Macroeconomic Dynamics, **17**, 2013, 1519–1524. Printed in the United States of America.
doi:10.1017/S1365100512000107

NOTES

NOTE ON ERGODIC CHAOS IN THE RSS MODEL

PAULO S. A. SOUSA
University of Porto

We show the possibility of ergodic chaos in the RSS model, due to Robinson, Solow, and Srinivasan. Moreover, under a relevant parametric regime, we analytically characterize the unique invariant probability measure that describes the statistical properties of a typical trajectory of capital stocks.

Keywords: RSS Model, Two-Sector Model, Turbulence, Ergodic Chaos

1. INTRODUCTION

It is a well established fact that complicated dynamics can arise in optimal growth models. However, the possibility of chaos in one of these models, the so-called RSS model due to Robinson, Solow, and Srinivasan, remained unsuspected for a very long time. In effect, Stiglitz (1968) studies this particular model in continuous time and concludes that, under linearity of the felicity function and unrestricted discount factor, the optimal program monotonically converges to a modified golden-rule stock. Khan and Mitra (2005a) revisit this model in discrete time and, under certain parametric restrictions, prove that his results are not universally valid, because there are certain parametric regimes under which the optimal program does not satisfy the monotonicity property, and could cycle even in the undiscounted case. Khan and Mitra (2005b) reinvigorate their previous results and show that the discrete-time version of the RSS model can even exhibit topological chaos if the discount factor is sufficiently small. This result is somewhat surprising, given the disarming aura of simplicity that surrounds the model. Later, Metcalf (2008) derives a similar result in an undiscounted setting. The sharp and rich discrepancy of results between the discrete-time and the continuous-time versions of the RSS model imposes upon us the task of deepening the study of this model as an important priority.

Mathematicians have long warned us that the existence of topological chaos does not necessarily predict the observability of complicated dynamics. Day and Shafer (1987) raise this criticism with crystal clarity:

The author acknowledges correspondence with M. Ali Khan regarding several aspects of the RSS model, especially, ergodic chaos. Address correspondence to: Paulo S. A. Sousa, LIAAD-INESC Porto LA, Faculdade de Economia, Universidade do Porto, Rua Dr. Roberto Frias, s/n, 4200-464 Porto, Portugal; e-mail: psads.g@gmail.com.

 1365-1005/12

> the mere existence of "mathematical chaos" does not tell how important irregular fluctuations are likely to be—even if the equations in question were accurate analogs of some real world processes. This is because in nonlinear dynamics long run behavior depends on initial conditions: although chaotic trajectories may exist for an unaccountable number of initial conditions this "scrambled" set *may* have zero measure. If this were true then "most" randomly chosen initial conditions would lead to periodic cycles.

The purpose of this paper is to give an answer to this criticism by focusing on the discounted version of the RSS model that is considered in Khan and Mitra (2005b). We show that, in addition to topological chaos, ergodic chaos is also possible. Furthermore, in relevant parametric ranges, we derive and characterize analytically the density of the unique invariant probability measure that describes the statistical behavior of almost all trajectories (emanating from different initial conditions). As we will see, for most initial capital stocks, the respective trajectories spend equal time below and above the modified golden rule.

2. THE TWO-SECTOR VERSION OF THE RSS MODEL

Time is measured in discrete periods $t \in \mathbf{N}$, where $\mathbf{N}$ is the set of non-negative integers. There is a single consumption good, which is produced by infinitely divisible labor and machines, using a Leontief technology. The production of a unit of the consumption good requires a unit of labor and a unit of machinery. In the investment-goods sector, the only input is labor, with $a > 0$ units of labor producing a single machine. Machines depreciate at the rate $0 < d < 1$. The amount of available labor is constant over time and normalized to unity. The *transition possibility set* characterizes the admissible production plans (x, x'), where x' represents the amount of machines in the next period (tomorrow) from the amount x available in the current period (today), and is formalized by

$$\Omega = \{(x, x') \in \mathbf{R}^2_+ : \; x' - (1-d)x \geq 0 \text{ and } a(x' - (1-d)x) \leq 1\}.$$

The number of machines that are produced is given by $z \equiv (x' - (1-d)x)$, and $z \geq 0$ and $az \leq 1$ respectively define constraints on reversibility of investment and the use of labor. For any $(x, x') \in \Omega$, one can define the amount y of the machines available for production in the consumption-goods sector through a correspondence $\Lambda : \Omega \to \mathbf{R}_+$, with

$$\Lambda(x, x') = \{y \in \mathbf{R}_+ : \; 0 \leq y \leq 1 \text{ and } y \leq 1 - a(x' - (1-d)x)\}.$$

Only the consumption good can promote welfare, in the model assumed to be linear and normalized, in the sense that y units of the consumption good yield a welfare level y. A *reduced-form utility function*, $u : \Omega \to \mathbf{R}_+$ with $u(x, x') = \max\{y \in \Lambda(x, x')\}$, measures the maximum welfare level that can be obtained today, if one starts with x machines today, and ends up with x' machines tomorrow. Intertemporal preferences are represented by the present value of the stream of welfare levels discounted at a factor $\rho \in (0, 1)$.

An *economy* is represented by the triple (a, d, ρ). A program from x_0 is a sequence $\{x(t), y(t)\}$ such that $x(0) = x_0$, and, for all $t \in \mathbf{N}$, $(x(t), x(t+1)) \in \Omega$ and $y(t) = \max\ \Lambda(x(t), x(t+1))$. A *program* $\{x(t), y(t)\}$ is simply a program from $x(0)$. For any program, there is an underlying gross *investment sequence* $\{z(t+1)\}$, defined by $z(t+1) = (x(t+1) - (1-d)x(t))$ for all $t \in \mathbf{N}$. It is straightforward to check that every program $\{x(t), y(t)\}$ is bounded by $\max\ \{x(0), 1/ad\} \equiv M(x(0))$, which implies that $\sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) < \infty$. A program $\{\overline{x}(t), \overline{y}(t)\}$ from x_0 is *optimal* when $\sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) \leq \sum_{t=0}^{\infty} \rho^t u(\overline{x}(t), \overline{x}(t+1))$ for every program $\{x(t), y(t)\}$ from x_0.

3. PRELIMINARIES

Consider a probability space (X, Σ, μ), where X is a set, Σ is a σ-algebra of subsets of X, and μ is a probability measure, and furthermore consider a Σ-measurable map $h : X \to X$. The probability measure μ is *invariant under h* if $\mu(E) = \mu(h^{-1}(E))$ for all $E \in \Sigma$. We say that the probability measure μ is *ergodic* if "$E \in \Sigma,\ h^{-1}(E) = E$" implies "$\mu(E) = 0$ or $\mu(E) = 1$." The map h is *nonsingular* if $\mu(h^{-1}(E)) = 0$ for all $E \in \Sigma$ such that $\mu(E) = 0$. The dynamical system (X, h) exhibits *ergodic chaos*, if there exists an absolutely continuous (with respect to Lebesgue measure) probability measure μ on X, which is invariant and ergodic under h.

At the heart of our proof is the *Frobenius–Perron operator*. In what follows, we will define this operator and state needed theorems for later reference.

DEFINITION 1. *Let (X, Σ, ν) be a measure space. If $h : X \to X$ is nonsingular, the unique operator $P : L^1 \to L^1$ defined by the equation*

$$\int_E Pf(x)\nu(dx) = \int_{h^{-1}(E)} f(x)\nu\ (dx), \quad \textit{for all } E \in \Sigma,$$

is called the Frobenius–Perron operator corresponding to h.

As discussed in Lasota and Mackey (1985, p. 38), if $X = [\alpha, \beta]$ is an interval on the real line $\mathbf{R}$, and $E = [\alpha, x]$, then an explicit form for Pf can be obtained from

$$Pf(x) = \frac{d}{dx}\left(\int_{h^{-1}(E)} f(s)\ ds\right).$$

In the same reference (pp. 46 and 55), Lasota and Mackey report two very useful results:

THEOREM 2. *Let (X, Σ, ν) be a measure space, $h : X \to X$ a nonsingular map, and P the Frobenius–Perron operator associated with h. Consider a nonnegative $f \in L^1$. Then a measure ν_f given by*

$$\nu_f(E) = \int_E f\nu\ (dx)$$

is invariant if and only if f is a fixed point of P.

THEOREM 3. *Let (X, Σ, ν) be a measure space, $h : X \to X$ a nonsingular map, and P the Frobenius–Perron operator associated with h. If h is ergodic, then there is at most one stationary density f_* of P.*

4. OPTIMAL ERGODIC CHAOS

Khan and Mitra (2009) show that the so-called check map, which is defined by

$$h(x) = \begin{cases} \frac{1}{a} - \xi x, & \text{if } x \in [0, 1] \\ (1-d)x, & \text{if } x \in [1, +\infty), \end{cases}$$

$\xi \equiv 1/a - (1-d)$, is the optimal policy function for the two-sector RSS model, provided that the discount factor is sufficiently small. Thus, to prove the existence of ergodic chaos in the RSS model, we only need to prove that the check map can exhibit ergodic chaos.

THEOREM 4. *Under the parametric restriction $\xi(1-d) = 1 + (1/\xi)(1-d) > 1$, there exists an invariant probability measure under the check map, μ, whose density is*

$$f(x) = \begin{cases} \dfrac{a\xi\,(1+ad)}{1+2a^2(-1+d)\,d\xi + a\,(-1+d-\xi+2d\xi)}, & \text{if } (1-d) \le x \le \hat{x} \\ \dfrac{a\,(1+ad)}{1+2a^2(-1+d)\,d\xi + a\,(-1+d-\xi+2d\xi)}, & \text{if } \hat{x} < x \le \frac{1}{a} - \xi(1-d), \end{cases}$$

where $\hat{x} = 1/\,(1+ad)$ is the modified golden-rule stock.

Proof. We begin the proof by defining a conjectured probability density,

$$f(x) = \begin{cases} k_1, & \text{if } (1-d) \le x \le \hat{x} \\ k_2, & \text{if } \hat{x} < x \le \frac{1}{a} - \xi\,(1-d)\,, \end{cases}$$

with k_1 and k_2 positive.

Considering this density, the Frobenius–Perron operator when $x \le \hat{x}$ is

$$Pf(x) = \frac{d}{dx}\left(\int_{\frac{1-ax}{\xi a}}^{\frac{x}{1-d}} k_2 dt\right) = \frac{d}{dx}\left(k_2\left(\frac{x}{1-d} - \frac{1-ax}{a\xi}\right)\right)$$
$$= k_2\left(\frac{1}{1-d} + \frac{1}{\xi}\right).$$

By Theorem 2, μ is invariant under h if

$$Pf(x) = f(x) \Longleftrightarrow k_1 = \left(\frac{1}{1-d} + \frac{1}{\xi}\right)k_2. \tag{1}$$

https://doi.org/10.1017/S1365100512000107 Published online by Cambridge University Press

When $x > \hat{x}$, the Frobenius–Perron operator is

$$Pf(x) = \frac{d}{dx}\left(\int_{\frac{1-ax}{\xi a}}^{\hat{x}} k_1 dt + \int_{\hat{x}}^{\frac{1}{a}-\xi(1-d)} k_2 dt\right)$$
$$= \frac{d}{dx}\left(k_1\left(\frac{1}{1+ad} - \frac{1}{a\xi} + \frac{x}{\xi}\right) + k_2\left(\frac{1}{a} - \frac{1}{1+ad} - (1-d)\,\xi\right)\right) = \frac{k_1}{\xi}.$$

Again, by Theorem 2, the invariance of μ under h requires

$$Pf(x) = f(x) \iff \frac{k_1}{\xi} = k_2 \iff k_1 = \xi k_2. \qquad \textbf{(2)}$$

Equations (1) and (2) must be satisfied simultaneously. To escape from the trivial solution $k_1 = k_2 = 0$, the following equality must hold:

$$\xi = \frac{1}{1-d} + \frac{1}{\xi} \iff \left(\xi - \frac{1}{\xi}\right)(1-d) = 1.$$

This equality is verified given the parametric restriction assumed in the statement of the theorem.

We still have to guarantee that k_1 and k_2 for the nontrivial solution are both positive. To be a valid probability density, k_2 has to verify

$$\int_{1-d}^{\hat{x}} \xi k_2 \, dt + \int_{\hat{x}}^{\frac{1}{a}-\xi(1-d)} k_2 \, dt = 1.$$

This implies that

$$k_2 = \frac{a\,(1+ad)}{1 + 2a^2\,(-1+d)\,d\xi + a\,(-1+d-\xi+2d\xi)}.$$

Obviously, the numerator is positive. Let us show that the denominator

$$\gamma = 1 + 2a^2\,(-1+d)\,d\xi + a\,(-1+d-\xi+2d\xi)$$

is also positive. In effect,

$$\begin{aligned}\gamma &= 1 + 2a^2(-1+d)d\xi + a\,(-1+d-\xi+2d\xi)\\ &= 1 + 2a^2(-1+d)d\xi + a\left(-\frac{1}{a} + 2d\xi\right)\\ &= 2ad\xi - 2a^2(1-d)d\xi = 2ad\xi\,(1 - a\,(1-d))\\ &= 2a^2d\xi^2.\end{aligned}$$

Consequently, $k_2 > 0$. Because $k_1 = \xi k_2$, k_1 is also positive, as the restriction $\xi(1-d) > 1$ implies that $\xi > 0$. The proof is complete. ■

Based on this theorem, it is easy to prove the main result of this article.

https://doi.org/10.1017/S1365100512000107 Published online by Cambridge University Press

THEOREM 5. *Under the parametric restriction* $\xi(1-d) = 1+(1/\xi)(1-d) > 1$*, the check map exhibits ergodic chaos.*

Proof. By using the definition of ergodic measure, it is easy to establish that the invariant probability measure determined in Theorem 4 is ergodic under the check map. Consequently, under the given parametric restriction, the check map exhibits ergodic chaos. ■

By combining Theorem 4 and Theorem 5, and subsequently invoking Theorem 3, we can establish the following corollary.

COROLLARY 6. *The invariant probability measure indicated in Theorem 4 is unique.*

The results of this section can be obtained using a completely different approach [see Khan and Mitra (2012, Section 5)].

5. CONCLUDING REMARKS

In this paper, we demonstrate the existence of ergodic chaos in the RSS model under a specific parametric regime. This result sharpens the contrast between the discrete-time and the continuous-time versions of the model, and we hope that this now-discovered contrast will stimulate the scientific interest of the audience of this paper in this model.

Constituting only a first step forward into the study of ergodic chaos in the context of the RSS model, our study leaves open the two following important problems: first, the complete delineation of the parametric restrictions compatible with ergodic chaos; second, the full characterization of the underlying invariant probability measure whenever ergodic chaos exists.

REFERENCES

Day, R.H. and W. Shafer (1987) Ergodic fluctuations in deterministic economic models. *Journal of Economic Behavior and Organization* 8, 339–361.

Khan, M.A. and T. Mitra (2005a) On choice of technique in the Robinson–Solow–Srinivasan model. *International Journal of Economic Theory* 1, 83–110.

Khan, M.A. and T. Mitra (2005b) On topological chaos in the Robinson–Solow–Srinivasan model. *Economics Letters* 88, 127–133.

Khan, M.A. and T. Mitra (2009) Complicated Dynamics and Parametric Restrictions in the Robinson–Solow–Srinivasan (RSS) Model. Mimeo, Johns Hopkins University.

Khan, M.A. and T. Mitra (2012) Long-run optimal behavior in a two-sector Robinson–Solow–Srinivasan model. *Macroeconomic Dynamics* 16(Suppl. 1), 70–85.

Lasota, A. and M.C. Mackey (1985) *Probabilistic Properties of Deterministic Systems*. New York: Cambridge University Press.

Metcalf, C.J. (2008) The dynamics of the Stiglitz policy in the RSS model. *Chaos, Solitons and Fractals* 37, 652–661.

Stiglitz, J. (1968) A note on technical choice under full employment in a socialist economy. *Economic Journal* 78, 603–609.