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Remarks on generalized elliptic integrals

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We study the monotonicity for certain combinations of generalized elliptic integrals, thus generalizing analogous well-known results for classical complete elliptic integrals, and prove a conjecture put forward by Heikkala, Vamanamurthy and Vuorinen.

1. Introduction

Throughout this paper we adopt the notation and terminology of [10]. For $\operatorname{Re} x > 0$ let

$$\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{1.1}$$

be the classical Euler gamma function and psi function, respectively. For all $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ and for all $n \in \mathbb{N} \equiv \{0, 1, 2, 3, \dots\}$ we have [17, (12.12)]

$$\Gamma(z + n) = (z, n)\Gamma(z), \tag{1.2}$$

where $(z, 0) = 1$ for $z \neq 0$, and (z, n) is the *shifted factorial function* $(z, n) = z(z + 1)(z + 2)\cdots(z + n - 1)$ for $z \in \mathbb{C}$ and $n \in \mathbb{N} \setminus \{0\}$. Furthermore, the gamma function satisfies the reflection formula [17, (12.14)]

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)} \tag{1.3}$$

for all $z \notin \mathbb{Z} \equiv \{\dots, -2, -1, 0, 1, 2, \dots\}$. In particular, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

The *beta function* is defined for $\operatorname{Re} x > 0$ and $\operatorname{Re} y > 0$ by

$$B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1}\,\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}. \tag{1.4}$$

Given complex numbers a, b and c with $c \neq 0, -1, -2 \dots$, the *Gaussian hypergeometric function* is the analytic continuation to the slit plane $\mathbb{C} \setminus [1, \infty)$ of the series

$$F(a, b; c; z) = {}_2F_1(a, b; c; z) \equiv \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!}, \quad |z| < 1; \tag{1.5}$$

(a, n) is the shifted factorial function.

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The behaviour of the hypergeometric function near $z = 1$ in the three cases $\mathrm{Re}(a+b-c) < 0$, $a+b=c$ and $\mathrm{Re}(a+b-c) > 0$, respectively, is given by

$$\left.\begin{aligned} F(a,b;c;1) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \\ B(a,b)F(a,b;a+b;z) + \log(1-z) &= R(a,b) + O((1-z)\log(1-z)), \\ F(a,b;c;z) &= (1-z)^{c-a-b}F(c-a,c-b;c;z), \end{aligned}\right\} \tag{1.6}$$

where $R(a,b) = -2\gamma - \psi(a) - \psi(b)$ and

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.57721\cdots$$

is the*Euler–Mascheroni constant*. Note that $R(\frac{1}{2},\frac{1}{2}) = \log 16$. The above asymptotic formula for the *zero-balanced* case $a+b=c$ is due to Ramanujan [3, 4].

For $a,b,c > 0$ with $a+b \geqslant c$, a *generalized modular equation* of order (or degree) $p > 0$ is

$$\frac{F(a,b;c;1-s^2)}{F(a,b;c;s^2)} = p\frac{F(a,b;c;1-r^2)}{F(a,b;c;r^2)}, \quad 0 < r < 1. \tag{1.7}$$

This equation uniquely defines s.

Modular equations (1.7) were studied extensively by Ramanujan [5]. Many particular cases of (1.7) have been studied in the literature on both analytic number theory and geometric function theory (see, for example, [1, 2, 4, 5, 8, 12]). In particular, in 1995 Berndt *et al*. published an important paper [5] in which they studied the case $(a,b,c) = (a, 1-a, 1)$ and p an integer. After the publication of [5] many papers have appeared on modular equations (see, for example, [2, 4, 6, 9, 14, 16]).

To rewrite (1.7) in a slightly shorter form, we use the decreasing homeomorphism $\mu_{a,b,c} : (0,1) \to (0,\infty)$, defined by

$$\mu_{a,b,c}(r) = \frac{B(a,b)}{2}\frac{F(a,b;c;r'^2)}{F(a,b;c;r^2)}, \quad r \in (0,1), \tag{1.8}$$

for $a,b,c > 0$, $a+b \geqslant c$. We call $\mu_{a,b,c}$ the *generalized modulus* (cf. [12, (2.2), p. 60]). We can now write (1.7) as

$$\mu_{a,b,c}(s) = p\mu_{a,b,c}(r), \quad 0 < r < 1. \tag{1.9}$$

With $p = 1/K$, $K > 0$, the solution of (1.7) is then given by

$$s = \varphi_K^{a,b,c}(r) = \mu_{a,b,c}^{-1}\biggl(\frac{\mu_{a,b,c}(r)}{K}\biggr). \tag{1.10}$$

We call $\varphi_K^{a,b,c}$ the *(a,b,c)-modular function with degree $p = 1/K$* (see [2] and [5, (1.5)]).

In the case when $a < c$ we also use the notation

$$\mu_{a,c} = \mu_{a,c-a,c}, \quad \varphi_K^{a,c} = \varphi_K^{a,c-a,c}. \tag{1.11}$$

For $0 < a < \min\{c, 1\}$ and $0 < b < c \leqslant a + b$, define *the generalized complete elliptic integrals of the first and second kinds* (cf. [2, (1.9), (1.10), (1.3), (1.5)]) on $[0, 1]$ by

$$\mathcal{K} = \mathcal{K}_{a,b,c} = \mathcal{K}_{a,b,c}(r) \equiv \frac{B(a,b)}{2} F(a, b; c; r^2), \tag{1.12}$$

$$\mathcal{E} = \mathcal{E}_{a,b,c} = \mathcal{E}_{a,b,c}(r) \equiv \frac{B(a,b)}{2} F(a-1, b; c; r^2), \tag{1.13}$$

$$\mathcal{K}' = \mathcal{K}'_{a,b,c} = \mathcal{K}'_{a,b,c}(r) \equiv \mathcal{K}_{a,b,c}(r'), \qquad \mathcal{E}' = \mathcal{E}'_{a,b,c} = \mathcal{E}'_{a,b,c}(r) \equiv \mathcal{E}_{a,b,c}(r'), \tag{1.14}$$

for $r \in (0, 1)$, $r' = \sqrt{1 - r^2}$. The end values are defined by limits as r tends to 0^+ and 1^-, respectively. In particular, we define $\mathcal{K}_{a,c} = \mathcal{K}_{a,c-a,c}$ and $\mathcal{E}_{a,c} = \mathcal{E}_{a,c-a,c}$. Thus, by (1.6),

$$\mathcal{K}_{a,b,c}(0) = \mathcal{E}_{a,b,c}(0) = \frac{B(a,b)}{2} \tag{1.15}$$

and

$$\mathcal{K}_{a,b,c}(1) = \infty, \quad \mathcal{E}_{a,b,c}(1) = \frac{1}{2} \frac{B(a,b)B(c, c+1-a-b)}{B(c+1-a, c-b)}. \tag{1.16}$$

In the remainder of the paper, we continue the studies in [2, 10], generalize some results in [2], and give a positive answer to [10, conjecture 4.39(1)]. We now state our main results.

THEOREM 1.1. *For $0 < a, b < \min\{c, 1\}$ and $a + b \geqslant c$, let $B = B(a, b)$, $\mathcal{K} = \mathcal{K}_{a,b,c}$ and $\mathcal{E} = \mathcal{E}_{a,b,c}$. In part (viii) let $a + b \geqslant \max\{c, (c + 1 - ab)/2\}$, in part (ix) let $a + b = c$, in part (x) let $c \leqslant a + b < \min\{c + \frac{1}{2}, c + 1 - b/c\}$. Then the function*

(i) $f_1(r) \equiv [(c-a)(\mathcal{E} - \mathcal{K}) + (br^2 + ar'^2)\mathcal{K}]/r'^2$ *has positive Maclaurin coefficients and maps $[0, 1)$ onto $[aB/2, \infty)$,*

(ii) $f_2(r) \equiv [(c-a)(\mathcal{E} - \mathcal{K}) + b\mathcal{K}]/r'^2$ *has positive Maclaurin coefficients and maps $[0, 1)$ onto $[bB/2, \infty)$,*

(iii) $f_3(r) \equiv \{(c - a)(\mathcal{E} - \mathcal{K}) + [br^2 + (c - 1)r'^2]\mathcal{K}\}/r'^2$ *has positive Maclaurin coefficients and maps $[0, 1)$ onto $[(c - 1)B/2, \infty)$,*

(iv) $f_4(r) \equiv \mathcal{E}$ *has negative Maclaurin coefficients, except for the constant term, and is log-concave from $(0, 1)$ onto $(BB(c, c + 1 - a - b)/(2B(c + 1 - a, c - b)), B/2)$,*

(v) $f_5(r) \equiv r^{-2}[(\mathcal{E} - r'^2\mathcal{K})/r^2 - (c - b)B/(2c)]$ *has positive Maclaurin coefficients and maps $(0, 1)$ onto $(ab(c - b)B/(2c(c + 1)), B(cB(c, c + 1 - a - b) - (c - b)B(c + 1 - a, c - b))/(2cB(c + 1 - a, c - b)))$,*

(vi) $f_6(r) \equiv r^{-4}[(c - b)(\mathcal{K} - \mathcal{E}) - b(\mathcal{E} - r'^2\mathcal{K})]$ *has positive Maclaurin coefficients and maps $(0, 1)$ onto $(ab(c - b)B/(2c(c + 1)), \infty)$,*

(vii) $f_7(r) \equiv [(c - b)(\mathcal{K} - \mathcal{E}) - b(\mathcal{E} - r'^2\mathcal{K})]/(2r^2 \log 1/r')$ *is strictly increasing from $(0, 1)$ onto $(ab(c-b)B/(2c(c+1)), D)$, where $D = a/2$ if $a+b = c$ and $D = \infty$ if $a + b > c$,*

(viii) $f_8(r) \equiv [(c-b)(\mathcal{K}-\mathcal{E}) - b(\mathcal{E} - r'^2\mathcal{K})]/(2\log 1/r' - r^2)$ *is strictly increasing from* $(0,1)$ *onto* $(ab(c-b)B/(c(c+1)), D)$, *where* $D = a/2$ *if* $a+b=c$ *and* $D=\infty$ *if* $a+b>c$,

(ix) $f_9 \equiv r'^2(\mathcal{K}-\mathcal{E})/(r^2\mathcal{E})$ *is strictly decreasing from* $(0,1)$ *onto* $(0, b/c)$,

(x) $f_{10} \equiv r'^{2(c+1-a-b)}\mathcal{K}/\mathcal{E}$ *is strictly decreasing from* $(0,1)$ *onto* $(0,1)$.

Remark 1.2. If $a+b=1=c$, theorem 1.1(ii) reduces to [2, lemma 5.4(2)]. Parts (v), (viii), (ix) and (x) of theorem 1.1 generalize (10), (11), (6) and (2) of [2, lemma 5.2], respectively.

Theorem 1.3, below, gives a positive answer to [10, conjecture 4.39(1)], and hence generalizes [2, theorem 5.5(4)].

Theorem 1.3. *Let* $a>0, a<c<a+1/a$. *Then the function*

$$f(r) \equiv \frac{\mu_{a,c}(r)}{\log(1/r)}$$

is strictly increasing from $(0,1)$ *onto* $(1,\infty)$.

Remark 1.4. As pointed out by the referee, theorem 1.3 is essentially [11, 3.6(4)], which has an additional restriction $c \leqslant 1$. From the proof, we can see this additional restriction is not necessary. Actually, what is needed is $2a(c-a) \leqslant c$, which is the same as the condition needed in the proof of lemma 2.3(i), below.

2. Proof of the main results

In this section, we shall give some derivative formulae and lemmas, and prove the main results.

We use the standard notation for contiguous hypergeometric functions (cf. [15])

$$F = F(a,b;c;z), \quad F(a+) = F(a+1,b;c;z), \qquad F(a-) = F(a-1,b;c;z),$$

etc. We also let

$$v = v(z) = F, \quad u = u(z) = F(a-), \quad v_1 = v_1(z) = v(1-z) \quad \text{and} \quad u_1 = u_1(1-z).$$

The derivative of F can be written in the following different forms (see [10,15]):

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}z} = \frac{\mathrm{d}F}{\mathrm{d}z} &= \frac{ab}{c}F(a+,b+;c+)\\ &= \frac{a}{z}(F(a+)-F) = \frac{b}{z}(F(b+)-F) = \frac{c-1}{z}(F(c-)-F)\\ &= \frac{(c-a)u + (a-c+bz)v}{z(1-z)}\end{aligned} \tag{2.1}$$

and

$$\frac{\mathrm{d}\mathcal{E}_{a,b,c}}{\mathrm{d}r} = \frac{2(a-1)}{r}(\mathcal{K}_{a,b,c} - \mathcal{E}_{a,b,c}). \tag{2.2}$$

Lemma 2.1, below, is [10, theorem 4.3], while a more general version of this lemma appears in [7,13].

LEMMA 2.1 (Biernacki and Krzyz [7]; Ponnusamy and Vuorinen [13]). *Let*

$$\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad \sum_{n=0}^{\infty} b_n x^n$$

be two real power series converging on the interval $(-R, R)$. *If the sequence* $\{a_n/b_n\}$ *is increasing (decreasing), and* $b_n > 0$ *for all* n, *then the function*

$$f(x) = \frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n}$$

is also increasing (decreasing) on $(0, R)$.

LEMMA 2.2. *Let*

$$\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad \sum_{n=0}^{\infty} b_n x^n$$

be two real power series converging on the interval $(-R, R)$, $b_n > 0$ *for all* n, *and*

$$\lim_{x\to R} \sum_{n=0}^{\infty} a_n x^n = \lim_{x\to R} \sum_{n=0}^{\infty} b_n x^n = \infty.$$

(i) *If* $\lim_{n\to\infty} a_n/b_n = 0$, *then*

$$\lim_{x\to R} \frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n} = 0.$$

(ii) *If* $\lim_{n\to\infty} a_n/b_n = 1$, *then*

$$\lim_{x\to R} \frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n} = 1.$$

Proof.
(i) If $\lim_{n\to\infty} a_n/b_n = 0$, then, for any $\varepsilon > 0$, there exists a positive integer $M > 0$, such that, for all $n > M$, we have

$$-b_n\varepsilon < a_n < b_n\varepsilon.$$

Thus,

$$0 \leqslant \lim_{x\to R} \frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n} \leqslant \lim_{x\to R} \frac{\sum_{n=0}^{M} a_n x^n + \varepsilon \sum_{n=0}^{\infty} b_n x^n}{\sum_{n=0}^{\infty} b_n x^n} = \varepsilon;$$

hence, part (i) follows.

(ii) The proof of this part is similar to that of part (i). □

Part (i) of lemma 2.3, below, is essentially [10, 4.23(3)], while part (ii) is essentially [10, 4.21(6)], except for slight difference in the range of parameters a, b and c. The proof of part (i) is the same as that of [10, 4.23(3)], while that of part (ii) is more direct.

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LEMMA 2.3.

(i) *If* $0 < a < c < a+1/a$, *then the function* $f_1(r) \equiv r'^2 F(a, c-a; c; r'^2)/\log(1/r)$ *is strictly increasing from* $(0, 1)$ *onto* $(2/B(a, c-a), 2)$.

(ii) *If* $0 < a < \min\{c, 1\}$ *or* $1 \leqslant a < c < a^2/(a-1)$, *then* $f_2(r) \equiv r'^2 F(a, c-a; c; r^2)$ *is strictly decreasing from* $[0, 1)$ *onto* $(0, 1]$. *Moreover,* $f_2(r)$ *has negative Maclaurin coefficients, except for the constant term.*

Proof.
(i) From (1.5), we get

$$f_1(r) = 2\biggl(\sum_{n=0}^{\infty} \frac{(a,n)(c-a,n)}{n!(c,n)} r'^{2n}\biggr)\biggl(\sum_{n=0}^{\infty} \frac{1}{n+1} r'^{2n}\biggr)^{-1}. \tag{2.3}$$

The ratio of the coefficients equals

$$\frac{2(a,n)(c-a,n)(n+1)}{n!(c,n)} \equiv T_n.$$

Then

$$\frac{T_{n+1}}{T_n} = \frac{(a+n)(c-a+n)(n+2)}{(n+1)^2(c+n)} < 1,$$

since $(a+n)(c-a+n)(n+2) - (n+1)^2(c+n) = [a(c-a)-1]n + [2a(c-a)-c] < 0$ if $0 < a < c < a+1/a$. Thus, the monotonicity of $f_1(r)$ is obtained by lemma 2.1. From (2.3) the limit value $f_1(1^-)$ is clear. By Stirling's formula, $\lim_{n\to\infty} T_n = 1$. Hence, by lemma 2.2(ii), the limit $f_1(0^+) = 1$ follows.

(ii) From (1.5), we get

$$f_2(r) = (1-r^2)F(a, c-a; c; r^2) = 1 + g(r),$$

where $g(r) = \sum_{n=1}^{\infty} T_n r^{2n}$, $T_n = A_n - B_n$, and

$$A_n = \frac{(a,n)(c-a,n)}{n!(c,n)}, \qquad B_n = \frac{(a,n-1)(c-a,n-1)}{(n-1)!(c,n-1)}.$$

Then

$$\frac{A_n}{B_n} = \frac{(a+n-1)(c-a+n-1)}{n(c+n-1)} < 1,$$

since $(a+n-1)(c-a+n-1) - n(c+n-1) = -(n-1) + [a(c-a)-c] < 0$, if $0 < a < \min\{c, 1\}$ or $1 \leqslant a < c < a^2/(a-1)$. Thus, $f_2(r)$ has negative Maclaurin coefficients, except for the constant term. So the monotonicity of $f_2(r)$ is obtained. The limit $f_2(0^+)$ is clear. For the limit value $f_2(1^-)$, we rewrite $f_2(r)$ as

$$f_2(r) = \frac{F(a, c-a; c; r^2)}{1/(r'^2)} = \frac{\sum_{n=0}^{\infty} A_n r^{2n}}{\sum_{n=0}^{\infty} r^{2n}}.$$

By Stirling's formula, the ratio of the coefficients $A_n \to 0 (n \to \infty)$. Hence, by lemma 2.2(i) we have the limit $f_2(1^-) = 0$. □

Proof of theorem 1.1.
(i) From (2.1), we get

$$\frac{\mathrm{d}F}{\mathrm{d}z} = \frac{a}{z}(F(a+) - F) = \frac{(c-a)u + (a-c+bz)v}{z(1-z)}.$$

Putting $z = r^2$ and multiplying by $\frac{1}{2}B$, we get

$$f_1(r) = \tfrac{1}{2}aBF(a+1, b; c; r^2),$$

which proves the assertion. The limit $f_1(0^+) = \frac{1}{2}aB$ is clear, while the limit $f_1(1^-)$ follows from (1.6).

For parts (ii) and (iii), similarly to part (i), the proof follows from (2.1):

$$\frac{b}{z}(F(b+) - F) = \frac{(c-a)u + (a-c+bz)v}{z(1-z)} = \frac{c-1}{z}(F(c-) - F).$$

(iv) The negatives of Maclaurin coefficients, except for the constant term and the limiting values, are clear. Next, by (2.2),

$$\frac{1}{2(a-1)}\frac{\mathrm{d}}{\mathrm{d}r}\log\mathcal{E} = \frac{\mathcal{K}-\mathcal{E}}{r^2\mathcal{K}}\frac{r\mathcal{K}}{\mathcal{E}},$$

which is a product of two positive and increasing functions by [10, lemma 4.21(1)]. Hence, the log-concavity of f_4 follows.

(v) From the proof of [10, lemma 4.21(2)], we have

$$\frac{\mathcal{E} - r'^2\mathcal{K}}{r^2} = \frac{(c-b)B}{2c}F(a, b; c+1; r^2).$$

Thus,

$$f_5(r) = \frac{(c-b)B}{2c}\sum_{n=0}^{\infty}\frac{(a, n+1)(b, n+1)}{(n+1)!(c+1, n+1)}r^{2n},$$

from which the assertion follows. The limit $f_5(0^+)$ is clear, while the limit $f_5(1^-)$ follows from [10, lemma 4.21(2)].

(vi) From (1.12) and (1.13), we get

$$f_6(r) = \frac{(c-b)B}{2c}\sum_{n=0}^{\infty}\frac{(a, n+1)(b, n+1)}{(n)!(c+1, n+1)}r^{2n},$$

which proves the assertion. The limit $f_6(0^+)$ is clear, while the limit $f_6(1^-)$ follows from [10, lemma 4.21(2)].

(vii) From (1.12) and (1.13), we get

$$f_7(r) = \frac{(c-b)B}{2c}\left(\sum_{n=1}^{\infty}\frac{(a, n)(b, n)}{(n-1)!(c+1, n)}r^{2n}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n}r^{2n}\right)^{-1}.$$

The ratio of the coefficients equals

$$\frac{(a, n)(b, n)n}{(c+1, n)(n-1)!} \equiv T_n.$$

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Then

$$\frac{T_{n+1}}{T_n} = \frac{(a+n)(b+n)(n+1)}{(c+1+n)n^2} > 1,$$

since $(a+n)(b+n)(n+1) - (c+1+n)n^2 = (a+b-c)n^2 + (a+b+ab)n + ab > 0$. Therefore, the monotonicity follows immediately by lemma 2.1. The limit $f_7(0^+)$ is clear, while the limit $f_7(1^-)$ follows from [10, lemma 4.21(10)].

(viii) From (1.12) and (1.13), we get

$$f_8(r) = \frac{(c-b)B}{2c}\biggl(\sum_{n=1}^{\infty}\frac{(a,n)(b,n)}{(n-1)!(c+1,n)}r^{2n+2}\biggr)\biggl(\sum_{n=1}^{\infty}\frac{1}{n+1}r^{2n+2}\biggr)^{-1}.$$

The ratio of the coefficients equals

$$\frac{(a,n)(b,n)(n+1)}{(c+1,n)(n-1)!} \equiv T_n.$$

Then

$$\frac{T_{n+1}}{T_n} = \frac{(a+n)(b+n)(n+2)}{(c+1+n)(n+1)n} > 1,$$

since

$$(a+n)(b+n)(n+2) - (c+1+n)(n+1)n = (a+b-c)n^2 + [ab+2(a+b)-c-1]n + 2ab > 0$$

if $a+b \geqslant \max\{c, \frac{1}{2}(c+1-ab)\}$. So that the monotonicity follows immediately by lemma 2.1. The limit $f_8(0^+)$ is clear, while the limit $f_8(1^-)$ follows from [10, lemma 4.21(10)].

(ix) From (1.12) and (1.13), we get

$$f_9(r) = \biggl(\sum_{n=0}^{\infty}\frac{(a,n+1)(c-a,n+1)}{(a+n)(n+1)!(c,n+1)}r^{2n+2}\biggr)\biggl(\sum_{n=0}^{\infty}\frac{(c-a+1,n)(a,n)}{n!(c,n)}r^{2n+2}\biggr)^{-1}.$$

The ratio of the coefficients equals

$$\frac{b}{c+n} \equiv T_n,$$

which is decreasing in n. The limiting values are clear.

(x) From (1.12), (1.13) and (1.6), we get

$$f_{10}(r) = \biggl(\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{n!(c,n)}r^{2n}\biggr)\biggl(\sum_{n=0}^{\infty}\frac{(c-a+1,n)(c-b,n)}{n!(c,n)}r^{2n}\biggr)^{-1}.$$

The ratio of the coefficients equals

$$\frac{(a,n)(b,n)}{(c-a+1,n)(c-b,n)} \equiv T_n.$$

Then

$$\frac{T_{n+1}}{T_n} = \frac{(a+n)(b+n)}{(c-a+1+n)(c-b+n)} < 1,$$

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since

$$(a+n)(b+n)-(c-a+1+n)(c-b+n) = [2(a+b)-2c-1]n+[ca+(c+1)(b-c)] < 0$$

if $c \leqslant a+b < \min\{c+\frac{1}{2}, c+1-b/c\}$. The limit $f_{10}(0^+)$ is clear, while the limit $f_{10}(1^-)$ follows from (1.6). □

Proof of theorem 1.3. From (1.8), we get

$$f(r) = \frac{B(a,c-a)}{2}\frac{f_1(r)}{f_2(r)},$$

where $f_1(r)$, $f_2(r)$ are the functions in lemma 2.3. Thus, we can easily obtain the result of $f(r)$ from lemma 2.3. □

The following theorem is an analogue of [10, 4.5(3)] for $\varphi_{1/K}^{a,b,c}(r)$.

THEOREM 2.4. *Let $a,b,c > 0$, $a+b > c$ and $K > 1$. Then*

$$r < \varphi_K^{a,b,c}(r) < K^{1/(2(a+b-c))}r \tag{2.4}$$

and

$$(1/K)^{1/(2(a+b-c))}r < \varphi_{1/K}^{a,b,c}(r) < r \tag{2.5}$$

for all $r \in (0,1)$.

Proof. The inequality (2.4) is proved in [10, 4.5(3)].

For inequality (2.5), putting $s = \varphi_{1/K}^{a,b,c}(r) < r$, by [10, lemma 4.5(2)] we get

$$\begin{aligned} f(r) &= (r/r')^{2(a+b-c)}\mu_{a,b,c}(r) \\ &= (r/r')^{2(a+b-c)}\mu_{a,b,c}(s)/K < f(s) \\ &= (s/s')^{2(a+b-c)}\mu_{a,b,c}(s), \end{aligned}$$

so that

$$(s/s')^{2(a+b-c)} > (1/K)(r/r')^{2(a+b-c)},$$

that is

$$s/s' > (1/K)^{1/(2(a+b-c))}r/r'.$$

Hence,

$$s > (1/K)^{1/(2(a+b-c))}r(s'/r') > (1/K)^{1/(2(a+b-c))}r.$$

□

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