Math. Struct. in Comp. Science (2015), *vol.* 25, *pp.* 1432–1462.
doi:10.1017/S0960129514000085 First published online 16 February 2015

Weak symmetry breaking and abstract simplex paths

DMITRY N. KOZLOV

Department of Mathematics, University of Bremen,
28334 Bremen, Federal Republic of Germany
Email: dfk@math.uni-bremen.de

Received 24 April 2013; revised 7 March 2014

Motivated by questions in theoretical distributed computing, we develop the combinatorial theory of abstract simplex path subdivisions. Our main application is a short and structural proof of a theorem of Castañeda and Rajsbaum. This theorem in turn implies the solvability of the weak symmetry breaking task in the immediate snapshot wait-free model in the case when the number of processes is not a power of a prime number.

1. Introduction

The mathematical research presented here is motivated by problems of theoretical distributed computing. Even though, the computer science background is, strictly speaking, not necessary to read the definitions and the proofs in this paper, we feel it is beneficial to review the broader context, before delving into the description of our results. The reader who is only interested in the mathematical part, may skip our explanations and proceed directly to Section 3 at this point.

In theoretical distributed computing, one studies solvability of standard tasks under various computational models. The computational model which we consider here is the following. We have $n + 1$ *asynchronous* processes, which communicate with each other using the read/write shared memory, which in turn consists of atomic registers assigned to individual processes. The operations allowed for each process are *write* and *snapshot read*. The operation *write* writes whatever value the process wants into its register in the shared memory. The operation *snapshot read* reads the entire shared memory atomically.

Each process executes a wait-free protocol; with crash failures allowed. A *crash failure* means that the process which failed simply stops executing its protocol, rather than, say, sending the wrong information. The protocol is called *wait-free*, if, intuitively speaking, the processes are not allowed to *wait* for each other, i.e., to make their executions contingent on hearing from other processes. In practical terms, this means that when a process A did not hear from the process B for a while, and now needs to decide on an output value, its decision must be such that the total produced output will be valid, no matter whether the process B crashed, or whether it will suddenly revive its execution once A has chosen its output. Due to asynchrony, the process A cannot distinguish between these two options at this point of the execution.

For brevity, we refer to this entire computational model as *read/write wait-free* model. There is a large class of computational models, which have been proved to be equivalent to each other. Since this class includes the read/write wait-free model, all the statements pertaining to solvability of certain tasks proved for this model are actually quite general, and say something about the entire subject of wait-free asynchronous computation. We refer the interested reader to Attiya and Welch (2004), Herlihy (1991), Herlihy *et al.* (2014) and Herlihy and Shavit (1999) for further background.

Once the computational model is fixed, it becomes interesting to understand which tasks are solvable in this model. To gain structural insight, one concentrates on the questions of solvability of the so-called *standard tasks*. The standard task which is central to this paper is the so-called *weak symmetry breaking task*, which will be abbreviated to WSB. In this task, the processes have no input values, just their ID's. Their output values are 0 and 1. The task is solvable if there exists an read/write wait-free protocol such that in every execution in which all processes decide on an output value, not all processes decide on the same one. In addition, this protocol is required to be *rank-symmetric* in the sense which we now describe.

Let $I_1, I_2 \subseteq \{0, \dots, n\}$, such that $|I_1| = |I_2|$, and let $r : I_1 \to I_2$ be the order-preserving bijection. Assume E_1 is an execution of the protocol, in which the set of participating processes is I_1 and E_2 is an execution of the protocol, in which the set of participating processes is I_2. Then, the protocol is called rank-symmetric if for every $i \in I_1$, the process with ID i must decide on the same value as the process with ID $r(i)$. The protocol will certainly be rank-symmetric, if each process only compares its ID to the ID's of the other participating processes, and makes the final decision based on the relative rank of its ID.

Another important task is *renaming*. The renaming task, see Attiya *et al.* (1990), is defined as follows: the $n+1$ processes here have unique input names, taken from some large universe of inputs, and need to decide on unique output names from the set $\{0, \dots, K\}$. The processes also have process ID's labelled $0, \dots, n$, and it is requested for the protocol to be *anonymous*. The protocol is called anonymous if its execution does not depend on the process ID, only the process input value, and whatever information the process receives during the execution. In other words, two different processes should decide on an identical output value if they receive same inputs and same information from the outside world, even though they will have different ID's. It has been proved that for $n+1$ processes the WSB is equivalent to $(2n-1)$-renaming in the read/write wait-free model. So deciding whether WSB is solvable will also tell us whether $(2n-1)$-renaming for $n+1$ processes is solvable. We refer the reader to Attiya and Welch (2004), Herlihy *et al.* (2014) and Gafni *et al.* (2006) for further information on the renaming task.

2. Using simplex paths to construct a protocol

2.1. *Solvability of the weak symmetry breaking*

At present time, the solvability of WSB in the read/write wait-free model is completely understood. For some time, it was believed that WSB is not solvable for any number of processes. However, this turned out to be incorrect. The correct answer is

Weak Symmetry Breaking for $n+1$ processes is solvable in read/write wait-free model if and only if $n+1$ is not a prime power.

https://doi.org/10.1017/S0960129514000085 Published online by Cambridge University Press

While impossibility of WSB when $n+1$ is a prime power has been known for a while, the solvability of WSB when $n+1$ is *not* a prime power (the smallest example here is clearly when we have 6 processes) has involved a few turns in the literature. The final point in the matter was put by Castañeda and Rajsbaum (2012a), see also Attiya *et al.* (1990) for the prehistory. The important and interesting paper of Castañeda and Rajsbaum serves as a motivation and the entry point for the research presented in this article. We recommend that the reader acquaints himself with the contents of Castañeda and Rajsbaum (2012a).

There is a way of reducing the computability of WSB in read/write wait-free model to a purely mathematical existence question. It involves the anonymous computability theorem of Herlihy and Shavit (1999), so its complete description would be too technical for the current presentation. Therefore, for the sake of brevity, we opt to confine ourselves to presenting a mathematical formulation directly, and then referring the reader, who is interested in the better understanding of the connection between distributed computing and the topological context to the above mentioned work of Castañeda and Rajsbaum (2012a), as well as to Herlihy *et al.* (2014). The necessary background in combinatorial topology can be found in Kozlov (2008), while more on the topology of protocol complexes can be found in Kozlov (2012, 2013). In the remainder of this section we give an equivalent reformulation of the solvability of WSB in read/write wait-free model in terms of simplicial subdivisions.

2.2. *Notations*

First, we introduce some terminology. We let $[n]$ denote the set $\{0,\dots,n\}$, in particular, $[1]=\{0,1\}$ will denote the set of boolean values. For a boolean value c we let $\bar{c}$ denote the negation of c. For every $n \geqslant 1$ we let Δ^n denote the so-called standard n-simplex. This is a geometric simplicial complex, which has a unique n-simplex, spanned by the unit coordinate vectors in $\mathbb{R}^{n+1}$. The vertices of Δ^n are indexed by the elements of the set $[n]$. More generally, the k-dimensional boundary simplices of Δ^n are indexed by the subsets $I \subseteq [n]$, such that $|I| = k+1$; we shall denote such a boundary simplex Δ^I and identify its set of vertices with I. For every two subsets $I,J \subseteq [n]$, such that $|I| = |J|$, there exists a unique order-preserving bijection $r_{I,J} : I \to J$. This order-preserving bijection induces a linear isomorphism $\varphi_{I,J} : \Delta^I \to \Delta^J$.

A finite geometric simplicial complex $\mathrm{D}iv\,(\Delta^n)$ is called a *subdivison* of Δ^n if every simplex of $\mathrm{D}iv\,(\Delta^n)$ is contained in a simplex of Δ^n, and every simplex of Δ^n is the union of those simplices of $\mathrm{D}iv\,(\Delta^n)$ which it contains. Clearly, for every $I \subseteq [n]$, we have an induced subdivision $\mathrm{D}iv\,(\Delta^I)$. For each simplex $\sigma \in \mathrm{D}iv\,(\Delta^n)$, we let $\mathrm{s}upp\,(\sigma)$ denote the unique simplex τ of Δ^n such that the interior of σ is contained in the interior of τ. Finally, for an arbitrary simplicial complex K, we let $V(K)$ denote the set of vertices of K.

2.3. *Hereditary subdivisions and compliant labellings*

The next definition is crucial for the simplicial approach to the rank-symmetric protocols.

Definition 2.1. A subdivision $\mathrm{D}iv\,(\Delta^n)$ of the standard n-simplex Δ^n is called **hereditary** if for all $I,J \subseteq [n]$ such that $|I| = |J|$, the linear isomorphism $\varphi_{I,J}$ is also an isomorphism of the subdivision restrictions $\mathrm{D}iv\,(\Delta^I)$ and $\mathrm{D}iv\,(\Delta^J)$.

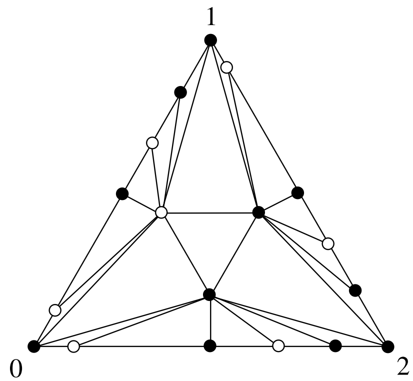

Fig. 1. A hereditary subdivision with a compliant binary labelling.

A labelling $\lambda : V(\mathrm{D}iv\,(\Delta^n)) \to L$, where L is an arbitrary set of labels, is called **compliant** if for all $I, J \subseteq [n]$ such that $|I| = |J|$, and all vertices $v \in V(\mathrm{D}iv\,(\Delta^I))$, we have

$$\lambda(\varphi_{I,J}(v)) = \lambda(v). \quad (1)$$

An example of a hereditary subdivision with a compliant binary labelling is shown in Figure 1.

Definition 2.2. Given a subdivision $\mathrm{D}iv\,(\Delta^n)$, the labelling $\chi : V(\mathrm{D}iv\,(\Delta^n)) \to [n]$ is called a *colouring* if

1. any two vertices of $\mathrm{D}iv\,(\Delta^n)$, which are connected by an edge, receive different colours;
2. for any $v \in V(\mathrm{D}iv\,(\Delta^n))$, we have $\chi(v) \in V(supp\,(v))$.

A subdivision which allows a colouring is called *chromatic*.

Clearly, not every subdivision is chromatic, however, if such a colouring exists, then it must be unique. The subdivision shown in Figure 1 is chromatic.

The next theorem provides the key bridge between the solvability of the WSB as a distributed computing task, and the mathematics of compliant labellings of hereditary chromatic subdivisions.

Theorem 2.3. The task WSB for $n+1$ processes is solvable in read/write wait-free model if and only if there exists a hereditary chromatic subdivision $\mathrm{D}iv\,(\Delta^n)$ of Δ^n and a binary compliant labelling $b : V(\mathrm{D}iv\,(\Delta^n)) \to \{0, 1\}$, such that for every n-simplex $\sigma \in \mathrm{D}iv\,(\Delta^n)$ the restriction map $b : V(\sigma) \to \{0, 1\}$ is surjective.

In general, if all vertices of an n-simplex $\sigma \in \mathrm{D}iv\,(\Delta^n)$ get the same label under b, one calls σ *monochromatic*, so the surjectivity condition is equivalent to saying that there are no monochromatic n-simplices under b. We shall also talk of 0-monochromatic and 1-monochromatic n-simplices when we need to specify the label assigned to the vertices of σ. The Theorem 2.3 is due to the work of Herlihy and Shavit. As we said earlier, we assume its proof for the purposes of this paper, and proceed with a purely mathematical analysis of the conditions.

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In Castañeda and Rajsbaum (2012a), they proved that if $n+1$ is not a prime power, there exists a hereditary chromatic subdivision $\mathrm{D}iv\,(\Delta^n)$ together with a binary compliant labelling $b : V(\mathrm{D}iv\,(\Delta^n)) \to \{0,1\}$, satisfying conditions of Theorem 2.3. This settles the solvability of WSB in read/write wait-free model when the number of processes is not a prime power. The idea of the proof of Castañeda and Rajsbaum is as follows:

— start with some $\mathrm{D}iv\,(\Delta^n)$ and some labelling b, which may have monochromatic simplices;
— connect the monochromatic simplices if possible by simplicial paths of even lengths;
— subdivide each path further, so as to eliminate the two monochromatic end n-simplices.

2.4. *Geometric simplex paths*

The hard part, and the crux of the proof in Castañeda and Rajsbaum (2012a) lies in a sophisticated and ingenious algorithm for the simplicial path subdivision. Let us define the necessary notions.

Definition 2.4. Let K be an arbitrary n-dimensional geometric simplicial complex. A **geometric simplex path** Σ is an ordered tuple $(\sigma_1,\dots,\sigma_l)$ of distinct n-simplices of K, such that $\sigma_i \cap \sigma_{i+1}$ is an $(n-1)$-dimensional simplex of K, for all $i = 1,\dots,l-1$.

We say that the path Σ has dimension n and length l. The best way to visualize a geometric simplex path is to imagine n-simplices glued along their boundary $(n-1)$-simplices to form a path. Walking along the path from σ_i to σ_{i+1} may be visualized as flipping over the $(n-1)$-simplex $\sigma_i \cap \sigma_{i+1}$. Given a path $\Sigma = (\sigma_1,\dots,\sigma_l)$, we define the *interior* of Σ to be the open set

$$int\,\Sigma := int\,\sigma_1 \cup \cdots \cup int\,\sigma_l \cup int\,(\sigma_1 \cap \sigma_2) \cup \cdots \cup int\,(\sigma_{l-1} \cap \sigma_l).$$

Definition 2.5. Assume $\mathrm{D}iv\,(\Delta^n)$ is a hereditary chromatic subdivision of the standard n-simplex and $b : V(\mathrm{D}iv\,(\Delta^n)) \to \{0,1\}$ is a binary labelling. A simplex path $\Sigma = (\sigma_1,\dots,\sigma_l)$ is said to be in *standard form* if

1. the number l is even,
2. the end simplices σ_1 and σ_l are 0-monochromatic,
3. the rest of the simplices $\sigma_2,\dots,\sigma_{l-1}$ are not monochromatic.

The following result is due to Castañeda and Rajsbaum (2012a), where it is formulated, using a slightly different language, as a collection of lemmas.

Theorem 2.6. Assume we are given a hereditary chromatic subdivision $\mathrm{D}iv\,(\Delta^n)$, a binary labelling $b : V(\mathrm{D}iv\,(\Delta^n)) \to \{0,1\}$, and an n-dimensional geometric simplex path Σ in standard form. Then there exists a chromatic subdivision $\mathcal{S}(\mathrm{D}iv\,(\Delta^n))$ of the geometric simplicial complex $\mathrm{D}iv\,(\Delta^n)$, such that

1. only the interior of Σ is subdivided;
2. it is possible to extend the binary labelling b to $\mathcal{S}(\mathrm{D}iv\,(\Delta^n))$ in such a way that $\mathcal{S}(\Sigma)$ has no monochromatic n-simplices.

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Note, that the obtained subdivision $\mathcal{S}(\mathrm{D}iv\,(\Delta^n))$ is automatically hereditary, since only the simplices in the interior of the path Σ, and hence in the interior of the standard simplex Δ^n are subdivided further.

The desire to understand this 'engine' of the entire proof of Castañeda and Rajsbaum has been the driving force behind the research in this paper. Our purpose here is twofold. On one hand we want to develop a self-contained mathematical theory of combinatorial simplex path subdivisions, motivated by questions in theoretical distributed computing. On the other hand, we want that our mathematical theory yields a simpler, more structural and more concise proof of Theorem 2.6. The rest of the paper is devoted to setting up and applying this combinatorial theory.

The specific plan for the paper is as follows. In Chapter 3 we introduce notations and define the main objects of study: the abstract simplex paths. In Chapter 4 we define different transformations of these abstract simplex paths which are of two basic types: the vertex and the edge expansions. In Chapter 5 more specific transformations are introduced, which we call summit and plateau moves. This terminology comes from the local shapes in the height graph of the path. In Chapter 6 we use this toolbox to prove our main theorem which says that any admissible abstract simplex path is reducible. We then obtain a new proof of Theorem 2.6 as a corollary of our combinatorial statement. Finally in Chapter 7 we collect further useful information, which is not directly needed for the proof of our main result.

An alternative approach to the proof of Castañeda and Rajsbaum has appeared in a recent work of Attiya, Castañeda, Herlihy and Paz, see Attiya *et al.* (2013). An interested reader should consult this paper as well for a complete picture.

3. The combinatorics of path subdivisions

In the previous motivational sections, we have perused the distributed computing context and reduced our question to a completely mathematical formulation. Within mathematics, all previous work has led one to study subdivisions of geometric simplicial complexes, see Theorems 2.3 and 2.6. In this paper, we make a further step from geometry to combinatorics. To this end, we shall define *abstract simplex paths*, as well as combinatorial operations corresponding to the geometric subdivisions. We shall then develop combinatorics of path subdivisions in a fully self-contained fashion, formally independent on the distributed computing or geometric context.

However, before we can proceed with our main definitions, we still need some notations from standard combinatorics, which we now introduce in Subsections 3.1 and 3.2.

3.1. *Tuples*

3.1.1. *Definition and notations.*

Definition 3.1. Let U be an arbitrary set. For an ordered set $S = \{s_1, \dots, s_k\}$, with $s_1 < \cdots < s_k$, an **S-tuple** T with elements from U is an ordered sequence $T = (a_{s_1}, \dots, a_{s_k})$, such that $a_{s_i} \in U$, for $1 \leqslant i \leqslant k$.

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For $s \in S$, we set $T_s := a_s$. An S-tuple $(a_{s_1}, \dots, a_{s_k})$ is said to have *length* k. We also write $U(T)$ to denote the universe set associated to an S-tuple T. We shall now go through further terminology involving S-tuples.

Intervals. For $i, j \in S$, $i \leqslant j$ and an S-tuple T, we set $T[i, j] := (T_i, \dots, T_j)$.

n-tuples. For each natural number n, we shall for the sake of brevity say *n-tuple* instead of $\{1, \dots, n\}$-tuple. Note, that in our notations, n-tuples and $[n]$-tuples are slightly different mathematical objects.

S-tuples equal to 0. We write $T = 0$, for an S-tuple T, if $0 \in U(T)$ and $T_s = 0$ for all $s \in S$.

Monochromatic S-tuples. An S-tuple T is called *monochromatic* if for some $w \in U(T)$ we have $T_s = w$, for all $s \in S$. If we need to be more specific we shall say *w-monochromatic.*

The height of an S-tuple. Assume T is an S-tuple with numerical elements, where we include the case of boolean values by interpreting them as numbers 0 and 1. We set $h(T) := \sum_{s \in S} T_s$, and call the value $h(T)$ the *height* of T.

The index set of occurrences of an element in an S-tuple. Given an S-tuple T, and an element $q \in U(T)$, we set

$$O(q, T) := (s \in S \mid T_s = q).$$

This is the index set of occurrences of q in T. We use the round brackets to emphasize that $O(q, T)$ is an ordered tuple, not just a set.

Number of occurrences of an element in an S-tuple. We let $\sharp(q, T)$ denote the number of occurrences of q in T, i.e., we set $\sharp(q, T) := |O(q, T)|$.

The last occurrence of an element in an S-tuple. We let $\mathrm{last(q, T)}$ denote the index of the last occurrence of q in T, i.e., $\mathrm{last(q, T) := max(O(q, T))}$. If $O(q, T) = \emptyset$, then we set $\mathrm{last(q, T)} := \infty$.

3.1.2. *Concatenation of tuples.* Given two disjoint ordered sets S' and S'', we can define a new ordered set $S = S' \cup S''$, by specifying the new order relation as follows:

$$s_1 < s_2, \text{ if } \begin{cases} s_1, s_2 \in S', \text{ and } s_1 < s_2; \\ s_1, s_2 \in S'', \text{ and } s_1 < s_2; \\ s_1 \in S' \text{ and } s_2 \in S''. \end{cases}$$

We write $S = S' \circ S''$.

Assume furthermore, that T' is an S'-tuple of length l', and T'' is an S''-tuple of length l''. We define a new S-tuple T by setting $T := T' \cup T''$ as a multi-set, and then ordering the elements according to the order on $S = S' \circ S''$. We call T the *concatenation* of T' and T'' and write $T = T' \circ T''$.

3.2. *The directed edge graph of the cube*

3.2.1. *Definition of Cube$_{[n]}$.* Assume $n \geqslant 1$, and let $\mathrm{Cube}_{[n]}$ denote the directed edge graph of the unit cube in $\mathbb{R}^{[n]}$. The vertices of $\mathrm{Cube}_{[n]}$ are labelled by all $[n]$-tuples of 0's and

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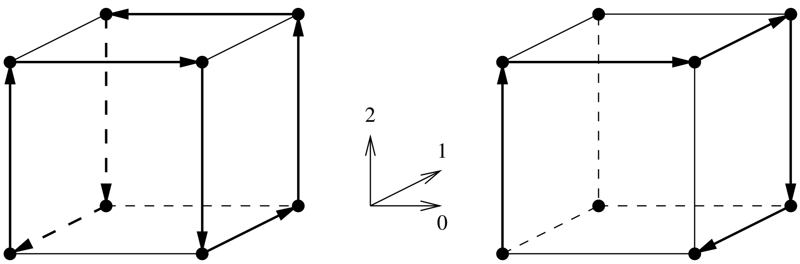

Fig. 2. A [2]-cube loop, and a [2]-cube 0-path.

1's, and edges connect those $[n]$-tuples which differ in exactly one coordinate. We shall view the $[n]$-tuples of 0's and 1's as functions $\alpha : [n] \to [1]$.

$\text{Cube}_{[n]}$ is a regular graph, where every vertex has both the in-degree and the out-degree equal to $n+1$. More specifically, for every vertex $v \in V(\text{Cube}_{[n]})$ and every $i \in [n]$, there exists a unique edge parallel to the ith axis, which has v as a source. In total, the graph $\text{Cube}_{[n]}$ has 2^{n+1} vertices and $(n+1) \cdot 2^{n+1}$ edges.

We shall use the standard graph terminology applied to the graph $\text{Cube}_{[n]}$ such as *directed cycle* and *directed path*. In particular, a *partial matching* on $\text{Cube}_{[n]}$ is a triple (A, B, φ), where A and B are disjoint sets of vertices of $\text{Cube}_{[n]}$ and $\varphi : A \to B$ is a bijection, such that for each $v \in A$, the vertices v and $\varphi(v)$ are connected by an edge (in this case they are of course automatically connected by 2 edges).

3.2.2. *Cube loops.*

Definition 3.2. An $[n]$-*cube loop* is a directed cycle in $\text{Cube}_{[n]}$, which contains the origin, and does not have any self-intersections.

Clearly, to describe an $[n]$-cube loop one can start at the origin and then list the directions of the edges as we trace the loop. This procedure will yield a t-tuple $(q_1, \dots, q_t)$, with $q_i \in [n]$, for all $1 \leqslant i \leqslant t$. Conversely, given such a t-tuple $Q = (q_1, \dots, q_t)$, it describes an $[n]$-cube loop if and only if the following conditions are satisfied:

1. *the cycle condition:* for every $l \in [n]$, the number $\sharp(l, Q)$ is even;
2. *no self-intersections:* for all $1 \leqslant i < j \leqslant t$, such that $(i, j) \neq (1, t)$, there exists $l \in [n]$, such that the number $\sharp(l, Q[i, j])$ is odd.

In the continuation, we shall usually describe an $[n]$-cube loop by taking a tuple $(q_1, \dots, q_t)$ satisfying the conditions above. An example of a [2]-cube loop is shown on the left-hand side of Figure 2. The corresponding t-tuple is $(2, 0, 2, 1, 2, 0, 2, 1)$.

3.2.3. *Cube paths.*

Definition 3.3. Assume $p \in [n]$. An $[n]$-**cube** p-**path** is a directed path I in the graph $\text{Cube}_{[n]}$, such that

— I starts at the origin, and ends in the vertex $(0, \dots, 0, 1, 0, \dots, 0)$, where 1 is in position p;

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— I does not have any self-intersections;
— I contains exactly one edge parallel to the pth axis.

In line with the situation of the $[n]$-cube loops, the $[n]$-cube p-path is given by a tuple $(q_1, \ldots, q_t)$, where $q_1, \ldots, q_t \in [n] \setminus \{p\}$, together with a number $1 \leqslant s \leqslant t-1$, subject to the following conditions:

1. *the path condition:* for every $l \in [n] \setminus \{p\}$, the number $\sharp(l, Q)$ is even;
2. *no self-intersections:* for all $1 \leqslant i < j \leqslant s$, there exists $l \in [n] \setminus \{p\}$, such that the number $\sharp(l, Q[i, j])$ is odd; the same is true for all $s+1 \leqslant i < j \leqslant t$.

An example of a [2]-cube 0-path is shown on the right-hand side of Figure 2. The corresponding t-tuple is $(2, 1, 2, 1)$ and $s = 1$.

3.3. *The abstract simplex paths*

3.3.1. *Definition and associated data.* Abstract simplex paths are the main objects of study in this paper.

Definition 3.4. Let k and n be positive integers. An *abstract simplex path* P is a triple (I, C, V), where I is an $[n]$-tuple of boolean values, C is a $(k-1)$-tuple of elements of $[n]$ and V is a $(k-1)$-tuple of boolean values. We request that the $(k-1)$-tuple C satisfies an additional 'no back-flip' condition: for all $i = 1, \ldots, k-2$, we have $C_i \neq C_{i+1}$.

Given an abstract simplex path $P = (I, C, V)$ as above, the number k is called its *length*, the number n is called its *dimension* and the tuple I is called *the initial simplex* of P. When we need to connect back to the path name we shall also write $I(P)$, $C(P)$ and $V(P)$.

Since the set $[n]$ indexes the vertices of an n-simplex, and the set [1] indexes boolean values, each pair (C_i, V_i) can be thought of as assigning a chosen boolean value V_i to a chosen vertex C_i of an n-simplex. So, the path P can be interpreted as starting from the initial $[n]$-tuple $I(P)$, and then changing specific boolean entries according to the pattern given by $C(P)$ and $V(P)$. We call a path P a 0-path if $I(P) = 0$ and $V(P) = 0$.

Next, we proceed to describe how to associate certain *data* to every abstract simplex path.

Definition 3.5. Assume $P = (I, C, V)$ is a path of dimension n and length k. Fix $1 \leqslant j \leqslant k$. For every $i \in [n]$, we set

$$e_i^j(P) := V_{\mathrm{last(i,C(1,j-1))}}.$$

If $\mathrm{last(i, C(1, j-1))} = \infty$, we set $e_i^j(P) := I_i$. We then set

$$R^j(P) := (e_0^j(P), \ldots, e_n^j(P)).$$

We call the $[n]$-tuples $R^1(P)$, ..., $R^k(P)$ the *simplices of the path* P. Finally, we set

$$R(P) := (R^1(P), \ldots, R^k(P)).$$

Note that for $j = 1$, the set $\{1, \ldots, j-1\}$ is empty and $\mathrm{last(i, C(1, 0))} = \infty$, for all i, implying that $R^1(P) := I(P)$.

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3.3.2. *Concatenation of paths.*

Definition 3.6. Assume that P and Q are paths of dimension n of lengths k and l. A path T of dimension n is called a *concatenation* of P and Q if

— $I(T) = I(P)$, $R^k(T) = R^{k+1}(T) = I(Q)$,
— $C(T) = C(P) \circ (p) \circ C(Q)$, for some $p \in [n]$,
— $V(T) = V(P) \circ (I(Q)_p) \circ V(Q)$.

Note that the fact that P in Definition 3.6 is a well-defined path implies that $p \neq C(P)_{k-1}$ and $p \neq C(Q)_1$. Since several conditions need to be satisfied, it might very well happen that there are no concatenations of P and Q. On the other hand, there could be several choices of p, for which all the conditions of Definition 3.6 are satisfied, so there may exist several concatenations of P and Q. Iterating Definition 3.6 we can talk about concatenations of any finite number of paths.

3.3.3. *The actions of symmetric groups on the sets of paths.* There is a natural group action of the symmetric group $\mathcal{S}_{[n]}$ of all permutations of the set $[n]$ on the set of all paths of dimension n: namely, for a permutation $\pi \in \mathcal{S}_{[n]}$, viewed as a bijection $[n] \to [n]$ and a path $P = (I, C, V)$ of dimension n and length k, we set $\pi(P) := (\pi(I), \pi(C), V)$, where

$$\pi(I) = (I_{\pi(0)}, \dots, I_{\pi(n)}),$$

and

$$\pi(C) := (\pi(C_1), \dots, \pi(C_{k-1})).$$

We may also consider the reflection action of $\mathcal{S}_2$ on the paths which switches the direction, which is easiest to describe in the $R(P)$-notations:

$$(R^1(P), \dots, R^k(P)) \mapsto (R^k(P), \dots, R^1(P)).$$

Alternatively, the nontrivial element of $\mathcal{S}_2$ takes the path $P = (I, C, V)$ to the path $\bar{P} = (\bar{I}, \bar{C}, \bar{V})$, where $\bar{I} = R^k(P)$,

$$\bar{C} = (C_{k-1}, \dots, C_1),$$

and

$$\bar{V} = (e^{k-1}_{C_{k-1}}, \dots, e^1_{C_1}).$$

3.3.4. *Atomic and admissible paths.* In this paper we are concerned with algorithms transforming certain types of paths, which we now proceed to define.

Definition 3.7. Let P be an abstract simplex path P of dimension n and length k.

1. The path P is called *atomic* if
   — k is even;
   — we have $I(P) = R^k(P) = 0$, and these are the only monochromatic simplices of P.
2. The path P is called *admissible* if it is a concatenation of atomic paths.

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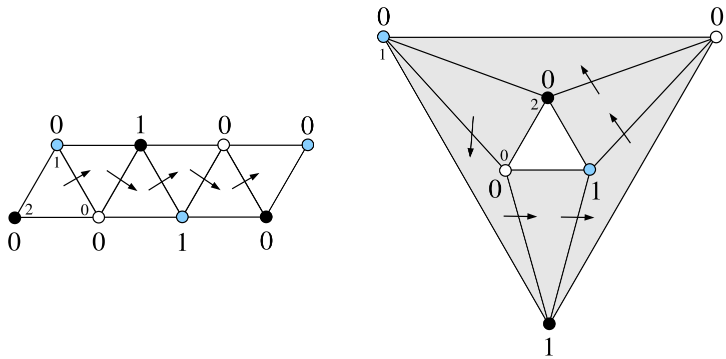

Fig. 3. Two geometric simplex paths with boolean labelling corresponding to the abstract simplex path on Figure 4.

Since for atomic and admissible paths P we have $I(P) = 0$, we shall at times for brevity skip it from the definition and just write $P = (C, V)$. Given an admissible path P, the set of atomic paths, whose concatenation is P, is uniquely defined. Furthermore, the actions of $\mathcal{S}_{[n]}$ and $\mathcal{S}_2$ from above restrict to the subsets of atomic and admissible paths.

It is often useful to complement the string notation of the path with a more graphic representation, by listing the parts of $[n]$-tuples $R^1(P)$, …, $R^k(P)$ where the values are changed at some point along the path, and indicating, using the downward arrows, the positions where the boolean values are changed. Many examples of atomic abstract simplex paths given in that form can be found in the Table A4.

3.3.5. *Connection between geometric simplex paths with boolean labels and abstract simplex paths.* To any given geometric simplex path with boolean labels $\Sigma = (\sigma_1, \dots, \sigma_l)$ one can associate a unique abstract simplex path $P = (I, C, V)$. To do that, simply take the $[n]$-tuples of labels of the simplices $\sigma_1, \dots, \sigma_l$ to be the tuples $R^1(P), \dots, R^l(P)$. Alternatively, we can take $I(P)$ to be the labels of σ_1, and then record in $C(P)$ the indices (colours) of the vertices which are flipped as we walk along the path, and record in $V(P)$ the labels assigned to the new vertices after each flip. Note, that under this correspondence, the geometric simplex paths in standard form will yield an atomic abstract simplex path.

Reversely, given an abstract simplex path, one can always associate to it the geometric simplex path with boolean labels, though this time the geometric path will not be unique. Figure 3 shows two different geometric simplex paths with boolean labels, which have the same associated abstract simplex path, described by Figure 4.

3.3.6. *The height data.* Further data, which we associate to an abstract simplex path P, is its *height graph* $h(P)$. Specifically, for a path P of length k, we set $h_i(P) := h(R^i(P))$, for $i = 1, \dots, k$. When P is atomic, we have $h_1(P) = h_k(P) = 0$, and $h_i(P) > 0$ for $1 < i < k$. The height graph of P is a partially marked graph imbedded in $\mathbb{R}^2$, whose set of vertices is $\{(i, h_i(P)) \,|\, i = 1, \dots, k\}$. The set of edges of $h(P)$ is described as follows: for each $i = 1, \dots, k-1$ we connect the dots $(i, h_i(P))$ and $(i+1, h_{i+1}(P))$ by an interval, and in addition, if $h_i(P) = h_{i+1}(P)$, i.e., if $V_i = e^i_{C_i}$, we mark the corresponding edge with

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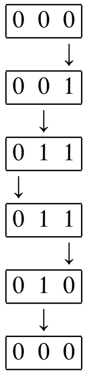

$$P = (I, C, V)$$
$$I = (0, 0, 0)$$
$$C = (2, 1, 0, 2, 1)$$
$$V = (1, 1, 0, 0, 0)$$

Fig. 4. The abstract simplex path corresponding to the geometric simplex paths with boolean labellings on Figure 3.

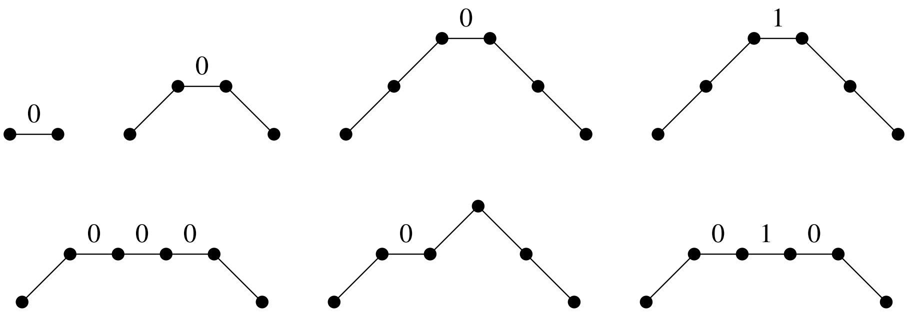

Fig. 5. The height graphs.

V_i. The examples of height graphs of atomic paths of length 2, 4 and 6 are shown on Figure 5; e.g., the height graph of the abstract simplex path in Figure 4 is the third graph in Figure 5.

For future reference, we introduce the height statistics $H(P) := (v_0, \ldots, v_{n+1})$, where $v_j = \sharp(j, (h_1(P), \ldots, h_k(P)))$. Since P is atomic, we have $v_0 = 2$, and $v_{n+1} = 0$. We order such height statistics using what is commonly known as the reverse lexicographic order on $(n+2)$-tuples, i.e., $(v_0, \ldots, v_{n+1}) \succ (\mu_0, \ldots, \mu_{n+1})$ if and only if there exists $0 \leqslant i \leqslant n+1$, such that $v_i > \mu_i$ and $v_{i+1} = \mu_{i+1}, \ldots, v_{n+1} = \mu_{n+1}$.

4. Combinatorial deformations of an abstract simplex path

At this point, we have introduced the abstract simplex paths and the associated data. We have seen different presentations for that data and we classified this information for all atomic paths of length atmost 6. Next, we describe two types of path deformations, which we call *path expansions*. The two specific types which we have in mind are called *vertex expansion* and *edge expansion*.

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4.1. *Vertex expansion*

4.1.1. *Combinatorial definition of the vertex expansion.* Assume $P = (I, C, V)$ is an abstract simplex path of length $k \geqslant 3$ and dimension n. The input data for the vertex expansion consists of:

— a number m, such that $2 \leqslant m \leqslant k-1$;
— an $[n]$-tuple $D = (d_0, \dots, d_n)$ of boolean values;
— an $[n]$-cube loop $Q = (q_1, \dots, q_t)$, such that $q_1 = C_{m-1}$ and $q_t = C_m$.

The number m is the position of the expansion. For convenience of notations, we set $(c_0, \dots, c_n) := R^m(P)$.

Definition 4.1. The vertex expansion of a path $P = (I, C, V)$ with respect to the input data (m, D, Q) is a new path $\widetilde{P}$ of dimension n defined as follows:
— $I(\widetilde{P}) = I$;
— $C(\widetilde{P})$ is obtained from C be inserting the tuple $(q_2, \dots, q_{t-1})$ between C_{m-1} and C_m;
— $V(\widetilde{P})$ is obtained from V by replacing V_{m-1} with the tuple $w_1, \dots, w_{t-1}$, where

$$w_i = \begin{cases} c_{q_i}, & \text{if } \sharp(q_i, (q_1, \dots, q_i)) \text{ is even,} \\ d_{q_i}, & \text{if } \sharp(q_i, (q_1, \dots, q_i)) \text{ is odd,} \end{cases} \tag{2}$$

for all $i = 1, \dots, t-1$.

We shall write $P(m, D, Q)$ to denote the obtained path $\widetilde{P}$. Clearly, path $P(m, D, Q)$ has length $k + t - 2$.

4.1.2. *Geometric interpretation of the vertex expansion.* Let us assume we have an abstract simplex path $P = (I, C, V)$. We want to see what a vertex expansion $P(m, D, Q)$ corresponds to geometrically. To do that, we need a specific subdivision of the interior of the simplex σ_m. There are several equivalent ways to view this subdivision. One could use so-called chromatic joins, see Castañeda and Rajsbaum (2012a) and Herlihy *et al.* (2014). Alternatively, one can use Schlegel diagrams, as was done in Kozlov (2012), see also Coxeter (1973) and Grünbaum (2003) for the polytope background. We prefer the latter, as then the fact that we actually have a subdivision is immediate. In this language, we simply replace the simplex σ_m by the Schlegel diagram S of the $(n+1)$-dimensional cross-polytope, see Kozlov (2012) for details. In this subdivision there are $n+1$ new vertices, which are in natural bijection with the vertices of σ_m: each vertex v of σ_m has a unique opposite vertex $op(v)$ in the corresponding cross-polytope. The vertex of S which is opposite to $v \in V(\sigma_m)$ gets the same colour as v, which immediately implies that the obtained subdivision is chromatic. The $[n]$-tuple D encodes the extension of the binary labelling to S as follows: let $v \in V(\sigma_m)$ have the colour ρ, then the label of $op(v)$ is d_ρ.

This describes the new subdivision of σ_m. The remaining variable Q describes how the new path $\widetilde{\Sigma}$ goes through this subdivision. In the old path, one simply flipped from σ_{m-1} to σ_m, and then on to σ_{m+1}. In the new subdivision, one has to connect σ_{m-1} to σ_{m+1} by flipping through S. To do that, we need to understand the combinatorics of S better. The standard fact about the $(n+1)$-dimensional cross-polytope is that it is

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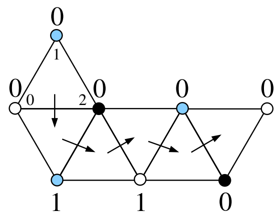

Fig. 6. A geometric simplex path corresponding to the abstract simplex path with $I = (0, 0, 0)$, $C = (1, 0, 1, 2, 0)$ and $V = (1, 1, 0, 0, 0)$.

dual to the $[n]$-cube, see Coxeter (1973) and Grünbaum (2003). One implication of that duality is that n-simplices of the cross-polytope correspond to the vertices of the $[n]$-cube, and a geometric n-simplex path on the boundary of the cross-polytope corresponds to a regular edge path on the boundary of an $[n]$-cube. The new path $\widetilde{\Sigma}$ is supposed to enter the Schlegel diagram S from one side, and then traverse it and exit on one of the other sides. Clearly, combinatorially we can think of it as an n-simplex loop which starts and ends at the n-simplex F of the cross-polytope, which is taken as the basis for the Schlegel diagram. Dually, this corresponds to an $[n]$-cube loop, cf. Definition 3.2, which is precisely the loop Q.

Finally, the identity (2) describes the change of the labels correctly, as the parity of the number $\sharp(q_i, (q_1, \ldots, q_i))$ tells us whether we get the vertex in the base face of the Schlegel diagram, this happens if the number is even, or we get one of the opposite vertices, lying in the interior of the subdivided simplex, which happens if the number is odd.

4.1.3. *An example.* Let us illustrate this interpretation by an example. Consider an abstract simplex path $P = (I, C, V)$, with $I = (0, 0, 0)$, $C = (1, 0, 1, 2, 0)$ and $V = (1, 1, 0, 0, 0)$. One of the corresponding geometric simplex paths is shown on Figure 6.

Consider the vertex expansion $P(m, D, Q)$, for $m = 3$, $D = (0, 0, 0)$, $Q = (0, 1, 2, 0, 2, 1)$. By Definition 4.1 we have $P(m, D, Q) = (I, \widetilde{C}, \widetilde{V})$, and we now proceed to determine $\widetilde{C}$ and $\widetilde{V}$. We have the following data which we substitute in this definition: $C_{m-1} = C_2$, $C_m = C_3$ and $t = 6$. The rule in Definition 4.1 says that to obtain $\widetilde{C}$ we need to insert $(q_2, \ldots, q_5) = (1, 2, 0, 2)$ between C_2 and C_3 in C. This gives us the answer $C = (1, 0, 1, 2, 0, 2, 1, 2, 0)$. Furthermore, to get $\widetilde{V}$ from V, we need to replace $V_2 = 1$ with the tuple $(w_1, \ldots, w_5)$. To determine that tuple, consider the table (3).

i	1	2	3	4	5
q_i	0	1	2	0	2
w_i	d_0	d_1	d_2	c_0	c_2

(3)

The second row in that table simply shows the relevant part of Q. The third row shows which value is assigned to w_i by the rule (2). Since $(c_0, c_1, c_2) = R^2(P) = (1, 1, 0)$ and $(d_0, d_1, d_2) = (0, 0, 0)$, we arrive at $(w_1, \ldots, w_5) = (0, 0, 0, 1, 0)$, which then yields the answer $\widetilde{V} = (1, 0, 0, 0, 1, 0, 0, 0, 0)$. An associated geometric simplex path is shown on Figure 7.

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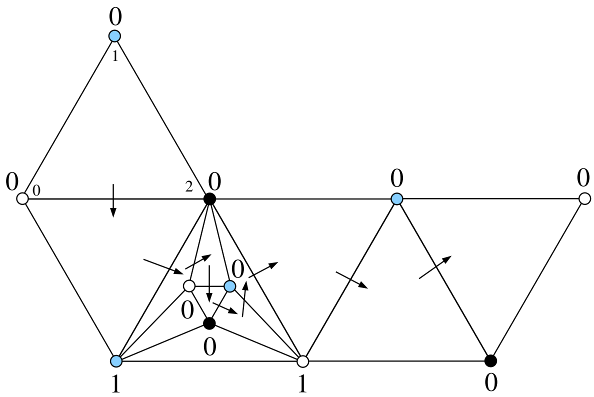

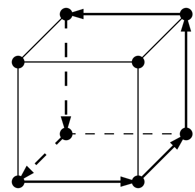

Fig. 7. A vertex expansion of the geometric simplex path from Figure 6, and the corresponding [2]-cube loop.

4.2. *Edge expansion*

4.2.1. *Combinatorial definition of the edge expansion.* Assume again that $P = (I, C, V)$ is an abstract simplex path of length $k \geqslant 4$ and dimension n. The input data for the edge expansion consists of:

— a number m, such that $2 \leqslant m \leqslant k-2$;
— an $([n] \setminus \{C_m\})$-tuple $D = (d_0, \dots, \widehat{d}_{C_m}, \dots, d_n)$ of boolean values;
— an $[n]$-cube C_m-path given by $Q = (q_1, \dots, q_t)$, such that $q_1 = C_{m-1}$ and $q_t = C_{m+1}$, together with a number $1 \leqslant s \leqslant t-1$.

For convenience, the alternative notation $D = (d_0, \dots, d_{C_m-1}, -, d_{C_m+1}, \dots, d_n)$ will also be used. Again, we set $(c_0, \dots, c_n) := R^m(P)$.

Definition 4.2. The edge expansion of a path P with respect to the input data (m, D, Q, s) is a new path $\widetilde{P}$ of dimension n defined as follows:

— $I(\widetilde{P}) = I$;
— $C(\widetilde{P})$ is obtained from C be inserting the tuple $(q_2, \dots, q_s)$ between C_{m-1} and C_m, and then inserting the tuple $(q_{s+1}, \dots, q_{t-1})$ between C_m and C_{m+1};
— $V(\widetilde{P})$ is obtained from V by replacing V_{m-1} with the tuple $w_1, \dots, w_s$, and then inserting the tuple $w_{s+1}, \dots, w_{t-1}$ between V_m and V_{m+1}, where

$$w_i = \begin{cases} c_{q_i}, & \text{if } \sharp(q_i, (q_1, \dots, q_i)) \text{ is even,} \\ d_{q_i}, & \text{if } \sharp(q_i, (q_1, \dots, q_i)) \text{ is odd,} \end{cases} \tag{4}$$

for all $i = 1, \dots, t-1$.

We shall write $P(m, D, Q, s)$ to denote the obtained path $\widetilde{P}$. Clearly, the path $P(m, D, Q, s)$ has length $k + t - 2$.

4.2.2. *Geometric interpretation of the edge expansion.* Let us describe the geometry behind the edge expansions. What happens in this case is that we first subdivide the $(n-1)$-simplex $\sigma_m \cap \sigma_{m+1}$, call this subdivision S. Then, we extend this to a subdivision of σ_m by

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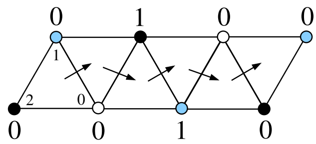

Fig. 8. A geometric simplex path corresponding to the abstract simplex path with $I = (0,0,0)$, $C = (2,1,0,2,1)$, $V = (1,1,0,0,0)$.

coning over S with the apex being the vertex $v_m = \sigma_m \setminus (\sigma_m \cap \sigma_{m+1})$, and we extend it to a subdivision of σ_{m+1} by coning over S with the apex at $v_{m+1} = \sigma_{m+1} \setminus (\sigma_m \cap \sigma_{m+1})$. In total, only the interiors of the simplices σ_m, σ_{m+1} and $\sigma_m \cap \sigma_{m+1}$ get subdivided, and one may view the whole thing as a suspension of S. The subdivision S itself is again chosen to be a Schlegel diagram of a cross-polytope of appropriate dimension: here we are dealing with the cross-polytope of dimension n. In this situation, C_m is the colour of the vertex v_m, which by the way is the same as the colour of the vertex v_{m+1}. The $[n] \setminus \{C_m\}$-tuple D gives the labels of the new vertices of the subdivision S.

In analogy to the analysis of the vertex expansion, we see that the $(n-1)$-simplices of S are in one-to-one correspondence with the vertices of the $[n] \setminus \{C_m\}$-cube. The n-simplices of the new subdivision of $\sigma_m \cup \sigma_{m+1}$ are obtained from the $(n-1)$-simplices of the Schlegel diagram S by adding either v_m or v_{m+1}, so there are exactly $2 \cdot (2^n - 1)$ of them. Thus, the new path $\widetilde{\Sigma}$, which is described by the edge expansion, will proceed as follows:

— it will enter the subdivision of σ_m, and flip for a while among the $(n-1)$-simplices of S, keeping v_m as one of the vertices;

— it will flip over to the subdivision of σ_{m+1}, swapping v_m for v_{m+1};

— finally, it will flip for a while among the $(n-1)$-simplices of S, keeping v_{m+1} as one of the vertices, and then exit at the same $(n-1)$-simplex of σ_{m+1} as Σ did.

This data is described by the variables Q and s. The part $(q_1, \ldots, q_s)$ of Q describes an $(n-1)$-simplex path in S corresponding to the n-simplex path in the total subdivision, with all simplices having v_m as a vertex.

4.2.3. *An example.* Again, we would like to illustrate how the edge expansion works by an example. Consider an abstract simplex path $P = (I, C, V)$, with $I = (0,0,0)$, $C = (2,1,0,2,1)$ and $V = (1,1,0,0,0)$. One of the corresponding geometric simplex paths is shown on Figure 8.

Consider the edge expansion $P(m, D, Q, s)$, where $m = 3$, $D = (-,0,1)$, $Q = (1,2,1,2)$ and $s = 2$. By Definition 4.2, we have $P(m, D, Q, s) = (I, \widetilde{C}, \widetilde{V})$, and we now determine the tuples $\widetilde{C}$ and $\widetilde{V}$. Here we have $q_s = q_2$, $C_m = C_3$, $t = 4$, so $q_{s+1} = q_{t-1} = q_3$. By Definition 4.2, to get $\widetilde{C}$ from C we need to insert q_2 between C_2 and C_3, and insert q_3 between C_3 and C_4. As a result we get $\widetilde{C} = (2,1,2,0,1,2,1)$. To get $\widetilde{V}$ from V, we need to replace V_2 with (w_1, w_2), and insert w_3 between V_3 and V_4. To calculate the tuple

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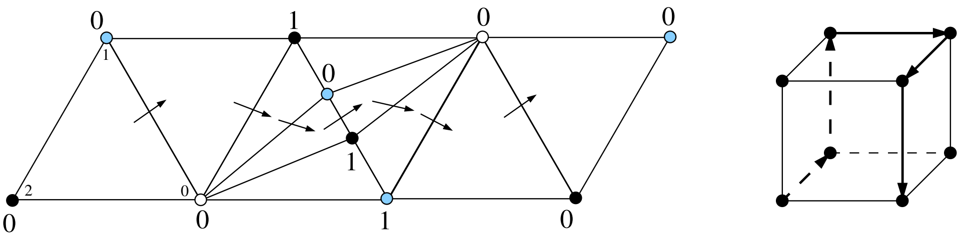

Fig. 9. An edge expansion of the geometric simplex path from Figure 8, and the corresponding [2]-cube loop.

(w_1, w_2, w_3) consider the table (5).

i	1	2	3
q_i	1	2	1
w_i	d_1	d_2	c_1

(5)

The second row in that table again shows the relevant part of Q. The third row shows which value is assigned to w_i by the rule (4). Since $(c_0, c_1, c_2) = R^3(P) = (1, 1, 0)$ and $(d_0, d_1, d_2) = (-, 0, 1)$, we arrive at $(w_1, w_2, w_3) = (0, 1, 1)$, which then yields the answer $\widetilde{V} = (1, 0, 1, 0, 1, 0, 0)$. The resulting geometric simplex path is shown on Figure 9.

4.3. *The exhaustive expansion*

4.3.1. *Combinatorial definition of the exhaustive expansion.* Let P be an abstract simplex path, and let $\widetilde{P}$ be either a vertex expansion $P(m, D, Q)$ or an edge expansion $P(m, D, Q, s)$. We describe how to associate a $[1] \times [n]$-array A to that pair $(P, \widetilde{P})$. To start with, in any case we set $A[0, -] := R^m(P)$. Proceeding to the second row: for a vertex expansion $P(m, D, Q)$ we set $A[1, -] := D$, while for an edge expansion $P(m, D, Q, s)$ we set $A[1, i] := D_i$, for all $i \in [n] \setminus \{C_m\}$ and $A[1, C_m] = V_m$.

Since the vertices of $\text{Cube}_{[n]}$ are indexed by all functions $\alpha : [n] \to [1]$, the $[1] \times [n]$-array of boolean values A can be used to associate boolean $[n]$-tuples to vertices of $\text{Cube}_{[n]}$. Specifically, to a vertex $\alpha : [n] \to [1]$ of $\text{Cube}_{[n]}$ we associate the $[n]$-tuple $A(\alpha) := (A[\alpha(0), 0], A[\alpha(1), 1], \dots, A[\alpha(n), n])$.

Let $M(A)$ denote the set of all $\alpha : [n] \to [1]$, such that $A(\alpha)$ is monochromatic. If $M(A) \neq \emptyset$ there are two possibilities. The first option is that $M(A)$ consists of two 'opposite' $[n]$-tuples, one is 0-monochromatic, and the other one is 1-monochromatic. The second option is that all $\alpha \in M(A)$ are b-monochromatic, where $b \in [1]$. In this case, the subgraph of $\text{Cube}_{[n]}$ induced by $M(A)$ is isomorphic to $\text{Cube}_{[m]}$, for some $m \leqslant n$; in particular, $|M(A)|$ is a power of 2.

Intuitively, we shall call the expansion $\widetilde{P}$ *exhaustive* if all vertices $\alpha \in M(A)$ either belong to $\widetilde{P}$, or can be matched (in the sense of graph theory) to each other. Unfortunately, it is somewhat technical to express the fact that a vertex α belongs to the new path $\widetilde{P}$. We now give the formal definition.

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Definition 4.3. Let the paths P and $\widetilde{P}$ be as above. The expansion $\widetilde{P}$ is called **exhaustive** if, there exists a partial matching on the vertices in $M(A)$, such that, for each unmatched $\alpha \in M(A)$, there exists i satisfying the following conditions:

— if $\widetilde{P}$ is a vertex expansion, then $1 \leqslant i \leqslant t-1$;
— if $\widetilde{P}$ is an edge expansion and $\alpha(C_m) = 0$, then $1 \leqslant i \leqslant s$;
— if $\widetilde{P}$ is an edge expansion and $\alpha(C_m) = 1$, then $s+1 \leqslant i \leqslant t$;
— for all $j \in [n]$, we have

$$\alpha(j) = \sharp(j,(q_1,\ldots,q_i)) \mod 2.$$

4.3.2. *Geometric interpretation of exhaustive vertex expansion.* Let us understand what it means geometrically for the vertex expansion to be exhaustive. For this, recall the framework of the Subsubsection 4.1.2, where the vertex expansion was interpreted as a replacement of an n-simplex σ_m by a Schlegel diagram S of an $(n+1)$-dimensional cross-polytope. The n-simplex σ_m corresponded to the face F of the cross-polytope, with regard to which the Schlegel diagram was taken.

In this case, the $[1] \times [n]$-array A contains all the labels of the vertices of that cross-polytope. All the functions $\alpha : [n] \to [1]$ correspond to n-faces of the cross-polytope, and, accordingly, the $[n]$-tuples $A(\alpha) = (A[\alpha(0), 0], \ldots, A[\alpha(n), n])$ correspond to labels of these faces. There are 2^{n+1} of these n-faces, and all but F, which corresponds to $\alpha \equiv 0$, are n-simplices in the Schlegel diagram S. Recall, that σ_m itself is not monochromatic, so the set $M(A)$ corresponds to all monochromatic simplices of S. Definition 4.3 then says precisely that all the monochromatic simplices in S, either belong to the new path, or can be matched pairwise, so that each two matched n-simplices share an $(n-1)$-dimensional boundary simplex.

4.3.3. *Geometric interpretation of exhaustive edge expansion.* Here we are in the framework discussed in the Subsubsection 4.2.2. We have two n-simplices σ_m and σ_{m+1} on our path. First the $(n-1)$-simplex $\sigma_m \cap \sigma_{m+1}$ is replaced by a Schlegel diagram, then this subdivision is extended to the interior of the simplices σ_m and σ_{m+1}. As in the case of vertex expansion, the edge expansion is exhaustive, if all monochromatic simplices in the new subdivision of $\sigma_m \cup \sigma_{m+1}$ are either on the new path, or can be matched to each other, having a common $(n-1)$-simplex in each matched pair.

Example 4.4. *An example of an exhaustive edge expansion.* As an example let us consider an atomic path P of dimension $n \geqslant 2$ and length 4, given by $P = ((0,1,0),(1,0,0))$. The height graph of this path is the second one on Figure 5. We consider the edge expansion $P(m, D, Q, s)$, with the data: $m = 2$, $D = (0, -, 1, \ldots, 1)$, $Q = (0,0)$, $s = 1$.

The graphic presentation of this expansion is shown in Figure 10. The reader is invited to compare it with Castañeda and Rajsbaum (2012a, Figure 16(b))

We can see that this edge expansion is exhaustive. Indeed, we have

$$A = \begin{array}{|c|c|c|c|c|}\hline 1 & 0 & 0 & \ldots & 0 \\ \hline 0 & 0 & 1 & \ldots & 1 \\ \hline \end{array},$$

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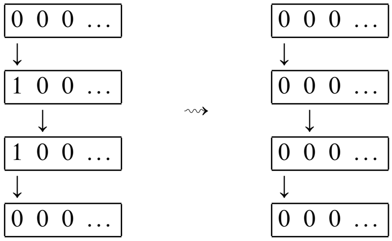

Fig. 10. An example of an exhaustive edge expansion.

and hence $|M(A)| = 2$. Both of the 0-monochromatic tuples α belong to the expansion, so the conditions of Definition 4.3 are satisfied. Also, we see that the obtained path $P(m, D, Q, s)$ is admissible.

5. The moves used by the algorithm

We now have all the technical tools needed to describe our algorithm, which in turn will be used to give a new proof of Theorem 2.6. Next, we proceed to define special cases of the vertex and edge expansions.

5.1. *The summit move*

5.1.1. *The definition.* Assume $P = (C, V)$ is an atomic path, and assume that we have a position i, such that $2 \leqslant i \leqslant k-1$, $h_i(P) > h_{i-1}(P)$ and $h_i(P) > h_{i+1}(P)$. We call such an index i a *summit* of P. We cannot have $h_i(P) = 1$, since that would imply $h_{i-1}(P) = h_{i+1}(P) = 0$, a contradiction. Assume therefore that $h_i(P) \geqslant 2$. Since the path has even length, changing the orientation of the path will change the parity of i, so up to the above $\mathcal{S}_2$-action we may assume that i is odd.

Since we are allowed to apply $\mathcal{S}_{[n]}$, we can furthermore assume without loss of generality that $(C_{i-1}, V_{i-1}) = (0, 1)$, $(C_i, V_i) = (1, 0)$ and $e_2^i = 0$ (this is because $R^i(P)$ is not 1-monochromatic). In particular, we have $R^i(P) = (1, 1, 0, e_3^i, \ldots, e_n^i)$.

With the data above, the *summit move on P at position i* is the vertex expansion $P(m, D, Q)$, where

— $m = i$;
— $D = (0, 0, 0, \bar{e}_3^i, \bar{e}_4^i, \ldots, \bar{e}_n^i)$;
— $Q = (0, 1, 2, 0, 2, 1)$.

The summit move is illustrated by Figure 11, and the corresponding height graph change is shown in Figure 12. We note that the vertex expansion discussed in Subsubsection 4.1.3 is an example of a summit move.

5.1.2. *The summit move is exhaustive and yields an admissible path.* The following is the crucial property of the summit move which makes it useful for our algorithm.

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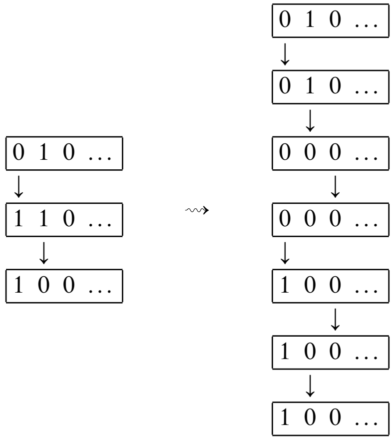

Fig. 11. The summit move.

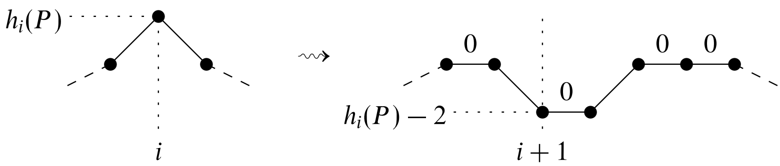

Fig. 12. The height graph change in a summit move.

Proposition 5.1. Assume that P is an admissible path, and $P(m, D, Q)$ is the path obtained from P using a summit move. Then

1. the vertex expansion $P(m, D, Q)$ is exhaustive;
2. the path $P(m, D, Q)$ is admissible.

Proof. We start by proving that $P(m, D, Q)$ is exhaustive. In the summit move situation we have

$$A = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 0 & e_3^i & \dots & e_n^i \\ \hline 0 & 0 & 0 & \bar{e}_3^i & \dots & \bar{e}_n^i \\ \hline \end{array}$$

In particular, $M(A)$ consists of two 0-monochromatic tuples, which are neighbours in $\text{Cube}_{[n]}$. If $h_i(P) = 2$, then both of these 0-monochromatic tuples are contained in the path $P(m, D, Q)$ at positions $i + 1$ and $i + 2$, see Figures 11 and 12. If $h_i(P) \geqslant 3$, then the path $P(m, D, Q)$ does not contain any of the two 0-monochromatic tuples, but these two tuples can be matched. In any case, the conditions of Definition 4.3 are satisfied, and $P(m, D, Q)$ is exhaustive.

Now we show that the path $P(m, D, Q)$ is admissible. Since P is atomic, we know that k is even. By construction, the path $P(m, D, Q)$ has length $k + 4$, so it is of even length as well. Furthermore, we have $I(P(m, D, Q)) = I(P) = 0$, and $R^{k+4}(P(m, D, Q)) = R^k(P) = 0$.

Clearly, since P is atomic, the path $P(m, D, Q)$ does not contain any 1-monochromatic simplices. If $h_i(P) \geqslant 3$, then $P(m, D, Q)$ does not contain any 0-monochromatic simplices either, hence $P(m, D, Q)$ is atomic. Finally, assume $h_i(P) = 2$. As mentioned above, in

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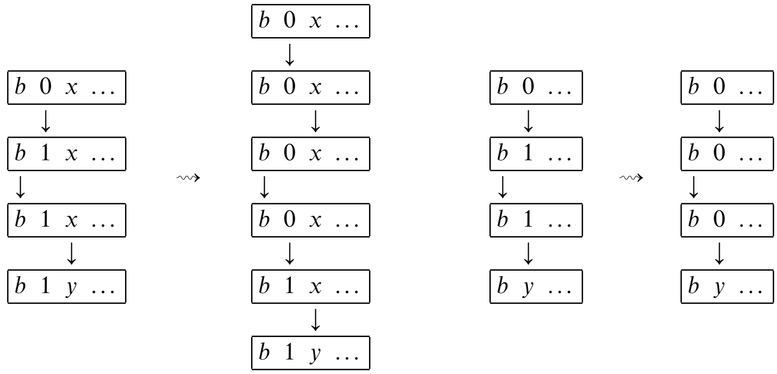

Fig. 13. The generic and the special plateau moves.

this case the simplices $i+1$ and $i+2$ are 0-monochromatic, so $P(m,D,Q)$ is not atomic. However, since i is odd, the subpaths $P(m,D,Q)[1,i+1]$ and $P(m,D,Q)[i+2,k]$ are atomic, and the path $P(m,D,Q)$ is a concatenation of them. Thus, we conclude that in any case the path $P(m,D,Q)$ is admissible. □

5.2. *The plateau move*

5.2.1. *The definition.* Assume again that $P=(C,V)$ is an atomic path. Assume we have a position i, such that $2 \leqslant i \leqslant k-2$, $h_{i+1}(P)=h_i(P)>h_{i-1}(P)$. In such a situation we say that we have a *plateau* at i. If $h_i(P)=1$, then $h_{i-1}(P)=0$, contradicting the choice of i. Thus we assume that $h_i(P) \geqslant 2$.

Since we are allowed to apply $\mathcal{S}_{[n]}$, we can assume without loss of generality that $(C_{i-1},V_{i-1})=(1,1)$, $(C_i,V_i)=(0,b)$ and $(C_{i+1},V_{i+1})=(p,y)$, where $p \in [n] \setminus \{0\}$. As a matter of fact, again due to the $\mathcal{S}_{[n]}$-action, we can assume that either $p=1$ or $p=2$. These two cases lead to two slightly different versions of the plateau move, which we consider separately.

5.2.2. *The generic plateau move.* *The case:* $p=2$. With the data above, the generic plateau move on P is the edge expansion $P(m,D,Q,s)$, where

— $m=i$;
— $D=(-,0,e_2^i,e_3^i,\ldots,e_n^i)$;
— $Q=(1,2,1,2)$;
— $s=2$.

The generic plateau move is shown on the left-hand side of Figure 13, and the corresponding height graph change is shown in Figure 14. Note, that the edge expansion considered in the Subsubsection 4.2.3 is an example of a generic plateau move.

5.2.3. *The special plateau move.* *The case:* $p=1$. With the data above, the special plateau move on P is the edge expansion $P(m,D,Q,s)$, where m and D are the same as in the generic case, and

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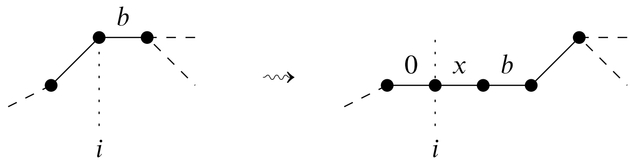

Fig. 14. The height graph change in a generic plateau move.

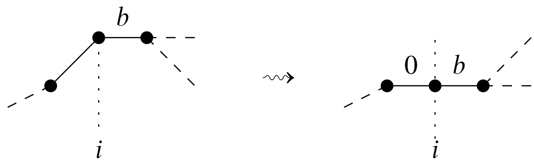

Fig. 15. The height graph change in a special plateau move.

— $Q = (1, 1)$;
— $s = 1$.

The special plateau move is shown on the right-hand side of Table 13, and the corresponding height graph change is shown in Figure 15.

5.2.4. *Plateau moves are exhaustive and yield an admissible path.* Similarly to the case of the summit move, we have the following result.

Proposition 5.2. Assume that P is an admissible path, and $P(m, D, Q, s)$ is the path obtained from P using a plateau move. Then

1. the edge expansion $P(m, D, Q, s)$ is exhaustive;
2. the path $P(m, D, Q, s)$ is admissible.

Proof. We start by proving that $P(m, D, Q, s)$ is exhaustive. Both for generic and for special plateau moves we have

$$A = \begin{array}{|c|c|c|c|c|c|} \hline b & 1 & e_2^i & e_3^i & \dots & e_n^i \\ \hline b & 0 & e_2^i & e_3^i & \dots & e_n^i \\ \hline \end{array}$$

In particular, $M(A)$ is empty, and hence $P(m, D, Q, s)$ is exhaustive, unless $b = e_2^i = \cdots = e_n^i$. So assume that $b = e_2^i = \cdots = e_n^i$. If $b = 1$, then $R^i(P)$ is 1-monochromatic, contradicting the assumption that P is atomic. If $b = 0$, then $h_i(P) = 1$, contradicting the choice of i. We see that in any case, the edge expansion $P(m, D, Q, s)$ is exhaustive.

Now we show that the path $P(m, D, Q, s)$ is admissible. Since P is atomic, we know that k is even. By construction, the path $P(m, D, Q, s)$ has length $k+2$ or k, so it is of even length as well. Furthermore, we have $I(P(m, D, Q, s)) = I(P) = 0$, and $R^l(P(m, D, Q, s)) = R^k(P) = 0$, where l is the length of $P(m, D, Q, s)$. Finally, it is clearly seen from Figures 14 and 15 that the new path $P(m, D, Q, s)$ does not contain any monochromatic simplices other than its end simplices, so it is admissible. □

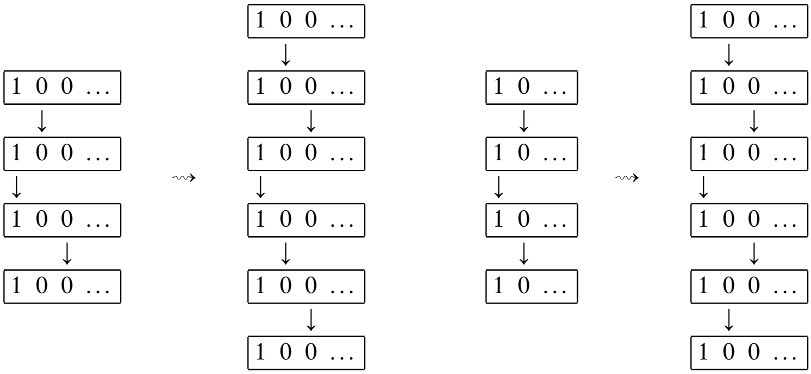

Fig. 16. The moves which fatten a unit: the left-hand side shows the case $C_{i-1} = 2$, while the right-hand side shows the case $C_{i-1} = 1$.

6. The main theorem

6.1. *Reducibility of low admissible paths*

Definition 6.1. A path P is called *reducible* if there exists a sequence of exhaustive vertex and edge expansions transforming it into a 0-path.

Lemma 6.2. Every admissible path P such that $\max_i h_i(P) = 1$ is reducible.

Proof. Clearly, it is enough to consider the case when $P = (C, V)$ is an atomic path, such that $\max_i h_i(P) = 1$. A number of facts follows at once from the assumption that $\max_i h_i(P) = 1$. Specifically:

— we have $V_1 = 1$, $V_2 = 0$, $V_{k-2} = 0$, $V_{k-1} = 0$;
— if $V_i = 1$, for some $3 \leqslant i \leqslant k-3$, then $V_{i-1} = V_{i+1} = 0$;
— after using $\mathcal{S}_{[n]}$-symmetry we can assume that $R^2(P) = \cdots = R^{k-1}(P) = (1, 0, \dots, 0)$.

To describe the reduction, we now describe a number of special vertex and edge expansions.

For the first two moves we assume that $3 \leqslant i \leqslant k-3$ is chosen so that $V_i = 1$; which of course implies that $V_{i-1} = V_{i+1} = 0$. Due to $\mathcal{S}_{[n]}$-action we may assume that $C_i = 0$, $C_{i-1} = 1$, and $C_{i+1} \in \{1, 2\}$.

Move 1: fatten a unit. These are edge expansions of P with the data

— $m = i$;
— $D = (-, 0, \dots, 0)$;
— $Q = (1, 2, 1, 2)$ if $C_{i-1} = 2$, $Q = (1, 2, 2, 1)$ if $C_{i-1} = 1$;
— $s = 2$.

This move is illustrated by Figure 16, and the corresponding height graph change is shown in Figure 17.

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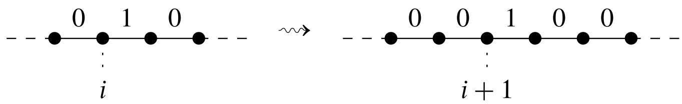

Fig. 17. The height graph change when fattening units.

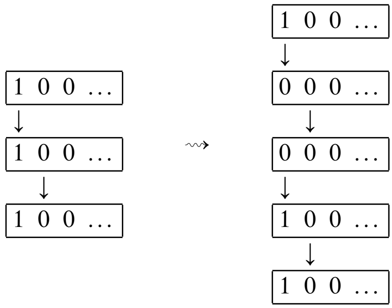

Fig. 18. The move eliminating a unit.

The resulting path $P(m, D, Q, s)$ has length $k + 2$, which is even, and Figure 17 shows that this path is atomic. In this edge expansion we have

$$A = \begin{array}{|c|c|c|c|} \hline 1 & 0 & \dots & 0 \\ \hline 1 & 0 & \dots & 0 \\ \hline \end{array},$$

hence $M(A)$ is empty and the expansion is exhaustive.

Move 2: eliminate a unit. Assume that i is odd. Eliminating a unit is a vertex expansion of P with the data

— $m = i$;
— $D = (0, 0, 1, 1, \dots, 1)$;
— $Q = (0, 1, 0, 1)$.

This move is illustrated by Figure 18, and the corresponding height graph change is shown in Figure 19.

The resulting path $P(m, D, Q)$ has length $k + 2$, which is even. Figure 19 shows that this path is admissible, since i is odd: it is a concatenation of atomic paths $P(m, D, Q)[1, i + 1]$

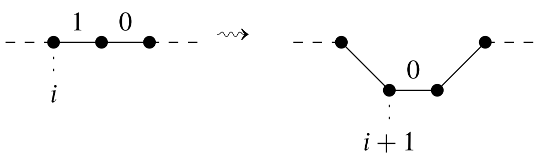

Fig. 19. The height graph change when eliminating units.

and $P(m, D, Q)[i+2, k]$. For this vertex expansion we have

$$A = \begin{array}{|c|c|c|c|c|}\hline 1 & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & 1 & \dots & 1 \\ \hline\end{array},$$

hence $|M(A)| = 2$. Both 0-monochromatic tuples belong to the expansion, so it is exhaustive.

Move 3: shorten zeroes. For this move we assume that $V_2 = V_3 = V_4 = 0$. Furthermore, due to $\mathcal{S}_{[n]}$-action, we may assume that $C_1 = 1$, $C_2 = 2$, $C_3 = 0$, and $C_4 \in \{2, 3\}$. The move is an edge expansion of P with the data

— $m = 3$;
— $D = (-, 0, 0, 1, 1, \dots, 1)$ if $C_4 = 2$; $D = (-, 0, 0, 0, 1, 1, \dots, 1)$ if $C_4 = 3$;
— $Q = (2, 1, 1, 2)$ if $C_4 = 2$; $Q = (2, 1, 3, 2, 1, 3)$ if $C_4 = 3$;
— $s = 2$ if $C_4 = 2$; $s = 3$ if $C_4 = 3$.

This move is illustrated by Figure 20, and the corresponding height graph changes are shown in Figure 21.

The resulting path $P(m, D, Q, s)$ has length $k+2$ or $k+4$, which is even. Figure 21 shows that this path is admissible: it is a concatenation of either 2 or 3 atomic paths. Furthermore, if $C_4 = 2$, we have

$$A = \begin{array}{|c|c|c|c|c|c|}\hline 0 & 1 & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & 0 & 1 & \dots & 1 \\ \hline\end{array}.$$

We have $|M(A)| = 4$. Two of the 0-monochromatic tuples belong to the expansion, and the other two can be matched to each other, so the expansion is exhaustive. If $C_4 = 3$, we have

$$A = \begin{array}{|c|c|c|c|c|c|c|}\hline 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & \dots & 1 \\ \hline\end{array},$$

hence $|M(A)| = 8$. One can see that 4 of the 0-monochromatic tuples belong to the expansion, and there is a complete matching on the remaining 4. Hence again the expansion is exhaustive.

We can now use these 3 types of moves to reduce an arbitrary atomic path P, such that $\max_i h_i(P) = 1$, to a 0-path. To start with, we can apply Move 1 to all even i, such that $V_i = 1$. When applying such a move we change the parity of the 1-edge at position i, see Figure 17, while all other 1-edges keep the same parity: those before i have the same index, and those after i will be shifted by 2. When we are done, all 1-edges start at odd positions. At this point, we can eliminate them one-by-one using Move 2. We will be left with having to consider the case of the atomic path P such that $V_2 = V_3 = \cdots = V_k = 0$. Now using Move 3, we can reduce the length of P, until $k = 4$, and this case was dealt with in the Example 4.4. □

6.2. *Reducibility of all admissible paths*

We are now ready to formulate our main theorem.

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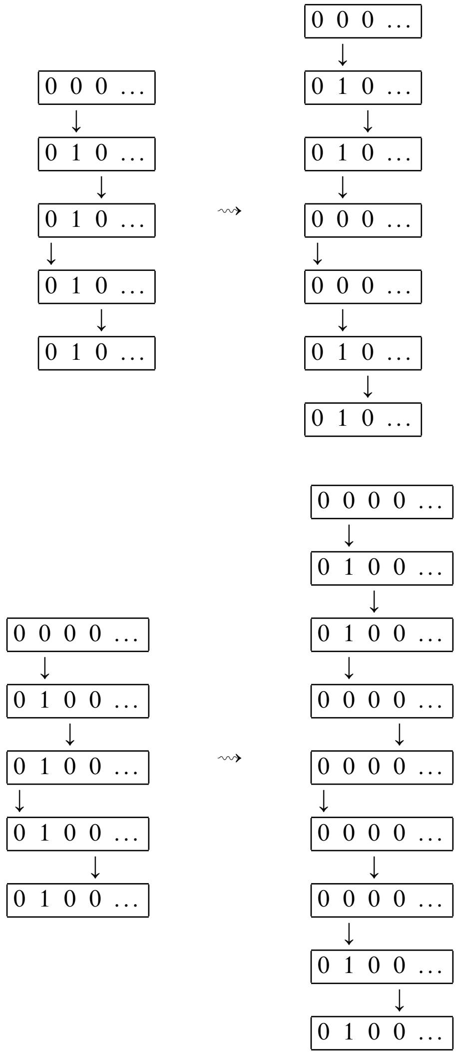

Fig. 20. The moves shortening zeroes.

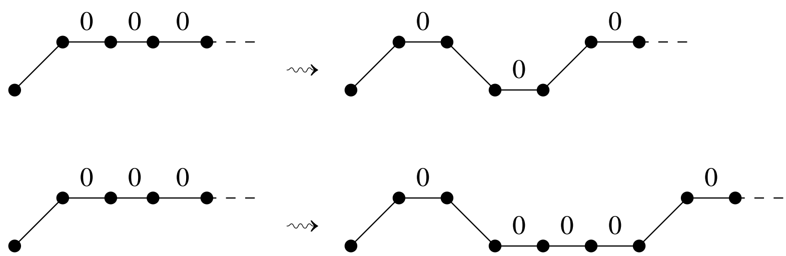

Fig. 21. The height graph changes when shortening zeroes.

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Theorem 6.3. Every admissible path is reducible.

Proof. Clearly, it is enough to consider the case when P is atomic. Assume that the statement of the theorem is false, and pick an atomic, but not reducible path P of length k, such that $H(P)$ is minimal with respect to the reverse lexicographic order. By Lemma 6.2 we can assume that $\max_i h_i(P) \geqslant 2$.

Assume first, that the path P has a summit i, $2 \leqslant i \leqslant k-1$. By reversing the direction of the path, if needed, we may assume that i is odd. Let $\widetilde{P}$ be the path resulting from P by applying the summit move at position i. By Proposition 5.2, that vertex expansion is exhaustive, and the obtained path is admissible.

If $h_i(P) \geqslant 3$, the path $\widetilde{P}$ is in fact atomic. On the other hand, as is visible from Figure 12, we have $H(P) \succ H(\widetilde{P})$, so by the choice of P, the path $\widetilde{P}$ must be reducible. This implies that P is reducible as well.

If $h_i(P) = 2$, the obtained path $\widetilde{P}$ is not atomic, since $R^{i+1}(\widetilde{P}) = R^{i+2}(\widetilde{P})$ are 0-monochromatic. However, both subpaths $\widetilde{P}_1 = \widetilde{P}[1, i+1]$ and $\widetilde{P}_2 = \widetilde{P}[i+2, k]$ are atomic. Since $H(P) \succ H(\widetilde{P}_1)$ and $H(P) \succ H(\widetilde{P}_2)$, both paths $\widetilde{P}_1$ and $\widetilde{P}_2$ are reducible, which implies that P is reducible as well.

We can thus assume that P has no summits. In this case, we pick i, $2 \leqslant i \leqslant k-1$, to be the minimal index such that $h_i(P) = \max_j h_j(P) \geqslant 2$. In particular, we have $h_i(P) > h_{i-1}(P)$, which, since P has no summits, implies $h_i(P) = h_{i+1}(P)$. Thus, the path P has a plateau at i, and we let $\widetilde{P}$ denote the path obtained from P by applying the corresponding plateau move. Here, we use the kind of plateau move which is appropriate to the situation, i.e., we use the specific plateau move if $C_{i-1} = C_{i+1}$, and we use the generic plateau move otherwise. In any case, by Proposition 5.1, this edge expansion is exhaustive, and the resulting path is atomic. On the other hand, we have $H(P) \succ H(\widetilde{P})$, implying by the choice of P, that the path $\widetilde{P}$ is reducible. This implies that the path P is reducible as well, leading to the contradiction to our original assumption. □

6.3. *Subdividing geometric simplex paths*

Having gained the understanding of how the vertex and edge expansions can be interpreted geometrically, we are now ready to apply our combinatorial theory to the geometric simplicial context. In fact, we derive Theorem 2.6 as a straightforward corollary of Theorem 6.3.

Proof of Theorem 2.6. Assume we are given an n-dimensional geometric simplex path Σ in the standard form, and let P be the associated abstract simplex path. Since Σ is in standard form, the path P is atomic. Theorem 6.3 states that there is a sequence of exhaustive vertex and edge expansions transforming the abstract simplex path P into an abstract simplex path $\widetilde{P}$, which is a 0-path of even length. It follows from our geometric interpretation of these expansions, that we obtain a subdivision $\widetilde{S}$ of $\mathrm{D}iv\,(\Delta^n)$, and an extension of the binary labelling b to $\widetilde{S}$, such that

1. only the interior of Σ is subdivided;
2. all monochromatic simplices in $\widetilde{S}$ are 0-monochromatic, some of them are matched in pairs, each pair having a common $(n-1)$-simplex, and the rest forms a geometric

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simplex path $\widetilde{\Sigma}$, which corresponds to the abstract simplex path $\widetilde{P}$, which is a 0-path of even length.

Since $\widetilde{P}$ has even length, also the 0-monochromatic simplices forming $\widetilde{\Sigma}$ can be broken in matched pairs, with n-simplices in each pair sharing an $(n-1)$-simplex.

It is now a simple fact, that two 0-monochromatic n-simplices σ_1 and σ_2, which are sharing an $(n-1)$-simplex $\sigma_1 \cap \sigma_2$, can be further subdivided to eliminate all monochromatic simplices, such that only the interior of σ_1, σ_2 and $\sigma_1 \cap \sigma_2$ are subdivided. To do that, simply subdivide $\sigma_1 \cap \sigma_2$ as a Schlegel diagram, cone over it with the apexes at $\sigma_1 \setminus (\sigma_1 \cap \sigma_2)$ and $\sigma_2 \setminus (\sigma_1 \cap \sigma_2)$, and take all the new labels to be 1. One may think of it as an edge expansion with $D = (1, \ldots, 1)$ without specifying m and Q. This is the same subdivision as in Castañeda and Rajsbaum (2012a, Figure 15).

This further subdivision of $\widetilde{S}$ leads to the subdivision $S(\mathrm{D}iv\,(\Delta^n))$, which has the desired properties. □

Appendix A.

A.1. *Simplified summit move*

One of the purposes of our algorithm in Section 5 was to demonstrate, that one can expand any atomic path to a path $\widetilde{P}$, which has a simple form, namely $\max_i h_i(\widetilde{P}) \leqslant 1$, using few types of moves: the summit move and the plateau moves. If instead the focus is on having as simple moves as possible, instead of as few as possible, it might be beneficial to consider the following additional move, which we call *simplified summit move*.

The simplified summit move on a path P at position i is the vertex expansion $P(m, D, Q)$, where

— $m = i$;
— $D = (0, 0, e_2^i, e_3^i, \ldots, e_n^i)$;
— $Q = (0, 1, 0, 1)$.

The simplified summit move is shown on Figure A1, and the corresponding height graph change is shown in Figure A2.

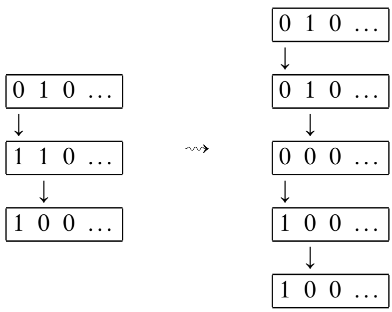

Fig. A 1. The simplified summit move.

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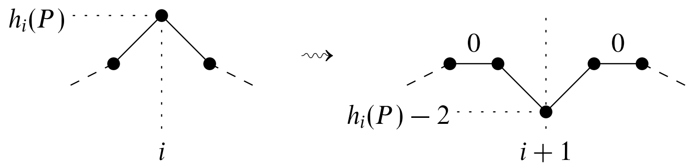

Fig. A 2. The height graph change in a simplified summit move.

Value of n	$\sharp$ atomic paths	$\sharp$ atomic paths mod reflection
$n=1$	2	2
$n=2$	8	7
$n \geqslant 3$	9	8

Fig. A 3. Number of atomic paths of length 6.

Proposition A.1. Assume that P is an admissible path, with summit at i such that $h_i(P) \geqslant 3$, and $P(m, D, Q)$ is the path obtained from P using a simplified summit move at position i. Then

1. the vertex expansion $P(m, D, Q)$ is exhaustive;
2. the path $P(m, D, Q)$ is admissible.

Proof. The proof is basically the same as that of Proposition 5.1. The only difference is that here we have

$$A = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & e_2^i & e_3^i & \dots & e_n^i \\ \hline 0 & 0 & e_2^i & e_3^i & \dots & e_n^i \\ \hline \end{array}.$$

In the simplified summit move situation, the set $M(A)$ is empty, there are no monochromatic simplices: at least one of the labels $e_2^i, \dots, e_n^i$ is equal to 0, since otherwise P would have a 1-monochromatic simplex, and at least one of them is equal to 1, since $h_i(P) \geqslant 3$. We conclude that the vertex expansion $P(m, D, Q)$ is exhaustive. It is admissible again due to the assumption $h_i(P) \geqslant 3$. □

The simplified summit move can be used in place of the regular summit move at position i, whenever $h_i(P) \geqslant 3$. It is only in the case $h_i(P) = 2$ that we need to resort to the full summit move, as otherwise the obtained path would not be admissible.

A.2. *Examples*

To have a collection of examples, let us analyse all atomic paths of length $k = 2, 4, 6$. We see, that up to the action of the symmetry group $\mathcal{S}_{[n]}$, there is only one atomic path of length 2, namely $P = ((0), (0))$, and there is only one atomic path of length 4, namely $P = ((0, 1, 0), (1, 0, 0))$. The numbers of atomic paths of length 6 are shown in Figure A3.

The representatives of all equivalence classes of atomic paths of length 6 under the $\mathcal{S}_{[n]}$-action are listed in the Figure A4. If, in addition, the reflection action is taken into account, the paths 3 and 4 in that table would be identified as well.

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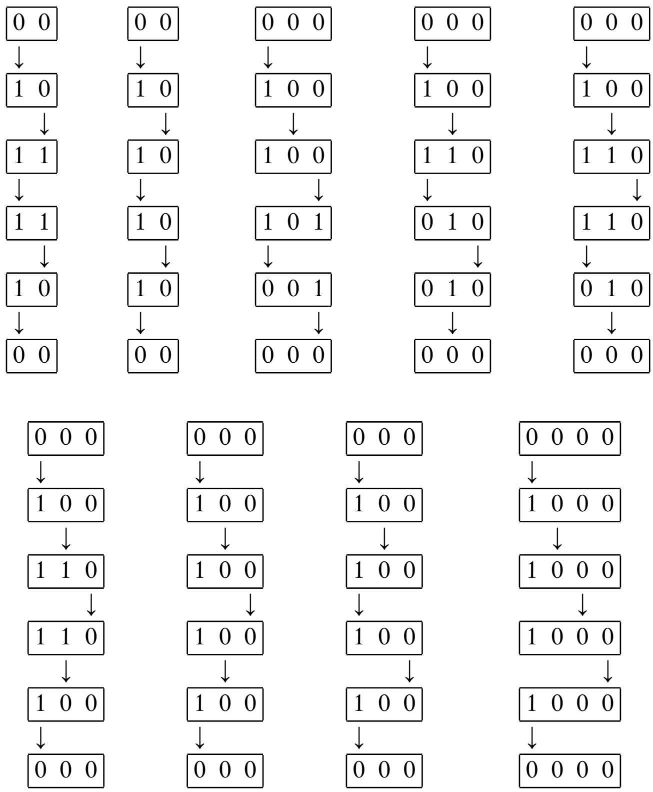

Fig. A 4. Atomic paths of length 6.

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