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The L^1-norm of exponential sums in $\mathbb{Z}^d$

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Abstract

Let A be a finite set of integers and $F_A(x) = \sum_{a \in A} \exp(2\pi i a x)$ be its exponential sum. McGehee, Pigno and Smith and Konyagin have independently proved that $\|F_A\|_1 \geqslant c \log |A|$ for some absolute constant c. The lower bound has the correct order of magnitude and was first conjectured by Littlewood. In this paper we present lower bounds on the L^1-norm of exponential sums of sets in the d-dimensional grid $\mathbb{Z}^d$. We show that $\|F_A\|_1$ is considerably larger than $\log |A|$ when $A \subset \mathbb{Z}^d$ has multidimensional structure. We furthermore prove similar lower bounds for sets in $\mathbb{Z}$, which in a technical sense are multidimensional and discuss their connection to an inverse result on the theorem of McGehee, Pigno and Smith and Konyagin.

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1. *Introduction*

We begin with a notational remark. Throughout the paper expressions of the form $Q \leqslant C$ are taken to mean that the quantity Q is less than an appropriately chosen absolute constant $C > 1$. We will therefore write counter-intuitive statements like $2C \leqslant C$. When the constant is less than 1 a lower case c is used.

For finite $A \subset \mathbb{Z}^d$ the *exponential sum of* A is

$$F_A(x) = \sum_{a \in A} e(a \cdot x),$$

where $\cdot$ is the usual dot product in $\mathbb{R}^d$, $e(t) = \exp(2\pi i t)$ and x lies in the d-dimensional

torus $\mathbb{T}^d$. The L^1-*norm* of F_A is given by

$$\|F_A\|_1 = \int_{x \in \mathbb{T}^d} |F_A(x)| \, dx.$$

We will also write

$$\langle f, g \rangle = \int_{x \in \mathbb{T}^d} f(x)\overline{g(x)} \, dx$$

for the inner product of two functions $f, g : \mathbb{T}^d \mapsto \mathbb{C}$.

J.E. Littlewood conjectured in 1948 [6] that for all finite sets $A \subset \mathbb{Z}$:

$$\|F_A\|_1 \geqslant c \log |A| .$$

The conjecture was proved in 1980 independently by O.C. McGehee, L. Pigno and B. Smith [9] and S.V. Konyagin [7].

THEOREM 1·1 (McGehee–Pigno–Smith, Konyagin). *Let A be a finite sets of integers. Then*

$$\|F_A\|_1 \geqslant c \log |A| .$$

Taking A to be a symmetric arithmetic progression about zero, and hence F_A the Dirichlet kernel, shows that the lower bound is of the correct order of magnitude [8].

The first proof works equally well when $A \subset \mathbb{Z}^d$. The order of magnitude of the lower bound is attained when A is an arithmetic progression in $\mathbb{Z}^d$. On the other hand, if A is the d-dimensional cube $\{(x_1, \ldots, x_d) : 1 \leqslant x_i \leqslant N \text{ for all } i\} \subset \mathbb{Z}^d$, then $\|F_A\|_1 = \|F_{\{1,\ldots,N\}}\|_1^d \geqslant (C \log N)^d$. It is therefore natural to ask whether a similar lower bound on $\|F_A\|_1$ holds when A has a genuinely multidimensional structure.

We answer this question to the affirmative, not only for sets in $\mathbb{Z}^d$, but also for sets in $\mathbb{Z}$. Our results present partial progress towards answering a question of W.T. Gowers on the L^1-norm of exponential sums in $\mathbb{Z}^2$, which will be stated below. They also help characterise sets of integers A for which $\|F_A\|_1$ is nearly minimal.

The first step is to quantify what we mean by 'genuinely multidimensional structure'. The most typical example that comes to mind is that of the d-dimensional cube, where as we have seen $\|F_A\|_1$ is roughly speaking $\log^d |A|$. The identity $\|F_A\|_1 = \|F_{\{1,\ldots,N\}}\|_1^d$ no longer holds when A is tweaked and taken to be $\{(a_1 + x_1, \ldots, a_d + x_d) : 1 \leqslant x_i \leqslant N \text{ for all } i\}$ for fixed integers $a_1, \ldots, a_d$. We study $\|F_A\|_1$ for sets that have a similar structure and show that in this case $\|F_A\|_1 \geqslant \log^{cd} |A|$. To keep the notation simple, here and most importantly in the proofs that follow, we will from now on set $d = 2$ or 3. Our methods can be generalised in a straightforward manner for $d > 3$. Considering the general case would make what already is a notation-heavy argument even more technical without adding anything to the method.

Let us now introduce some terminology, which will be helpful in pinning down an exact meaning for 'multidimensional structure'.

Definition. Let $j \in \{1, 2, 3\}$, $a_i \in \mathbb{Z}$ for $i \in \{1, 2, 3\} \setminus \{j\}$ and $A \subseteq \mathbb{Z}^3$. The intersection of A with the line $\{(x_1, x_2, x_3) : x_i = a_i \text{ for } i \in \{1, 2, 3\} \setminus \{j\}\}$ is a *row of A*.

Definition. Let $i \in \{1, 2, 3\}$, $a_i \in \mathbb{Z}$ and $A \subseteq \mathbb{Z}^3$. The intersection of A with the plane $\{(x_1, x_2, x_3) : x_i = a_i\}$ is a *planar slice of A*.

We call $A \subseteq \mathbb{Z}^2$ a *genuinely 2-dimensional* set, if its rows are either empty or large. We call $A \subseteq \mathbb{Z}^3$ a a *genuinely 3-dimensional* set, if its planar slices are either empty or a genuinely 2-dimensional set.

The first of our results asserts that, if A is genuinely 2-dimensional, then $\|F_A\|_1$ is considerably larger than $\log |A|$.

THEOREM 1·2. *Let $A \subset \mathbb{Z}^2$ be finite. Suppose that A consists of at least r rows of size at least s. Then*

$$\|F_A\|_1 \geqslant c \log s \left(\frac{\log r}{\log \log r} \right)^{1/2}.$$

The stated lower bound is probably not best possible. Gowers has asked whether $\|F_A\|_1 \geqslant c \log r \log s$ holds. Theorem 1·2 only gives $\|F_A\|_1 \geqslant \log s \log^{1/2-\varepsilon} r$ for all $\varepsilon > 0$ and sufficiently large A.

The method of proof of Theorem 1·2 can also be applied to subsets of $\mathbb{Z}$. To define 'multidimensional structure' in the integers we turn to a notion often used in additive problems.

Definition. Let A and B be sets in two additive groups. A map

$$\theta : A \longmapsto B$$

is a *Freiman isomorphism of degree k* if it is a bijection and $a_1 + \cdots + a_k = a_{k+1} + \cdots + a_{2k}$ holds if and only if $\theta(a_1) + \cdots + \theta(a_k) = \theta(a_{k+1}) + \cdots + \theta(a_{2k})$ holds for any choice of $a_1, \ldots, a_{2k} \in A$. We say A is *Freiman isomorphic of degree k* to B.

Our second main result asserts that if $A \subset \mathbb{Z}$ is Freiman isomorphic to a 3-dimensional set in $\mathbb{Z}^3$, then $\|F_A\|_1$ is considerably larger than $\log |A|$.

THEOREM 1·3. *Let $A \subset \mathbb{Z}^3$ be finite. Suppose that A consists of at least p planar slices each in turn consisting of at least r rows of size at least s. If $B \subset \mathbb{Z}$ is Freiman isomorphic of degree k to A, then*

$$\|F_B\|_1 \geqslant c \left(\frac{\log s \, \log r \, \log p}{\log \log s \, \log \log r \, \log \log p} \right)^{1/2},$$

provided that $k = 62 \log r \log s \log p$.

A helpful, if imprecise, way to rephrase the above is that $\|F_B\|_1 \geqslant \log^{3/2-\varepsilon} |B|$ for all $\varepsilon > 0$ whenever $B \subset \mathbb{Z}$ is isomorphic to a genuinely 3-dimensional set in $\mathbb{Z}^3$ and is sufficiently large. As a consequence we see that any sufficiently large set A where $\|F_A\|_1 \leqslant C \log |A|$ cannot have this particular 3-dimensional structure.

The lower bound in Theorem 1·3 is probably not best possible. Moreover, one suspects that the conclusion holds for smaller values of k. It is furthermore likely that if A is Freiman isomorphic to a 2-dimensional set in $\mathbb{Z}^2$, then $\|F_A\|_1 \geqslant \log^{1+\eta} |A|$ for some absolute $0 < \eta \leqslant 1$. The method we present is not strong enough to prove this.

The remaining sections are organised as follows. In Section 2 we prove a lemma that is central to the proof of both theorems. The lemma is a generalisation of a method developed by P.J. Cohen [2] to tackle Littlewood's conjecture and was later refined by H. Davenport [3] and S.K. Pichorides [10]. In Section 3 we prove Theorem 1·2 . In Section 4 we prove Theorem 1·3. Finally, in Section 5 we discuss how an inverse result for Theorem 1·1 may look and compare the suggested structure with that which comes out of Theorem 1·3.

2. *A method of Cohen, Davenport and Pichorides*

To prove Theorems 1·2 and 1·3 we will rely on a combination of techniques developed to tackle Littlewood's conjecture by Cohen [2], Davenport [3], Pichorides [**10**] and McGehee, Pigno and Smith [**9**]. The four aforementioned papers on the Littlewood conjecture concentrate on constructing a test function g that satisfies two properties: $\|g\|_\infty \leqslant 1$ and $\langle g, F_A \rangle \geqslant \log^\alpha |A|$ for some absolute constant α. This immediately gives $\log^\alpha |A| \leqslant \langle g, F_A \rangle \leqslant \|g\|_\infty \|F_A\|_1 \leqslant \|F_A\|_1$.

Our strategy to prove Theorem 1·2 is as follows. For simplicity let us assume that A consists of r rows $A_1, \ldots, A_r$ of size at least s, where $A_i \subset \{(x, n_i) : x \in \mathbb{Z}\}$ for some integers $n_1, \ldots, n_r$. Let Φ_{n_i} be the McGehee–Pigno–Smith test function for the exponential sum F_{A_i}. That is the function constructed by McGehee, Pigno and Smith that satisfies the two properties listed above for $\alpha = 1$. We will combine these to produce a better test function for A. This will be done by mirroring the method of Cohen, Davenport and Pichorides.

Cohen combined the exponentials $\{e(nx) : n \in A\}$ and obtained a test function which yields the value $\alpha = 1/8 - \varepsilon$ for all $\varepsilon > 0$. Davenport improved this to $\alpha = 1/4 - \varepsilon$ and Pichorides to $\alpha = 1/2 - \varepsilon$. The three arguments are rather similar. A closer look at the underlying method reveals that one can get the same result even when relaxing the most commonly used properties of exponentials to:

   (i) $|e(nx)| \leqslant 1$ for all n and x;
   (ii) $\langle e(nx) e(mx), e(kx) \rangle = 0$ unless $k = n + m$;
   (iii) $\langle e(nx), e(nx) \rangle \geqslant c$ for all n.

Our strategy is to replace the exponentials in the existing proofs by the Φ_{n_i}, which satisfy the first condition. The support of the Fourier transform $\widehat{\Phi_{n_i}}$ lies in the line that contains A_i and therefore the Φ_{n_i} also satisfy the following new versions of the later two conditions:

   (i) let k and l be positive integers. $\langle F_A, \Phi_{n_{i_1}} \Phi_{n_{i_2}} \cdots \Phi_{n_{i_k}} \overline{\Phi_{n_{i_{k+1}}} \Phi_{n_{i_{k+2}}} \cdots \Phi_{n_{i_{k+l}}}} \rangle = 0$ unless $n_{i_1} + \cdots + n_{i_k} - n_{i_{k+1}} - \cdots - n_{i_{k+l}} = n_\nu$ for some $1 \leqslant \nu \leqslant R$;
   (ii) $\langle \Phi_{n_i}, F_A \rangle = \langle \Phi_{n_i}, F_{A_i} \rangle \geqslant c \log s$.

As we will shortly see every step can still be carried out and we thus obtain Theorem 1·2. One way to describe this process is to say we will employ the McGehee–Pigno–Smith method in one dimension and the Cohen–Davenport–Pichorides in the other.

The Cohen–Davenport–Pichorides method is applicable when one considers Freiman isomorphisms. We will thus employ it in all three dimensions to prove Theorem 1·3. The details can be found in the two upcoming sections.

We begin with a technical result that is the main building block of the two proofs.

LEMMA 2·1. *Let R and d be positive integers, K a positive real number and $F : \mathbb{T}^d \mapsto \mathbb{C}$ be an integrable function. Suppose there are positive integers $n_1, \ldots, n_R$ and a collection of integrable functions $\Phi_{n_1}, \ldots, \Phi_{n_R}$ such that*:

(A) $\|\Phi_{n_i}\|_\infty \leqslant 1$ *for $1 \leqslant i \leqslant R$*;
(B) $\langle \Phi_{n_i}, F \rangle \geqslant K$ *for $1 \leqslant i \leqslant R$*;
(C) *let l be a positive integer.* $\langle F, \Phi_{n_{i_0}} \Phi_{n_{i_1}} \cdots \Phi_{n_{i_l}} \overline{\Phi_{n_{i_{l+1}}} \cdots \Phi_{n_{i_{2l}}}} \rangle = 0$ *for $1 \leqslant i_j \leqslant R$ unless $n_{i_0} + n_{i_1} + \cdots + n_{i_l} - n_{i_{l+1}} - \cdots - n_{i_{2l}} = n_\nu$ for some $1 \leqslant \nu \leqslant R$.*

Then there is a test function g such that:
   (i) $\|g\|_\infty \leqslant 1$;

(ii) *g is a linear combination of functions of the form $\Phi_{n_{i_0}} \cdots \Phi_{n_{i_k}} \overline{\Phi_{n_{i_k}} \cdots \Phi_{n_{i_{2k}}}}$ for some $k \leqslant 2 \log R$;*

(iii) $\langle g, F \rangle \geqslant c \, K \dfrac{\log^{1/2} R}{\log \log^{1/2} R}$.

In particular (i) *and* (iii) *imply*

$$\|F\|_1 \geqslant c \, K \frac{\log^{1/2} R}{\log \log^{1/2} R}.$$

The reader can think of the Φ_n as exponentials in order to gain some intuition. We will need two lemmata. The first is [**10**, lemma 1].

LEMMA 2·2 (Pichorides). *Let $t \geqslant 100$. Suppose the quantities P and Q satisfy $t + 2P \geqslant 0$ and $P^2 + Q^2 \leqslant t^4/4$. Then*

$$\left| 1 - \frac{1}{t} - \frac{P + iQ}{t^2} + \frac{(P + iQ)^2}{t^4} \right| + \frac{1}{4t^{3/2}} (t + 2P)^{1/2} \leqslant 1.$$

The second is also a result Pichorides ([**10**, lemma 2]) whose proof is essentially due to Davenport (cf. [**3**, lemma 3]).

LEMMA 2·3 (Davenport–Pichorides). *Let E and S be sets of positive integers. For $p \in S$ let $N(p)$ to be the number of elements of E that are greater than p.*

Let t be a positive integer and suppose that

$$t^4 \sum_{p \in S} N(p) \leqslant |E|.$$

Then there exist t integers $\{m_1, \ldots, m_t\}$ in E such that

$$p + (m_\alpha - m_\beta) + (m_\gamma - m_\delta) \notin E$$

for all $p \in S$ and $1 \leqslant \alpha \leqslant \beta \leqslant t$, $1 \leqslant \gamma < \delta \leqslant t$.

Furthermore $m_\alpha = n_{q(\alpha)}$, where $q(\alpha) \leqslant \alpha^4 \sum_{p \in S} N(p)$.

We now turn to proving Lemma 2·1.

Proof of Lemma 2·1. The proof is based on iteration. We will construct functions $g_1, g_2, \ldots$ that satisfy (i) and modified versions of (ii) and (iii):

(ii′) g_i *is a linear combination of functions of the form $\Phi_{n_{i_0}} \cdots \Phi_{n_{i_k}} \overline{\Phi_{n_{i_{k+1}}} \cdots \Phi_{n_{i_{2k}}}}$ for some $k \leqslant 2i$.*

(iii′) $\langle g_i, F \rangle \geqslant K (4t^{1/2})^{-1} \sum_{n=0}^{i-1} (1 - 1/t)^n$ *for some $t \geqslant 100$ to be chosen later.*

We set $g_1 = \Phi_{n_1}$, which satisfies (i), (ii′) and (iii′) as the sum is empty. We now inductively define

$$g_{i+1}(x) = g_i(x) \left(1 - \frac{1}{t} \right)$$

$$- g_i(x) \left(\frac{1}{t^2} \sum_{1 \leqslant i < j \leqslant t} \Phi_{m_i}(x) \overline{\Phi_{m_j}(x)} - \frac{1}{t^4} \left(\sum_{1 \leqslant i < j \leqslant t} \Phi_{m_i}(x) \overline{\Phi_{m_j}(x)} \right)^2 \right)$$

$$+ \frac{1}{4t^{3/2}} \sum_{1 \leqslant i \leqslant t} \Phi_{m_i}(x)$$

for some $m_1, \ldots, m_t$ carefully chosen from $\{n_1, \ldots, n_R\}$ in such a way that the inner product of the middle part with F is zero. For the time being we assume this can be done. We need to check that g_{i+1} satisfies (i), (ii′) and (iii′).

For (i) we apply Lemma 2·2. For any v set

$$P + iQ = \sum_{1 \leqslant i < j \leqslant t} \Phi_{m_i}(v)\overline{\Phi_{m_j}(v)}.$$

We observe that

$$P^2 + Q^2 = |P + iQ|^2 \leqslant \left(\sum_{1 \leqslant i < j \leqslant t} |\Phi_{m_i}(v)||\Phi_{m_j}(v)| \right)^2 \leqslant (t(t-1)/2)^2 < t^4/4$$

and that

$$0 \leqslant \left| \sum_{i=1}^t \Phi_{m_i}(v) \right|^2 = \sum_{i=1}^t |\Phi_{m_i}(v)|^2 + 2P \leqslant t + 2P.$$

The conditions of Lemma 2·2 are satisfied and so

$$
\begin{aligned}
|g_{i+1}(v)| &\leqslant |g_i(v)| \left| 1 - \frac{1}{t} - \frac{P+iQ}{t^2} + \frac{(P+iQ)^2}{t^4} \right| + \frac{1}{4t^{3/2}}(t+2P)^{1/2} \\
&\leqslant \left| 1 - \frac{1}{t} - \frac{P+iQ}{t^2} + \frac{(P+iQ)^2}{t^4} \right| + \frac{1}{4t^{3/2}}(t+2P)^{1/2} \\
&\leqslant 1.
\end{aligned}
$$

The last inequality coming from Lemma 2·2. Thus g_{i+1} satisfies (i). g_{i+1} by definition satisfies (ii′) and so we are left with (iii′).

It follows from our assumption on the middle part of g_{i+1} that

$$\langle g_{i+1}, F \rangle = \left(1 - \frac{1}{t} \right) \langle g_i, F \rangle + \frac{1}{4t^{3/2}} \sum_{i=1}^t \langle \Phi_{m_i}, F \rangle \geqslant \frac{K}{4t^{1/2}} \sum_{n=0}^i \left(1 - \frac{1}{t} \right)^n.$$

Once n becomes considerably bigger than t the terms $(1 - 1/t)^n \leqslant \exp(-n/t)$ become exponentially small and so add very little to the sum. We therefore iterate the process only t times and set $g = g_t$. It follows that the k appearing in (ii) can be taken to be $2t$.

$$\langle g, F \rangle \geqslant \frac{K}{4t^{1/2}} \sum_{n=1}^t \left(1 - \frac{1}{t} \right)^n \geqslant cKt^{1/2} \tag{2·1}$$

subject only to being able to repeat the iteration t times.

Our final task then becomes to prove that the m_i can indeed be chosen t times and get the largest possible value for t. This will be done by applying Lemma 2·3.

We start by labelling $m_1^{(i)}, \ldots, m_t^{(i)}$ the elements of $\{n_1, \ldots, n_R\}$ chosen in the ith iteration and recursively define the following sets:

$$S_1 = \{n_1\}, \quad S_{i+1} = S_i \cup T_i \cup U_i$$

where
$T_i = \{m_1^{(i)}, \ldots, m_t^{(i)}\}$ and
$U_i = \{p + (m_\alpha^{(i)} - m_\beta^{(i)}) + (m_\gamma^{(i)} - m_\delta^{(i)}) : p \in S_i, \ 1 \leqslant \alpha \leqslant \beta \leqslant t, \ 1 \leqslant \gamma < \delta \leqslant t\}.$

Let $E = \{n_1, \ldots, n_R\}$. It follows from condition (C) that the middle part of $\langle g_i, F \rangle$ is zero provided that $p + (m_\alpha^{(i)} - m_\beta^{(i)}) + (m_\gamma^{(i)} - m_\delta^{(i)}) \notin E$ for all $p \in S_{i-1}$, $1 \leqslant \alpha \leqslant \beta \leqslant t$ and $1 \leqslant \gamma < \delta \leqslant t$.

Applying Lemma 2·3 with $S = S_{i-1}$ we see that the $m_j^{(i)}$ can be chosen provided that

$$t^4 \sum_{p \in S_{i-1}} N(p) \leqslant R.$$

The sum on the left-hand side is estimated using the final conclusion of Lemma 2·3:

$$
\begin{aligned}
\sum_{p \in S_i} N(p) &= \sum_{p \in S_{i-1}} N(p) + \sum_{p \in T_{i-1}} N(p) + \sum_{p \in U_{i-1}} N(p) \\
&= \sum_{p \in S_{i-1}} N(p) + \sum_{\alpha=1}^{t} N(m_\alpha^{(i)}) + \sum_{p, \alpha, \beta, \gamma, \delta} N(p + (m_\alpha^{(i)} - m_\beta^{(i)}) + (m_\gamma^{(i)} - m_\delta^{(i)})) \\
&\leqslant \sum_{p \in S_{i-1}} N(p) + \sum_{\alpha=1}^{t} \alpha^4 \left(\sum_{p \in S_{i-1}} N(p) \right) + t^4 \sum_{p \in S_{i-1}} N(p) \\
&\leqslant t^5 \sum_{p \in S_{i-1}} N(p) \, .
\end{aligned}
$$

We used the fact that $m_\alpha^{(i)} - m_\beta^{(i)} + m_\gamma^{(i)} - m_\delta^{(i)} > 0$ so that $N(p + m_\alpha^{(i)} - m_\beta^{(i)} + m_\gamma^{(i)} - m_\delta^{(i)}) \leqslant N(p)$.

Observe that $\sum_{p \in S_1} N(p) = 1$. It follows by induction that

$$\sum_{p \in S_i} N(p) \leqslant t^{5i}.$$

The iteration is thus possible for t steps when $t^{5t} \leqslant R$. So we take

$$t = \left\lfloor \frac{\log R}{10 \log \log R} \right\rfloor.$$

Substituting this value of t in (2·1) gives conclusion (iii). Conclusion (ii) has been shown to hold for $k = 2t \leqslant 2 \log R$ and so has conclusion (i).

3. *Towards a 2-dimensional Littlewood conjecture*

We now prove Theorem 1·2. Loosely speaking the first dimension will be used to construct the Φ_n and the second to combine them and produce a better test function.

Proof of Theorem 1·2. We apply Lemma 2·1 to $F = F_A$. We take e_1, e_2 to be the standard basis of $\mathbb{Z}^2$ and translate A if necessary so that the coordinates of all its points are positive integers. We let $A_1, \ldots, A_R$ be the rows of A and $n_i = A_i \cdot e_2$ for $1 \leqslant i \leqslant R$.

We set Φ_{n_i} to be the McGehee–Pigno–Smith test function for F_{A_i}. By this we mean a function whose Fourier transform is supported on $\{u \in \mathbb{Z}^2 : u \cdot e_2 = n_i\}$ and which satisfies $\|\Phi_{n_i}\|_\infty \leqslant 1$ and $\langle F_{A_i}, \Phi_{n_i} \rangle \geqslant c \log |A_i| \geqslant c \log s$.

Hence the Φ_{n_i} satisfy conditions (A) and (B) for $K \geqslant c \log s$. Condition (C) is also satisfied as we see by examining the support of the Fourier transform of

$$\Phi_{n_{i_0}} \Phi_{n_{i_1}} \cdots \Phi_{n_{i_l}} \overline{\Phi_{n_{i_{l+1}}} \cdots \Phi_{n_{i_{2l}}}} :$$

it lies on the line $\{u \in \mathbb{Z}^2 : u \cdot e_2 = n_{i_0} + \cdots + n_{i_l} - n_{i_{l+1}} - \cdots - n_{i_{2l}}\}$. In particular

$$\langle F_A, \ \Phi_{n_{i_0}} \Phi_{n_{i_1}} \cdots \Phi_{n_{i_l}} \overline{\Phi_{n_{i_{l+1}}} \cdots \Phi_{n_{i_{2l}}}} \rangle = 0$$

unless $n_{i_0} + \cdots + n_{i_l} - n_{i_{l+1}} - \cdots - n_{i_{2l}} = n_\nu$ for some $1 \leqslant \nu \leqslant R$. The theorem follows from he final conclusion of Lemma 2·1 by observing that $R \geqslant r$.

We can of course take Φ_{n_i} to be the test function that satisfies $\langle \Phi_{n_i}, F_A \rangle = \|F_{A_i}\|_1$. Its Fourier transform is still supported on $\{u \in \mathbb{Z}^2 : u \cdot e_2 = n_i\}$ and hence everything we did above can be repeated to yield the following.

THEOREM 3·1. *Let $A \subset \mathbb{Z}^2$ be finite. Suppose that A consists of at least r rows of size at least s. Then*

$$\|F_A\|_1 \geqslant c \, \mu(s) \left(\frac{\log r}{\log \log r} \right)^{1/2}$$

with

$$\mu(s) = \min \|F_{A_i}\|_1,$$

where $A_1, A_2, \ldots$ are the rows of A.

4. *Multidimensional sets in $\mathbb{Z}$*

We repeat the same process to prove Theorem 1·3. We can no longer use the McGehee–Pigno–Smith test functions as their support is both very large and very difficult to analyse. It is furthermore unlikely that condition (C) in Lemma 2·1 holds. Instead we use the Cohen–Davenport–Pichorides test functions, which are Freiman isomorphism friendly because of conclusion (ii) in Lemma 2·1. In what follows for a set of integers S and a positive integer α we write

$$\alpha S = \{s_1 + \cdots + s_\alpha : s_i \in S\}.$$

Proof of Theorem 1·3. Translate A if necessary so that all three coordinates of its elements are positive. Let θ be the Freiman isomorphism between A and B and e_1, e_2, e_3 the standard basis of $\mathbb{Z}^3$. Suppose that $A_1, A_2, \ldots$ are the planar slices of A. For any i let a_i be the integer such that $A_i \subset \{u \in \mathbb{Z}^3 : u \cdot e_3 = a_i\}$. Each A_i consists of at least r rows $A_{i1}, A_{i2}, \ldots$ of size at least s. Let a_{ij} be the integer such that $A_{ij} \subset \{u \in \mathbb{Z}^3 : u \cdot e_3 = a_i, \ u \cdot e_2 = b_j^i\}$.

We construct a test function for $F_B = F_{\theta(A)}$ by three successive applications of Lemma 2·1.

We begin by applying Lemma 2·1 to get a test function for $F_{\theta(A_{ij})}$ for all pairs of indices $\{i, j\}$ for which A_{ij} is non-empty. Let $b_{ij}^{(1)}, b_{ij}^{(2)}, \ldots$ be the elements of $\theta(A_{ij})$. We set $n_l = b_{ij}^{(l)}$ and $\Phi_{n_l} = e(b_{ij}^{(l)})$ in Lemma 2·1. The Φ_{n_l} satisfy conditions (A), (B) with $K = 1$ and (C). Applying Lemma 2·1 we get a test function f_{ij} which satisfies

$$\langle F_{\theta(A_{ij})}, f_{ij} \rangle \geqslant c \left(\frac{\log s}{\log \log s} \right)^{1/2}$$

and $\|f_{ij}\|_\infty \leqslant 1$. Next we observe that the support of $\widehat{f_{ij}}$ lies in $(\alpha + 1)\theta(A_{ij}) - \alpha \, \theta(A_{ij})$ for some $\alpha \leqslant 2 \log s$. In particular it does not intersect $\theta(A \setminus A_{ij})$, for if $\theta(u) = \theta(u_0) + \cdots + \theta(u_\alpha) - \theta(u_{\alpha+1}) - \cdots - \theta(u_{2\alpha})$ for some $u \in A \setminus A_{ij}$ and $u_0, \ldots, u_{2\alpha} \in A_{ij}$, then $u = u_0 + \cdots + u_\alpha - u_{\alpha+1} - \cdots - u_{2\alpha}$ as θ is a Freiman isomorphism of degree k and $\alpha \leqslant k$. This

is impossible as the right hand side is supported on the line $\{u \in \mathbb{Z}^3 : u \cdot e_3 = a_i,\ u \cdot e_2 = b_j^i\}$, while the left hand is not. Hence

$$\langle F_{\theta(A_i)}, f_{ij}, \rangle = \langle F_{\theta(A_{ij})}, f_{ij} \rangle \geqslant c \left(\frac{\log s}{\log \log s} \right)^{1/2}.$$

Next we combine the f_{ij} to get a test function for $F_{\theta(A_i)}$. We set $n_j = b_j^i$ and $\Phi_{n_j} = f_{ij}$ in Lemma 2·1. The f_{ij} satisfy condition (A) and, as we saw above, (B) with $K \geqslant c(\log s / \log \log s)^{1/2}$. To check condition (C) note that the Fourier transform of

$$f_{ij_0} f_{ij_1} f_{ij_2} \cdots f_{ij_l} \overline{f_{ij_{l+1}} f_{ij_{l+2}} \cdots f_{ij_{2l}}}$$

is supported on

$$(\alpha + 1)\theta(A_{ij_0}) - \alpha\theta(A_{ij_0}) + (\alpha + 1)\theta(A_{ij_1}) - \alpha\theta(A_{ij_1}) + \cdots + (\alpha + 1)\theta(A_{ij_l})$$
$$-\alpha\theta(A_{ij_l}) - (\alpha + 1)\theta(A_{ij_{l+1}}) + \alpha\theta(A_{ij_{l+1}}) - \cdots - (\alpha + 1)\theta(A_{ij_{2l}}) + \alpha\theta(A_{ij_{2l}})$$

for $\alpha \leqslant 2 \log s$. Thus the inner product with $F_{\theta(A_i)}$ is zero unless $\theta(A_i)$ intersects the above sum-difference set. Note that $l \leqslant 2 \log r$ and that θ is a Freiman isomorphism of sufficiently large degree for this to happen only when the sum $b_{j_0}^i + b_{j_1}^i + \cdots + b_{j_l}^i - b_{j_{l+1}}^i - \cdots - b_{j_{2l}}^i$ equals b_j^i for some j.

By Lemma 2·1 we get a test function f_i that satisfies $\|f_i\|_\infty \leqslant 1$ and

$$\langle F_{\theta(A_i)}, f_i \rangle \geqslant c \left(\frac{\log s \ \log r}{\log \log s \ \log \log r} \right)^{1/2}.$$

The support of $\widehat{f_i}$ lies in $(\gamma + 1)\theta(A_i) - \gamma\theta(A_i)$ for some $\gamma \leqslant 12 \log r \log s$: the support of $\widehat{f_{ij}}$ lies in $(\alpha + 1)\theta(A_i) - \alpha\theta(A_i)$ for $\alpha \leqslant 2 \log s$ and we have to consider expressions of the form $f_{ij_0} f_{ij_1} f_{ij_2} \cdots f_{ij_\beta} \overline{f_{ij_{\beta+1}} f_{ij_{\beta+2}} \cdots f_{ij_{2\beta}}}$ for $\beta \leqslant 2 \log r$ and so γ can be taken to be $(\alpha + 1)\beta + \alpha(\beta + 1) = 2\alpha\beta + \alpha + \beta \leqslant 12 \log r \log s$. Thus the support of $\widehat{f_i}$ does not intersect $\theta(A \setminus A_i)$, for if $\theta(u) = \theta(u_0) + \cdots + \theta(u_\gamma) - \theta(u_{\gamma+1}) - \cdots - \theta(u_{2\gamma})$ for some $u \in A \setminus A_i$, $u_l \in A_i$ and $\gamma \leqslant 12 \log s \log r$, then, as θ is a Freiman isomorphism of degree $k \geqslant \gamma$, u would have to equal $u_0 + \cdots + u_\gamma - u_{\gamma+1} - \cdots - u_{2\gamma}$. This is impossible as the right-hand side lies on the plane $\{u \in \mathbb{Z}^3 : u \cdot e_3 = a_i\}$, while the left-hand does not. Hence

$$\langle F_{\theta(A)}, f_i, \rangle = \langle F_{\theta(A_i)}, f_i \rangle \geqslant c \left(\frac{\log s \ \log r}{\log \log s \ \log \log r} \right)^{1/2}.$$

Finally we combine the f_i to get a test function for $F_{\theta(A)}$. We let $n_i = a_i$ and $\Phi_{n_i} = f_i$. The f_i satisfy conditions (A) and (B) with $K \geqslant c(\log s \ \log r / (\log \log s \ \log \log r))^{1/2}$ in the statement of Lemma 2·1. To check condition (C) note that the Fourier transform of $f_{i_0} f_{i_1} f_{i_2} \cdots f_{i_l} \overline{f_{i_{l+1}} f_{i_{l+2}} \cdots f_{i_{2l}}}$ is supported on $(\gamma + 1)\theta(A_{i_0}) - \gamma\theta(A_{i_0}) + (\gamma + 1)\theta(A_{i_1}) - \gamma\theta(A_{i_1}) + \cdots + (\gamma + 1)\theta(A_{i_l}) - \gamma\theta(A_{i_l}) - (\gamma + 1)\theta(A_{i_{l+1}}) + \gamma\theta(A_{i_{l+1}}) - \cdots - (\gamma + 1)\theta(A_{i_{2l}}) + \gamma\theta(A_{i_{2l}})$ for $\gamma \leqslant 12 \log s \log r$. The inner product with $F_{\theta(A)}$ is zero unless $\theta(A)$ intersects the above sum-difference set, which is a subset of $(\delta + 1)\theta(A) - \delta\theta(A)$ for $\delta = 2l\gamma + l + \gamma \leqslant 62 \log p \log r \log s$. θ is a Freiman isomorphism of degree $k \geqslant \delta$, so this happens only if $a_{i_0} + a_{i_1} + \cdots + a_{i_l} - a_{i_{l+1}} - \cdots - a_{i_{2l}}$ equals a_i for some i. By Lemma 2·1 we get

$$\|F_{\theta(A)}\|_1 \geqslant c \left(\frac{\log s \ \log r \ \log p}{\log \log s \ \log \log r \ \log \log p} \right)^{1/2}.$$

Remark. One can extend this result to higher dimensions.

5. *Additive structure when $\|F_A\|_1$ is small*

In this final section we discuss the following question. Suppose $\|F_A\| \leqslant C \log |A|$ for $A \subset \mathbb{Z}$. Is there a particular structure A must have? We suggest a plausible structure and compare it with that implied by Theorem 1·3.

Determining the precise value of $\|F_A\|_1$ for a given A is hard. The Cauchy-Schwarz inequality shows that the L^1-norm is certainly bounded above by the L^2-norm, $\|F_A\|_2 = |A|^{1/2}$. This order of magnitude is attained when A is the lacunary sequence $\{2^i : 1 \leqslant i \leqslant N\}$. By an averaging argument one gets much denser random subsets of $\{1, 2, \ldots, N\}$ with $\|F_A\|_1 \geqslant cN^{1/2}$. In general sets with random like properties are expected to give rise to exponential sums with large L^1-norm. For example, if A is the set of the first N primes, then $\|F_A\|_1 \geqslant N^{1/2-\varepsilon}$ for all $\varepsilon > 0$ [12] and if A is the intersection of the support of the Möbius function with $\{1, 2, \ldots, N\}$, then $\|F_A\|_1 \geqslant N^{1/8-\varepsilon}$ [1].

At the other end of the spectrum we have structured sets. If A is the union of k arithmetic progressions, then by the triangle inequality $\|F_A\|_1 \leqslant C k \log |A|$. Furthermore, if A is a *proper d-dimensional arithmetic progression*

$$\{c + x_1 q_1 + \cdots + x_d q_d : 0 \leqslant x_i \leqslant N \text{ for } 1 \leqslant i \leqslant d\} \text{ for } c, q_i \in \mathbb{Z} \text{ for } 1 \leqslant i \leqslant d,$$

then $\|F_A\|_1 \leqslant (C \log |A|)^d$ (Theorem 3.3 in [11]). The above progression is called proper, if it has size $(N + 1)^d$.

Note however that not the whole of A needs to be structured. We can for example remove a subset X with $C \log^2 N$ elements from $\{1, 2, \ldots, N\}$ and still have

$$\|F_A\|_1 \leqslant \|F_{\{1,\ldots,N\}}\|_1 + \|F_B\|_1 \leqslant C \log N + \|F_B\|_2 = C \log N + C \log N \leqslant C \log |A|.$$

One can instead add a much larger set X. For example X can be a 2-dimensional arithmetic progression disjoint from $\{1, \ldots, N\}$. If X is Freiman 2-isomorphic to $\{1, \ldots, L\} \times \{1, \ldots, L\}$, where $L = \exp(\log^{1/2} N)$, then $\|F_X\|_1 \leqslant C \log N$. Thus $\|F_{\{1,\ldots,N\}\cup X}\|_1 \leqslant C \log |A|$.

Establishing a concrete relation between $\|F_A\|_1$ and the additive structure of A has not been possible so far. Even the simplest inverse theorem for sets A where $\|F_A\|_1$ is close to being minimal has been elusive. The following question arose in conversations with B.J. Green and is in accordance with a theorem of Green and T. Sanders on idempotent measures [5]. See also [4].

Question 5·1. Does there exists an absolute constant $1/2 \leqslant \eta < 1$ and a function $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ with the following property. Let $A \subset \mathbb{Z}$ be a finite set and K a positive constant. Suppose $\|F_A\|_1 \leqslant K \log |A|$. Then there exists a set $X \subset \mathbb{Z}$ of size at most $\exp\{\|F_A\|_1^{\eta}\}$, $g(K)$ arithmetic progressions $P_1, \ldots, P_{g(K)}$ and $\varepsilon_1, \ldots, \varepsilon_{g(K)} \in \{+1, -1\}$ such that

$$F_A = F_X + \sum_{i=1}^{g(K)} \varepsilon_i F_{P_i}.$$

The range of η comes from the example discussed above and Theorem 1·1. Taking A to be a 2-dimensional arithmetic progression Freiman 2-isomorphic to $\{1, \ldots, N\} \times \{1, \ldots, N\}$ suggests that $g(K)$ has to be exponential in K.

The results in this paper point to a slightly different direction. We have established that no sufficiently large set of integers A whose exponential sum has L^1-norm at most $C \log |A|$ can be Freiman isomorphic to a genuinely three dimensional set in $\mathbb{Z}^3$. This

puts a constraint on sets where $\|F_A\|_1$ is close to being minimal. Unfortunately it is not the case that such sets mainly consist of few long arithmetic progressions and a small set. The notion of dimensionality we have relied on is too restrictive to lead to such a conclusion.

Take for example the lacunary sequence $A = \{x_i = 2^i : 1 \leqslant i \leqslant N\}$. Its elements satisfy the recurrence relation $x_{i+1} = x_i + 2(x_i - x_{i-1})$. It follows that its image under a Freiman isomorphism θ of degree 3 also satisfies this relation. The y-coordinate of the elements of $\theta(A)$ is either constant (when $\theta(x_1) \cdot e_2 = \theta(x_2) \cdot e_2$) or distinct for all i. In other words either $\theta(A)$ is contained in a single row or it consists of $|A|$ singleton rows. In either case $\theta(A)$ is not a genuinely 3-dimensional set. Yet any subset $Y \subset A$ cannot be decomposed in fewer than $|Y|/2$ arithmetic progressions as A contains at most two consecutive elements of any arithmetic progression.

Lacunary sequences are very sparse, but the situation doesn't change when we consider dense sets as the following example demonstrates.

Let L be a large integer and P the first prime such that

$$\sum_{p \in \mathcal{P}} p^{-1} \geqslant 1/2$$

where $\mathcal{P}$ is the set of primes between L and P. Now let

$$N = \prod_{p \in \mathcal{P}} p$$

and

$$A = \bigcup_{p \in \mathcal{P}} A_p,$$

where A_p consists of all numbers in $\{1, \ldots, N\}$ that are congruent to 1 mod p.

A has large density in $\{1, \ldots, N\}$. To check this observe that

$$|A| = \left| \bigcup_{p \in \mathcal{P}} \left(A_p / \bigcup_{q \neq p} A_q \right) \right| = \sum_{p \in \mathcal{P}} \left| A_p / \bigcup_{q \neq p} A_q \right|.$$

We know that $|A_p| = N/p$ and $|A_p \cap A_q| = N/pq$. Hence

$$\left| A_p / \bigcup_{q \neq p} A_q \right| \geqslant \frac{N}{p} \left(1 - \sum_{q \in \mathcal{P} \setminus \{p\}} q^{-1} \right) \geqslant \frac{N}{2p}.$$

Which in turn implies that

$$|A| \geqslant \frac{N}{2} \sum_{p \in \mathcal{P}} p^{-1} \geqslant N/4.$$

Next we consider the image of A under a Freiman isomorphism of degree two. Freiman isomorphisms map arithmetic progressions in $\mathbb{Z}$ into lines in $\mathbb{Z}^3$ and hence $\theta(A)$ must be supported on a collection of lines $\{\theta(A_p) : p \in \mathcal{P}\}$. For every pair of indices $p \neq q$, $\theta(A_p) \cap \theta(A_q) = N/pq > 2$ and so the two lines must in fact be identical. Thus the image of A under any Freiman isomorphism lies in a single line in $\mathbb{Z}^3$. As a consequence $\theta(A)$ either lies in a single row or in $|A|$ different rows.

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