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SURVEY

A geometric approach to semi-dispersing billiards

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Abstract. We summarize the results of several recent papers, together with a few new results, which rely on a connection between semi-dispersing billiards and non-regular Riemannian geometry. We use this connection to solve several open problems about the existence of uniform estimates on the number of collisions, topological entropy and periodic trajectories of such billiards.

0. *Introduction*

While the first ideas of hyperbolicity of certain billiard systems go back to Krylov [**Kr**], the mathematical theory of semi-dispersing billiards originated with the works of Sinai [**Si4–6**] in connection with the foundations of statistical physics and the study of the hyperbolicity and ergodicity properties of such billiards. Since then the theory of semi-dispersing and dispersing (also called scattering or Sinai) billiards has grown in various directions, including the study of their ergodicity properties [**BuSi1, BLPS, KSS1, KSS2, Reh, Si5, SiCh1, Sim1, Sim2, SimWo**], the existence of stable and unstable manifolds, Markov partitions and other properties related to hyperbolicity [**BuSi2, BSC2, Ch3, Ef, KaSt, Le**], entropy and periodic orbits [**Bu1, ChMa, Ch1, Ch2, Ch4, Mo, Si1, SiCh2, St1, St2, Wo**], various statistical and symbolic properties, and limit theorems [**Bl, Bu2, BSC1, Ch5, GalOr, Tr, Yo**], quantum and other generalizations [**Be, Do, DoLi, DorSm, HaSh, CdV, Ve**], and many others (see also [**Si3, Si7, Ta**] and [**KoTr**] for reviews and more references).

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It has been known for a long time (see, for example, [**ZeKa**]) that the *dynamics* of a billiard in a rational polygon may be viewed as the *geometry* of its unfolding surface. Even in this simple situation the unfolding is not quite a Riemannian surface since in all but a few cases its metric is bound to have singularities (and away from the singularities the metric is flat). This object, however, is not at all pathological from the point of view of non-regular Riemannian geometry, the principal ideas and methods of which were developed by Alexandrov and his collaborators (see [**Al, AlB, AlSt, AlZa, Re**]) in the mid 1960's. Since then it has attracted the attention of many leading geometers, especially after the spectacular paper of Gromov [**Gr1**]. The cornerstone of the approach is the following observation. The well-known comparison theorems of Alexandrov and Toponogov show that there is a way to estimate the sectional curvature of a Riemannian manifold from above simply by comparing the geodesic triangles on the manifold and in a model space (a complete simply connected surface of constant curvature). However, since the procedure involves measurement of certain distances only, it may be considered a *definition* of a space whose curvature is bounded from above. This definition coincides with the usual one in the category of smooth Riemannian manifolds, but in fact makes sense for an arbitrary geodesic space (a metric space in which every two points may be connected by a geodesic).

Therefore, it would be natural to look for an unfolding space of an arbitrary semi-dispersing billiard inside the category of geodesic spaces. Soon it becomes clear that in order to reflect the dynamics of the billiard properly, the curvature of the space should be bounded from above by the initial curvature of the billiard's configuration space. This would allow one to view any semi-dispersing billiard as a finite factor of a geodesic flow on a space of bounded (non-positive in the case of billiards on manifolds of non-positive curvature) curvature (see [**BFK3**] for a detailed discussion). However, construction of such an object in general seems to be very difficult, if not impossible (see §5 for a detailed discussion). Instead, we construct an unfolding space for a given combinatorial class of trajectories, i.e. for all trajectories colliding with the same sequence of walls of the billiard. This space turns out to be a geodesic space of curvature bounded from above (in most applications, this is a non-positively curved space). Every billiard trajectory of the chosen combinatorial class uniquely corresponds to a geodesic in the unfolding space (we will identify them below) and, therefore, many questions concerning the dynamics of semi-dispersing billiards become purely geometric.

The purpose of this article is to summarize the results obtained in [**BFK1–3**]. All three papers address various problems in the theory of semi-dispersing billiards using the geometric approach outlined above. We will state all the main results from [**BFK1–3**] and outline some of the proofs. However, the emphasis of this paper is not on the rigorous proofs, which an interested reader may find in the articles mentioned above, but rather on the demonstration of the method, its power, and its limitations.

This paper is organized as follows. In §1 and 2 we summarize the results from [**BFK1**] and [**BFK3**] regarding the existence of uniform (i.e. independent of the choice of trajectory) estimates on the number of collisions in semi-dispersing billiards. In particular, we present the basic geometric construction of the unfolding space of a combinatorial class of trajectories. Next, we formulate and discuss a non-degeneracy condition that guarantees the existence of the estimates. As an application, we establish an explicit upper bound for the number of collisions that can take place in the infinite time interval $(-\infty, \infty)$ for hard-ball systems in an empty space of non-positive sectional curvature. (This upper bound depends only on the number of balls, and their maximum and minimum masses, and is independent of their radii, initial positions and velocities.) Also, in order to give some insight into our methods, we will outline the proofs of Theorem 2 and Theorem 3, and prove Theorem 4.

In §3 we discuss the results from [**BFK2**] regarding the finiteness of topological and metric entropy of an arbitrary non-degenerate semi-dispersing billiard on a manifold of non-positive curvature. In particular, the topological entropy of the billiard turns out to be bounded from above by a number which depends only on the number of walls and the non-degeneracy constant of the billiard. Also, we prove the existence and obtain an estimate on the limit of the topological entropy of the Lorentz gas while the radius of the scatterer tends to zero.

In §4 we present a general result which describes the structure of the set of periodic points of the billiard map (and periodic trajectories of the billiard flow) and gives an estimate of their number for semi-dispersing billiards on simply connected manifolds of non-positive curvature.

In §5 we discuss a difficult problem of non-regular Riemannian geometry already mentioned above—the existence of a 'universal' unfolding space for a non-degenerate semi-dispersing billiard.

Finally, in §6 we formulate a number of open questions related to the results and methods presented in this paper.

1. *Local uniform estimates on the number of collisions*

1.1. *Preliminaries.* Throughout this paper we denote by M an arbitrary C^2 Riemannian manifold without a boundary, with bounded sectional curvature and with the injectivity radius $\rho > 0$. Consider a collection of n geodesically convex subsets (*walls*) $B_i \subset M$, $i = 1, \ldots, n$ of M, such that their boundaries are C^1 submanifolds of codimension one. Let $B = M \backslash (\bigcup_{i=1}^{n} \mathrm{Int}(B_i))$, where $\mathrm{Int}(B_i)$ denotes the interior of the set B_i. A semi-dispersing billiard flow (or a semi-dispersing billiard system) acts on a certain subset of the unit tangent bundle to B (see, for example, [**Si3**] for more details). The projections of the orbits of that flow to B are called the billiard trajectories and correspond to free motions of particles inside B. Namely, the particle moves inside the set B with unit speed along a geodesic until it reaches one of the sets B_i (*collision*) where it reflects according to the law 'the angle of incidence is equal to the angle of reflection'. If it reaches one of the sets $B_i \bigcap B_j$, $i \neq j$, the trajectory is not defined after that moment. Any sequence of walls $K = \{B_{i_k}\}_{k=1}^{l}$ is called a combinatorial class of length l. Any curve $\Gamma \subset B$ determines a combinatorial class K_Γ of walls it has visited.

If a set A is isometrically embedded into two metric spaces N_1, N_2, we denote via $N_1 \bigcup^A N_2$ the result of gluing the spaces together along A. The definition of the unfolding space M_K of the billiard corresponding to a combinatorial class $K = \{B_{i_k}\}_{k=1}^{l}$ is rather straightforward: $M_K = M_0 \bigcup^{B_{i_1}} M_1 \bigcup^{B_{i_2}} M_2 \cdots \bigcup^{B_{i_l}} M_l$, where the M_i's are just distinct isometric copies of the manifold M, glued together in the order determined by the class

K. Since all the walls are convex, the fact that the curvature of M_K is bounded from above by the maximal sectional curvature of M follows from Reshetnyak's theorem (see [**Ba, Re**]). In order to simplify the notations, we will write M_Γ instead of M_{K_Γ}. The space M_K is an exact geometric model of the combinatorial class K because every billiard trajectory of this class uniquely corresponds to a geodesic in M_K. Namely, the geodesic is obtained by projection of the configuration space into the space $M_i \in M_K$, provided the trajectory already made exactly i collisions with the walls.

1.2. *Local estimates.* In the framework of the geometric model, the following generalization (a variable curvature counterpart) of the results of Galperin [**Ga1**] and Vaserstein [**Va**] becomes almost self-evident.

THEOREM 1. *For any trajectory of an arbitrary semi-dispersing billiard the number of collisions during any finite time interval is finite.*

Therefore, every finite piece of a trajectory of a semi-dispersing billiard makes only finitely many collisions. But, is there a sequence of pieces of trajectories of bounded length making more and more collisions or is the number of collisions bounded *uniformly*, at least for a short period of time? In fact, one can easily find cases of the former, but all such billiards look somewhat degenerate. This leads us to the following problem: *Find a general non-degeneracy condition that would guarantee the uniform boundedness of the number of collisions in a neighborhood of a given point of the billiard.* This problem was first posed by Sinai, who also gave a solution [**Si2**] for billiards in polyhedral angles (where no condition is necessary). The existence of such estimates is related to various properties of a billiard system. For example, Sinai–Chernov formulas [**Ch2, Si1**] for the metric entropy of billiards are proved under the assumption that such an estimate exists.

The following Theorem 2 proved in [**BFK1**], together with Definition 1.1 is a complete solution to this problem.

THEOREM 2. *Let a semi-dispersing billiard B with n walls be non-degenerate (see Definition 1.1) with constant C at a point x. Then there exists a neighborhood U_x of x such that every billiard trajectory entering U_x leaves it after making no more than*

$$(16(C+2))^{2(n-1)}$$

collisions with the walls.

As an immediate application of Theorem 2 we obtain the following global linear estimate.

COROLLARY 1.1. *For any non-degenerate semi-dispersing billiard there exists a constant P such that, for every t, every trajectory of the corresponding billiard flow makes no more than $P(t+1)$ collisions with the boundary in the time interval* $[0, t]$.

1.3. *Non-degeneracy condition.* Some non-degeneracy condition, prohibiting obvious counter-examples, is necessary for the existence of uniform local estimates. In [**BFK1**] we introduced the non-degeneracy condition used in Theorem 2. This condition is always satisfied for a system of hard balls in empty space (whereas other natural conditions are known to fail; for example, the condition that the normals to the walls be in general position). For a system of balls in a jar with concave walls our non-degeneracy condition is satisfied except for some special sets of radii, when it is possible to 'squeeze the balls tightly between the walls'. Actually, it is known that in those situations the system may have arbitrarily many collisions locally. The condition is the following.

Definition 1.1. A billiard B is *non-degenerate* in a subset $U \subset M$ (with constant $C > 0$), if for every $I \subset \{1, \dots, n\}$ and for every $y \in (U \bigcap B) \backslash (\bigcap_{j \in I} B_j)$,

$$\frac{\operatorname{dist}(y, \bigcap_{j \in I} B_j)}{\max_{k \in I} \operatorname{dist}(y, B_k)} \leq C,$$

whenever $\bigcap_{j \in I} B_j$ is non-empty.

A billiard B is called *non-degenerate at a point* $x \in B$ with constant C if it is non-degenerate in a neighborhood of x with the same constant, and *locally non-degenerate* with constant C if it is non-degenerate at every point with constant C.

We will say that B is *non-degenerate* if there exist $\delta > 0$ and $C > 0$ such that B is non-degenerate, with constant C, in any δ-ball.

Roughly speaking, the condition means that if a point is d-close to all the walls in I then it is Cd-close to their intersection. Formulated this way it is very easy to verify in many important cases, including the hard-ball gas models. However, in order to acquire some geometric insight, we notice that the condition is equivalent to the following geometric property: there exists a positive r such that, at every point, the unit tangent cone to B (which is a subset of the unit sphere in the tangent space to M) contains a ball of radius r. For flat M this means that every point of B is a vertex of a round cone of radius r which entirely belongs to B in some neighborhood of its vertex. As far as we know, 'the cone condition' was first formulated by Sinai. For compact billiard tables, these definitions can also be reformulated in the following way: the operations of taking tangent cone and intersection commute for any collection of the complements to the walls B_i. For non-compact tables, however, this definition guarantees the non-degeneracy at all points, but the constant C may deteriorate and have no positive lower bound.

1.4. *Outline of the proof of Theorem 2.* Reasoning by contradiction, we show that if a combinatorial class K_T of a billiard trajectory T were long enough then T could not be a length minimizer. On the other hand, by Alexandrov's theorem [**Re**], sufficiently short segments of geodesics in a space of curvature bounded from above are length minimizers, which produces a contradiction. Without loss of generality, we may assume that the intersection Q of all the walls of our billiard is non-empty.

Using induction by the number of walls n we may assume that every wall $B_{i_k} \in K_T$ has been visited sufficiently many, say N, times. Let x_i, $i = 1, \dots, N$, be the points of collisions of T with B_{i_k}, ordered with respect to K_T. We consider our trajectory T as a geodesic in M_T and will modify it in a shorter curve with the same endpoints. We

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will work with curves in our billiard table (not in M_T), and notice that a curve which visits the intersection of all the walls can be unfolded in M_T with prescribed lifts of its endpoints. For every pair x_i, x_{i+1} we can make our curve *shorter* simply by replacing the piece $T(x_i, x_{i+1})$ of the trajectory between the points x_i and x_{i+1} with the shortest curve with the same endpoints. This operation does not preserve the combinatorial class, but we repair it by forcing the curve to visit the intersection of all walls: pick $y \in Q$ and index i and replace $T(x_i, x_{i+1})$ with the two shortest curves connecting x_i and y, and y and x_{i+1}. Generally, the process will make our curve *longer*, but the non-degeneracy condition and some elementary (but essentially local, unless we are dealing with a Euclidean space, see the discussion in [**BFK3**]) considerations guarantee that, by a special choice of i (provided by the non-degeneracy condition), the ratio of what we gain due to the *lengthening* to what we get rid of in *every* shortening is uniformly bounded from above. Thus, if N is big enough to apply the shortenings sufficiently many times (and there is only one lengthening), the resulting curve will be shorter than T. (The detailed proof is presented in [**BFK1**].)

2. *Global estimates and hard-ball systems*

2.1. *Global estimates on the number of collisions.* As we have just pointed out, the argument above is local (unless M is flat), and so is the estimate on the number of collisions. Consider now a billiard on a manifold M of non-positive sectional curvature. If we could construct its universal unfolding space of non-positive curvature (see §5 for a rigorous definition) we would immediately obtain a *global* estimate on the number of collisions in the billiard. (The uniqueness of the geodesic connecting its endpoints implies that, since each wall is geodesically convex, there are no more collisions than the number of walls in the unfolding space.)

However, no construction of such a universal unfolding space is known in the general case. Nevertheless, one can make a similar idea work by the following modification of spaces M_T: for a combinatorial class $K = \{B_{i_1}, \dots, B_{i_l}\}$, consider the space $\tilde{M}_K = M_0 \bigcup^{B_{i_1}} M_1 \bigcup^{B_{i_2}} \cdots \bigcup^{B_{i_{l-1}}} M_{l-1} \bigcup^{B_{i_l}} M_0$, where M_k, $k = 0, \dots, l-1$, are distinct isometric copies of M, i.e. after sufficiently many collisions we 'close up' M_T by gluing the first and the last copies together.

Now we cannot guarantee *a priori* that this space is non-positively curved, since Reshetnyak's theorem is not applicable any more. Remember, however, that non-positiveness of curvature is a local property, so, in order to verify it, we only have to show that an excess of every *small* triangle is non-positive. However, such a triangle is contained in a small number of copies M_K, which follows from the proof of *local* estimates (we can now regard the sides of the triangle as 'generalized' billiard trajectories). Hence, if K is long enough, we may 'tear off the cycle of gluings' and the procedure will not affect the small triangle under consideration (i.e. the angles of the triangle will not decrease). Applying Reshetnyak's theorem to this resulting space, we see that the excess of the triangle is non-positive and therefore $\tilde{M}_K$ is a non-positively curved space. This is a contradiction since geodesics between fixed endpoints in such spaces are unique, while the development of T has returned to the same copy of M. This leads to the following.

THEOREM 3. *If M is a simply connected manifold of non-positive sectional curvature, $\bigcap_{i=1}^n B_i$ is non-empty, and B is locally non-degenerate with constant C, then every billiard trajectory in B has no more than*

$$(200(C+2))^{2n^2}$$

collisions in the infinite period of time $(-\infty, \infty)$.

2.2. *Hard-ball systems.* As an application of Theorem 3, we consider a system of hard balls in an arbitrary simply connected manifold of non-positive sectional curvature. In spite of the fact that the finiteness of the number of collisions in such a system in $\mathbb{R}^k$ has been known for a long time ([**Va, Ga1**], and much later [**Il**]), uniform estimates on the number of collisions were obtained only for the system of three identical balls in $\mathbb{R}^2$ [**MuCo, ThSa**] and for systems of particles on a line [**Ga2, SeVa**].

Theorem 3 allows us to prove the following.

THEOREM 4. *The maximal number of collisions that may occur in a system of N hard elastic balls (of arbitrary masses and radii) moving freely in a simply connected Riemannian space $\mathcal{M}$ of non-positive sectional curvature never exceeds*

$$\left(400N^2\frac{m_{\max}}{m_{\min}}\right)^{2N^4},$$

where $m_{\max}$ and $m_{\min}$ are, correspondingly, the maximal and the minimal masses in the system.

Remark. Theorem 3 was first established for $\mathbb{R}^k$ in [**BFK1**]. The results of [**BFK3**] allowed us to extend Theorem 3 to manifolds of non-positive curvature, and to get rid of the dependence on the radii that was present in [**BFK1**].

Proof. Consider a system of N balls of radii r_i and masses m_i, $i = 1, \dots, N$, moving freely in the space $\mathcal{M}$ and colliding with each other elastically. Without loss of generality we may assume that $\min_i m_i = 1$, $\max_i m_i = M$. Let ρ be the Riemannian metric on $\mathcal{M}$.

The dynamics of the system of hard balls is isomorphic to the dynamics of a certain billiard in the configuration space $\mathcal{M}^N$ (in which every ball is represented by its center) which is endowed with a Riemannian metric $\tilde{\rho}$,

$$\tilde{\rho}((x_1, \dots, x_n), (y_1, \dots, y_n)) = \left(\sum_{i=1}^N m_i \rho(x_i, y_i)^2\right)^{1/2}.$$

Notice that, providing that ρ is a metric of non-positive curvature, $\tilde{\rho}$ is a metric of non-positive curvature as well. The corresponding billiard is defined in the complement B of $N(N-1)/2$ bodies $B_{m,l}$, each of which corresponds to a pair of balls. Namely, for every $m, l = 1, \dots, N$, $m \neq l$,

$$B_{m,l} = \{(x_1, \dots, x_N) \in \mathcal{M}^N \mid \rho(x_m, x_l) \le r_m + r_l\}.$$

Every such body $B_{m,l}$ is isometric to a product of $\mathcal{M}^{N-2}$ with a convex set in $\mathcal{M}^2$ and, thus, is convex too.

Now we will check the uniform non-degeneracy condition for B.

Fix a set of walls I, and let $I_0 = \{m \mid (m,l) \in I\}$. Consider an arbitrary point $X_0 = (c_1, \ldots, c_N) \in \mathcal{M}^N \backslash (\bigcup_{(m,l)\in I} B_{m,l})$ and let $\delta = \max_{(m,l)\in I} \tilde{\rho}(X_0, B_{m,l})$. Our goal is to estimate $\tilde{\rho}(X_0, \bigcap_{(m,l)\in I} B_{m,l})$ via δ.

In order to do that, let us apply the following procedure: pick some $m_1 \in I_0$ and move all the balls $B_m, m \in I_0 \backslash \{m_1\}$, simultaneously and with equal velocities along the geodesics in $\mathcal{M}$, connecting the centers of B_m with the center of B_{m_1}, until every pair of balls B_{m_1}, B_m such that $(m_1, m) \in I$ intersect (if the center of one of the balls B_m reaches the center of B_{m_1}, we stop moving it any further, and continue to move the other balls). As a result, we obtain a point $X_1 \in \mathcal{M}^N$. Since we never have to move any ball in $\mathcal{M}$ more than by δ, we have $\tilde{\rho}(X_0, X_1) \leq MN\delta$. On the other hand, for every two geodesics γ_1, γ_2 in the simply connected space $\mathcal{M}$ of non-positive curvature the function $\rho(\gamma_1(t), \gamma_2(t))$ is convex. Therefore, distances between any pair of the balls will not increase, so that we still have $\max_{(m,l)\in I} \tilde{\rho}(X_1, B_{m,l}) \leq \delta$.

Next, we apply the same procedure to some $m_2 \in I_0 \backslash \{m_1\}$, obtaining a point $X_2 \in \mathcal{M}^N$ such that $\tilde{\rho}(X_1, X_2) \leq MN\delta$, etc. By construction, the last point $X_{|I_0|} \in \bigcap_{(m,l)\in I} B_{m,l}$ and $\tilde{\rho}(X_0, X_{|I_0|}) \leq \sum_{i=0}^{|I_0|-1} \tilde{\rho}(X_i, X_{i+1}) \leq MN^2\delta$. Therefore, it is shown that B is non-degenerate in the whole $\mathcal{M}^N$, with the constant MN^2.

Applying Theorem 3, we see that the number of collisions is not greater than $(200MN^2 + 2)^{2N^4} < (400MN^2)^{2N^4}$. □

2.3. *Generalized systems.* All our methods and results remain valid even if we drop the assumption that the boundaries of B_i are *hyper*-surfaces. Of course, in this case we have to change the definition of the outcome of a collision appropriately: it would not be uniquely defined any more, and we would require only the conservation of the tangential component of the velocity. In particular, Theorems 2 and 3 and Corollary 1.1 hold for singular trajectories as well (i.e. the trajectories that enter the intersections of several bodies and reflect in arbitrary directions preserving the component parallel to the tangent space of the intersection of the bodies at the point of collision). This also allows us to apply our results to particle systems, i.e. billiard systems of several balls of various masses and radii where some (mixed system) or all (pure system) of the balls may have zero radii (particles). In such systems multiple simultaneous collisions are allowed, as well as collisions with the intersections of several boundary components (for detailed definitions see [**SeVa**], which also generalizes estimates of [**Ga2**] for *pure* particle systems from the one-dimensional case to higher dimensions). In particular, Theorem 4 holds for arbitrary particle systems (with exactly the same estimate).

3. *Entropy estimates*

3.1. *Topological entropy.* Recall that there is a standard way to introduce a distance function in the unit tangent bundle TM to M (sometimes this distance function is called the Sasaki metric). This distance is used to define the topological entropy $h_{\text{top}}(f)$ of any transformation f of a subset of TM (for a definition of the topological entropy for

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transformations of non-compact spaces see, for example, [**PePi**], or the original paper by Bowen [**Bo**]). The topological entropy of the time-one map T^1 of the billiard flow will be called the topological entropy of the billiard.

Sinai–Chernov's formulas ([**Ch2, Si1**]; see also the excellent review in [**Ch4**]) imply the finiteness of the metric entropy of non-degenerate semi-dispersing billiards in $\mathbb{R}^n$ or $\mathbb{T}^n$ with respect to the Liouville measure (for various estimates of the metric entropy see also [**ChMa, Wo**]). However, little is known about the topological entropy of general semi-dispersing billiards. Most of the results known to the authors are proven only for two-dimensional semi-dispersing billiards (the connection between the topological entropy and the number of periodic points [**Ch1**] and the results of [**KaSt**]). The only results on the topological entropy of billiards of arbitrary dimension that we are aware of, are the fact that the topological entropy of polyhedral billiards is zero (see [**Ka**], [**GuHa**], also [**Ch2**] for a similar result about metric entropy), and the finiteness of topological entropy for billiards in the outside of several strictly convex and disjoined bodies in $\mathbb{R}^k$ [**St2**].

Denote via $H_M(t)$ the number of different homotopy classes that can be represented on the Riemannian manifold M by closed curves of length less than t. In [**BFK2**] we established the following estimate.

THEOREM 5. *The topological entropy of a compact non-degenerate semi-dispersing billiard with n walls on a manifold M of non-positive sectional curvature is less than or equal to*

$$(P+1)\log(n) + 2\varlimsup_{t\to\infty}\frac{\log(H_M(t))}{t},$$

where P is the constant from Corollary 1.1. In particular, the topological entropy is finite.

The result seems to be a purely dynamic one. However, its proof is based on Theorem 2 (in fact, on Corollary 1.1) as well as on certain geometric properties of the unfolding space. Let us outline the proof assuming for simplicity that the manifold M is simply connected.

Informally speaking, in order to estimate the topological entropy of the billiard, from above, by a constant A, we have to find a way to describe a billiard trajectory $\Gamma(t)$, $t \in [0, l]$, of length l with given precision, using an 'amount of data' which is no bigger than constant $\times\, e^{Al}$. We claim that such data is a triple (combinatorial class of Γ, $\Gamma(0)$, $\Gamma(l)$) where, by definition, we may know $\Gamma(0)$, $\Gamma(l)$ only approximately. Since the distance between two geodesics in a simply connected space of non-positive curvature is a convex function of time (a well-known fact for the usual Riemannian spaces, which is also true for Alexandrov spaces), any two geodesics with endpoints close in the configuration space are in fact uniformly close to each other in the phase space. (This is much harder to prove for singular spaces than for regular manifolds. The convexity immediately gives us the closeness in the configuration space, but the closeness in the phase space requires some additional work. See [**BFK2**] for details.) It means that the triple indeed determines a billiard trajectory with the necessary precision. Therefore, in order to estimate the entropy, we have to calculate the exponential speed of growth of the number of such triples, which is just the exponential speed of growth of the number of possible combinatorial classes as a function of the trajectory length. But according to Corollary 1.1, it is no greater than $(P+1)\log(n)$.

Let us call a point $x \in \widetilde{TB}$ *$\mathbb{Z}$-regular* if $T^i(x)$ belongs to the interior of TB for all $i \in \mathbb{Z}$. For example, almost all points of $\widetilde{TB}$ are $\mathbb{Z}$-regular with respect to the Liouville measure. Clearly the restriction of the time-one map T^1 to the set $TB_{\mathbb{Z}}$ of $\mathbb{Z}$-regular points in B is continuous, and its topological entropy is less than or equal to the topological entropy of T^1 on B. Thus, Theorem 5 together with the results of Pesin and Pitskel [**PePi**] concerning the variational principle for the continuous maps of non-compact spaces, yields the following.

COROLLARY 3.1. *Metric entropy, of a compact non-degenerate semi-dispersing billiard on any manifold of non-positive sectional curvature, with respect to any T^1-invariant probability measure μ such that $\mu(TB_{\mathbb{Z}}) = 1$, is finite. In particular, metric entropy is finite for any measure which is invariant with respect to the whole flow T^t.*

3.2. *Lorentz gas.* A Lorentz gas model is a billiard on $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$ with one wall which is a ball of radius $1/2 > r > 0$. For the last twenty years, this model has been studied extensively (see, for instance, [**Bl, BuSi3, Ch1, Ch5, Si8, Yo**]). In [**Ch1**] it was proven that the first return map to the boundary of the Lorentz gas billiard has infinite topological entropy, and that the metric entropy of the Lorentz gas billiard with respect to the Liouville measure converges to zero as $r \to 0$. In contrast to those results, in [**BFK2**] we prove the following.

THEOREM 6. *Denote by $h_r(k)$ the topological entropy of the Lorentz gas billiard described above. Then:*

1. *$h_r(k)$ is finite;*
2. *there exist $\lim_{r\to 0} h_r(k) = h_0(k)$, and $0 < h_0(k) < \infty$;*
3. $\ln(2k-1) \le h_0(k) \le h_0(k+1)$.

A computer aided computation shows that $h_0(2) = 1.526\ldots$.

4. *Periodic points and trajectories*

In [**BFK2**] we obtained a description of the set of periodic points and trajectories of semi-dispersing billiards. We call a curve $\nu(t)$ a periodic pseudo-trajectory of class $K = \{B_{i_1}, \ldots, B_{i_j}\}$ if it is a closed curve that consists of pieces of geodesics on M that connect some point $x_1 \in B_{i_1}$ with some point $x_2 \in B_{i_2}$, the point $x_2 \in B_{i_2}$ with some point $x_3 \in B_{i_3}, \ldots,$ the point $x_j \in B_{i_j}$ with the point $x_1 \in B_{i_1}$, and at each point x_k, $k = 1, \ldots, j$. The tangent vector to $\nu(t)$ changes according to the billiard rule with respect to B_{i_k}. (The difference between this and the usual trajectories is that a geodesic segment of a pseudo-trajectory between x_k and x_{k+1} may intersect some of the bodies B_i, $i = 1, \ldots, n$.) Notice that if Γ is any periodic trajectory then a periodic pseudo-trajectory close enough to Γ is also a periodic trajectory.

Our main result on periodic trajectories is the following.

THEOREM 7. *Let B be a semi-dispersing billiard on a simply connected manifold M of non-positive sectional curvature. Let K be some combinatorial class of trajectories.*

(Notice that, here, we do not require B to be compact or non-degenerate, unlike in our previous results.)

Then, the periodic trajectories of class K all have the same length and form a parallel family in the following sense. Any two periodic trajectories Γ_1 and Γ_2 of class C can be joined by a continuous curve Γ_t, $1 \le t \le 2$, of periodic pseudo-trajectories of type K, so that:

1. *the surface Σ_k, $k = 1, \dots, j$, formed by the pieces of trajectories Γ_t, $1 \le t \le 2$, between the kth and $(k+1)$st collisions is a piece of $\mathbb{R}^2$ isometrically embedded into M;*
2. *the intersections I_k of the boundary of B_{i_k} with the trajectories from the curve Γ_t, $1 \le t \le 2$, are isometrically embedded intervals of a straight line that connect the points $\Gamma_1 \bigcap B_{i_k}$ with the points $\Gamma_2 \bigcap B_{i_k}$;*
3. *inside of each flat surface Σ_k, $k = 1, \dots, j$, the pieces of trajectories from Γ_t, $1 \le t \le 2$, are parallel to each other.*

Therefore, periodic trajectories of the same combinatorial class always come in parallel families. For a typical semi-dispersing billiard such a family cannot contain more than one periodic trajectory.

COROLLARY 4.1. *For M as in Theorem 7:*

1. *if the curvature of M is strictly negative, every combinatorial class contains no more than one periodic trajectory;*
2. *if for some periodic trajectory Γ at least one of the walls it visits is strictly convex at the point of collision with the trajectory, then Γ is the only periodic trajectory in its combinatorial class.*

Let us call two periodic trajectories equivalent if they are parallel (in the sense explained in Theorem 7) and let us call two periodic points for the first return map to the boundary equivalent if the corresponding periodic billiard trajectories are equivalent.

Denote by P_k, $k \in \mathbb{N}$, the number of periodic points, and by $\tilde{P}_k$ the number of equivalence classes of periodic points of period k for the first return map to the boundary of the billiard B. Denote by P^t, $t \in \mathbb{R}^+$, the number of periodic trajectories, and by $\tilde{P}^t$ the number of equivalence classes of periodic trajectories, of the billiard flow of length less than or equal to t. Finally, $P(t)$ denotes the maximum number of collisions in time t.

Theorem 7 implies the following.

COROLLARY 4.2. *Let B be a semi-dispersing billiard on a simply connected manifold M of non-positive sectional curvature. Let*

$$\theta(m, x) = \begin{cases} 0, & x < 2 \\ m(m-1), & 2 \le x < 3 \\ m(m-1)^{x-1}(m-2), & x \ge 3 \end{cases}$$

for every $m \in \mathbb{N}$, $x \in \mathbb{R}^+$. Then, for every $k \in \mathbb{N}$, $t \in \mathbb{R}^+$:

1. *if the curvature of M is strictly negative or all the sets B_i, $i = 1, \dots, n$, are strictly convex then*

$$\tilde{P}_k = P_k \le \theta(n, k) \quad \textit{and} \quad \tilde{P}^t = P^t \le \theta(n, P(t+1));$$

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2. *otherwise, either*

$$\tilde{P}_k = P_k \le \theta(n,k) \quad \textit{and} \quad \tilde{P}^t = P^t \le \theta(n, P(t+1))$$

or

$$\tilde{P}_k \le \theta(n,k), \quad \tilde{P}^t \le \theta(n, P(t+1)) \quad \textit{but} \quad P^t = P_k = \infty.$$

In the case $M = \mathbb{R}^k$, for periodic points, and $M = \mathbb{R}^2$, for periodic trajectories, this assertion was proved in [**St1**].

Note that the estimate on P_k given in Corollary 4.2 cannot be improved since it is sharp in every billiard satisfying the so called Ikawa condition (H) [**Ik**] (all the bodies are disjoint and the convex hull of any two bodies does not have points in common with any of the other bodies).

5. *Universal unfolding space*

In proving all the results mentioned above, we have used two different geometric models of a semi-dispersing billiard; one presented in §1 and the other presented in §2. However, in both cases we have to use a separate model for each combinatorial class of billiard trajectories. Obviously, it would be desirable to construct a 'universal unfolding space' of the billiard such that every trajectory would correspond to a geodesic of that space. To be more precise, a universal unfolding space $\tilde{M}$ is a result of gluing together (along the sets B_i, $i = 1, \dots, n$) a finite number of copies of M so that:

1. Every copy is glued to exactly n other copies along each of the bodies B_i, $i = 1, \dots, n$. To be more precise, for every copy M_j:
   (a) there are n distinct copies M_j^i, $i = 1, \dots, n$, such that $M_j \bigcap M_j^i = B_i$;
   (b) if for some copy M_k, $M_j \bigcap M_k = B_i$ then $M_k = M_j^i$;
   (c) for any M_k, $M_k \bigcap M_j \subset B_i$, for some $i \in \{1, \dots, n\}$.
2. The curvature of $\tilde{M}$ is bounded by the maximal sectional curvature of M.

It is easy to show that the space $\tilde{M}$ may be constructed only if the billiard is non-degenerate. The most interesting case is when M is a manifold of non-positive sectional curvature, which we assume everywhere in this section. In [**BFK3**] we showed how to construct $\tilde{M}$ when the billiard has only two walls and also in the case when M is a surface. In both cases, the construction is elementary. For example, in the latter case, the construction is the following.

Fix a number K and consider a finite group Γ with n generators γ_i, $i = 1, \dots, n$, such that if a relation of the form $\gamma_{i_1}^{k_1} \dots \gamma_{i_l}^{k_l} = e$, $i_m \ne i_{m+1}$, $l \in \mathbb{N}$, $m = 1, \dots, (l-1)$, holds then necessarily $|k_1| + \dots + |k_l| > K$. An explicit example of such a group can be found in [**S**].

Consider $|\Gamma|$ copies of M, and denote them as M_g, $g \in \Gamma$. Consider another $|\Gamma|$ copies of M, and denote them as M^g, $g \in \Gamma$.

Now, let us glue together these $2|\Gamma|$ copies of M by performing the following operations: if $g_1 = \gamma_i g_2$, then we glue together M_{g_1} and M^{g_2} along the body B_i. Denote by $\tilde{M}$ the result of all these gluings. It is proven in [**BFK3**] that if K is 'big enough' then $\tilde{M}$ is a universal unfolding space for the billiard on the surface M outside of bodies B_i (where we can always assume that the intersections of more than two bodies are empty).

The problem of constructing a universal unfolding space in dimensions higher than two appears to be extremely difficult, even for the simplest possible case—the billiard in a simplex.

In a forthcoming paper [**BFKlK**] we plan to present a result that, in particular, will yield a construction of a universal unfolding space for polyhedral billiards in dimension three. Contrary to what might be anticipated, the construction is not at all elementary, in fact, it is essentially based on Thurston's theory of hyperbolic structures on three-dimensional manifolds. Here is a sketch of the construction.

Let S be a three-dimensional polyhedron. We start by constructing, for arbitrary $K \in \mathbb{N}$, a boundaryless pseudo-manifold M_K such that every edge (and, thus, every vertex) belongs to at least K copies of S (and, of course, M_K is constructed out of a finite number of isometric copies of S). If K is large enough M_K has non-positive curvature everywhere except maybe in the vertices. The rest of the proof consists of an 'unwrapping' of the space M_K into a space M with large links of all the vertices (and, thus, of a non-positive curvature).

We throw out convex polyhedral neighborhoods of all the vertices of M_K to form a space M_K^0. The 'unwrappings' of M_K correspond to finite covers of M_K^0. In order to construct the 'unwrapping' such that all the links would have no short homotopically non-trivial geodesics it is enough to find a finite index subgroup in $\pi_1(M_K^0)$ which does not contain the elements of $\pi_1(M_K^0)$ that correspond to the short closed geodesics in the links. The main difficulty in the proof is to show that such a subgroup exists. To overcome this we show that, for large enough K, $\pi_1(M_K^0)$ is a linear group, and, thus, is residually finite.

When K is large enough, M_K^0 has non-positive curvature and its boundary components are totally geodesic and non-positively curved. Let M_K^2 be the result of 'doubling' M_K^0, i.e. gluing two copies of M_K^0 along the boundary. We apply the uniformization theorem for Haken manifolds [**Ot, Th, MorBa**], to conclude that M_K^2 is hyperbolizable, and, therefore, $\pi_1(M_K^2)$ is linear [**Ma**]. This immediately implies that $\pi_1(M_K^0)$ is also linear.

The result of the construction described above is a pseudo-manifold, since the neighborhoods of the vertices are homeomorphic to cones over surfaces of high genus. In [**BFKlK**] we also give the necessary and sufficient conditions describing when it is possible to construct a boundaryless *manifold* (topological, of course) by gluing several copies of a given three-dimensional polyhedron.

6. *Open questions*

We conclude our survey with the formulation of several open questions, which are closely related to the results mentioned above.

Question 1. Are there universal unfolding spaces for arbitrary non-degenerate semi-dispersing billiards on non-positively curved manifolds of dimension greater than two?

Question 1a. Is it possible to construct a compact CAT(0) boundaryless pseudo-manifold by gluing together a finite number of copies of a given polyhedron S along the isometric faces?

It is quite possible that the answer to Question 1a (and, thus, to Question 1 as well) is negative for sufficiently high dimensions.

Question 2. What can be said about the topological, or even the metric entropy of degenerate compact semi-dispersing billiards, or of the billiards on manifolds without the non-positive curvature restriction? In particular, can it be infinite?

The question is open even for degenerate semi-dispersing billiards in Euclidean space. We strongly suspect that the introduction of even arbitrarily small amounts of positive curvature into a billiard on the Euclidean plane may produce a billiard with infinite topological entropy, which would be a nice demonstration of how *positive* curvature can force the entropy to become infinite.

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