

THE INTEGRAL HODGE CONJECTURE FOR 3-FOLDS OF KODAIRA DIMENSION ZERO

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Abstract We prove the integral Hodge conjecture for all 3-folds X of Kodaira dimension zero with $H^0(X, K_X)$ not zero. This generalizes earlier results of Voisin and Grabowski. The assumption is sharp, in view of counterexamples by Benoist and Ottem. We also prove similar results on the integral Tate conjecture. For example, the integral Tate conjecture holds for abelian 3-folds in any characteristic.

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The Hodge conjecture is true for all smooth complex projective 3-folds, by the Lefschetz (1, 1) theorem and the hard Lefschetz theorem [15, p. 164]. The integral Hodge conjecture is a stronger statement which fails for some 3-folds, in fact, for some smooth hypersurfaces in \mathbf{P}^4 , by Kollár [22]. Voisin made a dramatic advance by proving the integral Hodge conjecture for all uniruled 3-folds (or, equivalently, all 3-folds with Kodaira dimension $-\infty$) and all 3-folds X with trivial canonical bundle K_X and first Betti number zero [39]. Also, Grabowski proved the integral Hodge conjecture for abelian 3-folds [13, Corollary 3.1.9].

In this paper, we prove the integral Hodge conjecture for all smooth projective 3-folds X of Kodaira dimension zero with $h^0(X, K_X) > 0$ (hence equal to 1). This generalizes the results of Voisin and Grabowski in two directions. First, it includes all smooth projective 3-folds with trivial canonical bundle, not necessarily with first Betti number zero. For example, the integral Hodge conjecture holds for quotients of an abelian 3-fold by a free action of a finite group preserving a volume form and for volume-preserving quotients of a K3 surface times an elliptic curve. Second, our result includes any smooth projective 3-fold whose minimal model is a possibly singular variety with trivial canonical bundle; this extends the work of Horing–Voisin on singular 3-folds of this type [19, Proposition 3.18].

In contrast, Benoist and Ottem showed that the integral Hodge conjecture can fail for 3-folds of any Kodaira dimension ≥ 0 . In particular, it can fail for an Enriques surface times an elliptic curve; in that case, X has Kodaira dimension zero, and, in fact, the canonical bundle is torsion of order 2 [4]. So our positive result is sharp in a strong sense.

The proof here covers all cases (including abelian 3-folds) in a unified way, building on the arguments of Voisin and Höring–Voisin. In order to show that a given homology class is represented by an algebraic 1-cycle on X , we consider a family of surfaces of high degree in a minimal model of X . The 1-cycle we want cannot be found on most surfaces in the family, but it will appear on some surface in the family. This uses an analysis of Noether–Lefschetz loci, which depends on the assumption that $h^0(X, K_X) > 0$.

As an application of what we know about the integral Hodge conjecture, we prove the integral Tate conjecture for all rationally connected 3-folds and all 3-folds of Kodaira dimension zero with $h^0(X, K_X) > 0$ in characteristic zero (Theorem 6.1). Finally, we prove the integral Tate conjecture for abelian 3-folds in any characteristic (Theorem 7.1).

1. Notation

The *integral Hodge conjecture* for a smooth complex projective variety X asserts that every element of $H^{2i}(X, \mathbf{Z})$ whose image in $H^{2i}(X, \mathbf{C})$ is of type (i, i) is the class of an algebraic cycle of codimension i , that is, a \mathbf{Z} -linear combination of subvarieties of X . The *Hodge conjecture* is the analogous statement for rational cohomology and algebraic cycles with rational coefficients. The *integral Tate conjecture* for a smooth projective variety X over the separable closure F of a finitely generated field says: for k a finitely generated field of definition of X whose separable closure is F and l a prime number invertible in k , every element of $H^{2i}(X_F, \mathbf{Z}_l(i))$ fixed by some open subgroup of $\text{Gal}(F/k)$ is the class of an algebraic cycle over F with \mathbf{Z}_l coefficients. Although it does not hold for all varieties, this version of the integral Tate conjecture holds in more cases than the analogous statement over the finitely generated field k [37, § 1]. The *Tate conjecture* is the analogous statement with \mathbf{Q}_l coefficients.

On a normal variety Y , we use a natural generalization of the vector bundle of differential forms on a smooth variety, the sheaf $\Omega_Y^{[j]}$ of *reflexive differentials*:

$$\Omega_Y^{[j]} := (\Omega_Y^j)^{**} = i_*\Omega_U^j,$$

where $i: U \rightarrow Y$ is the inclusion of the smooth locus.

For a vector space V , $P(V)$ denotes the space of hyperplanes in V .

2. Examples

In this section, we discuss some examples of 3-folds satisfying our assumptions and how our proof works in various cases. One interesting point is the following dichotomy among 3-folds satisfying our assumptions. This dichotomy will not be used in the rest of the paper, but the proof of Proposition 2.1 develops some basic properties of these 3-folds that will be used.

Proposition 2.1. *Let X be a smooth projective complex 3-fold of Kodaira dimension zero with $h^0(X, K_X) > 0$ (hence equal to 1). Let Y be a minimal model of X . (Here, Y is a terminal 3-fold with K_Y trivial.) Then either $H^1(X, \mathcal{O}) = H^1(Y, \mathcal{O})$ is zero or Y is smooth (or both).*

Note that the integral Hodge conjecture for smooth projective 3-folds is a birationally invariant property [40, Lemma 15]. Therefore, to prove Theorem 4.1 (the integral Hodge

conjecture for X as above), we could assume that $H^1(X, \mathcal{O}) = 0$ or else that K_X is trivial (although we will not, in fact, divide up the proof of Theorem 4.1 that way). The case with $H^1(X, \mathcal{O}) = 0$ follows from the work of H\"oring and Voisin [19, Proposition 3.18] together with a relatively easy analysis of singularities below (Lemma 3.1). (To give examples of such 3-folds, there are many terminal hypersurface singularities in dimension 3, such as any isolated singularity of the form $xy + f(z, w) = 0$ for some power series f [30, Definition 3.1, Corollary 3.12] and X could be any resolution of a terminal quintic 3-fold in \mathbf{P}^4 .)

The case of smooth projective 3-folds X with K_X trivial (but $H^1(X, \mathcal{O})$ typically not zero) is harder and requires a thorough reworking of H\"oring and Voisin's arguments. We discuss examples of such varieties after the following proof.

Proof. (Proposition 2.1) By Mori, there is a minimal model Y of X [24, 2.14]. That is, Y is a terminal projective 3-fold whose canonical divisor K_Y is nef, with a birational map from X to Y . Terminal varieties are smooth in codimension 2, and so Y is smooth outside finitely many points. Since X has Kodaira dimension zero, the Weil divisor class K_Y is torsion by the abundance theorem for 3-folds, proved by Kawamata and Miyaoka [21]. Since $h^0(Y, K_Y) = h^0(X, K_X) > 0$, K_Y is trivial (and, hence, $h^0(X, K_X) = h^0(Y, K_Y) = 1$).

Since K_Y is linearly equivalent to zero, K_Y is, in particular, a Cartier divisor. Also, since Y is terminal, it has rational singularities [30, 3.8]; therefore, Y is Cohen–Macaulay. So the line bundle K_Y is the dualizing sheaf of Y . Since Y is a terminal 3-fold with K_Y Cartier, it has only hypersurface (hence lci) singularities, by Reid [30, Theorem 3.2]. Let S be a smooth ample Cartier divisor in Y ; then S is contained in the smooth locus of Y . Write $i: S \rightarrow X$ for the inclusion. By Goresky and MacPherson, $i_*: H_2(S, \mathbf{Z}) \rightarrow H_2(Y, \mathbf{Z})$ is surjective, using that Y has only lci singularities [11, p. 24].

For any scheme Y of finite type over the complex numbers, du Bois constructed a canonical object $\underline{\Omega}_Y^*$ in the filtered derived category of Y , isomorphic to the constant sheaf \mathbf{C}_Y in the usual derived category $D(Y_{\text{an}})$ [9]. For Y smooth, this is simply the de Rham complex. Write $\underline{\Omega}_Y^j$ in $D(Y)$ for the j th graded piece of $\underline{\Omega}_Y^*$ with respect to the given filtration, shifted j steps to the left; for Y smooth, this is the sheaf Ω_Y^j in degree zero. For Y proper over \mathbf{C} , the resulting spectral sequence

$$E_1^{pq} = H^q(Y, \underline{\Omega}_Y^p) \Rightarrow H^{p+q}(Y, \mathbf{C})$$

degenerates at E_1 [9, Theorem 4.5]. The associated filtration on $H^*(Y, \mathbf{C})$ is the Hodge filtration defined by Deligne.

The objects $\underline{\Omega}_Y^j$ need not be sheaves, even in our very special situation, where Y has terminal 3-fold hypersurface singularities. In particular, Steenbrink showed that $\underline{\Omega}_Y^1$ has nonzero cohomology in degree 1 (not just degree 0) for any isolated rational complete intersection 3-fold singularity other than a node or a smooth point [33, p. 1374].

In our case, because $H_2(S, \mathbf{Z}) \rightarrow H_2(Y, \mathbf{Z})$ is surjective, the pullback $H^2(Y, \mathbf{Q}) \rightarrow H^2(S, \mathbf{Q})$ is injective. By strict compatibility of pullback maps with the weight filtration, it follows that the mixed Hodge structure on $H^2(Y, \mathbf{Q})$ is pure of weight 2 [7]. By the discussion above, the graded pieces of the Hodge filtration on $H^2(Y, \mathbf{C})$ are $H^2(Y, \underline{\Omega}_Y^0)$, $H^1(Y, \underline{\Omega}_Y^1)$, and $H^0(Y, \underline{\Omega}_Y^2)$. Since Y is terminal (log canonical would be enough), it is du Bois, which means that $\underline{\Omega}_Y^0 \cong \mathcal{O}_Y$ [23].

Let $TY = (\Omega_Y^1)^*$, which is a reflexive sheaf on Y . The Lie algebra of the automorphism group of Y is $H^0(Y, TY)$. We have $TY \cong \Omega_Y^{[2]} \otimes K_Y^* \cong \Omega_Y^{[2]}$, the sheaf of reflexive 2-forms, since K_Y is trivial. So $H^0(Y, TY) \cong H^0(Y, \Omega_Y^{[2]})$. Du Bois’s object $\underline{\Omega}_Y^2$ in the derived category of Y has $\mathcal{H}^0(\underline{\Omega}_Y^2) \cong \Omega_Y^{[2]}$ since Y is klt [20, Theorems 5.4 and 7.12]. Since the cohomology sheaves of $\underline{\Omega}_Y^2$ are concentrated in degrees ≥ 0 , it follows that $H^0(Y, \Omega_Y^{[2]}) \cong H^0(Y, \underline{\Omega}_Y^2)$.

The polarization of $H^2(S_{t_0}, \mathbf{Q})$, by the intersection form, gives a canonical direct-sum decomposition of Hodge structures [38, Lemma 7.36]:

$$H^2(S_{t_0}, \mathbf{Q}) = H^2(Y, \mathbf{Q}) \oplus H^2(Y, \mathbf{Q})^\perp.$$

The restriction of this polarization gives a polarization of the Hodge structure $H^2(Y, \mathbf{Q})$; this can be described as the polarization of $H^2(Y, \mathbf{Q})$ given by the ample line bundle $H = \mathcal{O}(S)$ on Y .

In particular, the polarization of $H^2(Y, \mathbf{Q})$ gives an isomorphism $H^0(Y, \underline{\Omega}_Y^2) \cong H^2(Y, \underline{\Omega}_Y^0)^* \cong H^2(Y, \mathcal{O})^*$. By Serre duality and the triviality of K_Y , we have $H^2(Y, \mathcal{O})^* \cong H^1(Y, K_Y) \cong H^1(Y, \mathcal{O})$. Putting this all together, we have $H^0(Y, TY) \cong H^1(Y, \mathcal{O})$.

Thus, if $H^1(X, \mathcal{O})$ is not zero, then the identity component $\text{Aut}^0(Y)$ of $\text{Aut}(Y)$ has positive dimension. By the Barsotti–Chevalley theorem, $\text{Aut}^0(Y)$ is an extension of an abelian variety by a connected linear algebraic group [25, Theorem 8.27]. Any connected linear algebraic group over \mathbf{C} is unirational [25, Theorem 17.93]. Since Y has Kodaira dimension 0, it is not uniruled, and so it has no nontrivial action of a connected linear algebraic group. We conclude that $A := \text{Aut}^0(Y)$ is an abelian variety of positive dimension.

By Brion, extending the work of Nishi and Matsumura, any faithful action of an abelian variety on a normal quasi-projective variety has finite stabilizer groups [6, Theorem 2]. In our case, A preserves the singular locus of Y , which has dimension at most 0 because Y is a terminal 3-fold. Since A has positive dimension, the singular locus of Y must be empty. □

We conclude the section by giving examples of smooth projective 3-folds X with K_X trivial and $H^1(X, \mathcal{O}_X) \neq 0$, beyond the obvious examples: a K3 surface times an elliptic curve or an abelian 3-fold. The Beauville–Bogomolov structure theorem implies that X is a quotient of a variety Z of one of those special types by a free action of a finite group [3]. Knowing the integral Hodge conjecture for Z does not obviously imply it for X , which helps to motivate this paper.

Example 2.2. An action of a finite group G on a complex K3 surface S is said to be *symplectic* if G acts as the identity on $H^0(S, K_S) \cong \mathbf{C}$. Mukai (completing the earlier work of Nikulin) classified the finite groups that can act faithfully and symplectically on some K3 surface. In particular, the abelian groups that can occur are: \mathbf{Z}/a for $1 \leq a \leq 8$, $(\mathbf{Z}/2)^2$, $(\mathbf{Z}/2)^3$, $(\mathbf{Z}/2)^4$, $(\mathbf{Z}/3)^2$, $(\mathbf{Z}/4)^2$, $\mathbf{Z}/2 \times \mathbf{Z}/4$, and $\mathbf{Z}/2 \times \mathbf{Z}/6$ [29, Theorem 4.5(b) and note added in proof], [27, Theorem 0.6].

Let G be a nontrivial group on this list other than $(\mathbf{Z}/2)^3$ or $(\mathbf{Z}/2)^4$, and let G act symplectically on a K3 surface S . Let E be any complex elliptic curve. We can choose an embedding of G as a subgroup of E . Let $X = (S \times E)/G$, where G acts in the given way

on S and by translations on E . Then X is a smooth projective 3-fold with K_X trivial. Moreover, $H^1(X, \mathcal{O})$ is not zero because X maps onto the elliptic curve E/G . Finally, X is not the product of a K3 surface with an elliptic curve. So it is a new case for which Theorem 4.1 proves the integral Hodge conjecture.

Example 2.3. Theorem 4.1 also applies to some quotients of abelian 3-folds. For example, let S be a complex abelian surface, and let G be a finite abelian group with at most two generators which acts faithfully and symplectically on S as an abelian surface. Let E be any elliptic curve. Choose an embedding of G as a subgroup of E . Let $X = (S \times E)/G$, where G acts in the given way on S and by translations on E . Then X is a smooth projective 3-fold, K_X is trivial, and $H^1(X, \mathcal{O})$ is not zero because X maps onto the elliptic curve E/G . Here, X is not an abelian 3-fold, and so it is a new case for which this paper proves the integral Hodge conjecture. The simplest case is $G = \mathbf{Z}/2$, acting on any abelian surface S by ± 1 .

3. Terminal 3-folds

We here analyze the homology of the exceptional divisor of a resolution of an isolated rational 3-fold singularity (Lemma 3.1). This will be used in proving the integral Hodge conjecture for certain 3-folds whose minimal model is singular (Theorem 4.1). Benoist and Wittenberg used a similar argument in their work on the integral Hodge conjecture for real varieties while studying a 3-fold fibered over a curve [5, proof of Proposition 8.6].

Lemma 3.1. *Let Y be a complex 3-fold with isolated rational singularities. Let $\pi : X \rightarrow Y$ be a projective birational morphism with X smooth such that π is an isomorphism over the smooth locus of Y and the inverse image of the singular locus of Y is a divisor D in X with simple normal crossings. Then $H_2(D, \mathbf{Z})$ is generated by classes of algebraic 1-cycles on D .*

Here a complex projective curve C (possibly singular) has a fundamental class in $H_2(C, \mathbf{Z})$, which pushes forward to a class in $H_2(D, \mathbf{Z})$ when C is contained in a complex scheme D .

Proof. We start with the following result by Steenbrink [32, Lemma 2.14]. □

Lemma 3.2. *Let $\pi : X \rightarrow Y$ be a log resolution of an isolated rational singularity with exceptional divisor D . Then $H^i(D, \mathcal{O}) = 0$ for all $i > 0$.*

We continue the proof of Lemma 3.1. Let D_1, \dots, D_r be the irreducible components of D , which are smooth projective surfaces. Write $D_{i_0 \dots i_l}$ for an intersection $D_{i_0} \cap \dots \cap D_{i_l}$. We have an exact sequence of coherent sheaves on D :

$$0 \rightarrow \mathcal{O}_D \rightarrow \bigoplus_i \mathcal{O}_{D_i} \rightarrow \bigoplus_{i < j} \mathcal{O}_{D_{ij}} \rightarrow \bigoplus_{i < j < k} \mathcal{O}_{D_{ijk}} \rightarrow 0.$$

Taking cohomology gives a Mayer–Vietoris spectral sequence

$$E_1^{p,q} = \bigoplus_{i_0 < \dots < i_p} H^q(D_{i_0 \dots i_p}, \mathcal{O}) \Rightarrow H^{p+q}(D, \mathcal{O}).$$

$$\begin{array}{ccccccc}
 \oplus H^2(D_i, \mathcal{O}) & \longrightarrow & 0 & & 0 & & 0 \\
 \\
 \oplus H^1(D_i, \mathcal{O}) & \longrightarrow & \oplus H^1(D_{ij}, \mathcal{O}) & \longrightarrow & 0 & & 0 \\
 & & \searrow \text{---} & & & & \\
 \oplus H^0(D_i, \mathcal{O}) & \longrightarrow & \oplus H^0(D_{ij}, \mathcal{O}) & \longrightarrow & \oplus H^0(D_{ijk}, \mathcal{O}) & \longrightarrow & 0
 \end{array}$$

We have $H^2(D, \mathcal{O}) = 0$ by Lemma 3.2. It follows from the spectral sequence that each irreducible component D_i of D has $H^2(D_i, \mathcal{O}) = 0$.

There is also a Mayer–Vietoris spectral sequence for the integral homology of D :

$$\begin{array}{ccccccc}
 E_{p,q}^1 = \bigoplus_{i_0 < \dots < i_p} H_q(D_{i_0 \dots i_p}, \mathbf{Z}) & \Rightarrow & H_{p+q}(D, \mathbf{Z}) & & & & \\
 \\
 \oplus H_2(D_i, \mathbf{Z}) & \longleftarrow & \oplus H_2(D_{ij}, \mathbf{Z}) & \longleftarrow & 0 & & 0 \\
 \\
 \oplus H_1(D_i, \mathbf{Z}) & \longleftarrow & \oplus H_1(D_{ij}, \mathbf{Z}) & \longleftarrow & 0 & & 0 \\
 & & \swarrow \text{---} & & & & \\
 \oplus H_0(D_i, \mathbf{Z}) & \longleftarrow & \oplus H_0(D_{ij}, \mathbf{Z}) & \longleftarrow & \oplus H_0(D_{ijk}, \mathbf{Z}) & \longleftarrow & 0
 \end{array}$$

Finally, we have a Mayer–Vietoris spectral sequence converging to $H^*(D, \mathbf{C})$, which can be obtained from the integral homology spectral sequence by applying $\text{Hom}(\cdot, \mathbf{C})$. We have a map of spectral sequences from the one converging to $H^*(D, \mathbf{C})$ to the one converging to $H^*(D, \mathcal{O})$. Since $H^2(D, \mathcal{O}) = 0$, we know that the groups $E_\infty^{1,1}$ and $E_\infty^{2,0}$ are zero in the spectral sequence converging to $H^*(D, \mathcal{O})$. That is, the d_1 and d_2 differentials together map onto $\oplus H^1(D_{ij}, \mathcal{O})$ and $\oplus H^0(D_{ijk}, \mathcal{O})$. We will deduce that the d_1 and d_2 differentials together map onto $\oplus H^1(D_{ij}, \mathbf{C})$ and $\oplus H^0(D_{ijk}, \mathbf{C})$. In the first case, we are given that $d_1: \oplus H^1(D_i, \mathcal{O}) \rightarrow \oplus H^1(D_{ij}, \mathcal{O})$ is surjective, and we want to deduce that $d_1: \oplus H^1(D_i, \mathbf{C}) \rightarrow \oplus H^1(D_{ij}, \mathbf{C})$ is surjective. That follows from $d_1: \oplus H^1(D_i, \mathbf{C}) \rightarrow \oplus H^1(D_{ij}, \mathbf{C})$ being a morphism of Hodge structures of weight 1 so that $H^1(D_i, \mathbf{C}) = H^1(D_i, \mathcal{O}) \oplus \overline{H^1(D_i, \mathcal{O})}$ and this grading is compatible with the differential.

A similar argument applies to H^0 . First, the differential $d_1: \oplus H^0(D_{ij}, \mathbf{C}) \rightarrow \oplus H^0(D_{ijk}, \mathbf{C})$ maps isomorphically to $d_1: \oplus H^0(D_{ij}, \mathcal{O}) \rightarrow \oplus H^0(D_{ijk}, \mathcal{O})$. Also, by the comment about Hodge structures of weight 1, $E_2^{0,1}(\mathbf{C}) = \ker(\oplus H^1(D_i, \mathbf{C}) \rightarrow \oplus H^1(D_{ij}, \mathbf{C}))$ is the direct sum of $E_2^{0,1}(\mathcal{O}) = \ker(\oplus H^1(D_i, \mathcal{O}) \rightarrow \oplus H^1(D_{ij}, \mathcal{O}))$ and its conjugate, and so $E_2^{0,1}(\mathbf{C}) \rightarrow E_2^{0,1}(\mathcal{O})$ is surjective. Since $d_2: E_2^{0,1}(\mathcal{O}) \rightarrow E_2^{2,0}(\mathcal{O})$ is onto and $E_2^{0,1}(\mathbf{C}) \rightarrow E_2^{0,1}(\mathcal{O})$ is onto, it follows that $d_2: E_2^{0,1}(\mathbf{C}) \rightarrow E_2^{2,0}(\mathbf{C}) (= E_2^{2,0}(\mathcal{O}))$ is onto. That is, $E_\infty^{2,0}(\mathbf{C})$ as well as $E_\infty^{1,1}(\mathbf{C})$ are zero. Therefore, $H^2(D, \mathbf{C}) \rightarrow \oplus H^2(D_i, \mathbf{C})$ is injective. (In particular, the mixed Hodge structure on $H^2(D, \mathbf{Q})$ is pure of weight 2 [7].) By the universal coefficient theorem, it follows that $\oplus H_2(D_i, \mathbf{Q}) \rightarrow H_2(D, \mathbf{Q})$ is surjective.

The groups $H_0(D_{ijk}, \mathbf{Z})$ and $H_1(D_{ij}, \mathbf{Z})$ are torsion-free since D_{ijk} is a point or empty and D_{ij} is a smooth projective curve or empty. It follows that the subgroups $E_{2,0}^\infty(\mathbf{Z})$ and $E_{1,1}^\infty(\mathbf{Z})$ of these groups are also torsion-free. Since $\oplus H_2(D_i, \mathbf{Q}) \rightarrow H_2(D, \mathbf{Q})$ is surjective, those two E^∞ groups are zero after tensoring with the rationals, and so they are zero. Therefore, $\oplus H_2(D_i, \mathbf{Z}) \rightarrow H_2(D, \mathbf{Z})$ is surjective. Since $H^2(D_i, \mathcal{O}) = 0$, the Lefschetz (1, 1) theorem gives that the smooth projective surface D_i has $H_2(D_i, \mathbf{Z})$ spanned by algebraic cycles. We deduce that $H_2(D, \mathbf{Z})$ is spanned by algebraic cycles. \square

4. 3-folds of Kodaira dimension zero

In this section, we begin the proof of our main result on the integral Hodge conjecture, Theorem 4.1. We reduce the problem to a statement on the variation of Hodge structure associated with a family of surfaces of high degree in a minimal model of the 3-fold, to be proved in the next section (Proposition 5.3).

Theorem 4.1. *Let X be a smooth projective complex 3-fold of Kodaira dimension zero such that $h^0(X, K_X) > 0$. Then X satisfies the integral Hodge conjecture.*

Proof. Let Y be a minimal model of X . Then Y is terminal and hence has singular set of dimension at most zero. As in the proof of Proposition 2.1, K_Y is trivial (and hence $h^0(X, K_X) = h^0(Y, K_Y) = 1$).

For codimension-1 cycles, the integral Hodge conjecture always holds, by the Lefschetz (1, 1) theorem. It remains to prove the integral Hodge conjecture for codimension-2 cycles on X . This is a birationally invariant property for smooth projective varieties X [40, Lemma 15]. Therefore, we can assume that the birational map $X \dashrightarrow Y$ is a morphism and that X is whatever resolution of Y we like. Explicitly, we can assume that $X \rightarrow Y$ is an isomorphism over the smooth locus and that the fiber over each of the (finitely many) singular points of Y is a divisor with simple normal crossings. (We do this in order to apply Lemma 3.1.)

Let H be a very ample line bundle on Y and S a smooth surface in the linear system $|H|$. As shown in the proof of Proposition 2.1, the pushforward homomorphism $H_2(S, \mathbf{Z}) \rightarrow H_2(Y, \mathbf{Z})$ is surjective.

We assume that the Hilbert scheme \mathcal{H} of smooth surfaces in Y in the homology class of S is smooth, which holds if H is sufficiently ample. We are free to replace H by a large multiple in the course of the argument.

The following lemma was suggested by Schoen’s argument on the integral Tate conjecture [31, Theorem 0.5], combined with Voisin’s paper [39].

Lemma 4.2. *Let Y be a terminal projective complex 3-fold. Write S_{t_0} for the surface in Y corresponding to a point t_0 in \mathcal{H} , with inclusion $i: S_{t_0} \rightarrow Y$. Write $H_2(S_{t_0}, \mathbf{Z})_{\text{van}} = \ker(i_*: H_2(S_{t_0}, \mathbf{Z}) \rightarrow H_2(Y, \mathbf{Z}))$. By Poincaré duality, identify $H^2(S_{t_0}, \mathbf{Z})$ with $H_2(S_{t_0}, \mathbf{Z})$. Let C be a nonempty open cone in $H^2(S_{t_0}, \mathbf{R})_{\text{van}}$. Suppose that there is a contractible open neighborhood U of t_0 in \mathcal{H} such that every element of $H^2(S_t, \mathbf{Z})_{\text{van}} \cap C$ becomes a Hodge class on S_t for some t in U . Then every element of $H_2(Y, \mathbf{Z})$ whose image in $H_2(Y, \mathbf{C})$ is in $H_{1,1}(Y)$ is algebraic.*

Proof. By the proof of Proposition 2.1, the pushforward $H_2(S_{t_0}, \mathbf{Z}) \rightarrow H_2(Y, \mathbf{Z})$ is surjective, and so the pullback $H^2(Y, \mathbf{Q}) \rightarrow H^2(S_{t_0}, \mathbf{Q})$ is injective. Therefore, the Hodge structure on $H^2(S_{t_0}, \mathbf{Q})$ is pure of weight 2. Still following the proof of Proposition 2.1, the polarization of $H^2(S_{t_0}, \mathbf{Q})$ by the intersection form gives a canonical direct-sum decomposition of Hodge structures

$$H^2(S_{t_0}, \mathbf{Q}) = H^2(Y, \mathbf{Q}) \oplus H^2(Y, \mathbf{Q})^\perp.$$

In fact, this argument shows that the surjection $i_*: H_2(S_t, \mathbf{Q}) \rightarrow H_2(Y, \mathbf{Q})$ is split as a map of variations of \mathbf{Q} -Hodge structures over the space \mathcal{H} of smooth surfaces S . In particular, any element of $H_2(Y, \mathbf{Q}) \cap H_{1,1}(Y) \subset H_2(Y, \mathbf{C})$ is the image of some element in $H_2(S_{t_0}, \mathbf{Q})$ whose translate to every surface S_t is in $H_{1,1}(S_t)$. Therefore, for any element α of $H_2(Y, \mathbf{Z})$ that maps into $H_{1,1}(Y) \subset H_2(Y, \mathbf{C})$, there is a positive integer N and an element β of $H_2(S_{t_0}, \mathbf{Z})$ that lies in $H_{1,1}(S_t) \subset H_2(S_t, \mathbf{C})$ for every surface S_t such that $i_*\beta = N\alpha$.

Also, because $i_*: H_2(S_{t_0}, \mathbf{Z}) \rightarrow H_2(X, \mathbf{Z})$ is surjective, there is an element $v \in H_2(S_{t_0}, \mathbf{Z})$ (not necessarily a Hodge class) with $i_*v = \alpha$.

Let $u_0 = \beta - Nv$ in $H_2(S_{t_0}, \mathbf{Z})$. Then $i_*u_0 = 0$; that is, u_0 is in $H_2(S_{t_0}, \mathbf{Z})_{\text{van}}$. Let $T = u_0 + N \cdot H_2(S_{t_0}, \mathbf{Z})_{\text{van}} \subset H_2(S_{t_0}, \mathbf{Z})_{\text{van}}$. Since T is a translate of a subgroup of finite index in $H_2(S_{t_0}, \mathbf{Z})_{\text{van}}$, T has nonempty intersection with the open cone C in $H_2(S_{t_0}, \mathbf{R})_{\text{van}}$. Let u be an element of $C \cap T$. Because U is contractible, we can canonically identify $H_2(S_t, \mathbf{Z})$ with $H_2(S_{t_0}, \mathbf{Z})$ for all t in U . By our assumption on C , u becomes a Hodge class on $H_2(S_t, \mathbf{Z})$ for some t in U . By definition of T , we can write $u = u_0 + Nw$ for some w in $H_2(S_{t_0}, \mathbf{Z})_{\text{van}}$. We know that β in $H_2(S_{t_0}, \mathbf{Z})$ is a Hodge class in $H^2(S_t, \mathbf{Z})$ for all nearby surfaces S_t . Since u becomes a Hodge class in $H^2(S_t, \mathbf{Z})$, $\beta - u$ is a Hodge class in $H_2(S_t, \mathbf{Z})$, and $\beta - u = \beta - (u_0 + Nw) = \beta - (\beta - Nv + Nw) = N(v - w)$. So $v - w$ is a Hodge class in $H_2(S_t, \mathbf{Z})$. By the Lefschetz (1, 1) theorem, $v - w$ is algebraic on S_t . And we have $i_*(v - w) = i_*v = \alpha$. So α in $H_2(Y, \mathbf{Z})$ is algebraic. □

We will prove the hypothesis of Lemma 4.2 as Proposition 5.3. Given that, we now finish the proof of Theorem 4.1.

Let u be an element of $H_2(X, \mathbf{Z}) \cap H_{1,1}(X)$. Topologically, Y is obtained from X by identifying the fibers E_1, \dots, E_r over singular points of X to points. So we have an exact sequence

$$H_2(Y, \mathbf{Z}) \rightarrow \bigoplus_i H_2(E_i, \mathbf{Z}) \rightarrow H_2(X, \mathbf{Z}) \rightarrow H_2(Y, \mathbf{Z}) \rightarrow \bigoplus_i H_1(E_i, \mathbf{Z}).$$

By Lemma 3.1, $H_2(E_i, \mathbf{Z})$ is spanned by algebraic curves on E_i , for each i . The image of u in $H_2(Y, \mathbf{Z})$ is in $H_{1,1}(Y)$ and, hence, is in the image of the Chow group $CH_1(Y)$ by Lemma 4.2 and Proposition 5.3. (We use here that the integral Hodge conjecture holds on every smooth projective surface, by the Lefschetz (1, 1) theorem.) Since $X \rightarrow Y$ is an isomorphism outside a zero-dimensional subset of Y , it is clear that $CH_1(X) \rightarrow CH_1(Y)$ is surjective. Therefore, there is a 1-cycle α on X whose image in $H_2(X, \mathbf{Z})$ has the same image in $H_2(Y, \mathbf{Z})$ as u does. By the exact sequence above, $\alpha - u$ in $H_2(X, \mathbf{Z})$ is the image of some element of $\bigoplus H_2(E_i, \mathbf{Z})$. But $\bigoplus H_2(E_i, \mathbf{Z})$ is spanned by algebraic cycles on $\bigcup_i E_i$ by Lemma 3.1. Therefore, u is algebraic. □

5. The variation of Hodge structure associated with a family of surfaces

To complete the proof of Theorem 4.1, we need to show that the variation of Hodge structures on the family of surfaces in the 3-fold is as nontrivial as possible (Proposition 5.3). The first step is to rephrase the conclusion we want in terms of a cup product on a general surface in the family (Corollary 5.1), generalizing Proposition 1 in Voisin [39].

Let Y be a terminal complex projective 3-fold. (We will eventually assume that K_Y is trivial, but it seems clearer to formulate the basic arguments in greater generality.) Let H be a very ample line bundle on Y and S a smooth surface in the linear system $|H|$. (It follows that S is contained in the smooth locus of Y .) We are free to replace H by a large multiple in the course of the argument.

By the proof of Proposition 2.1, the Hodge structure on $H^2(Y, \mathbf{Q})$ is pure of weight 2, and the graded pieces of the Hodge filtration on $H^2(Y, \mathbf{C})$ are $H^2(Y, \mathcal{O})$, $H^1(Y, \underline{\Omega}_Y^1)$, and $H^0(Y, \underline{\Omega}_Y^2)$.

Define the vanishing cohomology $H^2(S, \mathbf{Z})_{\text{van}}$ to be the kernel of the pushforward homomorphism $i_*: H^2(S, \mathbf{Z}) \cong H_2(S, \mathbf{Z}) \rightarrow H_2(Y, \mathbf{Z})$. Likewise, write

$$\begin{aligned} H^2(S, \mathcal{O})_{\text{van}} &= \ker(H^2(S, \mathcal{O}) \rightarrow H^0(Y, \underline{\Omega}_Y^2)^*) \\ H^1(S, \Omega_S^1)_{\text{van}} &= \ker(H^1(S, \Omega^1) \rightarrow H^1(Y, \underline{\Omega}_Y^1)^*) \\ H^0(S, \Omega_S^2)_{\text{van}} &= \ker(H^0(S, \Omega^2) \rightarrow H^2(Y, \underline{\Omega}_Y^0)^*). \end{aligned}$$

These maps are Hodge-graded pieces of the surjection $H_2(S, \mathbf{C}) \rightarrow H_2(Y, \mathbf{C})$ (dual to the pullback $H^2(Y, \mathbf{C}) \rightarrow H^2(S, \mathbf{C})$), and so they are also surjective. By Serre duality, $H^2(Y, \underline{\Omega}_Y^0)^* = H^2(Y, \mathcal{O})^* \cong H^1(Y, K_Y)$. So we can also describe $H^0(S, \Omega_S^2)_{\text{van}}$ as the kernel of the pushforward $H^0(S, K_S) \rightarrow H^1(Y, K_Y)$.

The cohomology sheaves of $\underline{\Omega}_Y^j$ are in degrees ≥ 0 , and the zeroth cohomology sheaf is the sheaf $\Omega_Y^{[j]}$ of reflexive differentials because Y is klt [20, Theorems 5.4 and 7.12]. Also, $\underline{\Omega}_Y^j$ is concentrated in degrees from 0 to $3 - j$ since Y has dimension 3 [16, Théorème V.6.2]. It follows that $\underline{\Omega}_Y^3$ is the canonical sheaf K_Y .

We assume that the Hilbert scheme \mathcal{H} of smooth surfaces in Y in the homology class of S is smooth, which holds if H is sufficiently ample. Then $H^0(S, N_{S/Y})$ is the tangent space to \mathcal{H} at S . Let δ be the class of the extension $0 \rightarrow TS \rightarrow TY|_S \rightarrow N_{S/Y} \rightarrow 0$ in $H^1(S, N_{S/Y}^* \otimes TS)$. Then the product with δ is the Kodaira–Spencer map $H^0(S, N_{S/Y}) \rightarrow H^1(S, TS)$, which describes how the isomorphism class of S changes as S moves in Y .

For u in $H^1(S, TS)$, the product with u is a linear map

$$u \cdot : H^1(S, \Omega^1) \rightarrow H^2(S, \mathcal{O}).$$

The dual map

$$u \cdot : H^0(S, K_S) \rightarrow H^1(S, \Omega^1)$$

can also be described as the product with u . For λ in $H^1(S, \Omega^1)$, define

$$\mu_\lambda : H^0(S, N_{S/Y}) \rightarrow H^2(S, \mathcal{O})$$

by $\mu_\lambda(n) = (\delta n)\lambda$. This map describes the failure of $\lambda \in H^2(S, \mathbf{C})$ to remain a $(1, 1)$ class when the surface S is deformed in X . For λ in $H^1(S, \Omega^1)_{\text{van}}$, the map μ_λ lands in the

vanishing subspace $H^2(S, O)_{\text{van}}$ because $H^2(S, \mathbf{Q}) = H^2(Y, \mathbf{Q}) \oplus H^2(S, \mathbf{Q})_{\text{van}}$ as Hodge structures, where the Hodge structure on $H^2(Y, \mathbf{Q})$ is unchanged as S is deformed.

Corollary 5.1. *Let Y be a terminal projective complex 3-fold. Let H be a very ample line bundle on Y and S a smooth surface in the linear system $|H|$. Suppose that there is an element λ in $H^1(S, \Omega^1)_{\text{van}}$ such that the linear map*

$$\mu_\lambda: H^0(S, N_{S/Y}) \rightarrow H^2(S, O)_{\text{van}}$$

is surjective. Then there is a nonempty open cone C in $H^2(S_{t_0}, \mathbf{R})_{\text{van}}$ and a contractible open neighborhood U of t_0 in \mathcal{H} such that every element of $H^2(S_t, \mathbf{Z})_{\text{van}} \cap C$ becomes a Hodge class on S_t for some t in U .

For Y smooth and $H^2(Y, O) = 0$ (which implies that every element of $H_2(Y, \mathbf{Z})$ is a Hodge class), this was proved by Voisin [39, Proposition 1]. Voisin also formulated a statement similar to Corollary 5.1 in the case of uniruled 3-folds Y with $H^2(Y, O) \neq 0$ [39, Proposition 4].

Proof. The groups $H^2(S_t, \mathbf{Z})$ form a weight-2 variation of Hodge structures on the Hilbert scheme \mathcal{H} of smooth surfaces $S_t \subset Y$. Let U be a contractible open neighborhood of the given point t_0 in \mathcal{H} . We can canonically identify $H^2(S_t, \mathbf{C})$ with $H^2(S_{t_0}, \mathbf{C})$ for all $t \in U$. The surjectivity of μ_λ implies that the map from $\bigcup_{t \in U} H^{1,1}(S_t, \mathbf{R})_{\text{van}}$ to $H^2(S_{t_0}, \mathbf{R})_{\text{van}}$ is a submersion at λ , I claim.

To prove that this map is a submersion, we follow the argument of [39, Proposition 1], modified so as not to assume that $H^2(Y, O)$ is zero. Let $\pi: S_U \rightarrow U$ be the universal family of surfaces S_t , restricted to $t \in U$. Write H^2_{van} for the total space of the vector bundle $(R^2\pi_*\mathbf{C})_{\text{van}}$ over U , with fibers $H^2(S_t, \mathbf{C})_{\text{van}}$. The Gauss–Manin connection gives a trivialization of this bundle and, hence, a projection map from the total space to one fiber, $\tau: H^2_{\text{van}} \rightarrow H^2(S_{t_0}, \mathbf{C})_{\text{van}}$. Let $F^1H^2_{\text{van}}$ be the submanifold of H^2 whose fiber over each point $t \in U$ is the Hodge filtration

$$F^1H^2(S_t, \mathbf{C})_{\text{van}} = H^{2,0}(S_t)_{\text{van}} \oplus H^{1,1}(S_t)_{\text{van}} \subset H^2(S_t, \mathbf{C})_{\text{van}}.$$

Let $\tau_1: F^1H^2_{\text{van}} \rightarrow H^2(S_{t_0}, \mathbf{C})_{\text{van}}$ be the restriction of τ to $F^1H^2_{\text{van}}$. Let λ be an element of $H^1(S_{t_0}, \Omega^1)_{\text{van}}$ and $\tilde{\lambda}$ any lift of λ to $F^1H^2(S_{t_0}, \mathbf{C})_{\text{van}}$. By the proof of Voisin [39, Lemma 2] (modified since we are allowing $H^2(Y, O)$ to be nonzero), we have the following equivalence.

Lemma 5.2. *The map*

$$\mu_\lambda: H^0(S_{t_0}, N_{S_{t_0}/Y}) \rightarrow H^2(S_{t_0}, O)_{\text{van}}$$

is surjective if and only if τ_1 is a submersion at $\tilde{\lambda}$.

To relate Lemma 5.2 to cohomology with real coefficients, note that surjectivity of μ_λ is a Zariski open condition on λ in $H^1(S_{t_0}, \Omega^1)_{\text{van}}$. The vector space $H^1(S_{t_0}, \Omega^1)_{\text{van}}$ has a real structure, given by

$$H^{1,1}(S_{t_0})_{\mathbf{R}, \text{van}} = H^1(S_{t_0}, \Omega^1)_{\text{van}} \cap H^2(S_{t_0}, \mathbf{R})_{\text{van}} \subset H^2(S_{t_0}, \mathbf{C})_{\text{van}}.$$

Since we assume in this Corollary that μ_λ is surjective for one λ in $H^1(S_{t_0}, \Omega^1)_{\text{van}}$, it is surjective for some λ in $H^{1,1}(S_{t_0})_{\mathbf{R}, \text{van}}$.

In Lemma 5.2, take the lifting $\tilde{\lambda}$ to be λ itself. Then $\tilde{\lambda}$ is real, and so is $\tau_1(\tilde{\lambda})$. By our assumption on λ , Lemma 5.2 gives that τ_1 is a submersion at $\tilde{\lambda}$, and so the restriction

$$\tau_{1, \mathbf{R}}: H_{\mathbf{R}, \text{van}}^{1,1} \rightarrow H^2(S_{t_0}, \mathbf{R})_{\text{van}}$$

of τ_1 to $\tau_1^{-1}(H^2(S_{t_0}, \mathbf{R})_{\text{van}})$ is also a submersion. Here $\tau_1^{-1}(H^2(S_{t_0}, \mathbf{R})_{\text{van}})$ is identified with

$$\bigcup_{t \in U} F^1 H^2(S_t, \mathbf{C})_{\text{van}} \cap H^2(S_t, \mathbf{R})_{\text{van}} = \bigcup_{t \in U} H^{1,1}(S_t, \mathbf{R})_{\text{van}} =: H_{\mathbf{R}, \text{van}}^{1,1}.$$

Since $\tau_{1, \mathbf{R}}$ is a submersion at $\tilde{\lambda}$ on the real manifold $H_{\mathbf{R}, \text{van}}^{1,1}$ (a real vector bundle over U), the image of $\tau_{1, \mathbf{R}}$ contains a nonempty open subset of $H^2(S_{t_0}, \mathbf{R})_{\text{van}}$, as we wanted.

The image of $\tau_{1, \mathbf{R}}$ is a cone, and so it contains an open cone C in $H^2(S_{t_0}, \mathbf{R})_{\text{van}}$. Therefore, all elements of $H^2(S_{t_0}, \mathbf{Z})_{\text{van}}$ in the open cone C become Hodge classes on S_t for some t in U . Corollary 5.1 is proved. \square

Proposition 5.3. *Let Y be a terminal projective complex 3-fold with trivial canonical bundle. Write S_{t_0} for the surface in Y corresponding to a point t_0 in \mathcal{H} , with inclusion $i: S_{t_0} \rightarrow Y$. Write $H_2(S_{t_0}, \mathbf{Z})_{\text{van}} = \ker(i_*: H_2(S_{t_0}, \mathbf{Z}) \rightarrow H_2(Y, \mathbf{Z}))$. Then there is a nonempty open cone C in $H^2(S_{t_0}, \mathbf{R})_{\text{van}}$ and a contractible open neighborhood U of t_0 in \mathcal{H} such that every element of $H^2(S_{t_0}, \mathbf{Z})_{\text{van}} \cap C$ becomes a Hodge class on S_t for some t in U .*

Proof. Let H be a very ample line bundle on Y and let S be a smooth surface in $|nH|$ for a positive integer n . (We will eventually take n big enough and S general in $|nH|$.) Let $V = H^0(S, K_S)_{\text{van}}$ and $V' = H^0(Y, O(S))/H^0(Y, O)$. By the exact sequence of sheaves on Y

$$0 \rightarrow O_Y \rightarrow O(S) \rightarrow O(S)|_S \rightarrow 0,$$

we can view V' as a subspace of $H^0(S, O(S)|_S) = H^0(S, N_{S/Y})$. For n sufficiently large, the long exact sequence of cohomology gives an exact sequence

$$0 \rightarrow V' \rightarrow H^0(S, N_{S/Y}) \rightarrow H^1(Y, O) \rightarrow 0.$$

Likewise, by definition of V , we have an exact sequence

$$0 \rightarrow V \rightarrow H^0(S, K_S) \rightarrow H^1(Y, K_Y) \rightarrow 0,$$

where the pushforward map shown is the boundary map from the exact sequence of sheaves on Y :

$$0 \rightarrow K_Y \rightarrow K_Y(S) \rightarrow K_S \rightarrow 0.$$

We will only need to move S in its linear system (although $H^1(Y, O)$ need not be zero). That is, we will show that for a general $\lambda \in H^1(S, \Omega_S^1)_{\text{van}}$, the restriction of μ_λ to $V' \subset H^0(S, N_{S/Y})$ maps onto $H^2(S, O)_{\text{van}} = V^*$; by Corollary 5.1, that will finish the proof of Proposition 5.3. We will see that these two vector spaces have the same dimension,

using that K_Y is trivial, and so the argument just barely works. (For K_Y more positive, it would not work at all.)

Fix a trivialization of the canonical bundle K_Y . This gives an isomorphism between the two short exact sequences of sheaves above, in particular, an isomorphism $K_S \cong N_{S/Y}$ of line bundles on S . So we have an isomorphism between the two exact sequences of cohomology, including an isomorphism $V \cong V'$. In terms of this identification, μ_λ for λ in $H^1(S, \Omega^1)_{\text{van}}$ is a linear map $V' \rightarrow V^*$. For varying λ , this is equivalent to the pairing

$$\begin{aligned} \mu: V \times V' &\rightarrow H^1(S, \Omega^1)_{\text{van}} \\ \mu(v, v') &= v(v'\delta), \end{aligned}$$

which is symmetric. (Recall that δ is the class of the extension $0 \rightarrow TS \rightarrow TY|_S \rightarrow N_{S/Y} \rightarrow 0$ in $H^1(S, N_{S/Y}^* \otimes TS)$.) (Proof: this pairing is the restriction of a symmetric pairing $H^0(S, N_{S/Y}) \otimes H^0(S, N_{S/Y}) \rightarrow H^1(S, \Omega^1)$, given by $u \otimes v \mapsto uv\gamma\delta$, where $u, v \in H^0(S, N_{S/Y})$, and $\gamma \in H^0(S, N_{S/Y}^* \otimes (TS)^* \otimes \Omega_S^1)$ is the natural map $N_{S/Y} \otimes TS \cong \Omega_S^2 \otimes TS \rightarrow \Omega_S^1$ of bundles on S .) Serre duality $H^1(S, \Omega^1)_{\text{van}}^* \cong H^1(S, \Omega^1)_{\text{van}}$ gives a dual map

$$q = \mu^*: H^1(S, \Omega^1)_{\text{van}} \rightarrow S^2V^*.$$

We can think of q as a linear system of quadrics in the projective space $P(V^*)$ of lines in V . The condition that μ_λ from $V' \subset H^0(S, N_{S/Y})$ to $V^* = H^2(S, \mathcal{O})_{\text{van}}$ is surjective for generic $\lambda \in H^1(S, \Omega^1)_{\text{van}}$ is equivalent to the condition that the quadric defined by $q(\lambda)$ is smooth for generic λ . Thus, by Corollary 5.1, Proposition 5.3 will follow if we can show that the quadric $q(\lambda)$ is smooth for generic λ .

Note that we lose nothing by restricting the pairing μ to the subspaces $V \subset H^0(S, K_S)$ and $V' \subset H^0(S, N_{S/Y})$. Indeed, as discussed in the proof of Proposition 2.1, the Hodge structure $H^2(S, \mathbf{Q})$ is polarized by the intersection form, and the restriction $H^2(Y, \mathbf{Q}) \rightarrow H^2(S, \mathbf{Q})$ is injective, with image a sub-Hodge structure. Therefore, $H^2(S, \mathbf{Q})$ is the orthogonal direct sum of $H^2(Y, \mathbf{Q})$ and its orthogonal complement. This gives a splitting of each Hodge-graded piece of $H^2(S, \mathbf{C})$. For example, for $H^0(S, K_S)$, this gives the decomposition

$$H^0(S, K_S) \cong H^0(Y, \Omega_Y^2) \oplus H^0(S, K_S)_{\text{van}}.$$

Therefore, we also have a canonical decomposition of the isomorphic vector space $H^0(S, N_{S/Y})$. This is the decomposition

$$H^0(S, N_{S/Y}) \cong H^0(Y, TY) \oplus H^0(Y, \mathcal{O}(S))/H^0(Y, \mathcal{O}).$$

Thinking of $H^0(S, N_{S/Y})$ as the first-order deformation space of S in Y , these two subspaces correspond to moving S by automorphisms of Y and moving S in its linear system. The first type of move does not change the Hodge structure of S , and so it is irrelevant to our purpose (trying to make a given integral cohomology class on S into a Hodge class).

We use the following consequence of Bertini’s theorem from Voisin [39, Lemma 15], which we apply to our space V (identified with V') and $W = H^1(S, \Omega^1)$. (Note that we follow the numbering of statements in the published version of [39], not the preprint.)

Lemma 5.4. *Let $\mu: V \otimes V \rightarrow W$ be symmetric and let $q: W^* \rightarrow S^2V^*$ be its dual. For v in V , write $\mu_v: V \rightarrow W$ for the corresponding linear map. Think of q as a linear system of quadrics in $\mathbf{P}(V^*)$. Then the generic quadric in $\text{im}(q)$ is smooth if the following condition holds. There is no closed subvariety $Z \subset \mathbf{P}(V^*)$ contained in the base locus of $\text{im}(q)$ and satisfying*

$$\text{rank}(\mu_v) \leq \dim(Z)$$

for all $v \in Z$.

We have to show that such a subvariety Z does not exist for a general surface $S \in |nH|$ with n sufficiently divisible. We follow the outline of H\"oring and Voisin's argument ([19, after Lemma 3.35], extending [39, after Lemma 7] in the smooth case). After replacing the very ample line bundle H by a multiple if necessary, we can assume that $H^i(Y, \mathcal{O}(lH)) = 0$ for $i > 0$ and $l > 0$. We degenerate the general surface S to a surface with many nodes as follows. Consider a general symmetric $n \times n$ matrix A with entries in $H^0(Y, \mathcal{O}(H))$. Let S_0 be the surface in Y defined by the determinant of A in $H^0(Y, \mathcal{O}(nH))$. By the assumption of generality, S_0 is contained in the smooth locus of Y . By Barth [1], the singular set of S_0 consists of N nodes, where

$$N = \binom{n+1}{3} H^3.$$

Let $\mathcal{S} \rightarrow \Delta$ be a Lefschetz degeneration of surfaces $S_t \in |nH|$ over the unit disc Δ such that the central fiber S_0 has nodes x_1, \dots, x_N as singularities. The map q above makes sense for any smooth surface S in a 3-fold. Voisin showed that the limiting space

$$\lim_{t \rightarrow 0} \text{im}(q_t: H^1(S_t, \Omega_{S_t}^1) \rightarrow (H^0(S_t, K_{S_t}) \otimes H^0(S_t, \mathcal{O}(nH)))^*),$$

which is a linear subspace of $(H^0(S_0, K_{S_0}) \otimes H^0(S_0, \mathcal{O}(nH)))^*$, contains for each $1 \leq i \leq N$ the multiplication-evaluation map which is the composite

$$H^0(S_0, K_{S_0}) \otimes H^0(S_0, \mathcal{O}(nH)) \rightarrow H^0(S_0, K_{S_0}(nH)) \rightarrow K_{S_0}(nH)|_{x_i}$$

[39, Lemma 7].

Recall that we have identified $V = H^0(S, K_S)_{\text{van}}$ with $V' = H^0(Y, \mathcal{O}(S))/H^0(Y, \mathcal{O})$. When we degenerate a general surface S to the nodal surface S_0 , the base locus $B \subset \mathbf{P}(V^*)$ of $\text{im}(q)$ specializes to a subspace of the base locus B_0 of $\text{im}(q_0) \subset \mathbf{P}(V_0^*)$, where

$$V_0 = H^0(S_0, K_{S_0}) \cong V'_0 = H^0(S_0, \mathcal{O}_{S_0}(nH)).$$

Let W be the set of nodes of S_0 . By Voisin's lemma just mentioned, B_0 is contained in

$$C_0 := \{v \in \mathbf{P}(V_0^*) : v^2|_W = 0\}.$$

As a set, C_0 is a linear subspace:

$$\begin{aligned} C_0 &= \{v \in \mathbf{P}(V_0^*) : v|_W = 0\} \\ &= \mathbf{P}(H^0(S_0, K_{S_0} \otimes I_W)^*). \end{aligned}$$

By [19, equation (3.36)], extending [39, Corollary 3] in the smooth case (using only that $H^i(Y, \mathcal{O}(lH)) = 0$ for $i > 0$ and $l > 0$), we have $h^0(Y, K_Y(nH) \otimes I_W) \leq cn^2$ for some

constant c independent of n . Thus, the base locus B_0 of $\text{im}(q_0)$ has dimension at most cn^2 , for some constant c independent of n . By specializing, the base locus B of $\text{im}(q)$ also has dimension at most cn^2 for general surfaces S in $|nH|$.

By our assumption on the subvariety Z of B , for $v \in Z$, we have

$$\begin{aligned} \text{rank}(\mu_v: V \rightarrow H^1(S, \Omega^1)) &\leq \dim(Z) \\ &\leq cn^2. \end{aligned}$$

By the following lemma, it follows that $\dim(Z) \leq A$ for some constant A independent of n . This is Höring–Voisin’s [19, Lemma 3.37], extending [39, Lemma 12] in the smooth case. (As before, we follow the numbering from the published version of [39]. In our case, $H^2(Y, \mathcal{O})$ need not be zero, but that is not used in these proofs.)

Lemma 5.5. *Let Y be a Gorenstein projective 3-fold with isolated canonical singularities, H as above. For each positive integer n , let S be a general surface in $|nH|$ and define V, V', μ associated with S as above. Let c be any positive constant. Then there is a constant A such that the sets*

$$\Gamma = \{v \in V : \text{rank}(\mu_v) \leq cn^2\}$$

and

$$\Gamma' = \{v' \in V' : \text{rank}(\mu_{v'}) \leq cn^2\}$$

both have dimension bounded by A , independent of n .

By our assumption on the subvariety Z of B again, for $v \in Z$, we have

$$\begin{aligned} \text{rank}(\mu_v: V \rightarrow H^1(S, \Omega^1)) &\leq \dim(Z) \\ &\leq A. \end{aligned}$$

This implies that Z is empty by Lemma 5.6, to be proved next. But Z is a variety, so we have a contradiction. This completes the proof that the generic quadric in the linear system $\text{im}(q)$ is smooth. Proposition 5.3 is proved. \square

To complete the proof of Proposition 5.3 and, hence, Theorem 4.1, it remains to prove the following lemma.

Lemma 5.6. *Let Y be a terminal projective 3-fold with K_Y trivial, H as above. Let A be a positive integer. Let $S \in |nH|$ be general, with n large enough (depending on A). Let $V = H^0(S, K_S)_{\text{van}}$ and $\mu_v: V' \rightarrow H^1(S, \Omega^1)$ be the product with an element $v \in V$, as defined above. Then the set*

$$W = \{v \in V : \text{rank}(\mu_v) < A\}$$

is equal to \emptyset .

Proof. We have to modify the proof of Voisin’s Lemma 13 [39] to allow Y to be singular and also to have $H^2(Y, \mathcal{O})$ not zero. We use Höring and Voisin’s ideas on how to deal with Y being singular by working on the smooth surface S as far as possible [19, proof of Proposition 3.22].

Let S be a smooth surface in $|nH|$. Consider the following exact sequences of vector bundles on S , constructed from the normal bundle sequence of S in Y :

$$0 \rightarrow \Omega_S^1(nH) \rightarrow \Omega_Y^2|_S(2nH) \rightarrow K_S(2nH) \rightarrow 0$$

and

$$0 \rightarrow \mathcal{O}_S \rightarrow \Omega_Y^1|_S(nH) \rightarrow \Omega_S^1(nH) \rightarrow 0.$$

Let δ_1 and δ_2 be the resulting boundary maps:

$$\delta_1: H^0(S, K_S(2nH)) \rightarrow H^1(S, \Omega_S^1(nH))$$

and

$$\delta_2: H^1(S, \Omega_S^1(nH)) \rightarrow H^2(S, \mathcal{O}).$$

Let $\delta = \delta_2 \circ \delta_1: H^0(S, K_S(2nH)) \rightarrow H^2(S, \mathcal{O})$.

Lemma 5.7. *The image of δ is $H^2(S, \mathcal{O})_{\text{van}}$, for large enough n and any S as above.*

Proof. We first show that δ_1 is surjective. By the long exact sequence of cohomology associated with the first exact sequence above, it suffices to show that $H^1(S, \Omega_Y^2|_S(2nH))$ is zero. In terms of the sheaf $\Omega_Y^{[2]}$ of reflexive differentials, we have an exact sequence of sheaves on Y :

$$0 \rightarrow \Omega_Y^{[2]}(nH) \rightarrow \Omega_Y^{[2]}(2nH) \rightarrow \Omega_Y^2|_S(2nH) \rightarrow 0.$$

By Serre vanishing on Y , both $H^1(Y, \Omega_Y^{[2]}(2nH))$ and $H^2(Y, \Omega_Y^{[2]}(nH))$ vanish for large n , and so $H^1(S, \Omega_Y^2|_S(2nH)) = 0$ for all smooth S in $|nH|$ with n large.

Next, the long exact sequence involving δ_2 shows that the cokernel of δ_2 is contained in $H^2(S, \Omega_Y^1|_S(nH))$. Since $K_S = nH|_S$ by the adjunction formula, the dual of that H^2 space is $H^0(S, TY|_S) = H^0(S, \Omega_Y^2|_S)$.

By the exact sequence

$$0 \rightarrow \Omega_Y^{[2]}(-nH) \rightarrow \Omega_Y^{[2]} \rightarrow \Omega_Y^2|_S \rightarrow 0$$

of sheaves on Y , we have an exact sequence

$$H^0(X, \Omega_Y^{[2]}(-nH)) \rightarrow H^0(Y, \Omega_X^{[2]}) \rightarrow H^0(S, \Omega_Y^{[2]}|_S) \rightarrow H^1(Y, \Omega_Y^{[2]}(-nH)).$$

Since Y is normal, the sheaf $\Omega_Y^{[2]}$ is reflexive, and $\dim(Y) > 1$, the groups on the left and right are zero for n large. (Consider an embedding of Y into some \mathbf{P}^N and use Serre vanishing and Serre duality on \mathbf{P}^N , as in [17, proof of Corollary III.7.8].) So the map $H^0(X, \Omega_Y^{[2]}) \rightarrow H^0(S, \Omega_Y^{[2]}|_S)$ is an isomorphism. By the results above on δ_1 and δ_2 , this gives an exact sequence

$$H^0(S, K_S(2nH)) \xrightarrow{\delta} H^2(S, \mathcal{O}) \rightarrow H^0(Y, \Omega_Y^{[2]})^*.$$

Finally, we need to rephrase this in terms of du Bois’s object $\underline{\Omega}_Y^2$ in the derived category of Y . The cohomology sheaves of $\underline{\Omega}_Y^j$ are in degrees ≥ 0 , and the zeroth cohomology sheaf is $\Omega_Y^{[j]}$ because Y is klt [20, Theorems 5.4 and 7.12]. So there is a natural map $\Omega_Y^{[2]} \rightarrow \underline{\Omega}_Y^2$

in $D(Y)$. Because the other cohomology sheaves of $\underline{\Omega}_Y^2$ are in degrees > 0 , it is immediate that the map $H^0(Y, \Omega^{[2]}) \rightarrow H^0(Y, \underline{\Omega}_Y^2)$ is an isomorphism. So the previous paragraph yields an exact sequence:

$$H^0(S, K_S(2nH)) \xrightarrow{\delta} H^2(S, \mathcal{O}) \rightarrow H^0(Y, \underline{\Omega}_Y^2)^*.$$

Equivalently, the image of δ is $H^2(S, \mathcal{O})_{\text{van}}$. □

Assume that $v \in V$ satisfies the condition that $\text{rank}(\mu_v) < A$. Using that n is sufficiently large, Höring and Voisin show that $\delta(H^0(S, \mathcal{O}_S(3nH)))$ is orthogonal to v with respect to Serre duality [19, after Proposition 3.40]. By Lemma 5.7, $H^2(S, \mathcal{O})_{\text{van}}$ is orthogonal to v . Since $V = H^0(S, K_S)_{\text{van}}$ is dual to $H^2(S, \mathcal{O})_{\text{van}}$, it follows that $v = 0$. Lemma 5.6 is proved. This also completes the proofs of Proposition 5.3 and Theorem 4.1. □

6. The integral Tate conjecture for 3-folds

We now prove the integral Tate conjecture for 3-folds in characteristic zero that are rationally connected or have Kodaira dimension zero with $h^0(X, K_X) > 0$ (Theorem 6.1). In any characteristic, we will prove the integral Tate conjecture for abelian 3-folds (Theorem 7.1).

Theorem 6.1. *Let X be a smooth projective 3-fold over the algebraic closure of a finitely generated field of characteristic zero. If X is rationally connected or it has Kodaira dimension zero with $h^0(X, K_X) > 0$ (hence equal to 1), then X satisfies the integral Tate conjecture.*

Proof. We start by proving the following known lemma.

Lemma 6.2. *Let X be a smooth projective variety over the separable closure k_s of a finitely generated field k . For codimension-1 cycles on X , the Tate conjecture implies the integral Tate conjecture.*

Proof. For a prime number l invertible in k and a positive integer r , the Kummer sequence

$$0 \rightarrow \mu_{l^r} \rightarrow G_m \xrightarrow{l^r} G_m \rightarrow 0$$

of étale sheaves on X gives a long exact sequence of cohomology and, hence, an exact sequence involving the Picard and Brauer groups:

$$0 \rightarrow \text{Pic}(X)/l^r \rightarrow H_{\text{et}}^2(X, \mu_{l^r}) \rightarrow \text{Hom}(\mathbf{Z}/l^r, \text{Br}(X)) \rightarrow 0.$$

Writing $NS(X)$ for the group of divisors modulo algebraic equivalence, we have $\text{Pic}(X)/l^r = NS(X)/l^r$ because the group of k_s -points of an abelian variety is l -divisible. Since $NS(X)$ is finitely generated, taking inverse limits gives an exact sequence:

$$0 \rightarrow NS(X) \otimes \mathbf{Z}_l \rightarrow H^2(X, \mathbf{Z}_l(1)) \rightarrow \text{Hom}(\mathbf{Q}_l/\mathbf{Z}_l, \text{Br}(X)) \rightarrow 0.$$

The last group is automatically torsion-free. It follows that the Tate conjecture implies the integral Tate conjecture in the case of codimension-1 cycles. □

Lemma 6.3. *Let X be a smooth projective 3-fold over the algebraic closure \bar{k} of a finitely generated field k of characteristic zero. Suppose that the Tate conjecture holds for codimension-1 cycles on X and that the integral Hodge conjecture holds on $X_{\mathbf{C}}$ for some embedding $\bar{k} \hookrightarrow \mathbf{C}$. Then the integral Tate conjecture holds for X (over \bar{k}).*

Proof. By Lemma 6.2, the integral Tate conjecture holds for codimension-1 cycles on X . It remains to prove integral Tate for 1-cycles on X . Let $u \in H^4(X, \mathbf{Z}_l(2))$ be a Tate class; that is, u is fixed by $\text{Gal}(\bar{k}/l)$ for some finite extension l of k . Let H be an ample line bundle on X . Multiplication by the class of H is an isomorphism from $H^2(X, \mathbf{Q}_l(1))$ to $H^4(X, \mathbf{Q}_l(2))$, by the hard Lefschetz theorem. So there is a positive integer N with $Nu = Hv$ for some $v \in H^2(X, \mathbf{Z}_l(1))$. Because the isomorphism from $H^2(X, \mathbf{Q}_l(1))$ to $H^4(X, \mathbf{Q}_l(2))$ is Galois-equivariant, v is a Tate class (this works even if there is torsion in $H^2(X, \mathbf{Z}_l(1))$ because we are considering Tate classes over \bar{k}). By our assumptions, v is algebraic, that is, a \mathbf{Z}_l -linear combination of classes of subvarieties of X . So $Hv = Nu$ is algebraic and thus a \mathbf{Z}_l -linear combination of classes of curves on X .

In particular, Nu is a \mathbf{Z}_l -linear combination of Hodge classes in $H^4(X_{\mathbf{C}}, \mathbf{Z})$. Since the subgroup of Hodge classes is a summand in $H^4(X_{\mathbf{C}}, \mathbf{Z})$, it follows that u is a \mathbf{Z}_l -linear combination of Hodge classes in $H^4(X_{\mathbf{C}}, \mathbf{Z})$. Since the integral Hodge conjecture holds for $X_{\mathbf{C}}$, u is a \mathbf{Z}_l -linear combination of classes of curves on X . □

We prove Theorem 6.1 using Lemma 6.3. The integral Hodge conjecture holds for rationally connected 3-folds by Voisin [39, Theorem 2] and for 3-folds X with Kodaira dimension zero and $h^0(X, K_X) = 1$ by Theorem 4.1, generalizing Voisin [39, Theorem 2]. It remains to check the Tate conjecture in codimension 1 for X over \bar{k} . That is clear if $h^{0,2}(X) = 0$; then all of $H^2(X_{\mathbf{C}}, \mathbf{Z})$ is algebraic by the Lefschetz (1, 1) theorem, and so all of $H^2(X, \mathbf{Z}_l(1))$ is algebraic. That covers the case where X is rationally connected.

It remains to prove the Tate conjecture in codimension 1 for a 3-fold X over \bar{k} of Kodaira dimension zero with $h^{0,2}(X) > 0$. Let Y be a minimal model of X ; then Y is terminal and has torsion canonical bundle. By Höring and Peternell, generalizing the Beauville–Bogomolov structure theorem to singular varieties, there is a projective variety Z with canonical singularities and a finite morphism $Z \rightarrow Y$, étale in codimension one, such that Z is a product of an abelian variety, (singular) irreducible symplectic varieties, and (singular) Calabi–Yau varieties in a strict sense [18, Theorem 1.5]. Their theorem is stated over \mathbf{C} , but that implies the statement over \bar{k} . Höring and Peternell build on the earlier work by Druel and Greb–Guenancia–Kebekus [8, 14].

Since Y has dimension 3 and $h^0(Z, \Omega^{[2]}) \geq h^0(Y, \Omega^{[2]}) = h^0(X, \Omega^2) > 0$, the only possibilities are: Z is an abelian 3-fold or the product of an elliptic curve and a K3 surface with canonical singularities. (A strict Calabi–Yau 3-fold Z has $h^0(Z, \Omega^{[2]}) = 0$, by definition.) So there is a resolution of singularities Z_1 of Z which is either an abelian 3-fold or the product of an elliptic curve and a smooth K3 surface. Since we have a dominant rational map $Z_1 \dashrightarrow X$, the Tate conjecture in codimension 1 for X will follow from the same statement for Z_1 [36, Theorem 5.2].

It remains to prove the Tate conjecture in codimension 1 for Z_1 , which is either an abelian 3-fold or the product of a K3 surface and an elliptic curve over \bar{k} . Faltings proved the Tate conjecture in codimension 1 for all abelian varieties over number fields

[10], extended to all finitely generated fields of characteristic zero by Zarhin. Finally, the Tate conjecture holds for K3 surfaces in characteristic zero, by Tankeev [34]. Since $H^2(S \times E, \mathbf{Q}_l) \cong H^2(S, \mathbf{Q}_l) \oplus H^2(E, \mathbf{Q}_l)$ for a K3 surface S and an elliptic curve E , the Tate conjecture in codimension 1 holds for $S \times E$. \square

7. The integral Tate conjecture for abelian 3-folds in any characteristic

We now prove the integral Tate conjecture for abelian 3-folds in any characteristic. In characteristic zero, we have already shown this in Theorem 6.1. However, it turns out that a more elementary proof works in any characteristic, modeled on Grabowski’s proof of the integral Hodge conjecture for complex abelian 3-folds [13, Corollary 3.1.9]. More generally, we show that the integral Tate conjecture holds for 1-cycles on all abelian varieties of dimension g if the “minimal class” $\theta^{g-1}/(g-1)!$ is algebraic on every principally polarized abelian variety (X, θ) of dimension g (Proposition 7.2).

Theorem 7.1. *Let X be an abelian 3-fold over the separable closure of a finitely generated field. Then the integral Tate conjecture holds for X .*

Proof. The argument is based on Beauville’s Fourier transform for Chow groups of abelian varieties, inspired by Mukai’s Fourier transform for derived categories. Write $CH^*(X)_{\mathbf{Q}}$ for $CH^*(X) \otimes \mathbf{Q}$. Let X be an abelian variety of dimension g over a field k , with dual abelian variety $\widehat{X} := \text{Pic}^0(X)$, and let $f: X \times \widehat{X} \rightarrow X$ and $g: X \times \widehat{X} \rightarrow \widehat{X}$ be the projections. The Fourier transform $F_X: CH^*(X)_{\mathbf{Q}} \rightarrow CH^*(\widehat{X})_{\mathbf{Q}}$ is the linear map

$$F_X(u) = g_*(f^*(u) \cdot e^{c_1(L)}),$$

where L is the Poincaré line bundle on $X \times \widehat{X}$ and $e^{c_1(L)} = \sum_{j=0}^{2g} c_1(L)^j / j!$. For k separably closed, define the Fourier transform $H^*(X, \mathbf{Q}_l(*)) \rightarrow H^*(\widehat{X}, \mathbf{Q}_l(*))$ by the same formula.

By Beauville, the Fourier transform sends $H^j(X, \mathbf{Z}_l(a))$ to $H^{2g-j}(X, \mathbf{Z}_l(a+g-j))$, and this map is an isomorphism [2, Proposition 1]. By contrast, it is not clear whether the Fourier transform can be defined integrally on Chow groups; that actually fails over a general field, by Esnault [26, § 3.1]. Beauville’s proof (for complex abelian varieties) uses that the integral cohomology of an abelian variety is an exterior algebra over \mathbf{Z} , and the same argument works for the \mathbf{Z}_l -cohomology of an abelian variety over any separably closed field.

Next, let $\theta \in H^2(X, \mathbf{Z}_l(1))$ be the first Chern class of a principal polarization on an abelian variety X . Then we can identify \widehat{X} with X , and the Fourier transform satisfies

$$F_X(\theta^j / j!) = (-1)^{g-j} \theta^{g-j} / (g-j)!$$

[2, Lemme 1]. Here, $\theta^j / j!$ lies in $H^{2j}(X, \mathbf{Z}_l(j))$ (although it is not obviously algebraic, meaning the class of an algebraic cycle with \mathbf{Z}_l coefficients). Finally, let $h: X \rightarrow Y$ be an isogeny, and write $\widehat{h}: \widehat{Y} \rightarrow \widehat{X}$ for the dual isogeny. Then the Fourier transform switches pullback and pushforward, in the sense that for $u \in CH^*(Y)_{\mathbf{Q}}$,

$$F_X(h^*(u)) = \widehat{h}_*(F_Y(u))$$

[2, Proposition 3(iii)].

The following is the analog for the integral Tate conjecture of Grabowski’s argument on the integral Hodge conjecture [13, Proposition 3.1.8].

Proposition 7.2. *Let k be the separable closure of a finitely generated field. Suppose that for every principally polarized abelian variety (Y, θ) of dimension g over k , the minimal class $\theta^{g-1}/(g-1)! \in H^{2g-2}(Y, \mathbf{Z}_l(g-1))$ is algebraic. Then the integral Tate conjecture for 1-cycles holds for all abelian varieties of dimension g over k .*

Proof. Let X be an abelian variety of dimension g over k , and let u be a Tate class in $H^2(X, \mathbf{Z}_l(1))$ (meaning that u is fixed by some open subgroup of the Galois group). The Tate conjecture holds for codimension-1 cycles on abelian varieties over k , by Tate [35], Faltings [10], and Zarhin. This implies the integral Tate conjecture for codimension-1 cycles on X , by Lemma 6.2. So u is a \mathbf{Z}_l -linear combination of classes of line bundles and, hence, of ample line bundles.

For each ample line bundle L on X , there is a principally polarized abelian variety (Y, θ) and an isogeny $h: X \rightarrow Y$ with $c_1(L) = h^*\theta$ [28, Corollary 1, p. 234]. Then the Fourier transform of $c_1(L)$ is given by

$$\begin{aligned} F_X(c_1(L)) &= F_X(h^*\theta) \\ &= \widehat{h}_*(F_Y(\theta)) \\ &= (-1)^{g-1} \widehat{h}_*(\theta^{g-1}/(g-1)!). \end{aligned}$$

By assumption, $\theta^{g-1}/(g-1)!$ in $H^{2g-2}(Y, \mathbf{Z}_l(g-1))$ is algebraic (with \mathbf{Z}_l coefficients). Since the pushforward preserves algebraic classes, the equality above shows that $F_X(c_1(L))$ is algebraic. By the previous paragraph, it follows that the Fourier transform of any Tate class in $H^2(X, \mathbf{Z}_l(1))$ is algebraic in $H^{2g-2}(\widehat{X}, \mathbf{Z}_l(g-1))$.

Since the Fourier transform is Galois-equivariant and is an isomorphism from $H^2(X, \mathbf{Z}_l(1))$ to $H^{2g-2}(\widehat{X}, \mathbf{Z}_l(g-1))$, it sends Tate classes bijectively to Tate classes. This proves the integral Tate conjecture for 1-cycles on \widehat{X} and, hence, for 1-cycles on every abelian variety of dimension g over k . □

We now return to the proof of Theorem 7.1. Let k be the separable closure of a finitely generated field, and let X be an abelian 3-fold over k . We want to prove the integral Tate conjecture for X .

By Proposition 7.2, it suffices to show that for every principally polarized abelian 3-fold (X, θ) over k , the class $\theta^2/2$ in $H^4(X, \mathbf{Z}_l(2))$ is algebraic. (This is clear for $l \neq 2$.) A general principally polarized abelian 3-fold X over k is the Jacobian of a curve C of genus 3. In that case, choosing a k -point of C determines an embedding of C into X , and the cohomology class of C on X is $\theta^2/2$ by Poincaré’s formula [15, p. 350]. (Poincaré proved this for Jacobian varieties over \mathbf{C} , but that implies the same formula in l -adic cohomology for Jacobians in any characteristic.) By the specialization homomorphism on Chow groups [12, Proposition 2.6, Example 20.3.5], it follows that $\theta^2/2$ is algebraic for every principally polarized abelian 3-fold over k . Theorem 7.1 is proved. □

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