theta) = 0 \quad \text{for } s \in \mathbb{C}_{\mp}, \quad \theta \in \mathbb{S}^1, \tag{4.86 b}$$

where

$$Q^{\perp}_{+,\pm}(s,\theta), Q^{\perp}_{-,\mp}(s,\theta) \quad \text{for } s \in \mathbb{C}_{\mp} \quad \text{are defined by (4.77), (4.82)}, \tag{4.86 c}$$

$$Q^{\perp}_{+,\pm}(\infty,\theta) \stackrel{\text{def}}{=} I, \qquad Q^{\perp}_{-,\mp}(\infty,\theta) \stackrel{\text{def}}{=} I; \tag{4.86 d}$$

$$S^{\perp}(\cdot,\theta) \in C(\mathbb{R}, GL(n,\mathbb{C})), \tag{4.87 a}$$

$$\|S^{\perp}(\cdot,\theta) - I\|_{\alpha,\varepsilon'} \leqslant n\varepsilon^{-1} 2^{(7+5\varepsilon)/2} \rho^{\varepsilon'-\varepsilon} \|a\|_{\alpha,1+\varepsilon,\rho} \exp(n\varepsilon^{-1} 2^{5/2+\varepsilon} \rho^{-\varepsilon} \|a\|_{\alpha,1+\varepsilon,\rho}), \tag{4.87 b}$$

where $\theta \in \mathbb{S}^1$, $0 \leqslant \varepsilon' \leqslant \varepsilon$;

$$Q^{\perp}_{i,j}(\cdot,\theta) \in C(\mathbb{R}, GL(n,\mathbb{C})), \tag{4.88 a}$$

$$\begin{aligned} &\max(\|Q^{\perp}_{i,j}(\cdot,\theta) - I\|_{\alpha,\varepsilon'}, \|(Q^{\perp}_{i,j}(\cdot,\theta))^{-1} - I\|_{\alpha,\varepsilon'}) \\ &\qquad \leqslant n c_{18}(\alpha,\varepsilon,\varepsilon',\varepsilon'') \rho^{\varepsilon''-\varepsilon} \|a\|_{\alpha,1+\varepsilon,\rho} (1-\delta)^{-1}, \end{aligned} \tag{4.88 b}$$

where $i \in \{-,+\}$, $j \in \{-,+\}$, $\theta \in \mathbb{S}^1$, $0 \leqslant \varepsilon' < \min(1,\varepsilon'')$.

Remark. The estimates (4.87) are valid without the assumption (4.49).

Proposition 4.2B. *Let the assumptions of Proposition 4.1B be valid. Then the formulae (4.83)–(4.86) are valid and the following estimates hold:*

$$S^{\perp}(\cdot,\theta) \in C(\mathbb{R}, GL(n,\mathbb{C})), \tag{4.89 a}$$

$$\|S^{\perp}(\cdot,\theta) - I\|_{\beta,\varepsilon} \leqslant 2n(2+r)^{\varepsilon} c_4(r) \|b\|_{\beta,0} \exp(4nr\|b\|_{\beta,0}), \tag{4.89 b}$$

$$Q^{\perp}_{i,j}(\cdot,\theta) \in C(\mathbb{R}, GL(n,\mathbb{C})), \tag{4.90 a}$$

$$\begin{aligned} &\max(\|Q^{\perp}_{i,j}(\cdot,\theta) - I\|_{\beta,\varepsilon'}, \|(Q^{\perp}_{i,j}(\cdot,\theta))^{-1} - I\|_{\beta,\varepsilon'}) \\ &\qquad \leqslant n c_{21}(\beta,\varepsilon,\varepsilon')(2+r)^{\varepsilon} c_4(r) \|b\|_{\beta,0} (1-\delta^{1/2})^{-1}, \end{aligned} \tag{4.90 b}$$

where $i \in \{-,+\}$, $j \in \{-,+\}$, $\theta \in \mathbb{S}^1$, $\varepsilon \geqslant 0$, $0 \leqslant \varepsilon' < \min(1,\varepsilon)$.

Remark. The estimates (4.89) are valid without the assumption (4.58).

The proof of Propositions 4.2A and 4.2B consists of the following.

The formulae (4.83)–(4.85) follow from (4.71)–(4.76).

The estimates (4.87) follow from (3.25), (3.27), (4.76 a). The estimates (4.88 a), (4.88 b) for $Q^{\perp}_{i,j} - I$ and the properties (4.86), under assumptions of Proposition 4.1A, follow from (4.55) for $\psi_{\pm} - I$, $\lambda \in T$, (4.77), (4.78), (A.63), (A.73), (A.75). The proof of (4.88 b) for $(Q^{\perp}_{i,j})^{-1} - I$ is similar to the proof of (4.55) for $(\psi_{\pm})^{-1} - I$.

The estimates (4.89) follow from (3.32), (3.33) (4.76 a). The estimates (4.90 a), (4.90 b) for $Q^{\perp}_{i,j} - I$ and the properties (4.86), under assumptions of Proposition 4.1B, follow from (4.60 a) for $\psi_{\pm} - I$, (4.77), (4.78), (A.69), (A.70), (A.73), (A.74). The proof of (4.90 b) for $(Q^{\perp}_{i,j})^{-1} - I$ is similar to the proof of (4.60 a) for $(\psi_{\pm})^{-1} - I$.

5. Inverse scattering for the two-dimensional X-ray connection equation

Proposition 5.1. *Let the assumptions of Proposition 4.1A or Proposition 4.1B be valid. Then at fixed $\theta \in \mathbb{S}^1$ the scattering matrix $S^{\perp}(\cdot,\theta)$ on $\mathbb{R}$ uniquely determines $Q^{\perp}_{+,\pm}(\cdot,\theta)$ on $\bar{\mathbb{C}}_{\mp}$ (as functions with the properties (4.85 a), (4.86 a), (4.86 b), (4.86 d)), $Q^{\perp}_{-,\pm}(\cdot,\theta)$ on $\bar{\mathbb{C}}_{\pm}$ (as functions with the properties (4.85 b), (4.86 a), (4.86 b), (4.86 d)) and $R(\cdot,\theta)$ on $\mathbb{R}^2$ (using (4.83), (4.84)).*

The problem of finding $Q^{\perp}_{+,\pm}(\cdot,\theta)$ with the properties (4.85 a), (4.86 a), (4.86 b), (4.86 d) from $S^{\perp}(\cdot,\theta)$ and the problem of finding $Q^{\perp}_{-,\pm}(\cdot,\theta)$ with the properties (4.85 b), (4.86 a), (4.86 d) from $S^{\perp}(\cdot,\theta)$ are regular Riemann conjugation problems with fixed normalization. It is well known that any regular Riemann conjugation problem with fixed normalization has, at most, one solution (see [**11**]).

Proposition 5.2. *Let the assumptions of Proposition 4.1A or Proposition 4.1B be valid. Then the functions $Q^{\perp}_{+,-}(\cdot,\theta)$, $Q^{\perp}_{-,-}(\cdot,\theta)$ on $\mathbb{R}$ at fixed $\theta \in \mathbb{S}^1$ satisfy the equations*

$$Q^{\perp}_{+,-}(\cdot,\theta) - \frac{1}{2\mathrm{i}} H_{+}((S^{\perp}(\cdot,\theta) - I)Q^{\perp}_{+,-}(\cdot,\theta)) = I, \tag{5.1 a}$$

$$Q^{\perp}_{-,-}(\cdot,\theta) + \frac{1}{2\mathrm{i}} H_{-}((S^{\perp}(\cdot,\theta) - I)Q^{\perp}_{-,-}(\cdot,\theta)) = I \tag{5.1 b}$$

(where we use the estimates (4.87)–(4.90), (A.74)). If, in addition,

$$nc_{15}(\alpha,\varepsilon',0)\|S^{\perp}(\cdot,\theta) - I\|_{\alpha,\varepsilon'} < 1 \tag{5.2}$$

for some $\alpha \in \,]0,1[$, $\varepsilon' > 0$, then equations (5.1) are uniquely solvable in $C^{\alpha,0}(\mathbb{R},\mathcal{M}_{n\times n})$ by the method of successive approximations.

The deduction of (5.1) consists of the following. From (4.85) it follows that

$$Q^{\perp}_{+,+}(\cdot,\theta) - I = Q^{\perp}_{+,-}(\cdot,\theta) - I + (S^{\perp}(\cdot,\theta) - I)Q^{\perp}_{+,-}(\cdot,\theta), \tag{5.3 a}$$

$$Q^{\perp}_{-,+}(\cdot,\theta) - I = Q^{\perp}_{-,-}(\cdot,\theta) - I + (S^{\perp}(\cdot,\theta) - I)Q^{\perp}_{-,-}(\cdot,\theta). \tag{5.3 b}$$

Applying the operator H_{+} to the both sides of (5.3 a) and the operator H_{-} to the both sides of (5.3 b), using (4.86 a), (4.86 b), (4.86 d), (4.87)–(4.90) and properties of the Cauchy integral we obtain (5.1 a), (5.1 b).

Proposition 5.2 completes Proposition 5.1 by an effective method for determination of $Q^{\perp}_{+,-}$, $Q^{\perp}_{-,-}$ and, as a corollary, of $Q^{\perp}_{+,+}$, $Q^{\perp}_{-,+}$ from $S^{\perp}$, at least, under additional assumption (5.2).

Under assumptions of Proposition 4.1A or Proposition 4.1B, we consider

$$\tilde{\psi}_{\pm}(z,\lambda) = (g(z))^{-1}\psi_{\pm}(z,\lambda), \quad \lambda \in \bar{D}_{\pm}, \tag{5.4 a}$$

$$\tilde{a}_{+}(z) = (g(z))^{-1}a_{+}(z)g(z) + (g(z))^{-1}\partial_{\bar{z}}g(z), \tag{5.4 b}$$

$$\tilde{a}_{-}(z) = (g(z))^{-1}a_{-}(z)g(z) + (g(z))^{-1}\partial_{z}g(z), \tag{5.4 c}$$

$$\tilde{a}_{0}(z) = (g(z))^{-1}a_{0}(z)g(z), \tag{5.4 d}$$

where

$$g(z) = \psi_{-,0}(z), \quad z \in \mathbb{C}. \tag{5.4 e}$$

The collection $(\tilde{a}_+, \tilde{a}_-, \tilde{a}_0)$ is obtained from $(a_+,\ a_-,\ a_0)$ by the gauge transform given by the function $g(z) = \psi_{-,0}(z)$.

From (4.48 b), (5.4 c), (5.4 e) it follows that

$$\tilde{a}_-(z) \equiv 0, \quad z \in \mathbb{C}. \tag{5.5}$$

From the equation (4.28) for $\psi_\pm(z,\lambda)$, $0 < |\lambda|^{\pm 1} \leqslant 1$, and the formulae (5.4), (5.5) it follows that $\tilde{\psi}_\pm(z,\lambda)$, $0 < |\lambda|^{\pm 1} \leqslant 1$, satisfy the equation

$$(\lambda \partial_z + \lambda^{-1}\partial_{\bar{z}})\tilde{\psi} + \tilde{v}(z,\lambda)\tilde{\psi} = 0, \quad z \in \mathbb{C}, \quad \lambda \in \mathbb{C}\backslash 0, \tag{5.6}$$

where

$$\tilde{v}(z,\lambda) = \lambda^{-1}\tilde{a}_+(z) + \tilde{a}_0(z). \tag{5.7}$$

Under assumptions of Proposition 4.1A, the following is valid:

$$\psi_{-,0} - I \in C^{\beta,\varepsilon'}(\mathbb{C}, \mathcal{M}_{n\times n}), \tag{5.8 a}$$

$$(\psi_{-,0})^{-1} - I \in C^{\beta,\varepsilon'}(\mathbb{C}, \mathcal{M}_{n\times n}), \tag{5.8 b}$$

$$\partial_{\bar{z}}\psi_{-,0} \in C^{\alpha,\varepsilon''}(\mathbb{C}, \mathcal{M}_{n\times n}), \tag{5.8 c}$$

$$\tilde{a}_0, \tilde{a}_+ \in C^{\alpha,\varepsilon''}(\mathbb{C}, \mathcal{M}_{n\times n}), \tag{5.9}$$

where $0 < \beta < 1$, $0 < \varepsilon' < \min(\varepsilon, 1)$, $0 < \varepsilon'' < \min(1+\varepsilon, 2)$.

Under assumptions of Proposition 4.1B, the following is valid:

$$\psi_{-,0} - I \in C^{\beta,1}(\mathbb{C}, \mathcal{M}_{n\times n}), \tag{5.10 a}$$

$$(\psi_{-,0})^{-1} - I \in C^{\beta,1}(\mathbb{C}, \mathcal{M}_{n\times n}), \tag{5.10 b}$$

$$\partial_{\bar{z}}\psi_{-,0} \in L^\infty(\mathbb{C}, \mathcal{M}_{n\times n}), \tag{5.10 c}$$

$$\partial_{\bar{z}}\psi_{-,0} \in C^{\alpha,2}(\Omega_{r,\delta}, \mathcal{M}_{n\times n}), \tag{5.10 d}$$

$$\tilde{a}_0, \tilde{a}_+ \in L^\infty(\mathbb{C}, \mathcal{M}_{n\times n}), \tag{5.11 a}$$

$$\tilde{a}_0, \tilde{a}_+ \in C^{\alpha,2}(\Omega_{r,\delta}, \mathcal{M}_{n\times n}), \tag{5.11 b}$$

$$\tilde{a}_0(z) \equiv 0, \quad \partial_z\tilde{a}_+(z) \equiv 0 \quad \text{for } |z| > r, \tag{5.11 c}$$

where $0 < \beta < 1$, $0 < \delta < r$; $\Omega_{r,\delta}$ is defined by (A.120).

The property (5.8 a) follows from (4.48 a), (3.4 a), $d = 2$, (4.55) and Lemma A.10_a. The property (5.8 b) follows, for example, from (5.8 a) and the property $\det \psi_{-,0} \neq 0$ (according to (4.52 a). The property (5.8 c) follows from (4.48 a), (3.4 a), $d = 2$, (5.8 a) and Lemmas A.3 and A.11_a. The property (5.9) follows from (5.4 b), (5.4 d), (5.4 e), (3.4 a), $d = 2$, (5.8).

The property (5.10 a) follows from (4.48 a), (3.4 b), $d = 2$, (4.60 b) and Lemma A.10_b. The property (5.10 b) follows, for example, from (5.10 a) and the property $\det \psi_{-,0} \neq 0$ (according to (4.52 a)). The properties (5.10 c), (5.10 d) follow from (4.48 a), (3.4 b), $d = 2$, (5.10 *a*) and Lemmas A.3 and A.11$_b$. The properties (5.11) follow from (5.4 *b*), (5.4 *d*), (5.4 *e*), (3.4 *b*), $d = 2$, (5.10), (4.48 *b*).

Under assumptions of Proposition 4.1A or Proposition 4.1B, from Proposition 3.1, the formulae (5.4), (5.5) and properties (5.8)–(5.11), (4.48 *b*) it follows that the scattering matrix for the equation (4.28), $\lambda \in T$, coincides with the scattering matrix for the equation (5.6), $\lambda \in T$.

From (4.50)–(4.53), (5.4 *a*) it follows that at fixed $z \in \mathbb{C}$,

$$\tilde\psi_\pm(z,\cdot) \in C(\bar D_\pm, GL(n,\mathbb{C})), \tag{5.12 a}$$

$$\tilde\psi_+(z,\lambda) = \tilde\psi_-(z,\lambda)R(z,\lambda), \quad \lambda \in T, \tag{5.12 b}$$

$$\left(\frac{\partial}{\partial\bar\lambda}\right)\tilde\psi_+(z,\lambda) = 0, \quad \lambda \in D_+, \tag{5.12 c}$$

$$\left(\frac{\partial}{\partial\bar\lambda}\right)\tilde\psi_-(z,\lambda) = 0, \quad \lambda \in D_-\backslash\infty, \tag{5.12 d}$$

$$\tilde\psi_-(z,\infty) = I. \tag{5.12 e}$$

From (5.6) for $\tilde\psi_+(z,\lambda)$, $0 < |\lambda| \leqslant 1$, (5.12 *a*), (5.12 *c*) it follows that

$$\tilde a_+(z) = -(\partial_{\bar z}\tilde\psi_{+,0}(z))(\tilde\psi_{+,0}(z))^{-1}, \tag{5.13 a}$$

where

$$\tilde\psi_{+,0}(z) = \tilde\psi_+(z,0). \tag{5.13 b}$$

From (5.6) for $\tilde\psi_-(z,\lambda)$, $0 < |\lambda| \leqslant 1$, (5.12 *a*), (5.12 *d*), (5.12 *e*) it follows that

$$\tilde a_0(z) = -\partial_z\tilde\psi_{-,1}(z), \tag{5.14 a}$$

where

$$\tilde\psi_-(z,\lambda) = I + \lambda^{-1}\tilde\psi_{-,1}(z) + O(\lambda^{-2}) \quad \text{as } \lambda \to \infty. \tag{5.14 b}$$

Proposition 5.3. *Let the assumptions of Proposition 4.1A or Proposition 4.1B be valid. Then at fixed $z \in \mathbb{C}$ the function $R(z,\cdot)$ on T uniquely determines $\tilde\psi_\pm(z,\cdot)$ on $\bar D_\pm$ (as functions with the properties (5.12)). In turn, $\tilde\psi_\pm$ on $\mathbb{C}\times\bar D_\pm$ uniquely determine $\tilde a_0$, $\tilde a_+$ on $\mathbb{C}$ (using (5.6) or (5.13), (5.14)).*

The problem of finding $\tilde\psi_\pm(z,\cdot)$ with the properties (5.12) from $R(z,\cdot)$ is a regular Riemann conjugation problem with fixed normalization. Any regular Riemann conjugation problem with fixed normalization has, at most, one solution (see [**11**]).

Proposition 5.4. *Let the assumptions of Proposition 4.1A or Proposition 4.1B be valid. Then the function $\tilde\psi_-(z,\cdot)$ on T at fixed $z \in \mathbb{C}$ satisfies the equation*

$$\tilde\psi_-(z,\cdot) - C_-(\tilde\psi_-(z,\cdot)(R(z,\cdot) - I)) = I, \tag{5.15}$$

where

$$C_-f(\lambda) = -\frac{1}{2\pi \mathrm{i}}\int_T \frac{f(\xi)\,\mathrm{d}\xi}{\xi - \lambda(1+0)}, \quad \lambda \in T. \tag{5.16}$$

If, in addition,

$$n\|R(z,\cdot)-I\|_0<1, \tag{5.17}$$

then the equation (5.15) is uniquely solvable in $L^2(T,\mathcal{M}_{n\times n})$ *by the method of successive approximations.*

The deduction of (5.15) consists of the following. From (5.12 b) it follows that

$$\tilde\psi_+(z,\cdot)-I=\tilde\psi_-(z,\cdot)-I+\tilde\psi_-(z,\cdot)(R(z,\cdot)-I). \tag{5.18}$$

Applying the operator C_- to the both sides of (5.18), using (5.12 a), (5.12 c)–(5.12 e) and properties of the Cauchy integral we obtain (5.15). Proposition 5.4 completes Proposition 5.3 by an effective method for determination of $\tilde\psi_-$ and, as a corollary, $\tilde\psi_+$ from R, at least, under additional assumption (5.17).

Propositions 5.1–5.4 imply the following result.

Theorem 5.1. *Let the assumptions of Proposition 4.1A or Proposition 4.1B be valid. Then the scattering matrix* $S^\perp$ *on* $\mathbb{R}\times\mathbb{S}^1$ *for the equation (4.1) uniquely determines* $\tilde a_0$, $\tilde a_+$ *on* $\mathbb{C}$ *(and* $\tilde\psi_\pm$ *on* $\mathbb{C}\times\bar D_\pm$*).*

Theorem 5.1 and the formulae (5.4), (4.48 a) imply the following corollary.

Corollary 5.1. *Let the assumptions of Proposition 4.1A or Proposition 4.1B be valid and, in addition,* $a_-\equiv 0$ *on* $\mathbb{C}$*. Then the scattering matrix* $S^\perp$ *on* $\mathbb{R}\times\mathbb{S}^1$ *for the equation (4.1) uniquely determines* a_0, a_+ *on* $\mathbb{C}$.

Under assumptions (3.4 a) or (3.4 b), we consider further

$$\psi^{\pm,+}_\omega(x,\theta)=(\psi^\pm_0(x,\omega))^{-1}\psi^+(x,\theta), \tag{5.19 a}$$

$$\psi^{\pm,-}_\omega(x,\theta)=(\psi^\pm_0(x,\omega))^{-1}\psi^-(x,\theta), \tag{5.19 b}$$

$$a^\pm_{\omega,i}(x)=(\psi^\pm_0(x,\omega))^{-1}a_i(x)\psi^\pm_0(x,\omega)+(\psi^\pm_0(x,\omega))^{-1}\frac{\partial}{\partial x_i}\psi^\pm_0(x,\omega),\quad i=1,\dots,d, \tag{5.19 c}$$

$$a^\pm_{\omega,0}(x)=(\psi^\pm_0(x,\omega))^{-1}a_0(x)\psi^\pm_0(x,\omega), \tag{5.19 d}$$

where $x\in\mathbb{R}^d$, $\theta,\omega\in\mathbb{S}^{d-1}$, $\psi^\pm(x,\theta)$ are the wave functions for the equation (3.1), $\psi^\pm_0(x,\theta)$ are the wave functions for the equation

$$\theta\partial_x\psi_0+v_0(x,\theta)\psi_0=0,\quad x\in\mathbb{R}^d,\quad \theta\in\mathbb{S}^{d-1}, \tag{5.20}$$

where

$$v_0(x,\theta)=\sum_{i=1}^d\theta_i a_i(x). \tag{5.21}$$

In addition, if $a_0\equiv 0$ on $\mathbb{R}^d$, then $v_0\equiv v$, $\psi^\pm_0\equiv\psi^\pm$ on $\mathbb{R}^d\times\mathbb{S}^{d-1}$.

From (5.19 c), (5.20) for $\psi^\pm_0$, (5.21) (and the property $\det\psi^\pm_0\neq 0$) it follows that

$$\sum_{i=1}^d\omega_i a^\pm_{\omega,i}(x)\equiv 0,\quad x\in\mathbb{R}^d. \tag{5.22}$$

From the equation (3.1) for $\psi^\pm$ and the formulae (5.19) it follows that $\psi_\omega^{\pm,+}(x,\theta)$, $\psi_\omega^{\pm,-}(x,\theta)$ satisfy the equation

$$\theta\partial_x\psi_\omega^\pm + v_\omega^\pm(x,\theta)\psi_\omega^\pm = 0, \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}, \tag{5.23}$$

where

$$v_\omega^\pm(x,\theta) = \sum_{i=1}^{d} \theta_i a_{\omega,i}^\pm(x) + a_{\omega,0}^\pm(x). \tag{5.24}$$

In addition, the fact that $\psi^\pm$ and $\psi_0^\pm$ are the wave functions for (3.1) and (5.20), respectively, the estimate (3.26 a) or (3.33a) and the formulae (5.19 a), (5.19 b) imply that

$$\lim_{s\to-\infty} \psi_\omega^{\pm,+}(x+s\theta,\theta) = I, \quad \theta \neq \mp\omega, \tag{5.25 a}$$

$$\lim_{s\to+\infty} \psi_\omega^{\pm,-}(x+s\theta,\theta) = I, \quad \theta \neq \pm\omega, \tag{5.25 b}$$

$$S(x,\theta) = (\psi_\omega^{\pm,-}(x,\theta))^{-1}\psi_\omega^{\pm,+}(x,\theta), \tag{5.25 c}$$

where $x \in \mathbb{R}^d$, $\theta,\omega \in \mathbb{S}^{d-1}$, $S(x,\theta)$ is the scattering matrix for the equation (3.1).

Taking into account (5.25) we say that $\psi_\omega^{\pm,+}$, $\psi_\omega^{\pm,-}$ are the wave functions for the equation (5.23) and that the scattering matrix for (5.23) coincides with the scattering matrix for (3.1).

Under assumptions (3.4 a), from (5.19 a), (5.19 b), (5.19 d) and Proposition 3.1A it follows that

$$\psi_\omega^{i,j}(\cdot,\theta) \in C^{\alpha,0}(\mathbb{R}^d, GL(n,\mathbb{C})), \tag{5.26 a}$$

$$\|\psi_\omega^{i,j}(\cdot,\theta)\|_{\alpha,0} \leqslant A_1(n,\rho,\varepsilon,\|a\|_{\alpha,1+\varepsilon,\rho}), \tag{5.26 b}$$

$$a_{\omega,0}^i \in C^{\alpha,1+\varepsilon}(\mathbb{R}^d, \mathcal{M}_{n\times n}), \tag{5.27 a}$$

$$\|a_{\omega,0}^i\|_{\alpha,1+\varepsilon} \leqslant A_2(n,\rho,\varepsilon,\|a\|_{\alpha,1+\varepsilon,\rho}), \tag{5.27 b}$$

where $i,j \in \{-,+\}$, $\theta,\omega \in \mathbb{S}^{d-1}$, the bounds A_1, A_2 can be written explicitly.

Under assumptions (3.4 b), from (5.19 a), (5.19 b), (5.19 d), Proposition 3.1B (and the equations (3.1), (5.20) for $\psi^\pm$, $\psi_0^\pm$), it follows that

$$\psi_\omega^{i,j}(\cdot,\theta) \in C^{\beta,0}(\mathbb{R}^d, GL(n,\mathbb{C})), \tag{5.28 a}$$

$$\|\psi_\omega^{i,j}(\cdot,\theta)\|_{\beta,0} \leqslant A_3(n,\rho,\beta,\|b\|_{\alpha,0}), \tag{5.28 b}$$

$$a_{\omega,0}^i(x) = \chi_+(r-|x|)b_{\omega,0}^i(x), \quad x \in \mathbb{R}^d, \tag{5.28 c}$$

$$b_{\omega,0}^i \in C^{\beta,0}(\mathbb{R}^d, \mathcal{M}_{n\times n}), \tag{5.29 a}$$

$$\|b_{\omega,0}^i\|_{\beta,0} \leqslant A_4(n,\rho,\beta,\|b\|_{\alpha,0}), \tag{5.29 b}$$

where $i,j \in \{-,+\}$, $\theta,\omega \in \mathbb{S}^{d-1}$, $\beta = \min(\frac{1}{2},\alpha)$, the bounds A_3, A_4 can be written explicitly.

Under assumptions (3.4 a) or (3.4 b), $a_{\omega,j}^\pm$, $j = 1,\dots,d$, are generalized functions (distributions), in general.

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Under assumptions (3.4 *a*), $\alpha \in [1,2]$, from (5.19 *c*) and Proposition 3.1A it follows that

$$a^{\pm}_{\omega,i} \in C^{\alpha-1,0}(\mathbb{R}^d, \mathcal{M}_{n\times n}), \tag{5.30 a}$$

$$|a^{\pm}_{\omega,i}(x)| \leqslant c_1(\rho,\varepsilon,\pm\omega,x)A_5(n,\rho,\varepsilon,\|a\|_{\alpha,1+\varepsilon,\rho}), \quad x \in \mathbb{R}^d, \tag{5.30 b}$$

$$\|a^{\pm}_{\omega,i}\|_{\alpha-1,0} \leqslant A_6(n,\rho,\varepsilon,\|a\|_{\alpha,1+\varepsilon,\rho}), \tag{5.30 c}$$

where $i = 1,\dots,d$, $\omega \in \mathbb{S}^{d-1}$, the coefficients A_5, A_6 can be written explicitly.

Under assumptions (3.4 *b*), the formula (5.19 *c*) and Proposition 3.1B imply the following:

(1)

$$\text{if } 1 \leqslant \alpha \leqslant 2, \quad \text{then } a^{\pm}_{\omega,i} \in C(\mathcal{D}_{r,\delta,\omega}, \mathcal{M}_{n\times n}), \tag{5.31 a}$$

$$|a^{\pm}_{\omega,i}(x)| \leqslant \chi_+(\sqrt{r^2-(\pi_\omega x)^2} \pm (\pi^\omega x)\omega)\chi_+(r-|\pi_\omega x|) \times (r^2-(\pi_\omega x)^2)^{-1/2}A_7(d,n,r,\|b\|_{1,0}), \quad x \in \mathbb{R}^d, \tag{5.31 b}$$

$$\|a^{\pm}_{\omega,i}|_{D_{r,\delta,\omega}}\|_{\beta,0} \leqslant A_8(d,n,r,\delta,\beta,\|b\|_{\alpha,0}), \tag{5.31 c}$$

(2)

$$\text{if } \alpha = 2, \quad \text{then } a^{\pm}_{\omega,i} \in C^1(\mathcal{D}_{r,\delta,\omega}, \mathcal{M}_{n\times n}), \tag{5.31 d}$$

where $i = 1,\dots,d$, $\omega \in \mathbb{S}^{d-1}$, $\delta \in \,]0,r[$, $\beta \in [0,\min(\frac{1}{2},\alpha-1)]$, the coefficients A_7, A_8 can be written explicitly.

Under assumptions of Proposition 4.1A or Proposition 4.1B, the following formulae hold:

$$\psi^{\pm,+}_{\omega}(x,\theta) = (\tilde\psi^{\pm}_0(x,\omega))^{-1}\tilde\psi^{+}(x,\theta), \tag{5.32 a}$$

$$\psi^{\pm,-}_{\omega}(x,\theta) = (\tilde\psi^{\pm}_0(x,\omega))^{-1}\tilde\psi^{-}(x,\theta), \tag{5.32 b}$$

$$a^{\pm}_{\omega,i}(x) = (\tilde\psi^{\pm}_0(x,\omega))^{-1}\tilde a_i(x)\tilde\psi^{\pm}_0(x,\omega) + (\tilde\psi^{\pm}_0(x,\omega))^{-1}\left(\frac{\partial}{\partial x_i}\right)\tilde\psi^{\pm}_0(x,\omega), \quad i = 1,2, \tag{5.32 c}$$

$$a^{\pm}_{\omega,0}(x) = (\tilde\psi^{\pm}_0(x,\omega))^{-1}\tilde a_0(x)\tilde\psi^{\pm}_0(x,\omega), \tag{5.32 d}$$

where $x \in \mathbb{R}^2$, $\theta,\omega \in \mathbb{S}^1$, $\psi^{\pm,+}_{\omega}$, $\psi^{\pm,-}_{\omega}$, $a^{\pm}_{\omega,i}$, $i = 0,1,2$, are defined by (5.19) for $d = 2$, $\tilde a_i$, $i = 0,1,2$, are defined using (5.4 *b*)–(5.4 *d*), (4.30), $\tilde\psi^{\pm}$ are the wave functions for the equation (5.6), $\tilde\psi^{\pm}_0$ are the wave functions for the equation (5.6) with $\tilde a_0$ replaced by zero (and we use (4.25)). The formulae (5.32) follow from (5.19) and the formulae

$$\tilde\psi^{\pm}(x,\theta) = (g(x))^{-1}\psi^{\pm}(x,\theta), \tag{5.33 a}$$

$$\tilde\psi^{\pm}_0(x,\theta) = (g(x))^{-1}\psi^{\pm}_0(x,\theta), \tag{5.33 b}$$

where

$$g(x) = \psi_{-,0}(x), \quad x \in \mathbb{R}^2 \tag{5.33 c}$$

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(where the function g is the same as in (5.4), but written now, using the coordinates $x \in \mathbb{R}^2$). The formulae (5.33) follow from (5.4 *b*)–(5.4 *e*), (5.5), (5.8)–(5.11) and the definition of $\tilde{\psi}^{\pm}$, $\psi^{\pm}$, $\tilde{\psi}_0^{\pm}$, $\psi_0^{\pm}$.

Theorem 5.1 and the formulae (5.32) imply the following result.

Theorem 5.2. *Let the assumptions of Proposition 4.1A or Proposition 4.1B be valid. Then the scattering matrix $S^{\perp}$ on $\mathbb{R} \times \mathbb{S}^1$ for the equation (4.1) uniquely determines $\psi_{\omega}^{\pm,+}$, $\psi_{\omega}^{\pm,-}$ on $\mathbb{R}^2 \times \mathbb{S}^1$, $a_{\omega,i}^{\pm}$, $i = 0, 1, 2$, on $\mathbb{R}^2$ for any $\omega \in \mathbb{S}^1$.*

If

$$\sum_{i=1}^{d} \omega_i a_i(x) \equiv 0, \quad x \in \mathbb{R}^d, \quad \text{for some } \omega \in \mathbb{S}^{d-1}, \tag{5.34}$$

then

$$\psi_0^{\pm}(x, \omega) \equiv I, \quad x \in \mathbb{R}^d, \tag{5.35 a}$$

$$\psi_{\omega}^{\pm,+}(x, \theta) \equiv \psi^{+}(x, \theta), \quad \psi_{\omega}^{\pm,-}(x, \theta) \equiv \psi^{-}(x, \theta), \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}, \tag{5.35 b}$$

$$a_{\omega,i}^{\pm}(x) \equiv a_i(x), \quad i = 0, \dots, d, \quad x \in \mathbb{R}^d. \tag{5.35 c}$$

Theorem 5.2 and the formulae (5.34), (5.35) imply the following corollary.

Corollary 5.2. *Let the assumptions of Proposition 4.1A or Proposition 4.1B be valid and, in addition, (5.34), $d = 2$, holds. Then the scattering matrix $S^{\perp}$ on $\mathbb{R} \times \mathbb{S}^1$ for the equation (4.1) and the vector ω uniquely determine a_i, $i = 0, 1, 2$, on $\mathbb{R}^2$.*

Using Theorems 5.1 and 5.2, we obtain the following corollary.

Corollary 5.3. *Let the assumptions of Proposition 4.1A or Proposition 4.1B be valid for a collection $a = (a_0, a_1, a_2)$ and for a collection $a' = (a'_0, a'_1, a'_2)$. Let the scattering matrix $S^{\perp}$ for a coincides on $\mathbb{R} \times \mathbb{S}^1$ with the scattering matrix $S^{\perp}$ for a'. Then*

$$\psi'(x, \theta) = (h(x))^{-1} \psi(x, \theta), \quad \theta \in \Sigma \backslash \mathbb{S}^1, \tag{5.36 a}$$

$$\psi'^{\pm}(x, \theta) = (h(x))^{-1} \psi^{\pm}(x, \theta), \quad \theta \in \mathbb{S}^1, \tag{5.36 b}$$

$$a'_i(x) = (h(x))^{-1} a_i(x) h(x) + (h(x))^{-1} \left(\frac{\partial}{\partial x_i}\right) h(x), \quad i = 1, 2, \tag{5.36 c}$$

$$a'_0(x) = (h(x))^{-1} a_0(x) h(x) \tag{5.36 d}$$

for

$$h(x) = (\psi'_{-,0}(x))^{-1} \psi_{-,0}(x), \quad x \in \mathbb{R}^2 \tag{5.36 e}$$

(where we use the coordinates x, θ; $'$ denotes the correspondence to a'). In addition: under assumptions of Proposition 4.1A with fixed α and ε in (3.4 a),

$$h - I, h^{-1} - I \in C^{1+\alpha,\varepsilon}(\mathbb{R}^2, \mathcal{M}_{n\times n}), \tag{5.37 a}$$

$$\partial_i h \in C^{\alpha,1+\varepsilon}(\mathbb{R}^2, \mathcal{M}_{n\times n}), \quad \partial_i = \frac{\partial}{\partial x_i}, \quad i = 1, 2; \tag{5.37 b}$$

under assumptions of Proposition 4.1B with fixed α and r in (3.4 b),

$$h, h^{-1} \in C^{1+\alpha}(\bar{B}_r, \mathcal{M}_{n\times n}), \tag{5.38 a}$$

$$h \equiv I \quad \text{on } \mathbb{R}^2 \backslash B_r, \tag{5.38 b}$$

where $B_r = \{x \in \mathbb{R}^2 \mid |x| < r\}$, $\bar{B}_r = \{x \in \mathbb{R}^2 \mid |x| \leqslant r\}$.

If, in addition, a_i, a_i', $i = 0, 1, 2$, take values in $u(n)$, then h takes values in $U(n)$.

We obtain (5.36) using Theorem 5.1 and the formulae (5.4). The final statement of Corollary 5.3 follows from the formula

$$h(x) = (\psi_0'^{\pm}(x,\omega))^{-1}\psi_0^{\pm}(x,\omega), \quad \omega \in \mathbb{S}^1 \tag{5.39}$$

(this formula follows from Theorem 5.2) and Statement 3.1. Using (3.25), (3.26 a), (5.39) for different values of ω we obtain that

$$h - I, h^{-1} - I \in C^{0,\varepsilon}(\mathbb{R}^2, \mathcal{M}_{n\times n}). \tag{5.40}$$

Using (3.4 a), (5.36 c) and (5.40) we obtain (5.37). Using (3.32), (5.39) we obtain that

$$h, h^{-1} \in C(\mathbb{R}^2, \mathcal{M}_{n\times n}). \tag{5.41}$$

Using (3.4 b), (5.36 c) and (5.41) we obtain (5.38 a). Using (3.33 a), (5.39) for all $\omega \in \mathbb{S}^1$ we obtain (5.38 b).

6. Inverse scattering for the X-ray connection equation in dimension $d \geqslant 3$

Let S denote the scattering matrix for the collection $a = (a_0, a_1, \dots, a_d)$ (i.e. for the equation (3.1) with the coefficients $a_0(x), a_1(x), \dots, a_d(x)$).

Theorem 6.1. *For $d \geqslant 3$ the following statements are valid.*

(1) *Let a_0 satisfy (3.4 a), $0 < \alpha < 1$, $a_i \equiv 0$, $i = 1, \dots, d$. Then S on $T\mathbb{S}^{d-1}$ uniquely determines a_0 on $\mathbb{R}^d$.*

(2) *Let a_0 satisfy (3.4 b), $0 < \alpha < 1$, $a_i \equiv 0$, $i = 1, \dots, d$. Then S on $T\mathbb{S}^{d-1}$ uniquely determines a_0 on $\mathbb{R}^d$.*

(3) *Let $a_0 \equiv 0$, a_i, $i = 1, \dots, d$, satisfy (3.4 a), $\alpha = 2$. Then S on $T\mathbb{S}^{d-1}$ uniquely determines $a^{\pm}_{\omega,i}$, $i = 1, \dots, d$, on $\mathbb{R}^d$ for any $\omega \in \mathbb{S}^{d-1}$.*

(4) *Let $a_0 \equiv 0$, a_i, $i = 1, \dots, d$, satisfy (3.4 b), $\alpha = 2$. Then S on $T\mathbb{S}^{d-1}$ and r of (3.4 b) uniquely determine $a^{\pm}_{\omega,i}$, $i = 1, \dots, d$, on $\mathbb{R}^d$ for any $\omega \in \mathbb{S}^{d-1}$.*

Remark 6.1. In Theorem 6.1 we do not assume that the following is known: α and ε of (3.4 a) in item (1); α and r of (3.4 b) in item (2); ε of (3.4 a) in item (3). However, the *a priori* knowledge of the following is necessary for the reconstruction procedure contained in the proof of the relevant item: α, ε and an upper bound for $\|a\|_{\alpha,1+\varepsilon}$ for the case of item (1); α and an upper bound for $\|b\|_{\alpha,0}$ for the case of item (2); ε and an upper bound for $\|a\|_{2,1+\varepsilon}$ for the case of item (3); an upper bound for $\|b\|_{2,0}$ for the case of item (4), where $a = (a_0, a_1, \dots, a_d)$, $b = (b_0, b_1, \dots, b_d)$.

To prove Theorem 6.1 we use, in particular, Proposition 6.1, Statement 6.1, Proposition 6.2 and Proposition 6.3 given below.

Let

$$X_{\nu,s} = \{x \in \mathbb{R}^3 \mid \nu x = s\}, \quad \nu \in \mathbb{S}^2, \quad s \in \mathbb{R}, \tag{6.1}$$

$$T\mathbb{S}^1(X_{\nu,s}) = \{\gamma \in T\mathbb{S}^2 \mid \gamma \text{ lies in } X_{\nu,s}\}. \tag{6.2}$$

Let

$$\delta_a(n,\beta,\varepsilon,\varepsilon'',z,N) = nc_{18}(\beta,\varepsilon,0,\varepsilon'')(1+z)^{\varepsilon''-\varepsilon}2^{(2+\varepsilon)/2}N, \tag{6.3 a}$$

where $n \in \mathbb{N}$, $0 < \beta < 1$, $0 < \varepsilon'' < \varepsilon$, $z \geqslant 0$, $\mathbb{N} \geqslant 0$.

Let

$$\delta_b(n,\beta,\varepsilon,z,N) = 2n^2 c_{21}(\beta,\varepsilon,0)(2+z)^{\varepsilon}c_4(z)(1+c_{21}(\beta,\varepsilon,0)(2+z)^{\varepsilon}c_4(z))N^2, \tag{6.3 b}$$

where $n \in \mathbb{N}$, $0 < \beta \leqslant \frac{1}{2}$, $\varepsilon > 0$, $z \geqslant 0$, $N \geqslant 0$.

(We remind the reader that c_{18}, c_{21}, c_4 are defined in Appendix A.)

Proposition 6.1A. *Let a_i, $i = 0,1,2,3$, satisfy (3.4 a), $d = 3$ and $a = (a_0,a_1,a_2,a_3)$. Let $\nu \in \mathbb{S}^2$, $s \in \mathbb{R}$ and*

$$\delta_a(n,\beta,\varepsilon,\varepsilon'',|s|,\|a\|_{\beta,1+\varepsilon}) < 1 \tag{6.4 a}$$

for some β and ε'' such that $0 < \beta < 1$, $0 < \varepsilon'' < \varepsilon$. Then S restricted to $T\mathbb{S}^1(X_{\nu,s})$ uniquely determines $a^{\pm}_{\omega,0}(x)$ and $\sum_{i=1}^{3} a^{\pm}_{\omega,i}(x)\,\mathrm{d}x_i$ restricted to $X_{\nu,s}$ for any $\omega \in \mathbb{S}^2$, $\omega\nu = 0$.

Proposition 6.1B. *Let a_i, $i = 0,1,2,3$, satisfy (3.4 b), $d = 3$ and $b = (b_0,b_1,b_2,b_3)$. Let $\nu \in \mathbb{S}^2$, $s \in \mathbb{R}$ and*

$$\delta_b(n,\beta,\varepsilon,l(r,s),\|b\|_{\beta,0}) < 1, \qquad l(r,s) = \chi_+(r-|s|)\sqrt{r^2-s^2}, \tag{6.4 b}$$

for some β and ε such that $0 < \beta \leqslant \frac{1}{2}$, $\varepsilon > 0$. Then S restricted to $T\mathbb{S}^1(X_{\nu,s})$ uniquely determines $a^{\pm}_{\omega,0}(x)$ and $\sum_{i=1}^{3} a^{\pm}_{\omega,i}(x)\,\mathrm{d}x_i$ restricted to $X_{\nu,s}$ for any $\omega \in \mathbb{S}^2$, $\omega\nu = 0$.

Remark 6.2. Let a_i, $i = 0,1,2,3$, satisfy (3.4 b), $d = 3$. Let $\nu,\omega \in \mathbb{S}^2$, $\omega\nu = 0$, $s \in \mathbb{R}$, $|s| \geqslant r$. Then $a^{\pm}_{\omega,i}(x) \equiv 0$, $i = 0,1,2,3$, for $x \in X_{\nu,s}$. Therefore, Proposition 6.1B is non-trivial only if $|s| < r$.

Remark 6.3. If in Proposition 6.1A or 6.1B the number α of (3.4 a) or (3.4 b) is smaller than 1, then (as indicated in § 5) $a^{\pm}_{\omega,i}$, $i = 1,2,3$, are 'generalized' functions, in general. However, under assumptions of any item of Theorem 6.1, $a^{\pm}_{\omega,i}$, $i = 1,2,3$, are usual functions.

Proof of Propositions 6.1A and 6.1B. There is an orthogonal 3×3 matrix $M = (m_{ij})$ such that

$$X_{\nu,s} = \left\{x \in \mathbb{R}^3 \;\middle|\; x_i = \sum_{j=1}^{2} m_{ij}y_j + m_{i3}s,\ i = 1,2,3,\ y = (y_1,y_2) \in \mathbb{R}^2\right\}, \tag{6.5}$$

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where we consider y as Euclidean coordinates on $X_{\nu,s}$. In addition,

$$\{\theta \in \mathbb{S}^2 \mid \theta\nu = 0\} = \left\{\theta \in \mathbb{S}^2 \;\middle|\; \theta_i = \sum_{j=1}^{2} m_{ij}\xi_j,\ i = 1,2,3,\ \xi = (\xi_1, \xi_2) \in \mathbb{S}^1\right\}. \tag{6.6}$$

A solution $\psi(x,\theta)$, $\theta\nu = 0$, of (3.1), $d = 3$, restricted to $X_{\nu,s}$ and written as $\psi(y,\xi)$ satisfies the equation

$$\left.\begin{aligned} \xi\partial_y\psi + (\xi_1 u_1(y) + \xi_2 u_2(y) + a_0(y))\psi = 0,\\ u_j(y) = \sum_{i=1}^{3} m_{ij} a_i(y), \quad j = 1,2, \end{aligned}\right\} \tag{6.7}$$

where $a_i(y) = a_i|_{X_{\nu,s}}$ in the coordinates y, $i = 1,2,3$.

In addition,

$$\sum_{i=1}^{2} u_i(y)\,\mathrm{d}y_i = \sum_{i=1}^{3} a_i\,\mathrm{d}x_i\big|_{X_{\nu,s}} \text{ in the coordinates } y. \tag{6.8}$$

If $f \in C^{\beta,1+\varepsilon}(\mathbb{R}^3, \mathcal{M}_{n\times n})$, $0 < \beta < 1$, $\varepsilon > 0$, then

$$\|f|_{X_{\nu,s}}\|_{\beta,1+\varepsilon,1+s} \leqslant 2^{(1+\varepsilon)/2}\|f\|_{\beta,1+\varepsilon}, \tag{6.9}$$

where $f|_{X_{\nu,s}}$ is considered as a function of y.

If $f(x) = \chi_+(r - |x|)g(x)$, $x \in \mathbb{R}^3$, $g \in C^{\beta,0}(\mathbb{R}^3, \mathcal{M}_{n\times n})$, $0 < \beta < 1$, then

$$\left.\begin{aligned} f(y) = \chi_+(l(r,s) - |y|)g(y), \quad y \in \mathbb{R}^2,\\ \|g|_{X_{\nu,s}}\|_{\beta,0} \leqslant \|g\|_{\beta,0}, \end{aligned}\right\} \tag{6.10}$$

where $f(y) = f|_{X_{\nu,s}}$ considered as a function of y, $g(y) = g|_{X_{\nu,s}}$ considered as a function of y.

Under assumptions of Proposition 6.1A or Proposition 6.1B, from (6.7), (6.9), (6.10), the inequalities $|m_{i1}|+|m_{i2}| \leqslant \sqrt{2}$, $i = 1,2,3$, and Theorem 5.2 it follows that S restricted to $T\mathbb{S}^1(X_{\nu,s})$ uniquely determines

$$\left.\begin{aligned} a^{\pm}_{\xi,0}(y) &= (\psi^{\pm}_0(y,\xi))^{-1} a_0(y)\psi^{\pm}_0(y,\xi),\\ u^{\pm}_{\xi,i}(y) &= (\psi^{\pm}_0(y,\xi))^{-1} u_i(y)\psi^{\pm}_0(y,\xi) + (\psi^{\pm}_0(y,\xi))^{-1}\left(\frac{\partial}{\partial y_i}\right)\psi^{\pm}_0(y,\xi), \end{aligned}\right\} \tag{6.11}$$

$y \in \mathbb{R}^2$, $\xi \in \mathbb{S}^1$, $i = 1,2$, where $\psi^{\pm}_0$ are the wave functions for (6.7) with a_0 replaced by 0. In addition, $\psi^{\pm}_0(y,\xi)$ are the wave functions $\psi^{\pm}_0(x,\theta)$, $\theta\nu = 0$, for (5.20), $d = 3$, restricted to $X_{\nu,s}$ and written in the coordinates y, ξ.

Finally,

$$\left.\begin{aligned} a^{\pm}_{\xi,0}(y) = a^{\pm}_{\theta,0}\big|_{X_{\nu,s}} \text{ in the coordinates } y,\\ \sum_{i=1}^{2} u^{\pm}_{\xi,i}(y)\,\mathrm{d}y_i = \sum_{i=1}^{3} a^{\pm}_{\theta,i}\,\mathrm{d}x_i\big|_{X_{\nu,s}} \text{ in the coordinates } y \end{aligned}\right\} \tag{6.12}$$

for $\xi \in \mathbb{S}^1$ and $\theta \in \mathbb{S}^2$ related as in (6.6).

Propositions 6.1A and 6.1B are proved. □

Let

$$Y(q,\omega)=\{x\in\mathbb{R}^3 \mid x=q+s\omega,\ s\in\mathbb{R}\},\quad q\in\mathbb{R}^3,\quad \omega\in\mathbb{S}^2. \tag{6.13}$$

Statement 6.1. *Let $d=3$, $q\in\mathbb{R}^3$, $\omega,\nu,\nu'\in\mathbb{S}^2$, $\omega\nu=\omega\nu'=0$, $\nu\neq\nu'$, $j\in\{-,+\}$. Let $\mathcal{D}$ be an open neighbourhood in $\mathbb{R}^3$ of a point $x\in Y(q,\omega)$. Then $Y(q,\omega)=X_{\nu,q\nu}\cap X_{\nu',q\nu'}$ and the following is valid.*

(A) *If a_i, $i=1,2,3$, satisfy (3.4 a), $\alpha=1$, then*

$$\sum_{i=1}^{3} a^j_{\omega,i}(x)\,\mathrm{d}x_i|_{X_{\nu,q\nu}\cap\mathcal{D}} \quad \text{and} \quad \sum_{i=1}^{3} a^j_{\omega,i}(x)\,\mathrm{d}x_i|_{X_{\nu',q\nu'}\cap\mathcal{D}}$$

uniquely determine $a^j_{\omega,i}$, $i=1,2,3$, on $Y(q,\omega)\cap\mathcal{D}$.

(B) *If a_i, $i=1,2,3$, satisfy (3.4 b), $\alpha=1$, then*

$$\sum_{i=1}^{3} a^j_{\omega,i}(x)\,\mathrm{d}x_i|_{X_{\nu,q\nu}\cap\mathcal{D}},\qquad \sum_{i=1}^{3} a^j_{\omega,i}(x)\,\mathrm{d}x_i|_{X_{\nu',q\nu'}\cap\mathcal{D}}$$

and r of (3.4 b) uniquely determine $a^j_{\omega,i}$, $i=1,2,3$, on $Y(q,\omega)\cap\mathcal{D}$.

Remark 6.4. We remind the reader that $a^j_{\omega,i}$, $i=1,2,3$, are independent of a_0.

Let

$$C_{r,\omega}=\{x\in\mathbb{R}^d \mid |x|=r\}\cup\{x\in\mathbb{R}^d \mid |\pi_\omega x|=r\},\quad r>0,\quad \omega\in\mathbb{S}^{d-1}. \tag{6.14}$$

Remark 6.5. In item (B) of Statement 6.1 we assume that r of (3.4 b) is known in order to know *a priori* the set $C_{r,\omega}$ (containing all discontinuity points of $a^{\pm}_{\omega,i}$, $i=1,2,3$).

Statement 6.1 follows from elementary geometric facts and the properties (5.30), (5.31) of $a^{\pm}_{\omega,i}$, $i=1,2,3$.

Let

$$\left.\begin{array}{c} B_\tau=\{x\in\mathbb{R}^d \mid |x|<\tau\},\quad \tau\geqslant 0,\\ \bar{B}_\tau \text{ be the closure of } B_\tau \text{ in } \mathbb{R}^d. \end{array}\right\} \tag{6.15}$$

Let

$$\Omega_1(\mathcal{D})=\{\gamma\in T\mathbb{S}^{d-1} \mid \gamma \text{ intersects } \mathcal{D}\}, \tag{6.16 a}$$

$$\Omega_2(\mathcal{D},\omega)=\{\gamma\in\Omega_1(\mathcal{D}) \mid \gamma \text{ has the direction } \omega\}, \tag{6.16 b}$$

$$\Omega_3(\mathcal{D})=T\mathbb{S}^{d-1}\backslash\Omega_1(\mathcal{D}), \tag{6.16 c}$$

$$\Omega_4(\mathcal{D},\omega)=\{\gamma\in\Omega_3(\mathcal{D}) \mid \gamma \text{ has the direction } \omega\}, \tag{6.16 d}$$

where $\mathcal{D}$ is an open bounded convex domain in $\mathbb{R}^d$, $\omega\in\mathbb{S}^{d-1}$.

Consider the equation (with respect to z)

$$\delta_a(n,\beta,\varepsilon,\varepsilon'',z,N)=\kappa,\quad z\geqslant 0, \tag{6.17}$$

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where

$$n \in \mathbb{N}, \quad 0 < \beta < 1, \quad 0 < \varepsilon'' < \varepsilon, \quad N \geqslant 0, \quad 0 < \kappa < 1. \tag{6.18}$$

Under conditions (6.18), the following is valid:

$$\delta_a(n, \beta, \varepsilon, \varepsilon'', z_1, N) > \delta_a(n, \beta, \varepsilon, \varepsilon'', z_2, N) \tag{6.19 a}$$

for $z_2 > z_1$, $z_1 \geqslant 0$, $z_2 \geqslant 0$, $N > 0$;

$$\delta_a(n, \beta, \varepsilon, \varepsilon'', z, N) \to 0 \quad \text{as } z \to +\infty; \tag{6.19 b}$$

$$\delta_a(n, \beta, \varepsilon, \varepsilon'', z, 0) = 0, \quad z \geqslant 0. \tag{6.19 c}$$

Therefore, under conditions (6.18), the equation (6.17) has, at most, one root.

Under conditions (6.18), we use the following definition: let $z_a(n, \beta, \varepsilon, \varepsilon'', N, \kappa)$ be the root of (6.17), if it has a root; let $z_a(n, \beta, \varepsilon, \varepsilon'', N, \kappa) = 0$, if (6.17) has no root.

Consider the equation (with respect to z)

$$\delta_b(n, \beta, \varepsilon, z, N) = \kappa, \quad z \geqslant 0, \tag{6.20}$$

where

$$n \in \mathbb{N}, \quad 0 < \beta \leqslant \tfrac{1}{2}, \quad \varepsilon > 0, \quad N \geqslant 0, \quad 0 < \kappa < 1. \tag{6.21}$$

Under conditions (6.21), the following is valid:

$$\delta_b(n, \beta, \varepsilon, z_1, N) < \delta_b(n, \beta, \varepsilon, z_2, N) \tag{6.22 a}$$

for $z_1 < z_2$, $z_1 \geqslant 0$, $z_2 \geqslant 0$, $N > 0$;

$$\delta_b(n, \beta, \varepsilon, 0, N) = 0; \tag{6.22 b}$$

$$\delta_b(n, \beta, \varepsilon, z, N) \to +\infty \quad \text{as } z \to +\infty, \quad \text{for } N > 0; \tag{6.22 c}$$

$$\delta_b(n, \beta, \varepsilon, z, 0) = 0. \tag{6.22 d}$$

Therefore, under conditions (6.21), the equation (6.20) is uniquely solvable for $N > 0$ and (6.20) has no solution for $N = 0$.

Under conditions (6.21), we use the following definition: let $z_b(n, \beta, \varepsilon, N, \kappa)$ be the root of (6.20), for $N > 0$; let $z_b(n, \beta, \varepsilon, 0, \kappa) = +\infty$.

Proposition 6.2. *For $d = 3$, $\kappa \in]0, 1[$, the following statements are valid.*

(1) *Let a_0 satisfy (3.4 a), $0 < \alpha < 1$, $a_i \equiv 0$, $i = 1, 2, 3$, and $\|a_0\|_{\alpha, 1+\varepsilon} \leqslant N$. Let $\tau = z_a(n, \alpha, \varepsilon, \varepsilon'', N, \kappa)$, $0 < \varepsilon'' < \varepsilon$. Then S on $\Omega_3(B_\tau)$ uniquely determines a_0 on $\mathbb{R}^3 \backslash B_\tau$.*

(2) *Let a_0 satisfy (3.4 b), $0 < \alpha \leqslant \frac{1}{2}$, $a_i \equiv 0$, $i = 1, 2, 3$, and $\|b_0\|_{\alpha, 0} \leqslant N$, $r \leqslant r_1$. Let $\tau = \chi_+(r_1 - z_b)\sqrt{r_1^2 - z_b^2}$, $z_b = z_b(n, \alpha, \varepsilon, N, \kappa)$, $\varepsilon > 0$. Then S on $\Omega_3(B_\tau)$ uniquely determines a_0 on $\mathbb{R}^3 \backslash B_\tau$.*

(3) *Let* $a_0 \equiv 0$, a_i, $i = 1, 2, 3$, *satisfy (3.4 a),* $\alpha = 1$, $a = (a_0, a_1, a_2, a_3)$, *and* $\|a\|_{\beta,1+\varepsilon} \leqslant N$ *for some* $\beta \in]0, 1[$. *Let* $\tau = z_a(n, \beta, \varepsilon, \varepsilon'', N, \kappa)$, $0 < \varepsilon'' < \varepsilon$. *Then* S *on* $T\mathbb{S}^2$ *uniquely determines* $a^{\pm}_{\omega,i}$, $i = 1, 2, 3$, *on* $\mathbb{R}^3 \backslash B_\tau$ *for any* $\omega \in \mathbb{S}^2$.

(4) *Let* $a_0 \equiv 0$, a_i, $i = 1, 2, 3$, *satisfy (3.4 b),* $\alpha = 1$, $b = (b_0, b_1, b_2, b_3)$, *and* $\|b\|_{\beta,0} \leqslant N$ *for some* $\beta \in]0, \frac{1}{2}]$. *Let* $\tau = \chi_+(r - z_b)\sqrt{r^2 - z_b^2}$, $z_b = z_b(n, \beta, \varepsilon, N, \kappa)$, $\varepsilon > 0$. *Then* S *on* $T\mathbb{S}^2$ *and* r *uniquely determine* $a^{\pm}_{\omega,i}$, $i = 1, 2, 3$, *on* $\mathbb{R}^3 \backslash B_\tau$ *for any* $\omega \in \mathbb{S}^2$.

Proof of Proposition 6.2. Item (1) of Proposition 6.2 follows from Proposition 6.1A, the properties (6.19) and the definition of z_a. Item (2) of Proposition 6.2 follows from Proposition 6.1B, the properties (6.22), the definition of z_b and the following properties:

$$\left.\begin{aligned} l(r, s) &\leqslant l(r_1, s) \quad \text{for } r \leqslant r_1,\ r \geqslant 0,\ r_1 \geqslant 0,\ s \in \mathbb{R}, \\ l(r, s_1) &\geqslant l(r, s_2) \quad \text{for } r \geqslant 0,\ |s_2| \geqslant |s_1|. \end{aligned}\right\} \tag{6.23}$$

Proof of items (3) and (4) of Proposition 6.2.

Consider the following conditions for x and ω:

$$x \in \mathbb{R}^3 \backslash \bar{B}_\tau, \quad \omega \in \mathbb{S}^2, \tag{6.24 a}$$

$$x \in \mathbb{R}^3 \backslash (\bar{B}_\tau \cup C_{r,\omega}), \quad \omega \in \mathbb{S}^2 \tag{6.24 b}$$

(where $C_{r,\omega}$ is defined by (6.14), $d = 3$),

$$Y(x, \omega) \cap \bar{B}_\tau = \emptyset, \tag{6.25}$$

$$Y(x, \omega) \cap \bar{B}_\tau \neq \emptyset. \tag{6.26}$$

□

Statement 6.2.

(A) *Let the assumptions of item (3) of Proposition 6.2 be valid. Then* S *on* $\Omega_3(B_\tau)$ *uniquely determines* $a^{\pm}_{\omega,i}(x)$, $i = 1, 2, 3$, *for any* x *and* ω *satisfying (6.24 a), (6.25).*

(B) *Let the assumptions of item (4) of Proposition 6.2 be valid. Then* S *on* $\Omega_3(B_\tau)$ *and* r *uniquely determine* $a^{\pm}_{\omega,i}(x)$, $i = 1, 2, 3$, *for any* x *and* ω *satisfying (6.24 b), (6.25).*

Proof of Statement 6.2. From (6.25) it follows that there are $\nu, \nu' \in \mathbb{S}^2$, $s, s' \in \mathbb{R}$ such that

$$Y(x, \omega) = X_{\nu,s} \cap X_{\nu',s'}, \quad X_{\nu,s} \cap \bar{B}_\tau = \emptyset, \quad X_{\nu',s'} \cap \bar{B}_\tau = \emptyset \tag{6.27}$$

and, as a corollary,

$$|s| > \tau, \quad |s'| > \tau, \quad \nu\omega = \nu'\omega = 0, \quad \nu \neq \nu', \quad s = \nu x, \quad s' = \nu' x, \tag{6.28}$$

$$T\mathbb{S}^1(X_{\nu,s}) \subset \Omega_3(B_\tau), \quad T\mathbb{S}^1(X_{\nu',s'}) \subset \Omega_3(B_\tau). \tag{6.29}$$

Under the assumptions of items (A) or (B) of Statement 6.2, from Propositions 6.1A, 6.1B, Statement 6.1 and the formulae (6.27), (6.28), it follows that

$$S|_{T\mathbb{S}^1(X_{\nu,s})} \quad \text{and} \quad S|_{T\mathbb{S}^1(X_{\nu',s'})}$$

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(and r, under assumptions of item (B) of Statement 6.2) uniquely determine $a^{\pm}_{\omega,i}$ on $Y(x,\omega)$, $i = 0,1,2,3$, and, in particular, $a^{\pm}_{\omega,i}(x)$, $i = 1,2,3$. In addition, taking into account (6.29) we obtain Statement 6.2. The proof of Statement 6.2 is completed. □

Statement 6.3.

(A) *Let the assumptions of item (3) of Proposition 6.2 be valid. Let x and ω satisfy (6.24 a), (6.26). Then S on $\Omega_3(B_\tau) \cup \Omega_2(B_\tau,\omega)$ uniquely determines $a^{\pm}_{\omega,i}(x)$, $i = 1,2,3$.*

(B) *Let the assumptions of item (4) of Proposition 6.2 be valid. Let x and ω satisfy (6.24 b), (6.26). Then S on $\Omega_3(B_\tau) \cup \Omega_2(B_\tau,\omega)$ and r uniquely determine $a^{\pm}_{\omega,i}(x)$, $i = 1,2,3$.*

Proof of Statement 6.3. Let

$$Y^{\pm}(q,\theta) = \{x \in \mathbb{R}^3 \mid x = q + s\theta,\ s \in \mathbb{R}_{\pm} \cup 0\}, \quad q \in \mathbb{R}^3, \quad \theta \in \mathbb{S}^2, \tag{6.30}$$

$$P(q,\theta,\theta') = P^{+}(q,\theta,\theta') \cup P^{-}(q,\theta,\theta'), \tag{6.31}$$

where

$$P^{\pm}(q,\theta,\theta') = \cup_{t\in\mathbb{R}_{\pm}\cup 0} Y(q+t\theta,\theta'), \quad q \in \mathbb{R}^3, \quad \theta,\theta' \in \mathbb{S}^2, \quad \theta' \neq \theta,$$

where $\mathbb{R}_+ =]0,+\infty[$, $\mathbb{R}_- =]-\infty,0[$.

From (6.26) and (6.24 a) or (6.24 b) it follows that either

$$Y^{+}(x,\omega) \cap \bar{B}_\tau = \emptyset \tag{6.32 a}$$

or

$$Y^{-}(x,\omega) \cap \bar{B}_\tau = \emptyset. \tag{6.32 b}$$

Consider $\omega', \omega'' \in \mathbb{S}^2$ such that

$$Y(x,\omega') \cap \bar{B}_\tau = \emptyset, \quad Y(x,\omega'') \cap \bar{B}_\tau = \emptyset, \quad \omega'' \neq \omega', \quad \omega'' \neq -\omega'. \tag{6.33}$$

If (6.32 a) holds, then due to (6.33):

$$P^{+}(x,\omega,\omega') \cap \bar{B}_\tau = \emptyset, \qquad P^{+}(x,\omega,\omega'') \cap \bar{B}_\tau = \emptyset. \tag{6.34 a}$$

If (6.32 b) holds, then due to (6.33):

$$P^{-}(x,\omega,\omega') \cap \bar{B}_\tau = \emptyset, \qquad P^{-}(x,\omega,\omega'') \cap \bar{B}_\tau = \emptyset. \tag{6.34 b}$$

□

Lemma 6.1A. *Let the assumptions of item (A) of Statement 6.3 be valid. Let $\theta \in \{\omega',\omega''\}$, where ω', ω'' satisfy (6.33). Then the following is valid.*

(1) *If (6.32 a) holds, then S on $\Omega_3(B_\tau)$ uniquely determines $a^{\pm}_{\theta,i}(y)$, $i = 1,2,3$, for any $y \in P^{+}(x,\omega,\theta)$.*

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(2) *If (6.32 b) holds, then S on $\Omega_3(B_\tau)$ uniquely determines $a^{\pm}_{\theta,i}(y)$, $i=1,2,3$, for any $y\in P^-(x,\omega,\theta)$.*

Lemma 6.1B. *Let the assumptions of item (B) of Statement 6.3 be valid. Let $\theta\in\{\omega',\omega''\}$, where ω', ω'' satisfy (6.33). Then the following is valid.*

(1) *If (6.32 a) holds, then S on $\Omega_3(B_\tau)$ and r uniquely determine $a^{\pm}_{\theta,i}(y)$, $i=1,2,3$, for any $y\in P^+(x,\omega,\theta)\backslash C_{r,\theta}$.*

(2) *If (6.32 b) holds, then S on $\Omega_3(B_\tau)$ and r uniquely determine $a^{\pm}_{\theta,i}(y)$, $i=1,2,3$, for any $y\in P^-(x,\omega,\theta)\backslash C_{r,\theta}$.*

Items (1) of Lemmas 6.1A and 6.1B follow from (6.34 a) and Statement 6.2. Items (2) of Lemmas 6.1A and 6.1B follow from (6.34 b) and Statement 6.2.

Remark 6.6. Note that

$$P(x,\omega,\theta)\cap C_{r,\theta}=\emptyset\quad\text{for } |y'|>r,$$
$$P(x,\omega,\theta)\cap C_{r,\theta}=\{\xi\in\mathbb{R}^2 \mid |\xi|=r'\}\cup\{\xi\in\mathbb{R}^2 \mid |\pi_\zeta\xi|=r'\}\quad\text{for } |y'|\leqslant r,$$

where ξ are Euclidean coordinates on $P(x,\omega,\theta)$ with centre at the point y' which is the nearest to 0 in $\mathbb{R}^3$, $r'=\sqrt{r^2-|y'|^2}$, $\zeta=\xi(y'+\theta)-\xi(y')$ for $|y'|\leqslant r$, where $\theta\in\{\omega',\omega''\}$. Therefore, for the case of item (1) of Lemma 6.1B, $a^j_{\theta,i}$ on $P^+(x,\omega,\theta)\backslash C_{r,\theta}$ uniquely determines $a^j_{\theta,i}$ on $P^+(x,\omega,\theta)$ (for example, in $L^1_{\text{loc}}(P^+(x,\omega,\theta),\mathcal{M}_{n\times n})$) and for the case of item (2) of Lemma 6.1B, $a^j_{\theta,i}$ on $P^-(x,\omega,\theta)\backslash C_{r,\theta}$ uniquely determines $a^j_{\theta,i}$ on $P^-(x,\omega,\theta)$ (for example, in $L^1_{\text{loc}}(P^-(x,\omega,\theta),\mathcal{M}_{n\times n})$), where $i\in\{1,2,3\}$, $j\in\{-,+\}$.

Lemma 6.2.

(1) *Let the assumptions of item (1) of Lemma 6.1A or item (1) of Lemma 6.1B be valid. Then $a^j_{\theta,i}$, $i=1,2,3$, on $P^+(x,\omega,\theta)$ uniquely determine $\psi^{j,-}_\theta(\cdot,\omega)$ on $P^+(x,\omega,\theta)$, where $j\in\{-,+\}$.*

(2) *Let the assumptions of item (2) of Lemma 6.1A or item (2) of Lemma 6.1B be valid. Then $a^j_{\theta,i}$, $i=1,2,3$, on $P^-(x,\omega,\theta)$ uniquely determine $\psi^{j,+}_\theta(\cdot,\omega)$ on $P^-(x,\omega,\theta)$, where $j\in\{-,+\}$.*

Proof of Lemma 6.2. Under the assumptions of items (1) or (2) of Lemma 6.2, using (5.23), (5.24), (5.25 a), (5.25 b) we obtain that

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}+v^j_{\theta,y,\omega}(s)\right)\psi^{j,\pm}_{\theta,y,\omega}(s)=0, \tag{6.35}$$

$$\psi^{j,+}_{\theta,y,\omega}(s)\to I\quad\text{as } s\to-\infty, \tag{6.36 a}$$
$$\psi^{j,-}_{\theta,y,\omega}(s)\to I\quad\text{as } s\to+\infty, \tag{6.36 b}$$

where

$$v^j_{\theta,y,\omega}(s)=v^j_\theta(y+s\omega,\omega),\quad \psi^{j,\pm}_{\theta,y,\omega}(s)=\psi^{j,\pm}_\theta(y+s\omega,\omega),\quad y\in\mathbb{R}^3,\quad s\in\mathbb{R}. \tag{6.37}$$

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Item (1) of Lemma 6.2 follows from the following facts:

(i) $Y^+(y,\omega) \subset P^+(x,\omega,\theta)$ for $y \in P^+(x,\omega,\theta)$;

(ii) $a^j_{\theta,i}$, $i=1,2,3$, on $Y^+(y,\omega)$ uniquely determine $v^j_{\theta,y,\omega}$ on $[0,+\infty[$;

(iii) $v^j_{\theta,y,\omega}$ on $[0,+\infty[$ uniquely determines $\psi^{j,-}_{\theta,y,\omega}$ on $[0,+\infty[$ by means of (6.35), (6.36 b);

(iv) $\psi^{j,-}_\theta(y,\omega) = \psi^{j,-}_{\theta,y,\omega}(0)$.

Item (2) of Lemma 6.2 follows from the following facts:

(i) $Y^-(y,\omega) \subset P^-(x,\omega,\theta)$ for $y \in P^-(x,\omega,\theta)$;

(ii) $a^j_{\theta,i}$, $i=1,2,3$, on $Y^-(y,\omega)$ uniquely determine $v^j_{\theta,y,\omega}$ on $]-\infty,0]$;

(iii) $v^j_{\theta,y,\omega}$ on $]-\infty,0]$ uniquely determines $\psi^{j,+}_{\theta,y,\omega}$ on $]-\infty,0]$ by means of (6.35), (6.36 a);

(iv) $\psi^{j,+}_\theta(y,\omega) = \psi^{j,+}_{\theta,y,\omega}(0)$.

Remark 6.7.

(A) If a_i, $i=0,1,2,3$, satisfy the assumptions of item (3) of Proposition 6.2, then $v^j_{\theta,y,\omega}(s) = O(|s|^{-\varepsilon})$ as $|s| \to \infty$ (since $\omega \notin \{-\theta,\theta\}$). Therefore, for this case the proof of the uniqueness of $\psi^{j,\pm}_{\theta,y,\omega}$ defined by means of (6.35), (6.36) is standard for $\varepsilon > 1$ and consists of the following for $\varepsilon \leqslant 1$. If some $\psi^{j,\pm}_{\theta,y,\omega}$ satisfy (6.35), then

$$\psi^{j,\pm}_{\theta,y,\omega}(s) = \psi^{j,\pm}_\theta(y+s\omega,\omega)A^\pm, \quad A^\pm \in \mathcal{M}_{n\times n}, \tag{6.38}$$

since $\psi^{j,\pm}_{\theta,y,\omega}$ defined by means of (6.37) satisfy (6.35) and $\det \psi^{j,\pm}_\theta(y+s\omega,\omega) \neq 0$. Using (6.36), (6.38) one can show that $A^\pm = I$.

(B) If a_i, $i=0,1,2,3$, satisfy the assumptions of item (4) of Proposition 6.2, then $v^j_{\theta,y,\omega}(s) \equiv 0$ as $|s| \to \infty$ (since $\omega \notin \{-\theta,\theta\}$) and for this case the proof of the uniqueness of $\psi^{j,\pm}_{\theta,y,\omega}$ defined by means of (6.35), (6.36) is standard.

The proof of Lemma 6.2 is completed. □

Lemma 6.3.

(1) *Let the assumptions of item (1) of Lemma 6.2 be valid. Then*

$$S \text{ on } \Omega_2(B_\tau,\omega) \cup \Omega_4(B_\tau,\omega) \quad \text{and} \quad \psi^{j,-}_\theta(\cdot,\omega) \text{ on } P^+(x,\omega,\theta)$$

uniquely determine $\psi^{j,+}_\theta(\cdot,\omega)$ *on* $P^+(x,\omega,\theta)$, *where* $j \in \{-,+\}$.

(2) *Let the assumptions of item (2) of Lemma 6.2 be valid. Then*

$$S \text{ on } \Omega_2(B_\tau,\omega) \cup \Omega_4(B_\tau,\omega) \quad \text{and} \quad \psi^{j,+}_\theta(\cdot,\omega) \text{ on } P^-(x,\omega,\theta)$$

uniquely determine $\psi^{j,-}_\theta(\cdot,\omega)$ *on* $P^-(x,\omega,\theta)$, *where* $j \in \{-,+\}$.

Proof of Lemma 6.3. From (5.25 c) it follows that

$$\psi_\theta^{j,-}(x,\omega)S(x,\omega)=\psi_\theta^{j,+}(x,\omega). \tag{6.39}$$

The formulae (6.39), (3.25), (3.32) and definitions imply Lemma 6.3. The proof is completed. □

Lemma 6.4.

(1) *Let the assumptions of item (1) of Lemma 6.2 be valid. Then $\psi_\theta^{j,k}(\cdot,\omega)$ and $a_{\theta,i}^j(\cdot)$, $i=1,2,3$, on $P^+(x,\omega,\theta)$ uniquely determine*

$$\sum_{i=1}^3 a_{\omega,i}^k(y)\,\mathrm{d}y_i\big|_{P^+(x,\omega,\theta)} = (\psi_\theta^{j,k}(y,\omega))^{-1}\left(\left(\sum_{i=1}^3 a_{\theta,i}^j(y)\,\mathrm{d}y_i\right)\psi_\theta^{j,k}(y,\omega)+\sum_{i=1}^3\frac{\partial\psi_\theta^{j,k}(y,\omega)}{\partial y_i}\,\mathrm{d}y_i\right)\Bigg|_{P^+(x,\omega,\theta)}, \tag{6.40 a}$$

where $j,k\in\{-,+\}$.

(2) *Let the assumptions of item (2) of Lemma 6.2 be valid. Then $\psi_\theta^{j,k}(\cdot,\omega)$ and $a_{\theta,i}^j(\cdot)$, $i=1,2,3$, on $P^-(x,\omega,\theta)$ uniquely determine*

$$\sum_{i=1}^3 a_{\omega,i}^k(y)\,\mathrm{d}y_i|_{P^-(x,\omega,\theta)} = (\psi_\theta^{j,k}(y,\omega))^{-1} \times\left(\left(\sum_{i=1}^3 a_{\theta,i}^j(y)\,\mathrm{d}y_i\right)\psi_\theta^{j,k}(y,\omega)+\sum_{i=1}^3\frac{\partial\psi_\theta^{j,k}(y,\omega)}{\partial y_i}\,\mathrm{d}y_i\right)\Bigg|_{P^-(x,\omega,\theta)}, \tag{6.40 b}$$

where $j,k\in\{-,+\}$.

Proof of Lemma 6.4. Using (5.19) and that $\psi_0^\pm\equiv\psi^\pm$ on $\mathbb{R}^d\times\mathbb{S}^{d-1}$ for $a_0\equiv 0$ on $\mathbb{R}^d$ we obtain that, under assumptions of item (3) or (4) of Proposition 6.2

$$a_{\omega,i}^k(y)=\left(\psi_\theta^{j,k}(y,\omega))^{-1}(a_{\theta,i}^j(y)\psi_\theta^{j,k}(y,\omega)+\frac{\partial\psi_\theta^{j,k}(y,\omega)}{\partial y_i}\right), \tag{6.41}$$

where $y\in\mathbb{R}^3$, $\theta,\omega\in\mathbb{S}^2$, $j,k\in\{-,+\}$, $i\in\{1,2,3\}$. The formula (6.41) implies (6.40) and Lemma 6.4. The proof is completed. □

Lemmas 6.1–6.3 and Statement 6.1 imply Statement 6.3. The proof of Statement 6.3 is completed.

Statements 6.1 and 6.2 imply items (3) and (4) of Proposition 6.2. The proof of Proposition 6.2 is completed.

Let

$$a_{\tau,i}(x) = \chi_+(\tau - |x|)a_i(x), \quad x \in \mathbb{R}^d, \quad \tau > 0, \quad i = 0, 1, \dots, d, \tag{6.42 a}$$

and

$$S_{(\tau)} \text{ denote the scattering matrix for the collection } a_{(\tau)} = (a_{\tau,0}, \dots, a_{\tau,d}). \tag{6.42 b}$$

Let

$$a^{\pm}_{\omega,\tau,i}(x) = \chi_+(\tau - |x|)a^{\pm}_{\omega,i}(x), \quad x \in \mathbb{R}^d, \quad \omega \in \mathbb{S}^{d-1}, \quad \tau > 0, \quad i = 0, 1, \dots, d, \tag{6.43 a}$$

and

$$S^{\pm}_{\omega,(\tau)} \text{ denote the scattering matrix for the collection } a^{\pm}_{\omega,(\tau)} = (a^{\pm}_{\omega,\tau,0}, a^{\pm}_{\omega,\tau,1}, \dots, a^{\pm}_{\omega,\tau,d}). \tag{6.43 b}$$

Proposition 6.3. *Let a_i, $i = 0, 1, \dots, d$, satisfy (3.4 a) or (3.4 b). Let $\tau > 0$. Then the following are valid.*

(1) *τ, S on $\Omega_1(B_\tau)$ and a_i, $i = 0, 1, \dots, d$, on $\mathbb{R}^d \backslash B_\tau$ uniquely determine $S_{(\tau)}$ on $T\mathbb{S}^{d-1}$ by the formulae:*

$$S_{(\tau)}(\gamma) = I \quad \text{for } \gamma \in \Omega_3(B_\tau); \tag{6.44 a}$$

$$S_{(\tau)}(\gamma) = \psi^-(x + \sqrt{\tau^2 - x^2}\theta, \theta)S(\gamma)(\psi^+(x - \sqrt{\tau^2 - x^2}\theta, \theta))^{-1} \tag{6.44 b}$$

for $\gamma = (x, \theta) \in \Omega_1(B_\tau)$, where for $\psi^{\pm}$ the formulae (3.13) hold.

(2) *For $\alpha \geqslant 1$, τ, S on $\Omega_1(B_\tau)$ and $a^j_{\omega,i}$, $i = 0, 1, \dots, d$, on $\mathbb{R}^d \backslash B_\tau$ uniquely determine $S^j_{\omega,(\tau)}$ on $T\mathbb{S}^{d-1} \backslash (\Omega_2(B_\tau, \omega) \cup \Omega_2(B_\tau, -\omega))$ (where $j \in \{-, +\}$, $\omega \in \mathbb{S}^{d-1}$) by the formulae:*

$$S^j_{\omega,(\tau)}(\gamma) = I \quad \text{for } \gamma \in \Omega_3(B_\tau); \tag{6.45 a}$$

$$S^j_{\omega,(\tau)}(\gamma) = \psi^{j,-}_{\omega}(x + \sqrt{\tau^2 - x^2}\theta, \theta)S(\gamma)(\psi^{j,+}_{\omega}(x - \sqrt{\tau^2 - x^2}\theta, \theta))^{-1} \tag{6.45 b}$$

for $\gamma = (x, \theta) \in \Omega_1(B_\tau) \backslash (\Omega_2(B_\tau, \omega) \cup \Omega_2(B_\tau, -\omega))$, where

$$\left.\begin{aligned} \psi^{j,-}_{\omega}(x + s\theta, \theta) &= \left(T \exp \int_s^{+\infty} -v^j_{\omega}(x + t\theta, \theta)\, \mathrm{d}t\right)^{-1}, \\ \psi^{j,+}_{\omega}(x + s\theta, \theta) &= T \exp \int_{-\infty}^{s} -v^j_{\omega}(x + t\theta, \theta)\, \mathrm{d}t \end{aligned}\right\} \tag{6.46}$$

for $s \in \mathbb{R}$ and $(x, \theta) \in T\mathbb{S}^{d-1}$, $\theta \notin \{-\omega, \omega\}$.

The next remark is similar to Remark 6.7.

Remark 6.8.

(A) If a_i, $i = 0, 1, \dots, d$, satisfy (3.4 *a*), $\alpha = 1$, then $v^j_\omega(x + t\theta, \theta) = O(|t|^{-\varepsilon})$ as $|t| \to \infty$ ($\theta \notin \{-\omega, \omega\}$). Therefore, for this case the justification of the fact that the right-hand sides of (6.46) are well defined is standard for $\varepsilon > 1$ and consists of the following for $\varepsilon \leqslant 1$. There are the formulae:

$$\left.\begin{aligned} T\exp\int_s^r -v^j_\omega(x+t\theta,\theta)\,\mathrm{d}t\,\psi^{j,-}_\omega(x+s\theta,\theta) &= \psi^{j,-}_\omega(x+r\theta,\theta),\\ \psi^{j,-}_\omega(x+r\theta,\theta)\to I \quad \text{as } r\to+\infty, &\quad \theta\notin\{-\omega,\omega\}; \end{aligned}\right\} \tag{6.47 a}$$

$$\left.\begin{aligned} T\exp\int_r^s -v^j_\omega(x+t\theta,\theta)\,\mathrm{d}t\,\psi^{j,+}_\omega(x+r\theta,\theta) &= \psi^{j,+}_\omega(x+s\theta,\theta),\\ \psi^{j,+}_\omega(x+r\theta,\theta)\to I \quad \text{as } r\to-\infty, &\quad \theta\notin\{-\omega,\omega\}. \end{aligned}\right\} \tag{6.47 b}$$

These formulae imply the formulae (6.46) together with the fact that the right-hand sides of (6.46) are well defined for $\varepsilon > 0$.

(B) If a_i, $i = 0, 1, \dots, d$, satisfy (3.4 *b*), $\alpha = 1$, then $v^j_\omega(x + t\theta, \theta) \equiv 0$ as $|t| \to \infty$ ($\theta \notin \{-\omega, \omega\}$) and for this case the justification of the fact that the right-hand sides of (6.46) are well-defined is standard (in addition, the formulae (6.47) are also valid).

Proof of Proposition 6.3. The formula (6.44 *a*) follows from definitions. The formula (6.44 *b*) follows from (3.13) and the formula

$$\begin{aligned} S(\gamma) = T\exp\int_{\sqrt{\tau^2-x^2}}^{+\infty} -v(x+t\theta,\theta)\,\mathrm{d}t\,S_{(\tau)}(\gamma)&\\ \times T\exp\int_{-\infty}^{-\sqrt{\tau^2-x^2}} -v(x+t\theta,\theta)\,\mathrm{d}t&\\ \text{for } \gamma=(x,\theta)\in\Omega_1(B_\tau),& \end{aligned} \tag{6.48}$$

where (6.48) follows from (3.12) (for S and $S_{(\tau)}$).

The formula (6.45 *a*) follows from definitions. The formula (6.45 *b*) follows from (6.46) and the formula

$$\begin{aligned} S(\gamma) = T\exp\int_{\sqrt{\tau^2-x^2}}^{+\infty} -v^j_\omega(x+t\theta,\theta)\,\mathrm{d}t\,S^j_{\omega,(\tau)}(\gamma)&\\ \times T\exp\int_{-\infty}^{-\sqrt{\tau^2-x^2}} -v^j_\omega(x+t\theta,\theta)\,\mathrm{d}t&\\ \text{for } \gamma=(x,\theta)\in\Omega_1(B_\tau),\quad \theta\notin\{-\omega,\omega\},& \end{aligned} \tag{6.49}$$

where (6.46) follows from (6.47), to obtain (6.49) we use, in particular, (5.25 *c*) (the formula (6.49) is similar to (6.48)). The proof of Proposition 6.3 is completed. □

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Remark 6.9.

(A) If a_i, $i = 0, 1, \ldots, d$, satisfy (3.4 a), $\alpha > 1$, then (due to the properties (5.27 a), (5.30 a), the definitions (6.43) and Proposition 3.1B) $S^{\pm}_{\omega,(\tau)}$ is continuous on $T\mathbb{S}^{d-1}$ and, as a corollary, for $d \geqslant 2$, $S^{\pm}_{\omega,(\tau)}$ on $T\mathbb{S}^{d-1}\backslash(\Omega_2(B_\tau,\omega) \cup \Omega_2(B_\tau,-\omega))$ uniquely determines $S^{\pm}_{\omega,(\tau)}$ on $T\mathbb{S}^{d-1}$.

(B) If a_i, $i = 0, 1, \ldots, d$, satisfy (3.4 b), $\alpha > 1$, then for $\tau < r$ (due to the properties (5.28 c), (5.31 b), the definitions (6.43) and Proposition 3.1B) $S^{\pm}_{\omega,(\tau)}$ is continuous on $T\mathbb{S}^{d-1}$ and, as a corollary, for $d \geqslant 2$, $S^{\pm}_{\omega,(\tau)}$ on $T\mathbb{S}^{d-1}\backslash(\Omega_2(B_\tau,\omega) \cup \Omega_2(B_\tau,-\omega))$ uniquely determines $S^{\pm}_{\omega,(\tau)}$ on $T\mathbb{S}^{d-1}$.

Proof of Theorem 6.1 for $d = 3$. Proof of items (1), (2) of Theorem 6.1 for $d = 3$. For $d = 3$, item (1) of Theorem 6.1 follows from item (1) of Proposition 6.2, item (1) of Proposition 6.3 and item (2) of Theorem 6.1.

For $d = 3$, item (2) of Theorem 6.1 follows from item (2) of Proposition 6.2 and item (1) of Proposition 6.3 by the induction method. The step of the induction (the jth step, $j \in \mathbb{N}$) consists of the following.

(a) Due to item (2) of Proposition 6.2, $S_{(\tau_{j-1})}$ on $\Omega_3(B_{\tau_j})$ uniquely determines $a_{\tau_{j-1},0}$ on $\mathbb{R}^3\backslash B_\tau$ (i.e. a_0 on $B_{\tau_{j-1}}\backslash B_{\tau_j}$), where

$$\tau_j = \chi_+(\tau_{j-1}^2 - z_b^2)\sqrt{\tau_{j-1}^2 - z_b^2}, \qquad \tau_{j-1}^2 = r_1^2 - (j-1)z_b^2$$

(where r_1, z_b are the numbers of item (2) of Proposition 6.2).

(b) Due to item (1) of Proposition 6.3, $S_{(\tau_{j-1})}$ on $\Omega_1(B_{\tau_j})$ and a_0 on $B_{\tau_{j-1}}\backslash B_{\tau_j}$ uniquely determine $S_{(\tau_j)}$ on $T\mathbb{S}^2$. If $\tau_j = 0$, then the reconstruction of a_0 on $\mathbb{R}^3$ from S on $T\mathbb{S}^2$ is completed by the part (a) of the jth step.

Only a finite number of steps is necessary. Items (1), (2) of Theorem 6.1 for $d = 3$ are proved.

Proof of items (3), (4) of Theorem 6.1 for $d = 3$. For $d = 3$, items (3), (4) of Theorem 6.1 follows from items (3), (4) of Proposition 6.2, item (2) of Proposition 6.3, Remark 6.9, the formula (5.22) and the following statement.

Statement 6.4. *Let $d = 3$. Let $a_0 \equiv 0$, a_i, $i = 1, 2, 3$, satisfy (3.4 b), $\alpha = 1$, and $\sum_{i=1}^3 \omega_i a_i \equiv 0$ for some fixed $\omega \in \mathbb{S}^2$. Then S on $T\mathbb{S}^2$ and r of (3.4 b) uniquely determine a_i, $i = 1, 2, 3$, on $\mathbb{R}^3$.*

Proof of Statement 6.4. Statement 6.4 follows from item (4) of Proposition 6.2, the formulae (5.34), (5.35) and item (1) of Proposition 6.3 by the induction method. The step of the induction (the jth step, $j \in \mathbb{N}$) consists of the following.

(a) Due to item (4) of Proposition 6.2 and the formulae (5.34), (5.35), $S_{(\tau_{j-1})}$ on $T\mathbb{S}^2$ and τ_{j-1} uniquely determine a_i, $i = 1, 2, 3$, on $B_{\tau_{j-1}}\backslash B_{\tau_j}$, where

$$\tau_j = \chi_+(\tau_{j-1}^2 - z_b^2)\sqrt{\tau_{j-1}^2 - z_b^2}, \qquad \tau_{j-1}^2 = r^2 - (j-1)z_b^2$$

(where z_b is the number of item (4) of Proposition 6.2).

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(b) Due to item (1) of Proposition 6.3, $S_{(\tau_{j-1})}$ on $\Omega_1(B_{\tau_j})$ and a_i, $i = 1,2,3$, on $B_{\tau_{j-1}} \backslash B_{\tau_j}$ uniquely determine $S_{(\tau_j)}$ on $T\mathbb{S}^2$. If $\tau_j = 0$, then the reconstruction of a_i, $i = 1,2,3$, on $\mathbb{R}^3$ from S on $T\mathbb{S}^2$ is completed by the part (a) of the jth step. Only a finite number of steps is necessary. Statement 6.4 is proved. Items (3), (4) of Theorem 6.1 for $d = 3$ are proved.

The proof of Theorem 6.1 for $d = 3$ is completed. □

Proof of Theorem 6.1 for $d > 3$. Proof of items (1), (2) of Theorem 6.1 for $d > 3$. To determine $a_0(x')$ at a point $x' \in \mathbb{R}^d$ we consider in $\mathbb{R}^d$ a three-dimensional plane X containing x'. We consider in $T\mathbb{S}^{d-1}$ the subset $T\mathbb{S}^2(X)$, which is the set of all rays lying in X. We restrict S on $T\mathbb{S}^2(X)$. Due to items (1), (2) of Theorem 6.1 for $d = 3$, these data uniquely determines $a_0(x')$. Items (1), (2) of Theorem 6.1 for $d = 3$ are proved.

Proof of items (3), (4) of Theorem 6.1 for $d > 3$. To determine $a^{\pm}_{\omega,i}(x')$, $i = 1,\dots,d$, at a point $x' \in \mathbb{R}^d$ (for the case of item (4) we suppose that $x' \notin C_{r,\omega}$) for fixed $\omega \in \mathbb{S}^{d-1}$ we consider in $\mathbb{R}^d$ three-dimensional planes X_i, $i = 1,\dots,[d/2]$ (where $[d/2]$ is the integer part of $d/2$) such that:

$$Y(x',\omega) = \{x \in \mathbb{R}^d \mid x = x' + s\omega,\ s \in \mathbb{R}\} \subset X_i, \quad i = 1,\dots,[d/2];$$

the convex hull of $\cup_{i=1}^{[d/2]} X_i$ is $\mathbb{R}^d$. For each $i \in \{1,\dots,[d/2]\}$ we consider in $T\mathbb{S}^{d-1}$ the subset $T\mathbb{S}^2(X_i)$, which is the set of all rays lying in X_i. For each $i \in \{1,\dots,[d/2]\}$ we restricted S on $T\mathbb{S}^2(X_i)$ (for the case of item (4) we consider also r_i defined as the radius of the ball $B_r \cap X_i$). Due to items (3), (4) of Theorem 6.1 for $d = 3$, these data uniquely determine $a^{\pm}_{\omega}(x')\xi = \sum_{j=1}^{d} a^{\pm}_{\omega,j}(x')\xi_j$ for any $\xi \in T_{x'}X_i$ (the tangent space to X_i at x'). If for each $i \in \{1,\dots,[d/2]\}$ and each $\xi \in T_{x'}X_i$ the product $a^{\pm}_{\omega}(x')\xi$ is known, then $a^{\pm}_{\omega,j}(x')$, $j = 1,\dots,d$, are known. Items (3), (4) of Theorem 6.1 for $d > 3$ are proved. The proof of Theorem 6.1 for $d > 3$ is completed. □

Using items (3), (4) of Theorem 6.1 we obtain the following corollary.

Corollary 6.1. *Let the assumptions of item (3) for fixed ε in (3.4 a) or (4) for fixed r in (3.4 b) of Theorem 6.1 be valid for a collection $a = (a_0, a_1, \dots, a_d)$ and for a collection $a' = (a'_0, a'_1, \dots, a'_d)$. Let the scattering matrix S for a coincides on $T\mathbb{S}^{d-1}$ with the scattering matrix for a'. Then*

$$\psi'^{\pm}(x,\theta) = (h(x))^{-1}\psi^{\pm}(x,\theta), \quad \theta \in \mathbb{S}^{d-1}, \tag{6.50 a}$$

$$a'_i(x) = (h(x))^{-1}a_i(x)h(x) + (h(x))^{-1}\left(\frac{\partial}{\partial x_i}\right)h(x), \quad i = 1,\dots,d, \tag{6.50 b}$$

for

$$h(x) = (\psi'^{\pm}(x,\omega))^{-1}\psi^{\pm}(x,\omega), \quad x \in \mathbb{R}^d, \quad \omega \in \mathbb{S}^{d-1} \tag{6.50 c}$$

(where $'$ denotes the correspondence to a'). In addition: under assumptions of item (3) of Theorem 6.1 for fixed ε in (3.4 a),

$$h - I, \quad h^{-1} - I \in C^{3,\varepsilon}(\mathbb{R}^d, \mathcal{M}_{n\times n}), \tag{6.51 a}$$

$$\partial_i h \in C^{2,1+\varepsilon}(\mathbb{R}^d, \mathcal{M}_{n\times n}), \quad i = 1,\dots,d; \tag{6.51 b}$$

under assumptions of item (4) of Theorem 6.1 for fixed r in (3.4 b),

$$h, h^{-1} \in C^3(\bar{B}_r, \mathcal{M}_{n\times n}), \tag{6.52 a}$$

$$h \equiv I \quad \text{on } \mathbb{R}^d \backslash B_r. \tag{6.52 b}$$

If, in addition, a_i, a'_i, $i = 1, \dots, d$, take values in $u(n)$, then h takes values in $U(n)$.

To obtain (6.50) we use items (3), (4) of Theorem 6.1, the formulae (5.19) (and, for example, the fact that $a^+_{\omega,i}$, $i = 0, 1, \dots, d$, uniquely determine $\psi^{+,\pm}_\omega$).

Using (3.25) for $\psi^\pm$, (3.26 a), (6.50 c) for different values of ω we obtain that

$$h - I, \quad h^{-1} - I \in C^{0,\varepsilon}(\mathbb{R}^d, \mathcal{M}_{n\times n}). \tag{6.53}$$

Using (3.4 a), (6.50 b) and (6.53) we obtain (6.51).

Using (3.32) for $\psi^\pm$, (6.50 c) we obtain that

$$h, h^{-1} \in C(\mathbb{R}^d, \mathcal{M}_{n\times n}). \tag{6.54}$$

Using (3.4 b), (6.50 b) and (6.54) we obtain (6.52 a).

Using (3.33 a), (6.50 c) for all $\omega \in \mathbb{S}^{d-1}$ we obtain (6.52 b).

The final statement of Corollary 6.1 follows from the formula (6.50 c) and Statement 3.1.

Note now that Theorem 6.1 admits the following generalization.

Theorem 6.2. *For $d \geqslant 3$, we have the following results.*

(A) *Let a_i, $i = 0, 1, \dots, d$, satisfy (3.4 a), $\alpha = 2$. Then S on $T\mathbb{S}^{d-1}$ uniquely determines $a^\pm_{\omega,i}$, $i = 0, 1, \dots, d$, on $\mathbb{R}^d$ for any $\omega \in \mathbb{S}^{d-1}$.*

(B) *Let a_i, $i = 0, 1, \dots, d$, satisfy (3.4 b), $\alpha = 2$. Then S on $T\mathbb{S}^{d-1}$* and *r of (3.4 b) uniquely determine $a^\pm_{\omega,i}$, $i = 0, 1, \dots, d$, on $\mathbb{R}^d$ for any $\omega \in \mathbb{S}^{d-1}$.*

To obtain Theorem 6.2 we consider

$$\tilde{a}^\pm_{\omega,i}(x) = (\psi^\pm(x,\omega))^{-1} a_i(x) \psi^\pm(x,\omega) + (\psi^\pm(x,\omega))^{-1} \left(\frac{\partial}{\partial x_i}\right) \psi^\pm(x,\omega), \quad i = 1, \dots, d,$$

$$\tilde{a}^\pm_{\omega,0}(x) = (\psi^\pm(x,\omega))^{-1} a_0(x) \psi^\pm(x,\omega),$$

where $x \in \mathbb{R}^d$, $\omega \in \mathbb{S}^{d-1}$, $\psi^\pm$ are the wave functions for the equation (3.1).

We obtain, first, the following result.

Proposition 6.4. *For $d \geqslant 3$, the following statements are valid.*

(A) *Let a_i, $i = 0, 1, \dots, d$, satisfy (3.4 a), $\alpha = 2$. Then S on $T\mathbb{S}^{d-1}$ uniquely determines $\tilde{a}^\pm_{\omega,i}$, $i = 0, 1, \dots, d$, on $\mathbb{R}^d$ for any $\omega \in \mathbb{S}^{d-1}$.*

(B) *Let a_i, $i = 0, 1, \dots, d$, satisfy (3.4 b), $\alpha = 2$. Then S on $T\mathbb{S}^{d-1}$ and (3.4 b) uniquely determine $\tilde{a}^\pm_{\omega,i}$, $i = 0, 1, \dots, d$, on $\mathbb{R}^d$ for any $\omega \in \mathbb{S}^{d-1}$.*

The proof of Proposition 6.4 is similar to the proof of items (3), (4) of Theorem 6.1.

Theorem 6.2 follows from Proposition 6.4 and the fact that, under the assumptions in question, $\tilde{a}^s_{\omega',i}$, $i = 0, 1, \ldots, d$, on $\mathbb{R}^d$ for fixed $\omega' \in \mathbb{S}^{d-1}$ and $s \in \{+, -\}$ uniquely determine $a^{\pm}_{\omega,i}$, $i = 0, 1, \ldots, d$, on $\mathbb{R}^d$ for any $\omega \in \mathbb{S}^{d-1}$.

Using Theorem 6.2 we obtain the following generalization of Corollary 6.1.

Corollary 6.2. *Let the assumptions of item (A) for fixed ε in (3.4 a) or (B) for fixed r in (3.4 b) of Theorem 6.2 be valid for $a = (a_0, a_1, \ldots, a_d)$ and $a' = (a'_0, a'_1, \ldots, a'_d)$. Let the scattering matrix S for a coincides on $T\mathbb{S}^{d-1}$ with the scattering matrix for a'. Then the formulae (6.50) hold and, in addition, $a'_0 = h^{-1} a_0 h$. In addition: under assumptions of item (A) of Theorem 6.2 for fixed ε in (3.4 a), the formulae (6.51) hold; under assumptions of item (B) of Theorem 6.2 for fixed r in (3.4 b), the formulae (6.52) hold. If, in addition, a_i, a'_i, $i = 0, 1, \ldots, d$, take values in $u(n)$, then h takes values in $U(n)$.*

7. Non-trivial transparent $SU(2)$-connections in dimension $d = 2$

The equation (1.3) for $d = 2$ for any fixed complexified $\theta \in \Sigma = \{\theta \in \mathbb{C}^2 \mid \theta^2 = 1\}$, $\theta_2 \neq 0$, can be written in the form

$$(\zeta \nabla_1 - \nabla_2)\varphi(x, \zeta) = 0, \quad x \in \mathbb{R}^2, \quad \zeta \in \mathbb{C}, \tag{7.1}$$

where $\varphi(x, \zeta) = \psi(x, \theta)$, $\zeta = -\theta_1/\theta_2$. Due to [**19**], we have the following statement.

Let $\mu \in \mathbb{C}\backslash\mathbb{R}$ and f be a function of $x \in \mathbb{R}^2$ of the form

$$f(x) = p(z)/q(z), \quad \text{where } p, q \text{ are polynomials of } z = x_1 + \mu x_2, \tag{7.2}$$

where $q \not\equiv 0$. Let

$$\varphi(\cdot, \zeta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\zeta - \mu} \frac{\mu - \bar{\mu}}{1 + f\bar{f}} \begin{pmatrix} 1 & f \\ \bar{f} & f\bar{f} \end{pmatrix}, \quad \zeta \in \mathbb{C}, \tag{7.3}$$

$$a_1 \equiv 0, \qquad a_2 = \partial_1 \left(\frac{\mu - \bar{\mu}}{1 + f\bar{f}} \begin{pmatrix} 1 & f \\ \bar{f} & f\bar{f} \end{pmatrix} \right), \tag{7.4}$$

where $\partial_1 = \partial/\partial x_1$. Then the equation (7.1) (with φ, a_1, a_2 given by (7.3), (7.4)) is fulfilled for all $x \in \mathbb{R}^2$ and $\zeta \in \mathbb{C}$ and, in addition,

$$\left.\begin{aligned} \varphi(x, \bar{\zeta})^* = \varphi(x, \zeta)^{-1}, \quad \det \varphi(x, \zeta) = (\zeta - \bar{\mu})/(\zeta - \mu), \quad x \in \mathbb{R}^2, \quad \zeta \in \mathbb{C}, \\ a_2 \in C^{\infty,1}(\mathbb{R}^2, su(2)) \quad (\text{i.e. } a_2 \in C^{m,1}(\mathbb{R}^2, su(2)) \ \forall m \in \mathbb{N}), \end{aligned}\right\} \tag{7.5}$$

where $*$ denotes complex conjugate transpose.

Note that if, in addition,

$$l = \deg p - \deg q > 0, \tag{7.6}$$

then

$$\lim_{|x| \to \infty} \varphi(x, \zeta) = c(\zeta) = \begin{pmatrix} 1 & 0 \\ 0 & (\zeta - \bar{\mu})/(\zeta - \mu) \end{pmatrix}, \quad \zeta \in \mathbb{C}\backslash\mu, \tag{7.7}$$

$$a_2 \in C^{\infty, l+1}(\mathbb{R}^2, su(2)) \quad (\text{i.e. } a_2 \in C^{m, l+1}(\mathbb{R}^2, su(2)) \ \forall m \in \mathbb{N}). \tag{7.8}$$

Let

$$\chi(x,\zeta)=\varphi(x,\zeta)(c(\zeta))^{-1}, \tag{7.9}$$

where φ, c are given by (7.3), (7.7). Then the equation (7.1) with χ in place of φ and with a_1, a_2 given by (7.4) is fulfilled for all $x\in\mathbb{R}^2$ and $\zeta\in\mathbb{C}$ and, in addition,

$$\chi(\cdot,\zeta)=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{1}{s-\mu}\frac{\mu-\bar{\mu}}{}1+f\bar{f}\begin{pmatrix}1&0\\\bar{f}&0\end{pmatrix}+\frac{1}{\zeta-\bar{\mu}}\frac{\mu-\bar{\mu}}{1+f\bar{f}}\frac{1}{\zeta-\bar{\mu}}\frac{\mu-\bar{\mu}}{1+f\bar{f}}\begin{pmatrix}0&f\\0&-1\end{pmatrix}, \tag{7.10}$$

$$\chi(x,\bar{\zeta})^*=\chi(x,\zeta)^{-1},\quad \det\chi(x,\zeta)=1, \tag{7.11}$$

where $x\in\mathbb{R}^2$, $\zeta\in\mathbb{C}$.

For χ from the beginning defined by (7.10), it is a statement contained in [**18**] that the equation (7.1) with χ in place of φ and with a_1, a_2 given by (7.4) is fulfilled for all $x\in\mathbb{R}^2$ and $\zeta\in\mathbb{C}$. It seems, however, that the formula (7.9) relating a construction of [**19**] and a construction of [**18**] is new.

If (7.6) holds, then

$$\chi(x,\zeta)=I+O(|x|^{-l})\quad\text{as } |x|\to\infty \tag{7.12}$$

for any $\zeta\in\mathbb{C}\backslash(\mu\cup\bar{\mu})$ and uniformly in $\zeta\in\mathbb{R}$, where I is the 2×2 identity matrix and we consider $O(|x|^{-l})$ using the norm (2.6).

Let

$$\psi(x,\theta)=\chi(x,-\theta_1/\theta_2),\quad x\in\mathbb{R}^2,\quad \theta\in\Sigma=\{\theta\in\mathbb{C}^2\mid\theta^2=1\}, \tag{7.13}$$

where we assume that $\chi(x,\infty)=I$. Let (7.6) hold. Then the equation (1.3), $d=2$, with a_1, a_2, ψ given by (7.4), (7.13) is fulfilled for all $x\in\mathbb{R}^2$ and $\theta\in\Sigma$ and, in addition,

$$\psi(x,\theta)=I+O(|x|^{-l})\quad\text{as } |x|\to\infty \tag{7.14}$$

uniformly in $\theta\in\mathbb{S}^1$. Therefore, the considerations given in this section imply the following result.

Theorem 7.1. *For the equation (1.3), $d=2$, with a_1, a_2 given by (7.4), where (7.6) holds, the following is valid:*

(1) *the wave functions $\psi^{\pm}\equiv\psi$, where ψ is given by (7.10), (7.13) for $\theta\in\mathbb{S}^1$;*

(2) *the scattering matrix $S\equiv I$;*

(3) *the formulae (7.8), (7.14) hold;*

(4) *the gauge field $a=(a_1,a_2)$ considered up to the gauge transformations (1.6), $d=2$, $g\in C^1(\mathbb{R}^2,GL(2,\mathbb{C}))$, differs from $a'\equiv(0,0)$.*

Remark 7.1. A similar result is valid also if a_1, a_2 are given by (7.4), where $l=\deg p-\deg q<0$.

If $a = (a_1, a_2)$ is a gauge field of Theorem 7.1, then $a_1 \equiv 0$, $a_2 \in C^{\infty,l+1}(\mathbb{R}^2, su(2))$ and a generates a $SU(2)$-connection on η (by means of (1.2), $d = 2$), where η is the trivial vector bundle over $\mathbb{R}^2$ with the fibre $\mathbb{C}^2$. Since, in addition, $S \equiv I$, we say that this connection is transparent. Since, in addition, we have item (4) of Theorem 7.1, we say that this connection is non-trivial.

8. The attenuated X-ray transform as a reduction of the non-abelian Radon transform

Consider the equation (1.9) for the case when

$$n = 2, \quad a_i \equiv 0 \quad \text{for } i = 1, \dots, d, \quad a_0 = \begin{pmatrix} u_1 & v \\ 0 & u_2 \end{pmatrix}, \tag{8.1}$$

where

$$u_1,\ u_2,\ v \text{ are complex-valued sufficiently regular functions on } \mathbb{R}^d \text{ sufficiently rapidly vanishing at infinity,} \tag{8.2}$$

for example,

$$u_1, u_2, v \in C^{\alpha,1+\varepsilon}(\mathbb{R}^d, \mathbb{C}) \quad \text{for some } \alpha > 0 \text{ and } \varepsilon > 0. \tag{8.2$_a$}$$

For this case the equation (1.9) takes the form

$$\psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix},$$

$$\left.\begin{aligned} \theta\partial_x\psi_{11} + u_1(x)\psi_{11} + v(x)\psi_{21} = 0, \\ \theta\partial_x\psi_{12} + u_1(x)\psi_{12} + v(x)\psi_{22} = 0, \\ \theta\partial_x\psi_{21} + u_2(x)\psi_{21} = 0, \\ \theta\partial_x\psi_{22} + u_2(x)\psi_{22} = 0, \end{aligned}\right\} \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}, \tag{8.3}$$

and its solution ψ^+ specified by (1.5) is given by the formulae

$$\left.\begin{aligned} &\psi^+_{11}(x,\theta) = \exp[-D_{-\theta}u_1(x)], \\ &\psi^+_{12}(x,\theta) = -\exp[-D_{-\theta}u_1(x)] \\ &\qquad\qquad \times \int_{-\infty}^{0} \exp[D_{-\theta}u_1(x+t\theta) - D_{-\theta}u_2(x+t\theta)]v(x+t\theta)\,\mathrm{d}t, \\ &\psi^+_{21}(x,\theta) = 0, \\ &\psi^+_{22}(x,\theta) = \exp[-D_{-\theta}u_2(x)], \end{aligned}\right\} \quad \begin{aligned} &x \in \mathbb{R}^d, \\ &\theta \in \mathbb{S}^{d-1}, \end{aligned} \tag{8.4}$$

where D_θ is defined by (3.21). Therefore, under the assumptions (8.1), (8.2), the scattering matrix S for the equation (1.9) takes the form

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \tag{8.5}$$

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where

$$\left.\begin{aligned} S_{11}(\gamma) &= \exp[-Pu_1(\gamma)],\\ S_{12}(\gamma) &= -\exp[-Pu_2(\gamma)]P_{u_1-u_2}v(\gamma),\\ S_{21}(\gamma) &= 0,\\ S_{22}(\gamma) &= \exp[-Pu_2(\gamma)], \end{aligned}\right\} \quad \gamma \in TS^{d-1}, \tag{8.6}$$

where P is the classical X-ray transformation, P_μ is the attenuated X-ray transformation; the transformations P and P_μ are defined by the formulae

$$Pf(\gamma) = \int_{\mathbb{R}} f(x+s\theta)\,\mathrm{d}s, \tag{8.7}$$

$$P_\mu f(\gamma) = \int_{\mathbb{R}} \exp[-D_\theta \mu(x+s\theta)] f(x+s\theta)\,\mathrm{d}s, \tag{8.8}$$

where f and μ are complex-valued sufficiently regular functions on $\mathbb{R}^d$ sufficiently rapidly vanishing at infinity, f is a test function, μ is considered as a parameter (attenuation coefficient), $\gamma = (x,\theta) \in T\mathbb{S}^{d-1}$ presented by (3.17), D_θ is defined by (3.21).

Thus, under the assumptions (8.1), (8.2) the non-abelian Radon transform S of the collection a is reduced to the classical X-ray transforms Pu_1, Pu_2 and to the attenuated X-ray transform $P_{u_1-u_2}v$. The classical X-ray transformation P is a basic transformation of the transmission tomography; the attenuated X-ray transformation P_μ is a basic transformation of the emission tomography (see, for example, [**13**, **15**] and references given there). The theory of the classical X-ray transformation P is well-developed for a long time already (see, for example, [**7**, **13**]). Concerning results given in the literature for the attenuated X-ray transformation P_μ see [**4**, **13**, **15**] and references given there. Explicit inversion formulae for the attenuated X-ray transformation P_μ in dimension $d \geqslant 2$ were obtained only recently in [**15**] using techniques of [**6**, **12**, **14**].

Under the assumptions (8.1), (8.2_a), the formula (8.5), well-known results for the two-dimensional transformation P and the explicit inversion formula of [**15**] for the two-dimensional transformation P_μ imply that the non-abelian Radon transform S uniquely determines a_0.

Appendix A. Estimates for operators

We give, first, estimates for the operators D_θ, P_θ, $P_\theta^\perp$ defined by (3.21), (3.24), (4.23). We use χ_+ defined by (3.5) and π_θ, X_θ defined by (3.16).

Lemma A.1$_a$. *Let*

$$f \in C^{\alpha,1+\varepsilon}(\mathbb{R}^d, \mathcal{M}_{m\times n}), \tag{A.1}$$

$$\|f\|_{0,1+\varepsilon,\rho} \leqslant F_1, \tag{A.2}$$

$$\|f\|'_{\alpha,1+\varepsilon,\rho} \leqslant F_2, \tag{A.3}$$

where $0 < \alpha \leqslant 1$, $\varepsilon > 0$, $\rho > 0$, $m, n \in \mathbb{N}$. *Then*

$$|D_{-\theta} f(x)| \leqslant c_1(\rho,\varepsilon,\theta,x) F_1, \tag{A.4}$$

$$|D_{-\theta} f(x+y) - D_{-\theta} f(x)| \leqslant c_1(\rho,\varepsilon,\theta,x) F_2 |y|^\alpha, \tag{A.5}$$

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$$c_1(\rho,\varepsilon,\theta,x) \overset{\text{def}}{=} \frac{2^{(1+\varepsilon)/2}(1+\chi_+(\theta x))}{\varepsilon(\sqrt{2}\rho+|\pi_\theta x|-\theta x\chi_+(-\theta x))^\varepsilon}, \tag{A.6}$$

$$|D_{-\theta}f(x)-D_{-\theta'}f(x)| \leqslant c_2(\rho,\varepsilon,\beta,\theta,\theta',x)F|\theta-\theta'|^\beta, \tag{A.7}$$

$$c_2(\rho,\varepsilon,\beta,\theta,\theta',x) \overset{\text{def}}{=} \frac{2^{(3+\varepsilon)/2}}{\varepsilon}\left(\frac{3\varepsilon-2\beta}{\varepsilon-\beta}+\frac{3\chi_+(2\max(\theta x,\theta' x)-1)(2\max(\theta x,\theta' x))^\varepsilon}{(\sqrt{2}\rho+\min(|\pi_\theta x|,|\pi_{\theta'}x|))^\varepsilon}\right), \tag{A.8}$$

for $x,y\in\mathbb{R}^d$, $|y|\leqslant 1$, $\theta,\theta'\in\mathbb{S}^{d-1}$, $|\theta-\theta'|\leqslant 1$, $0<\beta<\varepsilon$, $\beta\leqslant\alpha$, $F=\max(F_1,F_2)$.

Remark A.1$_a$. Let (A.1), (A.2) be valid, where $\alpha=0$, $\varepsilon>0$, $\rho>0$, $m,n\in\mathbb{N}$. Then (A.4) holds and $D_{-\theta}f\in C(\mathbb{R}^d,\mathcal{M}_{m\times n})$ for $x\in\mathbb{R}^d$, $\theta\in\mathbb{S}^{d-1}$.

Lemma A.1$_b$. *Let*

$$f(x)=\chi_+(r-|x|)g(x) \quad \text{for } x\in\mathbb{R}^d, \tag{A.9}$$

where

$$g\in C(\mathbb{R}^d,\mathcal{M}_{m\times n}), \quad r>0, \quad m,n\in\mathbb{N},$$
$$|f(x)|\leqslant F_1 \quad \text{for } x\in\mathbb{R}^d \tag{A.10}$$

and

$$\left.\begin{aligned} |f(x+y)-f(x)|\leqslant F_{2,1}|y|^\alpha \quad \text{for fixed } \theta\in\mathbb{S}^{d-1},\\ \text{for } x,y\in\mathbb{R}^d,\quad |x|<r,\quad |x+y|<r,\quad \theta y=0,\quad |y|\leqslant 1,\quad 0<\alpha\leqslant 1.\end{aligned}\right\} \tag{A.11}$$

Then

$$|D_{-\theta}f(x)|\leqslant c_3(|\pi_\theta x|,r)F_1, \tag{A.12}$$

$$c_3(|\pi_\theta x|,r)\overset{\text{def}}{=}2\chi_+(r-|\pi_\theta x|)\sqrt{r^2-|\pi_\theta x|^2}, \tag{A.13}$$

$$|D_{-\theta}f(x+y)-D_{-\theta}f(x)|\leqslant c_4(r)F|y|^\beta, \tag{A.14}$$

$$c_4(r)\overset{\text{def}}{=}2^{3/2}r^{1/2}+2r, \tag{A.15}$$

for $x,y\in\mathbb{R}^d$, $\theta y=0$, $|y|\leqslant 1$, $F=\max(F_1,F_{2,1})$, $\beta=\min(\frac{1}{2},\alpha)$;

$$|D_{-\theta}f(x+y)-D_{-\theta}f(x)|\leqslant F_1|y| \quad \text{for } x,y\in\mathbb{R}^d,\quad \theta\in\mathbb{S}^{d-1},\quad \pi_\theta y=0. \tag{A.16}$$

Let (A.9), (A.10) be valid and, in addition,

$$\left.\begin{aligned} |f(x+y)-f(x)|\leqslant F_2|y|^\alpha\\ \text{for } x,y\in\mathbb{R}^d,\quad |x|<r,\quad |x+y|<r,\quad |y|\leqslant 1,\quad 0<\alpha<1.\end{aligned}\right\} \tag{A.17}$$

Then

$$|D_{-\theta}f(x)-D_{-\theta'}f(x)|\leqslant c_5(r,\alpha)F|\theta-\theta'|^\alpha, \tag{A.18}$$

$$c_5(r,\alpha)\overset{\text{def}}{=}2\max(\pi r,(2r)^{\alpha+1}), \tag{A.19}$$

$x \in \mathbb{R}^d$, $|x| \leqslant r$, $\theta, \theta' \in \mathbb{S}^{d-1}$, $|\theta - \theta'| \leqslant 1$, $F = \max(F_1, F_2)$;

$$|D_{-\theta} f(x) - D_{-\theta'} f(x)| \leqslant c_6(|x|, r, \alpha) F |\theta - \theta'|^\beta, \tag{A.20}$$

$$c_6(|x|, r, \alpha) \stackrel{\text{def}}{=} 2 \max(2^{3/2} r^{1/2} |x|^{1/2}, 2r(|x| + r)^\alpha), \tag{A.21}$$

$x \in \mathbb{R}^d$, $|x| \geqslant r$, $\theta, \theta' \in \mathbb{S}^{d-1}$, $|\theta - \theta'| \leqslant 1$, $F = \max(F_1, F_2)$, $\beta = \min(\frac{1}{2}, \alpha)$.

Remark A.1$_b$. If (A.9), (A.10) are valid, then (A.12) holds and $D_{-\theta} f \in C(\mathbb{R}^d, \mathcal{M}_{m \times n})$ for $x \in \mathbb{R}^d$ and (any) $\theta \in \mathbb{S}^{d-1}$.

Lemma A.2$_a$. *Under the assumptions of Lemma A.1$_a$, the following estimates hold:*

$$|P_\theta f(x)| \leqslant 2c_1(\rho, \varepsilon, \theta, \pi_\theta x) F_1, \tag{A.22}$$

$$|P_\theta f(x + y) - P_\theta f(x)| \leqslant 2c_1(\rho, \varepsilon, \theta, \pi_\theta x) F_2 |y|^\alpha, \tag{A.23}$$

$$|P_\theta f(x) - P_{\theta'} f(x)| \leqslant (c_2(\rho, \varepsilon, \beta, \theta, \theta', x) + c_2(\rho, \varepsilon, \beta, -\theta, -\theta', x)) F |\theta - \theta'|^\beta, \tag{A.24}$$

in addition, for $d = 2$,

$$|P_\theta^\perp f(s)| \leqslant 2c_7(\rho, \varepsilon, s) F_1, \tag{A.25}$$

$$|P_\theta^\perp f(s + \delta) - P_\theta^\perp f(s)| \leqslant 2c_7(\rho, \varepsilon, s) F_2 |\delta|^\alpha, \tag{A.26}$$

$$c_7(\rho, \varepsilon, s) \stackrel{\text{def}}{=} \frac{2^{(1+\varepsilon)/2}}{\varepsilon(\sqrt{2}\rho + |s|)^\varepsilon}, \tag{A.27}$$

$$|P_\theta^\perp f(s) - P_{\theta'}^\perp f(s)| \leqslant c_8(\varepsilon, \beta) F |\theta - \theta'|^\beta, \tag{A.28}$$

$$c_8(\varepsilon, \beta) \stackrel{\text{def}}{=} \frac{2^{(5+\varepsilon)/2}}{\varepsilon} \left(\frac{3\varepsilon - 2\beta}{\varepsilon - \beta} + 3 \left(\frac{2}{\sqrt{3}} \right)^\varepsilon \right), \tag{A.29}$$

where $x, y \in \mathbb{R}^d$, $|y| \leqslant 1$, $\theta, \theta' \in \mathbb{S}^{d-1}$, $|\theta - \theta'| \leqslant 1$, $0 < \beta < \varepsilon$, $\beta \leqslant \alpha$, $F = \max(F_1, F_2)$, $s, \delta \in \mathbb{R}$, $|\delta| \leqslant 1$.

Lemma A.2$_b$. *If (A.9)–(A.11) are valid, then*

$$|P_\theta f(x)| \leqslant c_3(|\pi_\theta x|, r) F_1, \tag{A.30}$$

$$|P_\theta f(x + y) - P_\theta f(x)| \leqslant c_4(r) F |y|^\beta \tag{A.31}$$

for $x, y \in \mathbb{R}^d$, $\theta \in \mathbb{S}^{d-1}$, $|y| \leqslant 1$, $F = \max(F_1, F_{2,1})$, $\beta = \min(\frac{1}{2}, \alpha)$.

If (A.9), (A.10), (A.17) are valid, then

$$|P_\theta f(x) - P_{\theta'} f(x)| \leqslant 2c_5(r, \alpha) F |\theta - \theta'|^\alpha \tag{A.32}$$

for $x \in \mathbb{R}^d$, $|x| \leqslant r$, $\theta, \theta' \in \mathbb{S}^{d-1}$, $|\theta - \theta'| \leqslant 1$, $F = \max(F_1, F_2)$;

$$|P_\theta f(x) - P_{\theta'} f(x)| \leqslant 2c_6(|x|, r, \alpha) F |\theta - \theta'|^\beta \tag{A.33}$$

for $x \in \mathbb{R}^d$, $|x| \geqslant r$, $\theta, \theta' \in \mathbb{S}^{d-1}$, $|\theta - \theta'| \leqslant 1$, $F = \max(F_1, F_2)$, $\beta = \min(\frac{1}{2}, \alpha)$.

In addition:

(i) *if (A.9)–(A.11) are valid for $d = 2$, then*

$$|P_\theta^\perp f(s)| \leqslant c_3(|s|, r)F_1, \tag{A.34}$$
$$|P_\theta^\perp f(s+\delta) - P_\theta^\perp f(s)| \leqslant c_4(r)F|\delta|^\beta, \tag{A.35}$$

$\theta \in \mathbb{S}^1$, $s, \delta \in \mathbb{R}$, $|\delta| \leqslant 1$, $F = \max(F_1, F_{2,1})$, $\beta = \min(\frac{1}{2}, \alpha)$;

(ii) *if (A.9), (A.10), (A.17) are valid for $d = 2$, then*

$$|P_\theta^\perp f(s) - P_{\theta'}^\perp f(s)| \leqslant c_9(|s|, r, \alpha)F|\theta - \theta'|^\beta, \tag{A.36}$$
$$c_9(|s|, r, \alpha) \overset{\text{def}}{=} 16\max(2r, 3r^{1+\alpha}), \tag{A.37}$$

$\theta, \theta' \in \mathbb{S}^1$, $|\theta - \theta'| \leqslant 1$, $s \in \mathbb{R}$, $F = \max(F_1, F_2)$, $\beta = \min(\frac{1}{2}, \alpha)$.

Lemmas A.1 and A.2 are proved in Appendix B.

In the next lemma we present an estimate for the product of two matrix-functions.

Lemma A.3. *Let*

$$\left.\begin{aligned} f &\in C^{\alpha,\sigma}(\mathbb{R}^d_Y, \mathcal{M}_{l\times m}),\\ g &\in C^{\alpha,\tau}(\mathbb{R}^d_Y, \mathcal{M}_{m\times n}),\end{aligned}\right\} \tag{A.38}$$

$0 < \alpha \leqslant 1$, $\sigma \geqslant 0$, $\tau \geqslant 0$, *Y is a non-zero subspace in $\mathbb{R}^d$, $l, m, n \in \mathbb{N}$. Then*

$$fg \in C^{\alpha,\sigma+\tau}(\mathbb{R}^d_Y, \mathcal{M}_{l\times n}), \tag{A.39}$$
$$\|fg\|_{0,\sigma+\tau,\rho} \leqslant m\|f\|_{0,\sigma,\rho}\|g\|_{0,\tau,\rho}, \tag{A.40}$$
$$\|fg\|_{\alpha,(Y),\sigma+\tau,\rho} \leqslant 2^{1+\min(\sigma,\tau)}m\|f\|_{\alpha,(Y),\sigma,\rho}\|g\|_{\alpha,(Y),\tau,\rho}, \tag{A.41}$$

$\rho \geqslant 1$.

This lemma is elementary.

We give now estimates for the operators $(D_{\mp\theta}v_\theta)^p$, $p \in \mathbb{N}$, and $P_\theta v_\theta$, where $D_{\mp\theta}v_\theta$, $P_\theta v_\theta$ are defined by (3.19), (3.20), (3.23), where $v(x,\theta)$ is defined by (3.3).

Lemma A.4$_a$. *Let (3.4 a) be valid and*

$$f \in C^{\alpha,0}(\mathbb{R}^d, \mathcal{M}_{n\times m}) \tag{A.42}$$

where $0 < \alpha \leqslant 1$. Then

$$|(D_{\mp\theta}v_\theta)^p f(x)| \leqslant \frac{1}{p!}(nc_1(\rho, \varepsilon, \pm\theta, x)\|a\|_{0,1+\varepsilon,\rho})^p\|f\|_0, \tag{A.43}$$

$$|(D_{\mp\theta}v_\theta)^p f(x+y) - (D_{\mp\theta}v_\theta)^p f(x)| \leqslant \frac{2^{(1+\varepsilon)(p+1)}}{p!}(nc_1(\rho,\varepsilon,\pm\theta,x)\|a\|_{\alpha,1+\varepsilon,\rho})^p \times \|f\|_{\alpha,0}|y|^\alpha, \quad \text{(A.44)}$$

$$|D_{\mp\theta}v_\theta f(x+y) - D_{\mp\theta}v_\theta f(x)| \leqslant 2nc_1(\rho,\varepsilon,\pm\theta,x)\|a\|_{\alpha,1+\varepsilon,\rho}\|f\|_{\alpha,0}|y|^\alpha, \quad \text{(A.45)}$$

$$\|(D_{\mp\theta}v_\theta)^p f\|_{\alpha,0} \leqslant \frac{2^{(1+\varepsilon)(p+1)}}{p!}\left(\frac{4n\|a\|_{\alpha,1+\varepsilon,\rho}}{\varepsilon\rho^\varepsilon}\right)^p \|f\|_{\alpha,0}, \quad \text{(A.46)}$$

$$|(D_{\mp\theta}v_\theta - D_{\mp\theta'}v_{\theta'})f(x)| \leqslant 2n\left(\frac{2}{\varepsilon\rho^\varepsilon} + c_2(\rho,\varepsilon,\beta,\pm\theta,\pm\theta',x)\right) \times \|a\|_{\alpha,1+\varepsilon,\rho}\|f\|_{\alpha,0}|\theta-\theta'|^\beta, \quad \text{(A.47)}$$

where

$$\|a\|_{\alpha,1+\varepsilon,\rho} = \sum_{i=0}^{d}\|a_i\|_{\alpha,1+\varepsilon,\rho}, \quad \text{(A.48)}$$

for $p \in \mathbb{N}$, $x, y \in \mathbb{R}^d$, $|y| \leqslant 1$, $\theta, \theta' \in \mathbb{S}^{d-1}$, $|\theta-\theta'| \leqslant 1$, $\rho \geqslant 1$, $0 < \beta < \varepsilon$, $\beta \leqslant \alpha$.

Remark A.2$_a$. Let (3.4 a), (A.42) be valid for $\alpha = 0$ and some $\varepsilon > 0$. Then (A.43) holds and $(D_{\mp\theta}v_\theta)^p f \in C^{0,0}(\mathbb{R}^d, \mathcal{M}_{n\times m})$ for $p \in \mathbb{N}$, $\theta \in \mathbb{S}^{d-1}$.

Lemma A.4$_b$. *Let (3.4 b), (A.42) be valid, where* $0 < \alpha \leqslant 1$. *Then*

$$|(D_{\mp\theta}v_\theta)^p f(x)| \leqslant \frac{1}{p!}(n, c_3(|\pi_\theta x|, r)\|b\|_0)^p\|f\|_0, \quad \text{(A.49)}$$

$$\|(D_{\mp\theta}v_\theta)^p f\|_{\beta,(X_\theta),0} \leqslant \left(\frac{(2r)^p}{p!} + \sum_{j=1}^{p}\frac{c_4(r)(2r)^{p-1}}{(j-1)!(p-j)!}\right)(n\|b\|_{\beta,0})^p\|f\|_{\beta,(X_\theta),0}, \quad \text{(A.50)}$$

$$|D_{\mp\theta}v_\theta f(x+y) - D_{\mp\theta}v_\theta f(x)| \leqslant n\|b\|_0\|f\|_0|y|, \qquad \pi_\theta y = 0, \quad \text{(A.51)}$$

$$\|(D_{\mp\theta}v_\theta)^2 f\|_{\beta,0} \leqslant c_{10}(r)(n\|b\|_{\beta,0})^2\|f\|_{\beta,0}, \quad \text{(A.52)}$$

$$c_{10}(r) \stackrel{\text{def}}{=} 4rc_4(r) + 2r^2 + 2r, \quad \text{(A.53)}$$

$$|(D_{\mp\theta}v_\theta - D_{\mp\theta'}v_{\theta'})f(x)| \leqslant 2nc_{11}(|x|, r, \alpha)\|b\|_{\alpha,0}\|f\|_{\alpha,0}|\theta-\theta'|^\beta, \quad \text{(A.54)}$$

$$c_{11}(|x|, r, \alpha) \stackrel{\text{def}}{=} \max(c_5(r,\alpha), c_6(|x|, r, \alpha)) + r, \quad \text{(A.55)}$$

where

$$\|b\|_{\alpha,0} = \sum_{i=0}^{d}\|b_i\|_{\alpha,0}, \qquad \|b\|_0 = \|b\|_{0,0}, \quad \text{(A.56)}$$

for $p \in \mathbb{N}$, $x, y \in \mathbb{R}^d$, $\theta, \theta' \in \mathbb{S}^{d-1}$, $|\theta-\theta'| \leqslant 1$, $\beta = \min(\frac{1}{2}, \alpha)$.

Remark A.2$_b$. Let (3.4 b), (A.42) be valid for $\alpha = 0$ and some $r \geqslant 0$. Then (A.49) holds and $(D_{\mp\theta}v_\theta)^p f \in C^{0,0}(\mathbb{R}^d, \mathcal{M}_{n\times m})$ for $p \in \mathbb{N}$, $\theta \in \mathbb{S}^{d-1}$.

Lemma A.4 is proved in Appendix B.

Lemma A.5$_a$. *Under the assumptions of Lemma A.4$_a$, the following estimates hold:*

$$|P_\theta v_\theta f(x)| \leqslant 2nc_1(\rho,\varepsilon,\theta,\pi_\theta x)\|a\|_{0,1+\varepsilon,\rho}\|f\|_0, \tag{A.57}$$

$$|P_\theta v_\theta f(x+y) - P_\theta v_\theta f(x)| \leqslant 4nc_1(\rho,\varepsilon,\theta,\pi_\theta x)\|a\|_{\alpha,1+\varepsilon,\rho}\|f\|_{\alpha,0}|y|^\alpha, \tag{A.58}$$

$$|P_\theta v_\theta f(x) - P_{\theta'} v_{\theta'} f(x)| \leqslant nc_{12}(\rho,\varepsilon,\beta,\theta,\theta',x)\|a\|_{\alpha,1+\varepsilon,\rho}\|f\|_{\alpha,0}|\theta-\theta'|^\beta, \tag{A.59}$$

$$\begin{aligned} c_{12}(\rho,\varepsilon,\beta,\theta,\theta',x) \overset{\text{def}}{=} 2(&c_2(\rho,\varepsilon,\beta,\theta,\theta',x) \\ &+ c_2(\rho,\varepsilon,\beta,-\theta,-\theta',x) + c_1(\rho,\varepsilon,\theta',\pi_{\theta'}x)), \end{aligned} \tag{A.60}$$

in addition, for $d=2$,

$$|P_\theta^\perp v_\theta f(s)| \leqslant 2nc_7(\rho,\varepsilon,s)\|a\|_{0,1+\varepsilon,\rho}\|f\|_0, \tag{A.61}$$

$$|P_\theta^\perp v_\theta f(s+\delta) - P_\theta^\perp v_\theta f(s)| \leqslant 4nc_7(\rho,\varepsilon,s)\|a\|_{\alpha,1+\varepsilon,\rho}\|f\|_{\alpha,0}|\delta|^\alpha, \tag{A.62}$$

$$\|P_\theta^\perp v_\theta f\|_{\alpha,\varepsilon,\sqrt{2}\rho} \leqslant 4n2^{(1+\varepsilon)/2}\varepsilon^{-1}\|a\|_{\alpha,1+\varepsilon,\rho}\|f\|_{\alpha,0}, \tag{A.63}$$

$$|P_\theta^\perp v_\theta f(s) - P_{\theta'}^\perp v_{\theta'} f(s)| \leqslant nc_{13}(\varepsilon,\beta)\|a\|_{\alpha,1+\varepsilon,\rho}\|f\|_{\alpha,0}|\theta-\theta'|^\beta, \tag{A.64}$$

$$c_{13}(\varepsilon,\beta) \overset{\text{def}}{=} 2(c_8(\varepsilon,\beta) + c_7(\rho,\varepsilon,0)), \tag{A.65}$$

where $x,y\in\mathbb{R}^d$, $|y|\leqslant 1$, $\theta,\theta'\in\mathbb{S}^{d-1}$, $|\theta-\theta'|\leqslant 1$, $0<\beta<\varepsilon$, $\beta\leqslant\alpha$, $s,\delta\in\mathbb{R}$, $|\delta|\leqslant 1$.

Lemma A.5$_b$. *Under the assumptions of Lemma A.4$_b$, the following estimates hold:*

$$|P_\theta v_\theta f(x)| \leqslant nc_3(|\pi_\theta x|,r)\|b\|_0\|f\|_0, \tag{A.66}$$

$$\|P_\theta v_\theta f\|_{\beta,0} \leqslant 2nc_4(r)\|b\|_{\beta,0}\|f\|_{\beta,(X_\theta),0}, \tag{A.67}$$

$$|P_\theta v_\theta f(x) - P_{\theta'} v_{\theta'} f(x)| \leqslant 4nc_{11}(|x|,r,\alpha)\|b\|_{\alpha,0}\|f\|_{\alpha,0}|\theta-\theta'|^\beta, \tag{A.68}$$

in addition, for $d=2$,

$$|P_\theta^\perp v_\theta f(s)| \leqslant nc_3(|s|,r)\|b\|_0\|f\|_0, \tag{A.69}$$

$$\|P_\theta^\perp v_\theta f\|_{\beta,0} \leqslant 2nc_4(r)\|b\|_{\beta,0}\|f\|_{\beta,(X_\theta),0}, \tag{A.70}$$

$$|P_\theta^\perp v_\theta f(s) - P_{\theta'}^\perp v_{\theta'} f(s)| \leqslant nc_{14}(|s|,r,\alpha)\|b\|_{\alpha,0}\|f\|_{\alpha,0}|\theta-\theta'|^\beta, \tag{A.71}$$

$$c_{14}(|s|,r,\alpha) \overset{\text{def}}{=} 2c_9(|s|,r,\alpha) + c_3(|s|,r), \tag{A.72}$$

where $x\in\mathbb{R}^d$, $\theta,\theta'\in\mathbb{S}^{d-1}$, $|\theta-\theta'|\leqslant 1$, $s\in\mathbb{R}$, $\beta=\min(\frac{1}{2},\alpha)$.

Lemma A.5 follows from Lemmas A.2, A.3.

We present now estimates for the operators H, $H_\pm$ defined by (4.24), (4.80), (4.82).

Lemma A.6. *Let $f\in C^{\alpha,\varepsilon}(\mathbb{R},\mathcal{M}_{m\times n})$, where $0<\alpha<1$, $\varepsilon>0$, $m,n\in\mathbb{N}$. Let $H_\pm f(s)$ for $s\in\mathbb{R}\cup\mathbb{C}_\pm$ be defined by (4.80), (4.82). Then*

$$\frac{\partial}{\partial\bar{s}}H_\pm f(s) = 0 \quad \text{for } s\in\mathbb{C}_\pm \tag{A.73}$$

and the following estimates hold:

$$\left.\begin{aligned} |H_\pm f(s)| &\leqslant c_{15}(\alpha,\varepsilon,\varepsilon')\|f\|_{\alpha,\varepsilon}(1+|s|)^{-\varepsilon'},\\ |H_\pm f(s+\delta)-H_\pm f(s)| &\leqslant c_{15}(\alpha,\varepsilon,\varepsilon')\|f\|_{\alpha,\varepsilon}|\delta|^\alpha(1+|s|)^{-\varepsilon'},\end{aligned}\right\} \tag{A.74}$$

$$\left.\begin{aligned} |H_\pm f(s)| &\leqslant c_{15}(\alpha,\varepsilon'',\varepsilon')\rho^{\varepsilon''-\varepsilon}\|f\|_{\alpha,\varepsilon,\rho}(1+|s|)^{-\varepsilon'},\\ |H_\pm f(s+\delta)-H_\pm f(s)| &\leqslant c_{15}(\alpha,\varepsilon'',\varepsilon')\rho^{\varepsilon''-\varepsilon}\|f\|_{\alpha,\varepsilon,\rho}|\delta|^\alpha(1+|s|)^{-\varepsilon'},\end{aligned}\right\} \tag{A.75}$$

$$c_{15}(\alpha,\varepsilon,\varepsilon') \text{ is a positive constant,}$$

for $s, s+\delta\in\mathbb{R}\cup\mathbb{C}_\pm$, $|\delta|\leqslant 1$, $0\leqslant\varepsilon'<\min(1,\varepsilon)$, $\varepsilon'<\varepsilon''\leqslant\varepsilon$, $\rho\geqslant 1$.

The formulae (A.73), (A.74) for $m,n\in\mathbb{N}$ follow from these formulae for $m=n=1$. For the latter case these formulae were given in Lemma I.3 of [**3**]. The estimates (A.75) follow from (A.74) and the inequalities

$$\|f\|_{\alpha,\varepsilon''}\leqslant\|f\|_{\alpha,\varepsilon'',\rho}\leqslant\rho^{\varepsilon''-\varepsilon}\|f\|_{\alpha,\varepsilon,\rho}, \tag{A.76}$$

$0<\alpha\leqslant 1$, $0<\varepsilon''\leqslant\varepsilon$, $\rho\geqslant 1$.

Lemma A.7. *Let* $f\in C^{\alpha,\varepsilon}(\mathbb{R},\mathcal{M}_{m\times n})$, *where* $0<\alpha<1$, $\varepsilon>0$, $m,n\in\mathbb{N}$. *Then*

$$Hf\in C^{\alpha,\varepsilon'}(\mathbb{R},\mathcal{M}_{m\times n}), \tag{A.77}$$

$$\|Hf\|_{\alpha,\varepsilon'}\leqslant c_{15}(\alpha,\varepsilon,\varepsilon')\|f\|_{\alpha,\varepsilon}, \tag{A.78}$$

$$\|Hf\|_{\alpha,\varepsilon'}\leqslant c_{15}(\alpha,\varepsilon'',\varepsilon')\rho^{\varepsilon''-\varepsilon}\|f\|_{\alpha,\varepsilon,\rho} \tag{A.79}$$

for $0\leqslant\varepsilon'<\min(1,\varepsilon)$, $\varepsilon'<\varepsilon''\leqslant\varepsilon$, $\rho\geqslant 1$;

$$\|Hf\|_0\leqslant 2(\pi\alpha)^{-1}\|f\|'_{\alpha,0}\delta^\alpha+2\pi^{-1}(|\ln\delta|+\ln r)\|f\|_0+c_{16}(\varepsilon,\beta)\|f\|_{0,\varepsilon}r^{-\beta}, \tag{A.80}$$

$$c_{16}(\varepsilon,\beta) \text{ is a positive constant}$$

for $0<\delta\leqslant 1$, $r\geqslant 1$, $0\leqslant\beta<\min(1,\varepsilon)$, *where* $\|f\|'_{\alpha,0}\stackrel{\text{def}}{=}\|f\|'_{\alpha,0,1}$.

The formulae (A.77)–(A.79) follow from (4.79), (A.74), (A.75).

The estimate (A.80) is proved in Appendix B.

We give now estimates for the operators $G_{\pm,\theta}v_\theta$ defined by (4.14)–(4.16) for $d=2$.

Lemma A.8$_a$. *Let (3.4 a), (A.42) be valid, where* $d=2$, $0<\alpha<1$. *Then*

$$|G_{\pm,\theta}v_\theta f(x)|\leqslant nc_{17}(\alpha,\varepsilon,\varepsilon',\varepsilon'')\rho^{\varepsilon''-\varepsilon}(1+|\theta^\perp x|)^{-\varepsilon'}\|a\|_{\alpha,1+\varepsilon,\rho}\|f\|_{\alpha,0}, \tag{A.81}$$

$$c_{17}(\alpha,\varepsilon,\varepsilon',\varepsilon'')\stackrel{\text{def}}{=}\varepsilon^{-1}2^{(1+\varepsilon'')/2}(\tfrac{3}{2}+2c_{15}(\alpha,\varepsilon'',\varepsilon')), \tag{A.82}$$

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$$|G_{\pm,\theta}v_\theta f(x+y) - G_{\pm,\theta}v_\theta f(x)| \leqslant nc_{18}(\alpha,\varepsilon,\varepsilon',\varepsilon'') \times \rho^{\varepsilon''-\varepsilon}(1+|\theta^\perp x|)^{-\varepsilon'}\|a\|_{\alpha,1+\varepsilon,\rho}\|f\|_{\alpha,0}|y|^\alpha, \quad (A.83)$$

$$c_{18}(\alpha,\varepsilon,\varepsilon',\varepsilon'') \stackrel{\text{def}}{=} \varepsilon^{-1}2^{(1+\varepsilon'')/2}(3+2c_{15}(\alpha,\varepsilon'',\varepsilon')), \quad (A.84)$$

$$\|G_{\pm,\theta}v_\theta f\|_{\alpha,0} \leqslant nc_{18}(\alpha,\varepsilon,0,\varepsilon'')\rho^{\varepsilon''-\varepsilon}\|a\|_{\alpha,1+\varepsilon,\rho}\|f\|_{\alpha,0}, \quad (A.85)$$

$$|G_{\pm,\theta}v_\theta f(x) - G_{\pm,\theta'}v_{\theta'}f(x)| \leqslant nc_{19}(\alpha,\varepsilon,\beta)(1+|x|^\varepsilon) \times \|a\|_{\alpha,1+\varepsilon,1}\|f\|_{\alpha,0}(1+|\ln|\theta-\theta'||)|\theta-\theta'|^\beta, \quad (A.86)$$

$c_{19}(\alpha,\varepsilon,\beta)$ is a positive constant,

where $x,y\in\mathbb{R}^2$, $|y|\leqslant 1$, $\theta,\theta'\in\mathbb{S}^1$, $|\theta-\theta'|\leqslant 1$, $0\leqslant\varepsilon'<\min(1,\varepsilon)$, $\varepsilon'<\varepsilon''\leqslant\varepsilon$, $0<\beta<\varepsilon$, $\beta\leqslant\alpha$, $\rho\geqslant 1$.

Lemma A.8$_b$. *Let (3.4 b), (A.42) be valid, where $d=2$, $0<\alpha<1$. Then*

$$|G_{\pm,\theta}v_\theta f(x)| \leqslant n(1+c_{15}(\beta,\varepsilon,\varepsilon'))(2+r)^\varepsilon c_4(r)(1+|\theta^\perp x|)^{-\varepsilon'} \times \|b\|_{\beta,0}\|f\|_{\beta,(X_\theta),0}, \quad (A.87)$$

$$|G_{\pm,\theta}v_\theta f(x+y) - G_{\pm,\theta}v_\theta f(x)| \leqslant n(2+c_{15}(\beta,\varepsilon,\varepsilon'))(2+r)^\varepsilon c_4(r) \times (1+|\theta^\perp x|)^{-\varepsilon'}\|b\|_{\beta,0}\|f\|_{\beta,(X_\theta),0}|y|^\beta, \quad y\in X_\theta, \quad (A.88)$$

$$\|G_{\pm,\theta}v_\theta f\|_{\beta,(X_\theta),0} \leqslant nc_{20}(\beta,\varepsilon,r)\|b\|_{\beta,0}\|f\|_{\beta,(X_\theta),0}, \quad (A.89)$$

$$\left.\begin{aligned} c_{20}(\beta,\varepsilon,r) &\stackrel{\text{def}}{=} c_{21}(\beta,\varepsilon,0)(2+r)^\varepsilon c_4(r),\\ c_{21}(\beta,\varepsilon,\varepsilon') &\stackrel{\text{def}}{=} 2+c_{15}(\beta,\varepsilon,\varepsilon'), \end{aligned}\right\} \quad (A.90)$$

$$|G_{\pm,\theta}v_\theta f(x+y) - G_{\pm,\theta}v_\theta f(x)| \leqslant n\|b\|_0\|f\|_0|y|, \qquad \pi_\theta y = 0, \quad (A.91)$$

$$\|(G_{\pm,\theta}v_\theta)^2 f\|_{\beta,0} \leqslant n^2 c_{22}(\beta,\varepsilon,r)(\|b\|_{\beta,0})^2\|f\|_{\beta,0}, \quad (A.92)$$

$$c_{22}(\beta,\varepsilon,r) \stackrel{\text{def}}{=} c_{20}(\beta,\varepsilon,r)(1+c_{20}(\beta,\varepsilon,r)), \quad (A.93)$$

$$|G_{\pm,\theta}v_\theta f(x) - G_{\pm,\theta'}v_{\theta'}f(x)| \leqslant nc_{23}(\beta,r)(1+|x|^{1/2}) \times \|b\|_{\beta,0}\|f\|_{\beta,0}(1+|\ln|\theta-\theta'||)|\theta-\theta'|^\beta, \quad (A.94)$$

$c_{23}(\beta,r)$ is a positive constant,

where $x,y\in\mathbb{R}^2$, $|y|\leqslant 1$, $\theta,\theta'\in\mathbb{S}^1$, $|\theta-\theta'|\leqslant 1$, $\varepsilon>0$, $0\leqslant\varepsilon'<\min(1,\varepsilon)$, $\beta=\min(\frac{1}{2},\alpha)$.

Lemma A.8 is proved in Appendix B.

We give now estimates for the operators $G_\lambda v_\lambda$, Ca_+, $\bar{C}a_-$ defined by (4.32)–(4.35), (4.43)–(4.46), where $v(z,\lambda)$, $a_\pm(z)$ are given by (4.29), (4.30).

Lemma A.9$_a$. *Let (3.4 a), $d=2$, be valid and*

$$f\in C^{\alpha,0}(\mathbb{C},\mathcal{M}_{n\times m}), \quad (A.95)$$

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where $0 < \alpha < 1$. *Then*

$$\|G_\lambda v_\lambda f\|_{\alpha,0} \leqslant nc_{18}(\alpha,\varepsilon,0,\varepsilon'')\rho^{\varepsilon''-\varepsilon}\|a\|_{\alpha,1+\varepsilon,\rho}\|f\|_{\alpha,0} \quad \text{for } |\lambda| \neq 1, \tag{A.96}$$

$$\|G_\lambda v_\lambda f\|_0 \leqslant 4n\varepsilon^{-1}\rho^{-\varepsilon}(1-|\lambda|^2)^{-1} \times (\|a_+\|_{0,1+\varepsilon,\rho} + |\lambda|\|a_0\|_{0,1+\varepsilon,\rho} + |\lambda|^2\|a_-\|_{0,1+\varepsilon,\rho})\|f\|_0 \quad \text{for } |\lambda| < 1, \tag{A.97}$$

$$\|(G_\lambda v_\lambda - Ca_+)f\|_0 \leqslant 4n\varepsilon^{-1}\rho^{-\varepsilon}(1-|\lambda|^2)^{-1}|\lambda| \times (\|a_0\|_{0,1+\varepsilon,\rho} + |\lambda|(\|a_+\|_{0,1+\varepsilon,\rho} + \|a_-\|_{0,1+\varepsilon,\rho}))\|f\|_0 \quad \text{for } |\lambda| < 1, \tag{A.98}$$

$$\|G_\lambda v_\lambda f\|_0 \leqslant 4n\varepsilon^{-1}\rho^{-\varepsilon}(1-|\lambda|^{-2})^{-1} \times (\|a_-\|_{0,1+\varepsilon,\rho} + |\lambda|^{-1}\|a_0\|_{0,1+\varepsilon,\rho} + |\lambda|^{-2}\|a_+\|_{0,1+\varepsilon,\rho})\|f\|_0 \quad \text{for } |\lambda| > 1, \tag{A.99}$$

$$\|(G_\lambda v_\lambda - \bar{C}a_-)f\|_0 \leqslant 4n\varepsilon^{-1}\rho^{-\varepsilon}(1-|\lambda|^{-2})^{-1}|\lambda|^{-1} \times (\|a_0\|_{0,1+\varepsilon,\rho} + |\lambda|^{-1}(\|a_+\|_{0,1+\varepsilon,\rho} + \|a_-\|_{0,1+\varepsilon,\rho}))\|f\|_0 \quad \text{for } |\lambda| > 1, \tag{A.100}$$

where $0 < \varepsilon'' \leqslant \varepsilon$, $\rho \geqslant 1$, $\lambda \in \mathbb{C}$.

Lemma A.9$_b$. *Let (3.4 b),* $d = 2$, *and (A.95) be valid, where* $0 < \alpha < 1$. *Then*

$$\|G_\lambda v_\lambda f\|_{\beta,0} \leqslant n(1 + c_{20}(\beta,\varepsilon,r))\|b\|_{\beta,0}\|f\|_{\beta,0}, \quad \text{for } |\lambda| \neq 1, \tag{A.101}$$

$$\|(G_\lambda v_\lambda)^2 f\|_{\beta,0} \leqslant n^2 c_{22}(\beta,\varepsilon,r)(\|b\|_{\beta,0})^2\|f\|_{\beta,0}, \quad \text{for } |\lambda| \neq 1, \tag{A.102}$$

$$\|G_\lambda v_\lambda f\|_0 \leqslant 4nr(1-|\lambda|^2)^{-1}(\|a_+\|_0 + |\lambda|\|a_0\|_0 + |\lambda|^2\|a_-\|_0)\|f\|_0 \quad \text{for } |\lambda| < 1, \tag{A.103}$$

$$\|(G_\lambda v_\lambda - Ca_+)f\|_0 \leqslant 4nr(1-|\lambda|^2)^{-1}|\lambda|(\|a_0\|_0 + |\lambda|(\|a_+\|_0 + \|a_-\|_0))\|f\|_0 \quad \text{for } |\lambda| < 1, \tag{A.104}$$

$$\|G_\lambda v_\lambda f\|_0 \leqslant 4nr(1-|\lambda|^{-2})^{-1}(\|a_-\|_0 + |\lambda|^{-1}\|a_0\|_0 + |\lambda|^{-2}\|a_+\|_0)\|f\|_0 \quad \text{for } |\lambda| > 1, \tag{A.105}$$

$$\|(G_\lambda v_\lambda - \bar{C}a_-)f\|_0 \leqslant 4nr(1-|\lambda|^{-2})^{-1}|\lambda|^{-1} \times (\|a_0\|_0 + |\lambda|^{-1}(\|a_+\|_0 + \|a_-\|_0)\|f\|_0 \quad \text{for } |\lambda| > 1, \tag{A.106}$$

where $\varepsilon > 0$, $\beta = \min(\frac{1}{2}, \alpha)$, $\lambda \in \mathbb{C}$.

Lemma A.10$_a$. *Let (3.4 a) be valid, where* $d = 2$, $0 < \alpha < 1$. *Let* $f \in L^\infty(\mathbb{C}, \mathcal{M}_{n\times m})$. *Then*

$$\|G_\lambda v_\lambda f\|_{\beta,\varepsilon'} \leqslant nc_{24}(\varepsilon,\varepsilon',\beta)\rho^{\varepsilon'-\varepsilon}(1+|\lambda|^2)(1-|\lambda|^2)^{-2} \times (\|a_+\|_{0,1+\varepsilon,\rho} + |\lambda|\|a_0\|_{0,1+\varepsilon,\rho} + |\lambda|^2\|a_-\|_{0,1+\varepsilon,\rho})\|f\|_0 \quad \text{for } |\lambda| < 1, \tag{A.107}$$

$$\|Ca_+ f\|_{\beta,\varepsilon'} \leqslant nc_{24}(\varepsilon,\varepsilon',\beta)\rho^{\varepsilon'-\varepsilon}\|a_+\|_{0,1+\varepsilon,\rho}\|f\|_0, \tag{A.108}$$

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$$\|G_\lambda v_\lambda f\|_{\beta,\varepsilon'} \leqslant nc_{24}(\varepsilon,\varepsilon',\beta)\rho^{\varepsilon'-\varepsilon}(1+|\lambda|^{-2})(1-|\lambda|^{-2})^{-2} \times (\|a_-\|_{0,1+\varepsilon,\rho} + |\lambda|^{-1}\|a_0\|_{0,1+\varepsilon,\rho} + |\lambda|^{-2}\|a_+\|_{0,1+\varepsilon,\rho})\|f\|_0 \quad \text{for } |\lambda|>1, \tag{A.109}$$

$$\|\bar{C}a_- f\|_{\beta,\varepsilon'} \leqslant nc_{24}(\varepsilon,\varepsilon',\beta)\rho^{\varepsilon'-\varepsilon}\|a_-\|_{0,1+\varepsilon,\rho}\|f\|_0, \tag{A.110}$$

$$c_{24}(\varepsilon,\varepsilon',\beta) \text{ is a positive constant,}$$

where $0 \leqslant \varepsilon' < \min(\varepsilon,1)$, $0<\beta<1$, $\rho \geqslant 1$, $\lambda \in \mathbb{C}$.

Lemma A.10$_b$. *Let (3.4b) be valid, where* $d=2$, $0<\alpha<1$. *Let* $f \in L^\infty(\mathbb{C},\mathcal{M}_{n\times m})$. *Then*

$$\|G_\lambda v_\lambda f\|_{\beta,1} \leqslant nc_{25}(\beta)\max(r^{1-\beta},r^2)(1+|\lambda|^2)(1-|\lambda|^2)^{-2} \times (\|a_+\|_0 + |\lambda|\|a_0\|_0 + |\lambda|^2\|a_-\|_0)\|f\|_0 \quad \text{for } |\lambda|<1, \tag{A.111}$$

$$\|Ca_+ f\|_{\beta,1} \leqslant nc_{25}(\beta)\max(r^{1-\beta},r^2)\|a_+\|_0\|f\|_0, \tag{A.112}$$

$$\|G_\lambda v_\lambda f\|_{\beta,1} \leqslant nc_{25}(\beta)\max(r^{1-\beta},r^2)(1+|\lambda|^{-2})(1-|\lambda|^{-2})^{-2} \times (\|a_-\|_0 + |\lambda|^{-1}\|a_0\|_0 + |\lambda|^{-2}\|a_+\|_0)\|f\|_0 \quad \text{for } |\lambda|>1, \tag{A.113}$$

$$\|\bar{C}a_- f\|_{\beta,1} \leqslant nc_{25}(\beta)\max(r^{1-\beta},r^2)\|a_-\|_0\|f\|_0, \tag{A.114}$$

$$c_{25}(\beta) \text{ is a positive constant,}$$

where $0<\beta<1$.

Lemmas A.9 and A.10 are proved in Appendix B.

We give, finally, estimates for the operators C', $\bar{C}'$ defined by the formulae

$$C'\varphi(z) = \partial_z C\varphi(z), \qquad \bar{C}'\varphi(z) = \partial_{\bar z}\bar{C}\varphi(z), \tag{A.115}$$

where C, $\bar{C}$ are defined by (4.45), (4.46), $\partial_z = \partial/\partial z$, $\partial_{\bar z} = \partial/\partial \bar z$, φ is a test function.

Lemma A.11$_a$. *Let* $\varphi \in C^{\alpha,\varepsilon}(\mathbb{C},\mathcal{M}_{m\times n})$, *where* $0<\alpha<1$, $\varepsilon>0$, $m,n\in\mathbb{N}$. *Then*

$$J\varphi \in C^{\alpha,\varepsilon'}(\mathbb{C},\mathcal{M}_{m\times n}), \tag{A.116}$$

$$\|J\varphi\|_{\alpha,\varepsilon'} \leqslant c_{26}(\alpha,\varepsilon,\varepsilon')\|\varphi\|_{\alpha,\varepsilon}, \tag{A.117}$$

$$c_{26}(\alpha,\varepsilon,\varepsilon') \text{ is a positive constant,}$$

where $J=C'$ *or* $J=\bar{C}'$, $0\leqslant\varepsilon'<\min(2,\varepsilon)$.

Lemma A.11$_b$. *Let*

$$\varphi(z) = \chi_+(r-|z|)\psi(z), \quad \psi \in C^{\alpha,0}(\mathbb{C},\mathcal{M}_{m\times n}), \tag{A.118}$$

$r>0$, $0<\alpha<1$, $m,n\in\mathbb{N}$. *Then*

$$J\varphi \in C^{\alpha,2}(\Omega_{r,\delta},\mathcal{M}_{m\times n}), \tag{A.119}$$

$$\Omega_{r,\delta} = \{z\in\mathbb{C} \mid ||z|-r| \geqslant \delta\}, \quad 0<\delta<r, \tag{A.120}$$

$$\|J\varphi\|_{\Omega_{r,\delta},\alpha,2} \leqslant c_{27}(\alpha,r,\delta)\|\psi\|_{\alpha,0}, \tag{A.121}$$

$$c_{27}(\alpha,r,\delta) \text{ is a positive constant,}$$

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$$|J\varphi(z)| \leqslant c_{28}(\alpha, r)(1+|z|)^{-2}\|\psi\|_{\alpha,0} \quad \text{for } z \in \mathbb{C}, \qquad \text{(A.122)}$$
$$c_{28}(\alpha, r) \text{ is a positive constant,}$$

where $J = C'$ or $J = \bar{C}'$.

Appendix B. Proofs of estimates for operators

Proof of Lemma A.1$_a$. First,

$$|D_{-\theta} f(x)| \overset{\text{(A.2)}}{\leqslant} F_1 I_1(x,\theta), \qquad \text{(B.1)}$$

$$\begin{aligned} I_1(x,\theta) &\overset{\text{def}}{=} \int_{-\infty}^{0} \frac{\mathrm{d}t}{(\rho + |\pi_\theta x + \theta(\theta x + t)|)^{1+\varepsilon}} \\ &\leqslant \int_{-\infty}^{0} \frac{\mathrm{d}t}{(\rho + 2^{-1/2}(|\pi_\theta x| + |\theta x + t|))^{1+\varepsilon}} \\ &= \int_{-\infty}^{\theta x} \frac{2^{(1+\varepsilon)/2}\,\mathrm{d}s}{(2^{1/2}\rho + |\pi_\theta x| + |s|)^{1+\varepsilon}} \\ &\overset{\text{def}}{=} I_2(x,\theta), \end{aligned} \qquad \text{(B.2)}$$

$$\left.\begin{aligned} I_2(x,\theta) &= \frac{2^{(1+\varepsilon)/2}}{\varepsilon(2^{1/2}\rho + |\pi_\theta x| - \theta x)^{\varepsilon}} \quad \text{for } \theta x \leqslant 0, \\ I_2(x,\theta) &\leqslant \frac{2^{(3+\varepsilon)/2}}{\varepsilon(2^{1/2}\rho + |\pi_\theta x|)^{\varepsilon}} \qquad\quad \text{for } \theta x > 0. \end{aligned}\right\} \qquad \text{(B.3)}$$

The estimate (A.4) follows from (B.1)–(B.3). Second,

$$\begin{aligned} |D_{-\theta} f(x+y) - D_{-\theta} f(x)| &\leqslant \int_{-\infty}^{0} |f(x+y+t\theta) - f(x+t\theta)|\,\mathrm{d}t \\ &\overset{\text{(A.3)}}{\leqslant} F_2 |y|^{\alpha} I_1(x,\theta). \end{aligned} \qquad \text{(B.4)}$$

The estimate (A.5) follows from (B.4), (B.2), (B.3). Third,

$$\begin{aligned} |D_{-\theta} f(x) - D_{-\theta'} f(x)| &\leqslant \int_{-\infty}^{0} |f(x+t\theta) - f(x+t\theta')|\,\mathrm{d}t \\ &= I_3(x,\theta,\theta') + I_4(x,\theta,\theta'), \end{aligned} \qquad \text{(B.5)}$$

$$\begin{aligned} I_3(x,\theta,\theta') &\overset{\text{def}}{=} \int_{-\infty}^{-|\theta-\theta'|^{-1}} |f(x+t\theta) - f(x+t\theta')|\,\mathrm{d}t \\ &\leqslant \int_{-\infty}^{-|\theta-\theta'|^{-1}} |f(x+t\theta)|\,\mathrm{d}t + \int_{-\infty}^{-|\theta-\theta'|^{-1}} |f(x+t\theta')|\,\mathrm{d}t \\ &\overset{\text{(A.2)}}{\leqslant} F_1(I_5(x,\theta,\theta') + I_5(x,\theta',\theta)), \end{aligned} \qquad \text{(B.6)}$$

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$$I_5(x,\theta,\theta') \overset{\text{def}}{=} \int_{-\infty}^{-|\theta-\theta'|^{-1}} \frac{\mathrm{d}t}{(\rho + 2^{-1/2}(|\pi_\theta x| + |\theta x + t|))^{1+\varepsilon}}$$
$$= \int_{-\infty}^{\theta x - |\theta-\theta'|^{-1}} \frac{2^{(1+\varepsilon)/2}\,\mathrm{d}s}{(2^{1/2}\rho + |\pi_\theta x| + |s|)^{1+\varepsilon}}, \tag{B.7}$$
$$I_5(x,\theta,\theta') = \frac{2^{(1+\varepsilon)/2}|\theta-\theta'|^{\varepsilon}}{\varepsilon(1 + (2^{1/2}\rho + |\pi_\theta x| - \theta x)|\theta-\theta'|)^{\varepsilon}} \quad \text{for } |\theta-\theta'|^{-1} \geqslant \theta x,$$

$$I_5(x,\theta,\theta') \leqslant \varepsilon^{-1} 2^{(1+3\varepsilon)/2}|\theta-\theta'|^{\varepsilon} \quad \text{for } \theta x|\theta-\theta'| \leqslant \tfrac{1}{2}, \tag{B.8 a}$$
$$I_5(x,\theta,\theta') \leqslant \frac{2^{(3+\varepsilon)/2}}{\varepsilon(2^{1/2}\rho + |\pi_\theta x|)^{\varepsilon}} \quad \text{everywhere}, \tag{B.8 b}$$

$$I_5(x,\theta,\theta') \leqslant \varepsilon^{-1}\biggl(2^{(1+3\varepsilon)/2} + \frac{2^{(3+\varepsilon)/2}\chi_+(2\theta x - 1)(2\theta x)^{\varepsilon}}{(2^{1/2}\rho + |\pi_\theta x|)^{\varepsilon}}\biggr)|\theta-\theta'|^{\varepsilon} \quad \text{for } |\theta-\theta'| \leqslant 1, \tag{B.9}$$

$$I_4(x,\theta,\theta') \overset{\text{def}}{=} \int_{-|\theta-\theta'|^{-1}}^{0} |f(x+t\theta) - f(x+t\theta')|\,\mathrm{d}t$$
$$\overset{(\text{A.3})}{\leqslant} \int_{-|\theta-\theta'|^{-1}}^{0} \frac{F_2|t(\theta-\theta')|^{\alpha}\,\mathrm{d}t}{(\rho + |x+t\theta|)^{1+\varepsilon}}$$
$$\overset{0<\beta<\varepsilon,\ \beta\leqslant\alpha}{\leqslant} \int_{-|\theta-\theta'|^{-1}}^{0} \frac{F_2|t|^{\beta}\,\mathrm{d}t}{(\rho + 2^{-1/2}(|\pi_\theta x| + |\theta x + t|))^{1+\varepsilon}}|\theta-\theta'|^{\beta}$$
$$\leqslant \int_{-\infty}^{\theta x} \frac{2^{(1+\varepsilon)/2}F_2|s-\theta x|^{\beta}\,\mathrm{d}s}{(2^{1/2}\rho + |\pi_\theta x| + |s|)^{1+\varepsilon}}|\theta-\theta'|^{\beta}$$
$$\leqslant 2^{(1+\varepsilon)/2}F_2(I_6(x,\theta) + I_7(x,\theta))|\theta-\theta'|^{\beta}, \tag{B.10}$$
$$I_6(x,\theta) \overset{\text{def}}{=} \int_{-\infty}^{\theta x} \frac{|s|^{\beta}\,\mathrm{d}s}{(2^{1/2}\rho + |\pi_\theta x| + |s|)^{1+\varepsilon}}$$
$$\leqslant \int_{-\infty}^{\theta x} \frac{\mathrm{d}s}{(2^{1/2}\rho + |\pi_\theta x| + |s|)^{1+\varepsilon-\beta}}$$
$$\leqslant \frac{2}{(\varepsilon-\beta)(2^{1/2}\rho + |\pi_\theta x|)^{\varepsilon-\beta}}, \tag{B.11}$$
$$I_7(x,\theta) = \int_{-\infty}^{\theta x} \frac{|\theta x|^{\beta}\,\mathrm{d}s}{(2^{1/2}\rho + |\pi_\theta x| + |s|)^{1+\varepsilon}}$$
$$\leqslant \frac{2}{\varepsilon}\biggl(\frac{1}{(2^{1/2}\rho + |\pi_\theta x|)^{\varepsilon-\beta}} + \frac{\chi_+(\theta x - 1)(\theta x)^{\beta}}{(2^{1/2}\rho + |\pi_\theta x|)^{\varepsilon}}\biggr), \tag{B.12}$$

where $0 < \beta < \varepsilon$, $\beta \leqslant \alpha$. The estimate (A.8) follows from (B.5), (B.6), (B.9)–(B.12). Lemma A.1_a is proved. □

Proof of Lemma A.1_b. Let

$$\Lambda^{-}_{x,\theta,r} = \{s \in\,]-\infty, 0] \mid |x + s\theta| \leqslant r\} \tag{B.13}$$

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and $|\Lambda^-_{x,\theta,r}|$ denote the length of $\Lambda^-_{x,\theta,r}$. Note that

$$|\Lambda^-_{x,\theta,r}| \leqslant 2\chi_+(r^2 - |\pi_\theta x|^2)\sqrt{r^2 - |\pi_\theta x|^2}. \tag{B.14}$$

To prove (A.12), under the assumptions (A.9), (A.10), it remains to note that

$$|D_{-\theta} f(x)| \leqslant \int_{\Lambda^-_{x,\theta,r}} F_1 \, \mathrm{d}s = F_1 |\Lambda^-_{x,\theta,r}|. \tag{B.15}$$

Consider now $\Lambda^-_{x,\theta,r}$ and $\Lambda^-_{x+y,\theta,r}$, $\theta y = 0$. Suppose that

$$|\Lambda^-_{x+y,\theta,r}| \leqslant |\Lambda^-_{x,\theta,r}|. \tag{B.16}$$

Then

$$\Lambda^-_{x+y,\theta,r} \subseteq \Lambda^-_{x,\theta,r}, \tag{B.17 a}$$

$$|\Lambda^-_{x,\theta,r}| - |\Lambda^-_{x+y,\theta,r}| \leqslant 2\sqrt{2r|y|}. \tag{B.17 b}$$

The proof of (B.17 b) consists of the following. If $|\Lambda^-_{x,\theta,r}| = 0$, then (B.17 b) holds. If $|\Lambda^-_{x,\theta,r}| > 0$, then

$$|\Lambda^-_{x,\theta,r}| - |\Lambda^-_{x+y,\theta,r}| \leqslant 2\big(\sqrt{r^2 - |\pi_\theta x|^2} - \sqrt{r^2 - (|\pi_\theta x| + h)^2}\big) \tag{B.18}$$

for some h such that $0 \leqslant h \leqslant |y|$, $|\pi_\theta x| + h \leqslant r$. Finally, for $0 \leqslant l \leqslant l + h \leqslant r$,

$$\begin{aligned}\sqrt{r^2 - l^2} - \sqrt{r^2 - (l+h)^2} &= \int_l^{l+h} \frac{t \, \mathrm{d}t}{\sqrt{r^2 - t^2}} \leqslant \int_{r-h}^{r} \frac{t \, \mathrm{d}t}{\sqrt{r^2 - t^2}} \\ &= \sqrt{r^2 - (r-h)^2} \leqslant \sqrt{2rh}.\end{aligned} \tag{B.19}$$

Taking into account (B.14), (B.17), to prove (A.14), under assumptions (A.9)–(A.11), (B.16), it remains to note that

$$\begin{aligned}|D_{-\theta} f(x+y) - D_{-\theta} f(x)| &\leqslant \int_{\Lambda^-_{x+y,\theta,r}} F_{2,1} |y|^\alpha \, \mathrm{d}s + \int_{\Lambda^-_{x,\theta,r} \setminus \Lambda^-_{x+y,\theta,r}} F_1 \, \mathrm{d}s \\ &= F_{2,1} |\Lambda^-_{x+y,\theta,r}| |y|^\alpha + F_1 (|\Lambda^-_{x,\theta,r}| - |\Lambda^-_{x+y,\theta,r}|).\end{aligned} \tag{B.20}$$

The estimate (A.14) for the general case follows from this estimate for the case (B.16). The estimate (A.16) follows from (A.9), (A.10) and the formula

$$\frac{\mathrm{d}}{\mathrm{d}t} D_{-\theta} f(x + t\theta) = f(x + t\theta). \tag{B.21}$$

Consider now $\Lambda^-_{x,\theta,r}$ and $\Lambda^-_{x,\theta',r}$. Consider, first, the case $|x| \leqslant r$. Suppose that

$$|\Lambda^-_{x,\theta,r}| \leqslant |\Lambda^-_{x,\theta',r}|. \tag{B.22}$$

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Then

$$\Lambda^-_{x,\theta,r} \subseteq \Lambda^-_{x,\theta',r}, \tag{B.23 a}$$

$$|\Lambda^-_{x,\theta',r}| - |\Lambda^-_{x,\theta,r}| \leqslant \pi r|\theta - \theta'|. \tag{B.23 b}$$

The proof of (B.23 b) consists of the following. Suppose that $d = 2$. Consider $\Lambda^-_{x,\theta(\varphi),r}$ for $x = (l, 0)$, $0 \leqslant l \leqslant r$, $\theta(\varphi) = -(\cos\varphi, \sin\varphi)$. We have

$$|\Lambda^-_{x,\theta(\varphi),r}| = -l\cos\varphi + \sqrt{r^2 - l^2\sin^2\varphi}, \tag{B.24 a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}\varphi}\right)|\Lambda^-_{x,\theta(\varphi),r}| = l\sin\varphi\left(1 - \frac{l\cos\varphi}{\sqrt{r^2 - l^2\sin^2\varphi}}\right), \tag{B.24 b}$$

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}\varphi}\right)|\Lambda^-_{x,\theta(\varphi),r}|\right| \leqslant 2r. \tag{B.24 c}$$

Therefore,

$$\begin{aligned} ||\Lambda^-_{x,\theta(\varphi'),r}| - |\Lambda^-_{x,\theta(\varphi),r}|| &\leqslant 2r|\varphi - \varphi'| \\ &\overset{|\varphi-\varphi'|\leqslant\pi}{\leqslant} 2r2\arcsin\tfrac{1}{2}|\theta(\varphi) - \theta(\varphi')| \\ &\leqslant \pi r|\theta(\varphi) - \theta(\varphi')|. \end{aligned} \tag{B.25}$$

Therefore, (B.23 b) holds for $d = 2$ and as a corollary for $d \geqslant 2$. From (A.9)–(A.11), (B.23) it follows that

$$\begin{aligned} &|D_{-\theta}f(x) - D_{-\theta'}f(x)| \\ &\quad\leqslant \int_{\Lambda^-_{x,\theta,r}} |f(x + s\theta) - f(x + s\theta')|\,\mathrm{d}s + \int_{\Lambda^-_{x,\theta',r}\setminus\Lambda^-_{x,\theta,r}} |f(x + s\theta')|\,\mathrm{d}s \\ &\quad\overset{2r|\theta-\theta'|\leqslant 1}{\leqslant} (2r)^{1+\alpha}F_2|\theta - \theta'|^\alpha + \pi F_1 r|\theta - \theta'|. \end{aligned} \tag{B.26}$$

On the other hand, from (A.12) it follows that

$$|D_{-\theta}f(x) - D_{-\theta'}f(x)| \leqslant 4F_1 4. \tag{B.27}$$

From (B.26), (B.27) it follows that (A.18) holds, under the assumptions (B.22) and, as a corollary, for the general case.

Consider, finally, $\Lambda^-_{x,\theta,r}$, $\Lambda^-_{x,\theta',r}$ for the case $|x| \geqslant r$. Suppose that (B.22) holds. Then

$$\Lambda^-_{x,\theta,r} \subseteq \Lambda^-_{x,\theta',r}, \tag{B.28 a}$$

$$|\Lambda^-_{x,\theta',r}| - |\Lambda^-_{x,\theta,r}| \leqslant 2\sqrt{2r|x||\theta - \theta'|}. \tag{B.28 b}$$

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The proof of (B.28) consists of the following. Suppose that $d=2$. Consider $\Lambda^-_{x,\theta(\varphi),r}$ for $x=(-l,0)$, $r\leqslant l$, $\theta(\varphi)=-(\cos\varphi,\sin\varphi)$. We have

$$\Lambda^-_{x,\theta(\varphi),r}=[-s_+,-s_-], \tag{B.29 a}$$

$$s_\pm=l\cos\varphi\pm\sqrt{r^2-l^2\sin^2\varphi}, \tag{B.29 b}$$

$$|\Lambda^-_{x,\theta(\varphi),r}|=2\sqrt{r^2-l^2\sin^2\varphi}\quad\text{for } |\sin\varphi|\leqslant r/l; \tag{B.29 c}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}\varphi}\right)s_-=l\sin\varphi\left(-1+\frac{l\cos\varphi}{\sqrt{r^2-l^2\sin^2\varphi}}\right)\geqslant 0, \tag{B.30 a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}\varphi}\right)s_+=l\sin\varphi\left(-1-\frac{l\cos\varphi}{\sqrt{r^2-l^2\sin^2\varphi}}\right)\leqslant 0, \tag{B.30 b}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}\varphi}\right)|\Lambda^-_{x,\theta(\varphi),r}|\leqslant 0\quad\text{for } 0\leqslant\varphi\leqslant\arcsin(r/l)\leqslant\pi/2; \tag{B.30 c}$$

$$\Lambda^-_{x,\theta(\varphi),r}=\emptyset\quad\text{for } |\sin\varphi|>r/l; \tag{B.31}$$

$$\begin{aligned}|\Lambda^-_{x,\theta(\varphi'),r}|-|\Lambda^-_{x,\theta(\varphi),r}| &\overset{0\leqslant\varphi'\leqslant\varphi\leqslant\pi/2\ (\text{B.19})}{\leqslant} 2\sqrt{2rl|\sin\varphi-\sin\varphi'|}\\ &\leqslant 2\sqrt{2rl|\theta(\varphi)-\theta(\varphi')|}.\end{aligned} \tag{B.32}$$

Using these formulae we obtain (B.28) for $d=2$ and, as a corollary, for $d\geqslant 2$. Using (B.28) we prove (A.21) in a similar way with (A.19). Lemma A.1_b is proved. □

Proof of Lemma A.2_a. The estimates (A.22)–(A.24) follow from (A.4), (A.5), (A.7) and the formulae

$$P_\theta f(x)=D_{-\theta}f(x)+D_\theta f(x), \tag{B.33 a}$$

$$P_\theta f(x)=P_\theta f(\pi_\theta x). \tag{B.33 b}$$

The estimates (A.25), (A.26) follow from (4.23), (A.22), (A.23). The estimate (A.28) follows from (A.24) and the formula

$$P^\perp_\theta f(s)-P^\perp_{\theta'}f(s)=P_\theta f(x(\theta,\theta',s))-P_{\theta'}f(x(\theta,\theta',s)), \tag{B.34}$$

where $x(\theta,\theta',s)=s\theta^\perp+t\theta=s\theta'^\perp-t\theta'$ for some $t=t(\theta,\theta',s)$, where, in particular,

$$|t|=|s||\theta-\theta'|/\sqrt{4-|\theta-\theta'|^2}\overset{|\theta-\theta'|\leqslant 1}{\leqslant}|s|/\sqrt{3}.$$

Lemma A.2_a is proved. □

Proof of Lemma A.2_b. The estimates (A.30), (A.31) follow from (A.12), (A.14), (B.33 b) and the formula

$$P_\theta f(x)=\lim_{s\to+\infty}D_{-\theta}f(x+s\theta). \tag{B.35}$$

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The estimates (A.34), (A.35) follow from (4.23), (A.30), (A.31). The estimates (A.32), (A.33) follow from (B.33 *a*), (A.18), (A.20). The estimate (A.36) follows from (A.32), (A.33), (A.34) and the formula (B.34), where, in particular,

$$|x| = |s|/\sqrt{1-|\theta-\theta'|^2/4} \overset{|\theta-\theta'|\leqslant 1}{\leqslant} 2|s|/\sqrt{3}.$$

Lemma A.2_b is proved. □

Proof of Lemma A.4. We use the formulae

$$(D_{-\theta}v_\theta)^p f(x) = \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} V_p(x,\theta,\underbrace{s_1,\dots,s_p}_{p}) f(x+(\underbrace{s_1+\cdots+s_p}_{p})\theta)\,\underbrace{\mathrm{d}s_1\cdots\mathrm{d}s_p}_{p}, \tag{B.36}$$

where $p\in\mathbb{N}$, $x\in\mathbb{R}^d$, $\theta\in\mathbb{S}^{d-1}$,

$$\begin{aligned} &V_p(x,\theta,s_1,\dots,s_p)\\ &\quad= \underbrace{v(x+s_p\theta,\theta)\times\cdots\times v(x+(\overbrace{s_j+\cdots+s_p}^{p-j+1})\theta,\theta)\times\cdots\times v(x+(s_1+\cdots+s_p)\theta,\theta)}_{p}; \end{aligned} \tag{B.37}$$

$$\begin{aligned} &V_p(x+y,\theta,s_1,\dots,s_p)f(x+y+(s_1+\cdots+s_p)\theta)\\ &\qquad - V_p(x,\theta,s_1,\dots,s_p)f(x+(s_1+\cdots+s_p)\theta) = Q_p = \sum_{j=0}^{p} Q_{p,j}, \end{aligned} \tag{B.38}$$

where $p\in\mathbb{N}$, $x,y\in\mathbb{R}^d$, $\theta\in\mathbb{S}^{d-1}$, $s_i\in\mathbb{R}$, $i=1,\dots,p$,

$$\begin{aligned} Q_{p,0} &= V_p(x+y,\theta,s_1,\dots,s_p)\\ &\quad\times(f(x+y+(s_1+\cdots+s_p)\theta)-f(x+(s_1+\cdots+s_p)\theta)), && \text{(B.39 }a\text{)}\\ Q_{p,1} &= V_{p-1}(x+y,\theta,s_2,\dots,s_p)\\ &\quad\times(v(x+y+(s_1+\cdots+s_p)\theta,\theta)-v(x+(s_1+\cdots+s_p)\theta,\theta))\\ &\quad\times f(x+(s_1+\cdots+s_p)\theta), && \text{(B.39 }b\text{)}\\ Q_{p,2} &= V_{p-2}(x+y,\theta,s_3,\dots,s_p)\\ &\quad\times(v(x+y+(s_2+\cdots+s_p)\theta,\theta)-v(x+(s_2+\cdots+s_p)\theta,\theta))\\ &\quad\times V_1(x,\theta,s_1,\dots,s_p)f(x+(s_1+\cdots+s_p)\theta) && \text{(B.39 }c\text{)} \end{aligned}$$

for $p\geqslant 2$,

$$\begin{aligned} Q_{p,j} &= V_{p-j}(x+y,\theta,s_{j+1},\dots,s_p)\\ &\quad\times(v(x+y+(s_j+\cdots+s_p)\theta,\theta)-v(x+(s_j+\cdots+s_p)\theta,\theta))\\ &\quad\times V_{j-1}(x,\theta,s_1,\dots,s_{j-2},s_{j-1}+\cdots+s_p)f(x+(s_1+\cdots+s_p)\theta) && \text{(B.39 }d\text{)} \end{aligned}$$

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for $3 \leqslant j \leqslant p$, where $V_0 = 1$;

$$\int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \prod_{k=1}^{p} \varphi\Big(x, \theta, \sum_{i=k}^{p} s_i\Big) \, \mathrm{d}s_1 \cdots \mathrm{d}s_p = \int_{0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_p} \mathrm{d}t_1 \cdots \mathrm{d}t_p \prod_{k=1}^{p} \varphi(x, \theta, t_k) = \frac{1}{p!}\Big(\int_{-\infty}^{0} \varphi(x, \theta, t) \, \mathrm{d}t\Big)^p \tag{B.40}$$

for $\varphi(x, \theta, \cdot) \in L^1(]-\infty, 0], \mathbb{C})$.

Under the assumptions (3.4 a), (A.42), $0 < \alpha \leqslant 1$, the following estimates hold:

$$|V_p(x, \theta, s_1, \ldots, s_p) f(x + (s_1 + \cdots + s_p)\theta)| \leqslant (n\|a\|_{0,1+\varepsilon,\rho})^p \|f\|_0 \prod_{k=1}^{p} \Big(\rho + \Big|x + \Big(\sum_{i=k}^{p} s_i\Big)\theta\Big|\Big)^{1+\varepsilon}, \tag{B.41}$$

$$|Q_{p,j}| \leqslant 2^{(1+\varepsilon)(p-j)} (n\|a\|_{\alpha,1+\varepsilon,\rho})^p \|f\|_{\alpha,0} |y|^{\alpha} \prod_{k=1}^{p} \Big(\rho + \Big|x + \Big(\sum_{i=k}^{p} s_i\Big)\theta\Big|\Big)^{1+\varepsilon}, \tag{B.42}$$

where $p \in \mathbb{N}$, $j \in \mathbb{N} \cup 0$, $0 \leqslant j \leqslant p$, $x, y \in \mathbb{R}^d$, $|y| \leqslant 1$, $\theta \in \mathbb{S}^{d-1}$, $\rho \geqslant 1$.

Under the assumptions (3.4 b), (A.42), $0 < \alpha \leqslant 1$, the following estimates hold:

$$|V_p(x, \theta, s_1, \ldots, s_p) f(x + (s_1 + \cdots + s_p)\theta)| \leqslant (n\|b\|_0)^p \|f\|_0 \prod_{k=1}^{p} \chi_+\Big(r - \Big|x + \Big(\sum_{i=k}^{p} s_i\Big)\theta\Big|\Big), \tag{B.43}$$

$$|Q_{p,0}| \leqslant (n\|b\|_0)^p \|f\|_{\beta,(X_\theta),0} |y|^{\beta} \prod_{k=1}^{p} \chi_+\Big(r - \Big|x + y + \Big(\sum_{i=k}^{p} s_i\Big)\theta\Big|\Big), \tag{B.44 a}$$

$$\begin{aligned}\Big|\int_{-\infty}^{0} \cdots \int_{-\infty}^{0} Q_{p,j} \, \mathrm{d}s_1 \cdots \mathrm{d}s_j\Big| &\leqslant n\Big| V_{p-j}(x + y, \theta, s_{j+1}, \ldots, s_p) \\ &\quad \times \int_{-\infty}^{0} (v(x + y + (s_j + \cdots + s_p)\theta, \theta) - v(x + (s_j + \cdots + s_p)\theta, \theta)) \\ &\quad \times g_{j-1}(x + (s_j + \cdots + s_p)\theta, \theta) \, \mathrm{d}s_j\Big|,\end{aligned} \tag{B.44 b}$$

where

$$g_{j-1}(x, \theta) = \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} V_{j-1}(x, \theta, s_1, \ldots, s_{j-1}) f(x + (s_1 + \cdots + s_{j-1})\theta) \, \mathrm{d}s_1 \cdots \mathrm{d}s_{j-1}, \tag{B.44 c}$$

$$\left|\int_{-\infty}^{0}(v(x+y+(s_j+t)\theta,\theta)-v(x+(s_j+t)\theta,\theta))h(x+(s_j+t)\theta)\,\mathrm{d}s_j\right|$$
$$\leqslant nc_4(r)\|b\|_{\beta,0}\|h\|_0|y|^{\beta}\quad \text{for } h\in C^{0,0}(\mathbb{R}^d,\mathcal{M}_{n\times m}),\qquad (\text{B.44}\,d)$$

where $p,j\in\mathbb{N}$, $j\leqslant p$, $V_0=1$, $x\in\mathbb{R}^d$, $y\in X_\theta$, $|y|\leqslant 1$, $\beta=\min(\frac{1}{2},\alpha)$.

Every estimate of Lemma A.4 for $(D_\theta v_\theta)^p f$ follows from the related estimate of Lemma A.4 for $(D_{-\theta}v_\theta)^p f$. Taking into account this fact we complete below the proof of Lemma A.4.

The estimate (A.43) follows from (B.36), (B.41), (B.40), (A.4). The estimate (A.44) follows from (B.36), (B.38), (B.42), (B.40) and the formula

$$\sum_{j=0}^{p}2^{(1+\varepsilon)(p-j)}=\frac{2^{(1+\varepsilon)(p+1)}-1}{2^{(1+\varepsilon)}-1}\overset{0\leqslant\varepsilon}{\leqslant}2^{(1+\varepsilon)(p+1)}. \qquad (\text{B.45})$$

The estimate (A.45) follows from (A.5), (A.41). The estimate (A.46) follows from (A.43), (A.44). The estimate (A.47) follows from (A.7), (A.4) and the formula

$$D_{-\theta}v_\theta f-D_{-\theta'}v_{\theta'}f=(D_{-\theta}-D_{-\theta'})v_\theta f+D_{-\theta'}(v_\theta-v_{\theta'})f. \qquad (\text{B.46})$$

The estimate (A.49) follows from (B.36), (B.43), (B.40), (A.12). The estimate (A.50) follows from (A.49), (B.36), (B.38), (B.44), (B.43), (B.40), (A.12). The estimate (A.51) follows from (B.21), (A.40). The estimate (A.52) follows from (A.49)–(A.51). The estimate (A.54) follows from (A.18), (A.20), (A.12), (B.46).

Lemma A.4 is proved. □

Proof of (A.80). We have

$$\|Hf\|_0\leqslant\sup_{s\in\mathbb{R}}\frac{1}{\pi}\int_{s-r}^{s+r}\frac{|f(t)-f(s)|}{|t-s|}\,\mathrm{d}t+\sup_{s\in\mathbb{R}}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{|f(t)|\chi_+(|t-s|-r)}{|t-s|}\,\mathrm{d}t, \qquad (\text{B.47})$$

$$\int_{s-r}^{s+r}\frac{|f(t)-f(s)|}{|t-s|}\,\mathrm{d}t\leqslant\left(\int_{s-r}^{s-\delta}+\int_{s+\delta}^{s+r}\right)\frac{\|f\|_0}{|t-s|}\,\mathrm{d}t+\int_{s-\delta}^{s+\delta}\frac{\|f\|'_{\alpha,0}|t-s|^{\alpha}}{|t-s|}\,\mathrm{d}t$$
$$=2\|f\|_0(\ln r-\ln\delta)+2\alpha^{-1}\|f\|'_{\alpha,0}\delta^{\alpha}, \qquad (\text{B.48})$$

$$\int_{-\infty}^{+\infty}\frac{|f(t)|\chi_+(|t-s|-r)}{|t-s|}\,\mathrm{d}t\leqslant\frac{1}{r^{\beta}}\int_{-\infty}^{+\infty}\frac{\|f\|_{0,\varepsilon}\chi_+(|t-s|-r)}{(1+|t|)^{\varepsilon}|t-s|^{1-\beta}}\,\mathrm{d}t$$
$$\leqslant\text{const.}(\varepsilon,\beta)r^{-\beta}\|f\|_{0,\varepsilon}. \qquad (\text{B.49})$$

The estimate (A.80) follows from (B.47)–(B.49). □

Proof of Lemma A.8. The estimate (A.81) follows from (4.22), (A.43) with $p=1$, (A.63), (A.79). The estimate (A.83) follows from (4.22), (A.45), (A.63), (A.79). The estimate (A.85) follows from (A.81), (A.83). The estimate (A.86) follows from (4.22), (A.47), (A.79), (A.80) with $\delta=|\theta-\theta'|$, $r=|\theta-\theta'|^{-1}$, (A.63), (A.64) and the formula

$$HP_\theta^{\perp}v_\theta f(\theta^{\perp}x)-HP_{\theta'}^{\perp}v_{\theta'}f(\theta'^{\perp}x)$$
$$=H(P_\theta^{\perp}v_\theta f-P_{\theta'}^{\perp}v_{\theta'}f)(\theta^{\perp}x)+(HP_{\theta'}^{\perp}v_{\theta'}f(\theta^{\perp}x)-HP_{\theta'}^{\perp}v_{\theta'}f(\theta'^{\perp}x)). \qquad (\text{B.50})$$

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The estimate (A.87) follows from (4.22), (A.49) with $p=1$, (A.69), (A.70), (A.78) and the estimate

$$\frac{(2+r)^{\varepsilon}}{(1+|s|)^{\varepsilon}} \geqslant \chi_{+}(r+1-|s|), \quad s \in \mathbb{R}. \tag{B.51}$$

The estimate (A.88) follows from (4.22), (A.49) and (A.50) with $p=1$, (A.69), (A.70), (A.78), (B.51). The estimate (A.89) follows from (A.87), (A.88). The estimate (A.91) follows from (4.22), (A.51). The estimate (A.92) follows from (A.89), (A.91). The estimate (A.94) follows from (4.22), (A.54), (B.50), (A.79), (A.80), (A.69)–(A.71).

Lemma A.8 is proved. □

Proof of Lemma A.9. The estimate (A.96) follows from (A.85), (4.39), (4.40), the boundedness of $G_\lambda v_\lambda f(z)$ with respect to λ at fixed z and the maximum principle for holomorphic functions.

The proof of (A.97), (A.98) consists of the following:

$$\begin{aligned}
|G_\lambda v_\lambda f(z)| &\leqslant n\pi^{-1}\|f\|_0 \int_{\mathbb{C}} \frac{|\lambda||a_-(\zeta)| + |a_0(\zeta)| + |\lambda|^{-1}|a_+(\zeta)|}{|\lambda(\bar z - \bar\zeta) - \lambda^{-1}(z-\zeta)|}\,\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}} \\
&\overset{|\lambda|<1}{\leqslant} n\pi^{-1}\|f\|_0 \int_{\mathbb{C}} \frac{|\lambda|^2|a_-(\zeta)| + |\lambda||a_0(\zeta)| + |a_+(\zeta)|}{(1-|\lambda|^2)|z-\zeta|}\,\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}} \\
&\leqslant n\pi^{-1}(1-|\lambda|^2)^{-1} \\
&\qquad \times (\|a_+\|_{0,1+\varepsilon,\rho} + |\lambda|\|a_0\|_{0,1+\varepsilon,\rho} + |\lambda|^2\|a_-\|_{0,1+\varepsilon,\rho})\|f\|_0 \\
&\qquad \times \int_{\mathbb{C}} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|z-\zeta|(\rho+|\zeta|)^{1+\varepsilon}},
\end{aligned} \tag{B.52}$$

$$\begin{aligned}
|(G_\lambda v_\lambda - Ca_+)f(z)| &\leqslant n\pi^{-1}\|f\|_0 \int_{\mathbb{C}} \frac{|\lambda||a_0(\zeta)| + |\lambda|^2(|a_+(\zeta)| + |a_-(\zeta)|)}{|(z-\zeta) - \lambda^2(\bar z - \bar\zeta)|}\,\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}} \\
&\overset{|\lambda|<1}{\leqslant} n\pi^{-1}(1-|\lambda|^2)^{-1}|\lambda| \\
&\qquad \times (\|a_0\|_{0,1+\varepsilon,\rho} + |\lambda|(\|a_+\|_{0,1+\varepsilon,\rho} + \|a_-\|_{0,1+\varepsilon,\rho}))\|f\|_0 \\
&\qquad \times \int_{\mathbb{C}} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|z-\zeta|(\rho+|\zeta|)^{1+\varepsilon}},
\end{aligned} \tag{B.53}$$

$$\begin{aligned}
\int_{\mathbb{C}} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|z-\zeta|(\rho+|\zeta|)^{1+\varepsilon}} &= \biggl(\int_{|\zeta|\leqslant|z-\zeta|} + \int_{|\zeta|\geqslant|z-\zeta|}\biggr) \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|z-\zeta|(\rho+|\zeta|)^{1+\varepsilon}} \\
&\leqslant \int_{|\zeta|\leqslant|z-\zeta|} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|\zeta|(\rho+|\zeta|)^{1+\varepsilon}} + \int_{|\zeta|\geqslant|z-\zeta|} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|z-\zeta|(\rho+|z-\zeta|)^{1+\varepsilon}} \\
&\leqslant 2\int_{\mathbb{C}} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|\zeta|(\rho+|\zeta|)^{1+\varepsilon}} \\
&= 4\pi\varepsilon^{-1}\rho^{-\varepsilon}.
\end{aligned} \tag{B.54}$$

The proof of (A.99), (A.100) is completely similar to the proof of (A.97), (A.98).

The estimate (A.101) follows from (A.89), (A.91), (4.39), (4.40), the boundedness of $G_\lambda v_\lambda f(z)$ with respect to λ at fixed z and the maximum principle for holomorphic functions. Further, using (A.92) we obtain (A.102).

The proof of (A.103)–(A.106) is similar to the proof of (A.97)–(A.100); in this case, instead of (B.54) we use the estimate

$$
\begin{aligned}
&\int_{\mathbb{C}} \frac{\chi_+(r-|\zeta|)}{|z-\zeta|}\,\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}} \\
&\qquad = \left(\int_{|\zeta|\leqslant|z-\zeta|} + \int_{|\zeta|\geqslant|z-\zeta|}\right) \frac{\chi_+(r-|\zeta|)}{|z-\zeta|}\,\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}} \\
&\qquad \leqslant \int_{|\zeta|\leqslant|z-\zeta|} \frac{\chi_+(r-|\zeta|)}{|\zeta|}\,\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}} + \int_{|\zeta|\geqslant|z-\zeta|} \frac{\chi_+(r-|z-\zeta|)}{|z-\zeta|}\,\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}} \\
&\qquad \leqslant 2\int_{|\zeta|<r} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|\zeta|} \\
&\qquad = 4\pi r.
\end{aligned} \tag{B.55}
$$

Lemma A.9 is proved. □

Proof of Lemma A.10. The estimate (A.107) follows from (B.52), (B.54) and the following estimates

$$
\begin{aligned}
\int_{\mathbb{C}} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|z-\zeta|(\rho+|\zeta|)^{1+\varepsilon}} &\leqslant \int_{|\zeta|\leqslant|z-\zeta|} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|z/2|^{\varepsilon'}|\zeta|^{1-\varepsilon'}(\rho+|\zeta|)^{1+\varepsilon}} \\
&\qquad + \int_{|\zeta|\geqslant|z-\zeta|} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|z-\zeta|(\rho+|z/2|)^{\varepsilon'}(\rho+|z-\zeta|)^{1+\varepsilon-\varepsilon'}} \\
&\leqslant \frac{2^{1+\varepsilon'}}{|z|^{\varepsilon'}} \int_{\mathbb{C}} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|\zeta|(\rho+|\zeta|)^{1+\varepsilon-\varepsilon'}} \\
&= \frac{2^{2+\varepsilon'}\pi}{|z|^{\varepsilon'}(\varepsilon-\varepsilon')\rho^{\varepsilon-\varepsilon'}},
\end{aligned}
$$

$$
\begin{aligned}
&|G_\lambda v_\lambda f(z+h) - G_\lambda v_\lambda f(z)| \\
&\qquad \leqslant n\pi^{-1}\|f\|_0 \int_{\mathbb{C}} \frac{|h||\lambda^{-1}-\lambda|(|\lambda||a_-(\zeta)| + |a_0(\zeta)| + |\lambda|^{-1}|a_+(\zeta)|)}{|\lambda(\bar z+\bar h-\bar\zeta) - \lambda^{-1}(z+h-\zeta)||\lambda(\bar z-\bar\zeta) - \lambda^{-1}(z-\zeta)|}\,\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}} \\
&\qquad \overset{|\lambda|<1}{\leqslant} n\pi^{-1}\|f\|_0|h| \int_{\mathbb{C}} \frac{(1+|\lambda|^2)(|a_+(\zeta)| + |\lambda||a_0(\zeta)| + |\lambda|^2|a_-(\zeta)|)}{(1-|\lambda|^2)^2|z+h-\zeta||z-\zeta|}\,\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}} \\
&\qquad \leqslant n\pi^{-1}(1+|\lambda|^2)(1-|\lambda|^2)^{-2}(\|a_+\|_{0,1+\varepsilon,\rho} + |\lambda|\|a_0\|_{0,1+\varepsilon,\rho} + |\lambda|^2\|a_-\|_{0,1+\varepsilon,\rho}) \\
&\qquad\qquad \times \|f\|_0|h| \int_{\mathbb{C}} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|z+h-\zeta||z-\zeta|(\rho+|\zeta|)^{1+\varepsilon}},
\end{aligned} \tag{B.56}
$$

$$
\begin{aligned}
&\int_{\mathbb{C}} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|z+h-\zeta||z-\zeta|(\rho+|\zeta|)^{1+\varepsilon}} \\
&\qquad = \left(\int_{|z+h-\zeta|\leqslant|z-\zeta|} + \int_{|z+h-\zeta|\geqslant|z-\zeta|}\right) \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|z+h-\zeta||z-\zeta|(\rho+|\zeta|)^{1+\varepsilon}}
\end{aligned}
$$

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$$\overset{0<\delta<1}{\leqslant} \int_{|z+h-\zeta|\leqslant|z-\zeta|} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|z+h-\zeta|^{2-\delta}|h/2|^{\delta}(\rho+|\zeta|)^{1+\varepsilon}} + \int_{|z+h-\zeta|\geqslant|z-\zeta|} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|z-\zeta|^{2-\delta}|h/2|^{\delta}(\rho+|\zeta|)^{1+\varepsilon}}$$
$$\leqslant \frac{2^{\delta}}{|h|^{\delta}} \int_{\mathbb{C}} \left(\frac{1}{|z+h-\zeta|^{2-\delta}} + \frac{1}{|z-\zeta|^{2-\delta}}\right) \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{(\rho+|\zeta|)^{1+\varepsilon}}, \qquad \text{(B.57)}$$

$$\int_{\mathbb{C}} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|z-\zeta|^{2-\delta}(\rho+|\zeta|)^{1+\varepsilon}} \leqslant \frac{2^{1+\varepsilon'}}{|z|^{\varepsilon'}} \int_{\mathbb{C}} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|\zeta|^{2-\delta}(\rho+|\zeta|)^{1+\varepsilon-\varepsilon'}} \leqslant \frac{2^{1+\varepsilon'}}{|z|^{\varepsilon'}\rho^{\varepsilon-\varepsilon'}} \int_{\mathbb{C}} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|\zeta|^{2-\delta}(\rho+|\zeta|)}. \qquad \text{(B.58)}$$

The estimate (A.108) follows from (A.107), (A.98). The proof of (A.109), (A.110) is completely similar to the proof of (A.107), (A.108).

The proof of (A.111)–(A.114) is similar to the proof of (A.107)–(A.110); in this case, instead of (B.54), (B.57), (B.58) we use (B.55) and the estimates

$$\int_{\mathbb{C}} \frac{\chi_{+}(r-|\zeta|)}{|z-\zeta|}\,\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}} \overset{r\leqslant|z|/2}{\leqslant} \frac{2}{|z|} \int_{|\zeta|\leqslant r} \mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}} = \frac{2\pi r^{2}}{|z|}, \qquad \text{(B.59)}$$

$$\int_{\mathbb{C}} \frac{\chi_{+}(r-|\zeta|)\,\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|z+h-\zeta||z-\zeta|} \overset{0<\delta<1}{\leqslant} \frac{2^{\delta}}{|h|^{\delta}} \int_{\mathbb{C}} \left(\frac{1}{|z+h-\zeta|^{2-\delta}} + \frac{1}{|z-\zeta|^{2-\delta}}\right) \chi_{+}(r-|\zeta|)\,\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}, \qquad \text{(B.60)}$$

$$\int_{\mathbb{C}} \frac{\chi_{+}(r-|\zeta|)}{|z-\zeta|^{2-\delta}}\,\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}} \leqslant 2 \int_{|\zeta|\leqslant r} \frac{\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}}}{|\zeta|^{2-\delta}} = \frac{4\pi r^{\delta}}{\delta}, \qquad \text{(B.61)}$$

$$\int_{\mathbb{C}} \frac{\chi_{+}(r-|\zeta|)}{|z-\zeta|^{2-\delta}}\,\mathrm{d}\zeta_{\mathrm{R}}\,\mathrm{d}\zeta_{\mathrm{I}} \overset{r\leqslant|z|/2}{\leqslant} \frac{2^{2-\delta}\pi r^{2}}{|z|^{2-\delta}}. \qquad \text{(B.62)}$$

Lemma A.10 is proved. □

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