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Homotopy commutativity in Hermitian symmetric spaces

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Abstract

Ganea proved that the loop space of $\mathbb{C}P^n$ is homotopy commutative if and only if $n = 3$. We generalize this result to that the loop spaces of all irreducible Hermitian symmetric spaces but $\mathbb{C}P^3$ are not homotopy commutative. The computation also applies to determining the homotopy nilpotency class of the loop spaces of generalized flag manifolds G/T for a maximal torus T of a compact, connected Lie group G.

1. Introduction

A fundamental problem on H-spaces is to find whether or not a given H-space is homotopy commutative. This was intensely studied for finite H-spaces, and a complete answer was given by Hubbuck [15] such that if a connected finite H-space is homotopy commutative, then it is homotopy equivalent to a torus. As for infinite H-spaces, the problem should be studied by fixing a class of infinite H-spaces because there are too many classes of infinite H-spaces, each of which has its own special features.

In [8], Ganea studied the homotopy nilpotency of the loop spaces of complex projective spaces, and in particular, he proved that the loop space of the complex projective space $\mathbb{C}P^n$ is homotopy commutative if and only if $n = 3$. Then, we continue this work to study the homotopy commutativity of the loop spaces of homogeneous spaces. Recently, Golasiński [9] showed that the loop spaces of some homogeneous spaces such as complex Grassmannians are homotopy nilpotent. However, their homotopy nilpotency class is not computed: it is not even proved that they are homotopy commutative or not. In this paper, we study the homotopy commutativity of the loop spaces of Hermitian symmetric spaces, which generalizes Ganea's result and makes Golasiński's result more concrete. Hermitian symmetric spaces were first studied by Cartan [5], who classified them by means of his classification [4] of Riemannian symmetric spaces. The work of Borel and de Siebethal on subgroups of maximal rank in compact Lie groups [2] gives a simpler proof of Cartan's classification result. It states that every Hermitian symmetric space is a product of irreducible ones in the following table.

AIII	$U(m+n)/U(m)\times U(n)$	$(m, n \geq 1)$
BDI	$SO(n+2)/SO(2)\times SO(n)$	$(n \geq 3)$
CI	$Sp(n)/U(n)$	$(n \geq 4)$
DIII	$SO(2n)/U(n)$	$(n \geq 4)$
EIII	$E_6/Spin(10)\cdot T^1$	$(Spin(10)\cap T^1 \cong \mathbb{Z}/4)$
EVII	$E_7/E_6\cdot T^1$	$(E_6\cap T^1 \cong \mathbb{Z}/3)$

Then, we only need to consider the loop spaces of irreducible Hermitian symmetric spaces. Now, we state the main theorem.

Theorem 1.1 *The loop spaces of all irreducible Hermitian symmetric spaces but $\mathbb{C}P^3$ are not homotopy commutative.*

Theorem 1.1 will be proved by a case-by-case analysis of irreducible Hermitian symmetric spaces. Our main tools for the analysis are rational homotopy theory (Section 2) and Steenrod operations (Section 3). The rational homotopy technique also applies to flag manifolds, so that we can prove the following, where the definition of the homotopy nilpotency will be given in Section 2.

Theorem 1.2 *Let G be a compact connected non-trivial Lie group with maximal torus T. Then, the loop space of the flag manifold G/T is homotopy nilpotent of class 2.*

2. Rational homotopy

In this section, we apply rational homotopy theory to prove that the loop spaces of irreducible Hermitian symmetric spaces of type CI, DIII, and EVII are not homotopy commutative. We also consider the homotopy nilpotency of the loop spaces of flag manifolds. By [7, Proposition 13.16] and the adjointness of Whitehead products and Samelson products, we have the following criterion for a loop space not being homotopy commutative.

Lemma 2.1. *Let $(\Lambda V, d)$ be the minimal Sullivan model of a simply connected CW complex of finite type X. If there is $x \in V$ such that*

$$dx \not\equiv 0 \mod \Lambda^3 V,$$

then ΩX is not homotopy commutative.

In order to apply Lemma 2.1, we will use the following lemma.

Lemma 2.2. *Let X, Y be simply connected spaces such that*

$$H^*(X;\mathbb{Q}) = \mathbb{Q}[x_1, \dots, x_m] \quad \text{and} \quad H^*(Y;\mathbb{Q}) = \mathbb{Q}[y_1, \dots, y_n].$$

If a map $f: X \to Y$ is injective in rational cohomology, then there is a Sullivan model of the homotopy fiber of f such that

$$(\Lambda(x_1, \dots, x_m, z_1, \dots, z_n), d), \quad dx_i = 0, \quad dz_i = f^*(y_i).$$

Proof. By the Borel transgression theorem, $H^*(\Omega Y;\mathbb{Q}) = E(z_1, \dots, z_n)$ such that $\tau(z_i) = y_i$, where $E(z_1, \dots, z_n)$ denotes the exterior algebra generated by $z_1, \dots, z_n$ and τ denotes the transgression. Let F denote the homotopy fiber of the map f. Then, the sequence

$$(\Lambda(x_1, \dots, x_m), 0) \xrightarrow{\text{incl}} (\Lambda(x_1, \dots, x_m, z_1, \dots, z_n), d) \xrightarrow{\text{proj}} (\Lambda(z_1, \dots, z_n), 0)$$

is a model of the principal fibration $\Omega Y \to F \to X$, where $dx_i = 0$ and $dz_i = f^*(y_i)$. Thus, the statement is proved. □

Proposition 2.3. *The loop spaces of $Sp(n)/U(n)$ and $SO(2n)/U(n)$ are not homotopy commutative.*

Proof. First, we consider $Sp(n)/U(n)$. Recall that the cohomology of $BU(n)$ and $BSp(n)$ are given by

$$H^*(BU(n);\mathbb{Z}) = \mathbb{Z}[c_1, \dots, c_n] \quad \text{and} \quad H^*(BSp(n);\mathbb{Z}) = \mathbb{Z}[q_1, \dots, q_n],$$

where c_i and q_i are the Chern classes and the symplectic Pontrjagin classes. Then as in [22, Chapter III, Theorem 5.8], the natural map $q: BU(n) \to BSp(n)$ satisfies

$$q^*(q_i) = \sum_{k+l=2i} (-1)^{i+k} c_k c_l,$$

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where $c_0 = 1$ and $c_i = 0$ for $i > n$. Then by Lemma 2.2, there is a Sullivan model of $Sp(n)/U(n)$ such that

$$(\Lambda(c_1, \dots, c_n, r_1, \dots r_n), d), \quad dc_i = 0, \quad dr_i = \sum_{k+l=2i} (-1)^{i+k} c_k c_l,$$

where $c_0 = 1$ and $c_i = 0$ for $i > n$. Hence, the minimal model of $Sp(n)/U(n)$ is given by

$$(\Lambda(c_1, c_3, \dots, c_{2n-2[n/2]-1}, r_{[n/2]+1}, \dots r_n), d)$$
$$dc_i = 0, \quad dr_i \equiv \sum_{k+l=i-1} (-1)^{i+1} c_{2k+1} c_{2l+1} \mod (c_1, c_3, \dots, c_{2n-2[n/2]-1})^4,$$

where $c_0 = 1$, $c_i = 0$ for $i > n$. Thus, modulo $(c_1, c_3, \dots, c_{2n-2[n/2]-1})^4$,

$$dr_{n-1} \equiv c_{n-1}^2 \quad (n \text{ is even}), \quad dr_n \equiv c_n^2 \quad (n \text{ is odd}).$$

Therefore, by Lemma 2.1, $\Omega(Sp(n)/U(n))$ is not homotopy commutative.

Next, we consider $SO(2n)/U(n)$. The rational cohomology of $BSO(2n)$ is given by

$$H^*(BSO(2n); \mathbb{Q}) = \mathbb{Q}[p_1, \dots, p_{n-1}, e],$$

where p_i is the i-the Pontrjagin classes and e is the Euler class. By [22, Chapter III, Lemma 5.15 and Theorem 5.17]. Then, the natural map $r \colon BU(n) \to BSO(2n)$ satisfies

$$r^*(p_i) = \sum_{k+l=2i} (-1)^k c_k c_l \quad \text{and} \quad r^*(e) = c_n,$$

where $c_0 = 1$ and $c_i = 0$ for $i > n$. Thus arguing as above, we can see that the minimal model of $SO(2n)/U(n)$ coincides with that of $Sp(n-1)/U(n-1)$, implying that $\Omega(SO(2n)/U(n))$ is not homotopy commutative. □

Proposition 2.4. *The loop space of $E_7/E_6 \cdot T^1$ is not homotopy commutative.*

Proof. As in the proof of [27, Lemma 2.1], we have

$$H^*(BE_7; \mathbb{Q}) = \mathbb{Q}[x_4, x_{12}, x_{16}, x_{20}, x_{24}, x_{28}, x_{36}]$$
$$H^*(B(E_6 \cdot T^1); \mathbb{Q}) = \mathbb{Q}[u, v, w, x_4, x_{12}, x_{16}, x_{24}],$$

where $|x_i| = i$, $|u| = 2$, $|v| = 10$ and $|w| = 18$. Moreover, the natural map $j \colon B(E_6 \cdot T^1) \to BE_7$ satisfies $j^*(x_i) = x_i$ for $i = 4, 12, 16, 24$ and $j^*(x_i) \equiv z_i \mod (x_4, x_{12}, x_{16}, x_{24})$ for $i = 20, 28, 36$, where

$$z_{20} = v^2 - 2uv \qquad z_{28} = -2vw + 18u^5 w - 6u^6 v + u^{14}$$
$$z_{36} = w^2 + 20u^4 vw - 18u^9 w + 2u^{13} v.$$

Then by Lemma 2.2, there is a Sullivan model of $E_7/E_6 \cdot T^1$ such that

$$(\Lambda(u, v, w, x_4, x_{12}, x_{16}, x_{24}, y_3, y_{11}, y_{15}, y_{19}, y_{23}, y_{27}, y_{36}), d),$$

where $du = dv = dw = 0$ and $dy_i = x_{i+1}$ for $i = 3, 11, 15, 23$ and $dy_i \equiv z_{i+1} \mod (x_4, x_{12}, x_{16}, x_{24})$. Thus, we can easily see that the minimal model of $E_7/E_6 \cdot T^1$ is given by $(\Lambda(u, v, w, y_{19}, y_{27}, y_{36}), d)$ such that $du = dv = dw = 0$ and $dy_i = z_{i+1}$ for $i = 19, 27, 36$. Therefore by Lemma 2.1, $\Omega(E_7/E_6 \cdot T^1)$ is not homotopy commutative as stated. □

We consider the homotopy nilpotency of flag manifolds. Let X be an H-group. Let $\gamma : X \wedge X \to X$ denote the reduced commutator map, and let $\gamma_n = \gamma \circ (\gamma_{n-1} \wedge 1_X)$ for $n \geq 2$ and $\gamma_1 = 1_X$. Recall from [28, Definition 2.6.2] that X is called *homotopy nilpotent of class* $< n$ if $\gamma_n \simeq *$. Let honil (X) denote the homotopy nilpotency class of X. Then, X is homotopy commutative if and only if honil $(X) \leq 1$.

Proposition 2.5. *Let G be a Lie group, and let K be a subgroup of G. Then*

$$\text{honil}\,(\Omega(G/K)) \leq \text{honil}\,(K) + 1.$$

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Proof. There is a homotopy fibration $G/K \to BK \to BG$, and so the result follows from [1, Theorem 3.3]. □

Hopkins [14, Corollary 2.2] proved that a connected finite H-space is homotopy nilpotent whenever it is torsion free in homology. Then for a compact connected Lie group G and its closed subgroup K, it follows from Corollary 2.5 that $\Omega(G/K)$ is homotopy nilpotent whenever K is torsion free in homology (cf. [9, Proposition 2.2]). In particular, we obtain that the loop space of the flag manifold G/T is homotopy nilpotent, where T is a maximal torus of G. Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Clearly, we may assume G is simply connected. Since T is homotopy commutative and non-contractible, we have honil $(T) = 1$. Then by Corollary 2.5, honil $(\Omega(G/T)) \leq 2$, and so it remains to show that $\Omega(G/T)$ is not homotopy commutative. It is well known that the natural map $H^*(BG;\mathbb{Q}) \to H^*(BT;\mathbb{Q})^W$ is an isomorphism and

$$H^*(BT;\mathbb{Q})^W = \mathbb{Q}[x_1, \ldots, x_n],$$

where W is the Weyl group of G. Since G is simply connected, $H^*(BG;\mathbb{Q}) = 0$ for $* \leq 3$ and $H^4(BG;\mathbb{Q}) \neq 0$. Then, we may assume $|x_1| = 4$. By Lemma 2.2, there is a Sullivan model of G/T such that

$$(\Lambda(t_1, \ldots, t_n, y_1, \ldots, y_n), d), \quad dt_i = 0, \quad dy_i = x_i,$$

where $t_1, \ldots, t_n$ are generators of $H^*(BT;\mathbb{Q})$ which are of degree 2. Since all x_i are decomposable by degree reasons, this is the minimal model of G/T. Moreover, x_1 is a quadratic polynomial in $t_1, \ldots, t_n$. Then by Lemma 2.1, G/T has non-trivial Whitehead product, implying that $\Omega(G/T)$ is not homotopy commutative. □

3. Steenrod operation

In this section, we prove that the loop spaces of the irreducible Hermitian symmetric spaces of type AIII, BDI, EIII are not homotopy commutative by applying the following lemma. The lemma was proved by Kono and Ōshima [21] when A and B are spheres and p is odd, and its variants are used in [10, 11, 12, 13, 17, 18, 19, 20, 26]. For an augmented graded algebra A, let QA^n denote the module of indecomposables of dimension n.

Lemma 3.1. *Let X be a path-connected space X, let $\alpha\colon \Sigma A \to X$, $\beta\colon \Sigma B \to X$ be maps, and let p be a prime. Suppose the following conditions hold:*

(1) there are $a, b \in H^*(X;\mathbb{Z}/p)$ such that $\alpha^*(a) \neq 0$, $\beta^*(b) \neq 0$, and

  (a) $\alpha^*(b) = 0$ or $\beta^*(a) = 0$ for $p = 2$,
  (b) $A = B$, $\alpha = \beta$ and $a = b$ for $|a| = |b|$ and p odd;

(2) there are $x \in H^*(X;\mathbb{Z}/p)$ and a Steenrod operation θ such that $\theta(x)$ is decomposable and includes the term $ab \neq 0$;
(3) $\dim QH^*(X;\mathbb{Z}/p) = 1$ for $* = |a|, |b|$;
(4) θ acts trivially on $H^*(\Sigma A \times \Sigma B;\mathbb{Z}/p)$.

Then, the Whitehead product $[\alpha, \beta]$ in X is non-trivial.

Proof. Suppose $[\alpha, \beta] = 0$. Then, there is a homotopy commutative diagram

$$\begin{array}{ccc} \Sigma A \vee \Sigma B & \xrightarrow{\alpha\vee\beta} & X \\ {\scriptstyle \mathrm{incl}}\downarrow & & \| \\ \Sigma A \times \Sigma B & \xrightarrow{\mu} & X. \end{array}$$

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By the conditions (1), (2), and (3), the $H^{|a|}(\Sigma A;\mathbb{Z}/p)\otimes H^{|b|}(\Sigma B;\mathbb{Z}/p)$-part of $\mu^*(\theta(x))$ is

$$\mu^*(ab)=(\alpha^*(a)\times 1+1\times\beta^*(a))(\alpha^*(b)\times 1+1\times\beta^*(b))$$
$$=\begin{cases}2\alpha^*(a)\times\beta^*(b) & |a|=|b| \text{ and } p \text{ odd}\\ \alpha^*(a)\times\beta^*(b) & \text{otherwise,}\end{cases}$$

implying $\mu^*(\theta(x))\neq 0$. By the condition (4), we have $\mu^*(\theta(x))=\theta(\mu^*(x))=0$. Then, we obtain a contradiction, implying $[\alpha,\beta]\neq 0$, as stated. □

Let $G_{m,n}=U(m+n)/U(m)\times U(n)$. Since $G_{m,n}\cong G_{n,m}$, we may assume $m\leq n$. Let $j\colon G_{m,n}\to BU(m)$ denote the natural map. Then since $m\leq n$, the map j is a $(2m+1)$-equivalence. Let $g_i\colon S^{2i}\to BU(m)$ denote a generator of $\pi_{2i}(BU(m))\cong\mathbb{Z}$ for $i=1,\dots,m$. Then, since j is a $(2m+1)$-equivalence, there is a map $\bar{g}_i\colon S^{2i}\to G_{m,n}$ such that $j\circ\bar{g}_i=g_i$ for each $i\leq m$. Thus

$$j\circ[\bar{g}_k,\bar{g}_l]=[j\circ\bar{g}_k,j\circ\bar{g}_l]=[g_k,g_l].$$

So if $[g_k,g_l]\neq 0$, then $[\bar{g}_k,\bar{g}_l]\neq 0$, implying that $\Omega G_{m,n}$ is not homotopy commutative. We can find a non-trivial Whitehead product $[g_k,g_l]$ by using the result of Bott [3], but here we use Lemma 3.1 instead.

Recall from [22, Chapter III, Theorem 6.9] that the cohomology of $G_{m,n}$ is given by

$$H^*(G_{m,n};\mathbb{Z})=\mathbb{Z}[c_1,\dots,c_m,\bar{c}_1,\dots,\bar{c}_n]/\left(\sum_{i+j=k}c_i\bar{c}_j \mid k\geq 1\right)$$

such that $j^*(c_i)=c_i$ for each i, where $c_0=\bar{c}_0=1$, $c_i=0$ for $i>m$, $\bar{c}_j=0$ for $j>n$ and the cohomology of $BU(m)$ is as in the proof of Proposition 2.3. We say that a cohomology class $x\in H^k(X;\mathbb{Z}/p)$ is *mod p spherical* if there is a map $\alpha\colon S^k\to X$ such that $\alpha^*(x)\neq 0$. We denote the mod p reduction of an integral cohomology class by the same symbol x.

Lemma 3.2. *If p is a prime, then c_i is mod p spherical for $i\leq p$.*

Proof. By [22, Chapter IV, Lemma 5.8], $g_i^*(c_i)=\pm(i-1)!u_{2i}$, where u_{2i} is a generator of $H^{2i}(S^{2i};\mathbb{Z})\cong\mathbb{Z}$. Then the proof is done. □

Proposition 3.3. *The loop space of $G_{m,n}$ for $m,n\geq 2$ is not homotopy commutative.*

Proof. As observed above, it suffices to show $[g_k,g_l]\neq 0$ for some k,l. First, we consider the $m=2$ case. By Lemma 3.2, $c_1,c_2\in H^*(G_{2,n};\mathbb{Z}/2)$ are mod 2 spherical. By the Wu formula, $\mathrm{Sq}^2c_2=c_1c_2\neq 0$ in $H^*(BU(2);\mathbb{Z}/2)$. Then by Lemmas 3.1 and 3.2, $[g_1,g_2]\neq 0$.

Next, we consider the $m>2$ case. Take any odd prime p with $m/2<p\leq m$, where such an odd prime exists by Bertrand's postulate. Let $k=m/2$ for m even and $k=(m+1)/2$ for m odd. By Lemma 3.2, c_k and c_{m-k+1} are mod p spherical. By the mod p Wu formula proved by Shay [24], $\mathcal{P}^1c_{m-p+2}$ is decomposable and includes the term

$$-(m+1)c_kc_{m-k+1}$$

in $H^*(BU(m);\mathbb{Z}/p)$. So if $m+1\not\equiv 0 \mod p$, then $[g_k,g_{m-k+1}]\neq 0$. Now we suppose $m+1\equiv 0 \mod p$. Then, we must have $m=2p-1$. So if there is another prime q in $(m/2,m]$, then $m+1\not\equiv 0 \mod q$. So the above argument for the $m+1\not\equiv 0 \mod p$ case works, and thus, $[g_k,g_{m-k+1}]\neq 0$. Hence, we aim to show that there are two primes in $(m/2,m]$. Recall from [25] that the Ramanujan prime R_n is the least integer k such that for each $x\geq k$, there are at least n primes in the interval $(x/2,x]$. It is proved in [25] that R_n exists for each n and $R_2=11$. Then, it remains the cases $m=2\cdot 3-1=5$ and $m=2\cdot 5-1=9$, and

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we have $5/2 < 3, 5 \leq 5$ and $9/2 < 5, 7 \leq 9$. Thus, there are at least two primes in $(m/2, m]$, completing the proof. □

Let $Q_n = SO(n+2)/SO(2) \times SO(n)$.

Proposition 3.4. *The loop space of Q_n for $n \geq 2$ is not homotopy commutative.*

Proof. There is a homotopy fibration

$$S^1 = SO(2) \to SO(n+2)/SO(n) \xrightarrow{q} Q_n. \tag{3.1}$$

Then the projection $q\colon SO(n+2)/SO(n) \to Q_n$ is injective in π_* for $* \geq 2$, and so by the naturality of Whitehead products, it is sufficient to show that there is a non-trivial Whitehead products in $\pi_*(SO(n+2)/SO(n))$ for some $* \geq 2$. Let $\iota\colon S^n = SO(n+1)/SO(n) \to SO(n+2)/SO(n)$ denote the inclusion. Then, Ōshima [23] proved that the Whitehead product $[\iota, \iota] \in \pi_{2n-1}(SO(n+2)/SO(n))$ is non-trivial whenever $n+1$ is not the power of 2. Thus, we obtain that ΩQ_n is not homotopy commutative if $n+1$ is not the power of 2.

Suppose $n = 2m-1$. Then as in [16], the cohomology of Q_n is given by

$$H^*(Q_n; \mathbb{Z}) = \mathbb{Z}[t, e]/(t^m - 2e, e^2), \quad \mathrm{Sq}^2 e = te,$$

where $|t| = 2$ and $|e| = 2m$. Since Q_n is simply connected, the Hurewicz theorem implies that t is mod 2 spherical. Let $B = S^{n-1} \cup_2 e^n$. Then, $SO(n+2)/SO(n) = \Sigma B \cup e^{2n+1}$, so that

$$H^*(SO(n+2)/SO(n); \mathbb{Z}/2) = E(x_n, x_{n+1}), \quad |x_i| = i.$$

Let $j\colon \Sigma B \to Q_n$ denote the composition of the inclusion $\Sigma B \to SO(n+2)/SO(n)$ and the projection $q\colon SO(n+2)/SO(n) \to Q_n$. Then by the Gysin sequence for the fibration (3.1), we get $j^*(e) = x_{n+1}$. Thus by Lemma 3.1, we obtain that Q_n has non-trivial Whitehead product, implying ΩQ_n is not homotopy commutative. □

Proposition 3.5. *The loop space of $E_6/Spin(10) \cdot T^1$ is not homotopy commutative.*

Proof. As in [16], the mod 2 cohomology of $E_6/Spin(10) \cdot T^1$ is given by

$$H^*(E_6/Spin(10) \cdot T^1; \mathbb{Z}/2) = \mathbb{Z}/2[t, w']/(tw'^2, t^{12} + w'^3), \quad \mathrm{Sq}^2 w' = tw',$$

where $|t| = 2$ and $|w'| = 8$. Since $E_6/Spin(10) \cdot T^1$ is simply connected, the Hurewicz theorem implies that t is mod 2 spherical. We can deduce from Conlon's result [6] that $\pi_*(E_6/Spin(10), F_4/Spin(9)) = 0$ for $* \leq 31$. In particular,

$$H^*(E_6/Spin(10); \mathbb{Z}/2) \cong H^*(F_4/Spin(9); \mathbb{Z}/2) \quad (* \leq 30).$$

Note that $F_4/Spin(9)$ is the Cayley plane $\mathbb{O}P^2$. Then since $\mathbb{O}P^2 = S^8 \cup e^{16}$, a generator $u \in H^8(F_4/Spin(9); \mathbb{Z}/2) \cong \mathbb{Z}/2$ is mod 2 spherical, and so a generator $v \in H^8(E_6/Spin(10)) \cong \mathbb{Z}/2$ is mod 2 spherical too. By the Gysin sequence associated with the fibration $S^1 \to E_6/Spin(10) \xrightarrow{q} E_6/Spin(10) \cdot T^1$, we can see that $q^*(w') = v$, implying w' is mod 2 spherical. Thus by Lemma 3.1, we obtain that $E_6/Spin(10) \cdot T^1$ has a non-trivial Whitehead product, and so $\Omega(E_6/Spin(10) \cdot T^1)$ is not homotopy commutative. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Combine Propositions 2.3, 2.4, 3.3, 3.4, 3.5 and the result of Ganea [8] on the homotopy commutativity of the loop space of $\mathbb{C}P^n$ mentioned in Section 1. □

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