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NECESSARY AND SUFFICIENT CONDITIONS FOR THE STOCHASTIC COMPARISON OF JACKSON NETWORKS

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External and internal monotonicity properties for Jackson networks have been established in the literature with the use of coupling constructions. Recently, Lopez et al. derived necessary and sufficient conditions for the (strong) stochastic comparison of two-station Jackson networks with increasing service rates, by constructing a certain Markovian coupling. In this article, we state necessary and sufficient conditions for the stochastic comparison of L-station Jackson networks in the general case. The proof is based on a certain characterization of the stochastic order for continuous-time Markov chains, written in terms of their associated intensity matrices.

1. INTRODUCTION

Questions concerning external and/or internal monotonicity properties of queuing networks have attracted the interest of many investigators in the literature (see, e.g., Lindvall [3,4], Shanthikumar and Yao [9,10], and Lopez, Martinez, and Sanz [5]).

Lindvall [3] considered the problem of the stochastic comparison of Jackson networks with identical transition probabilities and derived easily verifiable sufficient conditions. Shanthikumar and Yao [9–11] studied mainly internal monotonicity and convexity questions for certain classes of closed Jackson networks. The main tool in these works is the so-called coupling method. Lindvall [3] is the standard

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reference on coupling, whereas Thorisson [14] studied more theoretical issues. Shaked and Shanthikumar [8] summarized many results about various stochastic orders and their work contains many applications in diverse fields. The works of Szekli [13] and Stoyan [12] have many queuing applications.

Recently, Lopez et al. [5] derived necessary and sufficient conditions for the stochastic comparison of two, two-station Jackson networks with increasing service rates, by constructing a certain Markovian coupling. Their method, although interesting, is quite involved. In this article, we extend their conditions for the stochastic comparison of L-station Jackson networks in the general case (i.e., without assuming increasing service rates). The proof is simple, based on a characterization of the stochastic comparison of two continuous-time Markov chains via their associated intensity matrices. This characterization has been established by Massey [6] and extended by Brandt and Last [1] in their work on the stochastic domination of processes defined in partially ordered spaces. Recently, this approach has been successfully used by Miyazawa and Taylor [7] and Economou [2] for obtaining tractable stochastic bounds for nontractable queuing networks with batch transfers.

2. THE NECESSARY AND SUFFICIENT CONDITIONS

Let $(E,\leq)$ be a countable partially ordered set. A set $\Gamma \subseteq E$ is said to be increasing if $x \in \Gamma$ implies that $\{y : y \geq x\} \subseteq \Gamma$. Consider two Markov semigroups $P(t) = (p_{xy}(t) : x, y \in E)$ and $P'(t) = (p'_{xy}(t) : x, y \in E)$. The semigroup P is said to be stochastically dominated by P' (denoted by $P \leq_{\text{st}} P'$) if $p_{x.}(t) \leq_{\text{st}} p_{y.}(t)$ for all $x \leq y$ and $t \geq 0$. This is equivalent to $\mu P(t) \leq_{\text{st}} \nu P'(t)$, for all initial distributions μ and ν on E such that $\mu \leq_{\text{st}} \nu$ and $t \geq 0$.

Massey [6] proved a useful characterization of the stochastic domination for two Markov semigroups defined on $(E,\leq)$ in terms of their associated intensity matrices. More specifically, he considered two continuous-time Markov chains on $(E,\leq)$ with semigroups P and P' and intensity matrices Q and Q', respectively, and showed that the following two conditions are necessary and sufficient for $P \leq_{\text{st}} P'$:

(i) For every $x, y \in E$ and $\Gamma \subseteq E$ increasing,

$$x \leq y \text{ and } y \notin \Gamma \Rightarrow \sum_{z\in\Gamma} q(x,z) \leq \sum_{z\in\Gamma} q'(y,z). \tag{1}$$

(ii) For every $x, y \in E$ and $\Gamma \subseteq E$ increasing,

$$x \leq y \text{ and } x \in \Gamma \Rightarrow \sum_{z\in\Gamma^c} q(x,z) \geq \sum_{z\in\Gamma^c} q'(y,z). \tag{2}$$

We will now use conditions (1) and (2) to establish conditions for the stochastic domination of two L-station Jackson networks. We generalize the framework and notation of Lindvall [3] slightly by allowing the service rates to depend on the num-

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ber of customers at each station. The dynamics of such a network are given by the following:

(i) Customers arrive from outside at station k according to a Poisson process at rate β_k,
(ii) The service rate at station k is $\delta_k(x_k)$ whenever there are x_k present customers at station k.
(iii) After finished service at station k, a customer goes to station $m \neq k$ with probability γ_{km} or leaves the system with probability $\gamma_k = 1 - \sum_{m\neq k}\gamma_{km}$.
(iv) All of the random quantities in items (i)–(iii) are independent.

The state of the network at each time is described by a vector $\mathbf{x} = (x_1, x_2, \ldots, x_L)$, where x_k denotes the number of customers at station k. The stochastic process describing the number of customers at the various stations is a continuous-time Markov chain with state space $E = \mathbf{Z}_+^L$ and transition rates

$$\begin{aligned} q(\mathbf{x},\mathbf{x}+\mathbf{e}_m) &= \beta_m, & m &= 1,2,\ldots,L,\\ q(\mathbf{x},\mathbf{x}-\mathbf{e}_k) &= \delta_k(x_k)\gamma_k, & k &= 1,2,\ldots,L,\\ q(\mathbf{x},\mathbf{x}-\mathbf{e}_k+\mathbf{e}_m) &= \delta_k(x_k)\gamma_{km}, & k,m &= 1,2,\ldots,L, \end{aligned}$$

where by $\mathbf{e}_k$ we denote the k-unit vector with 1 in the kth position and 0 elsewhere. The partial order in E is the coordinatewise order. We are now in position to state the main result of the article.

THEOREM 1: *Consider two L-station Jackson networks with Markov semigroups P and P′ and parameters* $(\beta_m, \delta_k(\cdot), \gamma_{km}, \gamma_k)$ *and* $(\beta'_m, \delta'_k(\cdot), \gamma'_{km}, \gamma'_k)$, *respectively. The following conditions are sufficient for* $P \leq_{\mathrm{st}} P'$:

(i) For every $m = 1,2,\ldots,L$, $A \subseteq \{1,2,\ldots,L\}$ *and* $0 \leq s_k \leq t_k$, $k \in A$,

$$\beta_m + \sum_{k\in A}\delta_k(s_k)\gamma_{km} \leq \beta'_m + \sum_{k\in A}\delta'_k(t_k)\gamma'_{km}. \tag{3}$$

(ii) For every $k = 1,2,\ldots,L$, $A \subseteq \{1,2,\ldots,L\}$ *and* $s_k \geq 0$,

$$\delta_k(s_k)\left(\gamma_k + \sum_{m\in A}\gamma_{km}\right) \geq \delta'_k(s_k)\left(\gamma'_k + \sum_{m\in A}\gamma'_{km}\right). \tag{4}$$

PROOF: It suffices to prove that (3) and (4) imply conditions (1) and (2).

For verifying condition (1), consider $\mathbf{x},\mathbf{y} \in \mathbf{Z}_+^L$ and Γ increasing such that $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{y} \notin \Gamma$. Then, we have that $\mathbf{x} \notin \Gamma$ also. We will use the following notation. For $\mathbf{z} \notin \Gamma$, let $\Gamma(\mathbf{z}) = \{i : \mathbf{z} + \mathbf{e}_i \in \Gamma\} \subseteq \{1,2,\ldots,L\}$. It is easy to see that $\mathbf{z} \leq \mathbf{w} \notin \Gamma$ implies $\Gamma(\mathbf{z}) \subseteq \Gamma(\mathbf{w})$. We also denote the indicator function of a set $A \subseteq \mathbf{Z}_+^L$ by $I_A(\mathbf{z})$ and the complement of A by A^c.

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The only states in Γ that can be reached in a single transition from $\mathbf{x}$ under P are of the form $\mathbf{x}+\mathbf{e}_m$ or $\mathbf{x}-\mathbf{e}_k+\mathbf{e}_m$. Therefore, we have

$$\begin{aligned}\sum_{\mathbf{z}\in\Gamma} q(\mathbf{x},\mathbf{z}) &= \sum_{m=1}^{L} q(\mathbf{x},\mathbf{x}+\mathbf{e}_m)I_\Gamma(\mathbf{x}+\mathbf{e}_m)\\ &\quad+\sum_{k=1}^{L}\sum_{m=1}^{L} q(\mathbf{x},\mathbf{x}-\mathbf{e}_k+\mathbf{e}_m)I_\Gamma(\mathbf{x}-\mathbf{e}_k+\mathbf{e}_m)\\ &= \sum_{m\in\Gamma(\mathbf{x})}\beta_m+\sum_{k=1}^{L}\sum_{m\in\Gamma(\mathbf{x}-\mathbf{e}_k)}\delta_k(x_k)\gamma_{km}. \end{aligned}\tag{5}$$

Similarly,

$$\sum_{\mathbf{z}\in\Gamma} q'(\mathbf{y},\mathbf{z}) = \sum_{m\in\Gamma(\mathbf{y})}\beta'_m+\sum_{k=1}^{L}\sum_{m\in\Gamma(\mathbf{y}-\mathbf{e}_k)}\delta'_k(y_k)\gamma'_{km}.\tag{6}$$

Subtracting (5) from (6) and taking into account that $\Gamma(\mathbf{x})\subseteq\Gamma(\mathbf{y})$ and $\Gamma(\mathbf{x}-\mathbf{e}_k)\subseteq\Gamma(\mathbf{y}-\mathbf{e}_k)$, we obtain

$$\begin{aligned}\sum_{\mathbf{z}\in\Gamma} q'(\mathbf{y},\mathbf{z})-\sum_{\mathbf{z}\in\Gamma} q(\mathbf{x},\mathbf{z}) &= \sum_{m\in\Gamma(\mathbf{y})\setminus\Gamma(\mathbf{x})}\beta'_m+\sum_{m\in\Gamma(\mathbf{x})}(\beta'_m-\beta_m)\\ &\quad+\sum_{k=1}^{L}\Bigg(\sum_{m\in\Gamma(\mathbf{y}-\mathbf{e}_k)\setminus\Gamma(\mathbf{x}-\mathbf{e}_k)}\delta'_k(y_k)\gamma'_{km}\\ &\qquad+\sum_{m\in\Gamma(\mathbf{x}-\mathbf{e}_k)}(\delta'_k(y_k)\gamma'_{km}-\delta_k(x_k)\gamma_{km})\Bigg)\\ &\geq \sum_{m\in\Gamma(\mathbf{x})}(\beta'_m-\beta_m)\\ &\quad+\sum_{k=1}^{L}\sum_{m\in\Gamma(\mathbf{x}-\mathbf{e}_k)}(\delta'_k(y_k)\gamma'_{km}-\delta_k(x_k)\gamma_{km}). \end{aligned}$$

Since $\Gamma(\mathbf{x}-\mathbf{e}_k)\subseteq\Gamma(\mathbf{x})$, we can interchange summations on the right-hand side of the above and write it equivalently as

$$\begin{aligned}\sum_{\mathbf{z}\in\Gamma} q'(\mathbf{y},\mathbf{z})-\sum_{\mathbf{z}\in\Gamma} q(\mathbf{x},\mathbf{z}) \geq \sum_{m\in\Gamma(\mathbf{x})}\Bigg(&\Bigg(\beta'_m+\sum_{k:m\in\Gamma(\mathbf{x}-\mathbf{e}_k)}\delta'_k(y_k)\gamma'_{km}\Bigg)\\ &-\Bigg(\beta_m+\sum_{k:m\in\Gamma(\mathbf{x}-\mathbf{e}_k)}\delta_k(x_k)\gamma_{km}\Bigg)\Bigg). \end{aligned}\tag{7}$$

However, now (7) and (3) imply (1) immediately.

For condition (2), consider $\mathbf{x},\mathbf{y}\in\mathbf{Z}_+^L$ and Γ increasing such that $\mathbf{x}\leq\mathbf{y}$ and $\mathbf{x}\in\Gamma$. Then, $\mathbf{y}\in\Gamma$. For $\mathbf{z}\in\Gamma$, let $\Gamma^c(\mathbf{z})=\{i:\mathbf{z}-\mathbf{e}_i\notin\Gamma\}\subseteq\{1,2,\ldots,L\}$. Then,

$\mathbf{z} \le \mathbf{w}$ and $\mathbf{z} \in \Gamma$ imply $\Gamma^c(\mathbf{w}) \subseteq \Gamma^c(\mathbf{z})$. Now, the only states in Γ^c that can be reached in a single transition from $\mathbf{x}$ under P are of the form $\mathbf{x} - \mathbf{e}_k$ or $\mathbf{x} - \mathbf{e}_k + \mathbf{e}_m$. We have

$$\begin{aligned}\sum_{\mathbf{z}\in\Gamma^c} q(\mathbf{x},\mathbf{z}) &= \sum_{k=1}^{L} q(\mathbf{x},\mathbf{x}-\mathbf{e}_k) I_{\Gamma^c}(\mathbf{x}-\mathbf{e}_k) \\ &\quad + \sum_{k=1}^{L}\sum_{m=1}^{L} q(\mathbf{x},\mathbf{x}-\mathbf{e}_k+\mathbf{e}_m) I_{\Gamma^c}(\mathbf{x}-\mathbf{e}_k+\mathbf{e}_m) \\ &= \sum_{k\in\Gamma^c(\mathbf{x})} \delta_k(x_k)\gamma_k + \sum_{m=1}^{L}\sum_{k\in\Gamma^c(\mathbf{x}+\mathbf{e}_m)} \delta_k(x_k)\gamma_{km} \end{aligned} \tag{8}$$

and, similarly,

$$\sum_{\mathbf{z}\in\Gamma^c} q'(\mathbf{y},\mathbf{z}) = \sum_{k\in\Gamma^c(\mathbf{y})} \delta_k'(y_k)\gamma_k' + \sum_{m=1}^{L}\sum_{k\in\Gamma^c(\mathbf{y}+\mathbf{e}_m)} \delta_k'(y_k)\gamma_{km}'. \tag{9}$$

Subtracting (9) from (8) and taking into account that $\Gamma^c(\mathbf{y}) \subseteq \Gamma^c(\mathbf{x})$ and $\Gamma^c(\mathbf{y}+\mathbf{e}_m) \subseteq \Gamma^c(\mathbf{x}+\mathbf{e}_m)$ for all $m = 1,2,\ldots,L$, we obtain

$$\begin{aligned}\sum_{\mathbf{z}\in\Gamma^c} q(\mathbf{x},\mathbf{z}) - \sum_{\mathbf{z}\in\Gamma^c} q'(\mathbf{y},\mathbf{z}) &= \sum_{k\in\Gamma^c(\mathbf{x})\setminus\Gamma^c(\mathbf{y})} \delta_k(x_k)\gamma_k + \sum_{k\in\Gamma^c(\mathbf{y})} (\delta_k(x_k)\gamma_k - \delta_k'(y_k)\gamma_k') \\ &\quad + \sum_{m=1}^{L}\Bigg(\sum_{k\in\Gamma^c(\mathbf{x}+\mathbf{e}_m)\setminus\Gamma^c(\mathbf{y}+\mathbf{e}_m)} \delta_k(x_k)\gamma_{km} \\ &\qquad + \sum_{k\in\Gamma^c(\mathbf{y}+\mathbf{e}_m)} (\delta_k(x_k)\gamma_{km} - \delta_k'(y_k)\gamma_{km}')\Bigg) \\ &\ge \sum_{k\in\Gamma^c(\mathbf{y})} (\delta_k(x_k)\gamma_k - \delta_k'(y_k)\gamma_k') \\ &\quad + \sum_{m=1}^{L}\sum_{k\in\Gamma^c(\mathbf{y}+\mathbf{e}_m)} (\delta_k(x_k)\gamma_{km} - \delta_k'(y_k)\gamma_{km}'). \end{aligned} \tag{10}$$

CLAIM:

(i) $\mathbf{x} \le \mathbf{y}$, $\mathbf{x} \in \Gamma$, *and* $k \in \Gamma^c(\mathbf{y})$ *implies* $x_k = y_k$.
(ii) $\mathbf{x} \le \mathbf{y}$, $\mathbf{x} \in \Gamma$, *and* $k \in \Gamma^c(\mathbf{y}+\mathbf{e}_m)$ *implies* $x_k = y_k$.

For part (i), we have that $k \in \Gamma^c(\mathbf{y})$ implies $\mathbf{y} - \mathbf{e}_k \in \Gamma^c$. If $x_k < y_k$, then $\mathbf{x} \le \mathbf{y} - \mathbf{e}_k$. However, then Γ increasing implies $\mathbf{x} \in \Gamma^c$, contradiction. Therefore $x_k = y_k$. Part (ii) is immediate from part (i).

Using the claim, (10) is reduced to

$$\sum_{\mathbf{z}\in\Gamma^c} q(\mathbf{x},\mathbf{z}) - \sum_{\mathbf{z}\in\Gamma^c} q'(\mathbf{y},\mathbf{z}) \geq \sum_{k\in\Gamma^c(\mathbf{y})} (\delta_k(x_k)\gamma_k - \delta'_k(x_k)\gamma'_k) + \sum_{m=1}^{L} \sum_{k\in\Gamma^c(\mathbf{y}+\mathbf{e}_m)} (\delta_k(x_k)\gamma_{km} - \delta'_k(x_k)\gamma'_{km}). \quad \textbf{(11)}$$

Because $\Gamma^c(\mathbf{y}+\mathbf{e}_m) \subseteq \Gamma^c(\mathbf{y})$, we can interchange summations on the right-hand side of the above and write it equivalently as

$$\sum_{\mathbf{z}\in\Gamma^c} q(\mathbf{x},\mathbf{z}) - \sum_{\mathbf{z}\in\Gamma^c} q'(\mathbf{y},\mathbf{z}) \geq \sum_{k\in\Gamma^c(\mathbf{y})} \left(\delta_k(x_k)\left(\gamma_k + \sum_{m:k\in\Gamma^c(\mathbf{y}+\mathbf{e}_m)} \gamma_{km}\right) - \delta'_k(x_k)\left(\gamma'_k + \sum_{m:k\in\Gamma^c(\mathbf{y}+\mathbf{e}_m)} \gamma'_k\right)\right). \quad \textbf{(12)}$$

However, now (12) and (4) imply (2). ■

Unfortunately the conditions of Theorem 1 are not necessary in general for $P \leq_{\text{st}} P'$. However, they are necessary if we limit the class of networks under consideration. To this end, we will first give necessary conditions for $P \leq_{\text{st}} P'$ in the general case.

THEOREM 2: *Consider two L-station Jackson networks with Markov semigroups P* and P' and parameters $(\beta_m, \delta_k(\cdot), \gamma_{km}, \gamma_k)$ *and* $(\beta'_m, \delta'_k(\cdot), \gamma'_{km}, \gamma'_k)$*, respectively. The following conditions are necessary for* $P \leq_{\text{st}} P'$*:*

(i) For every $m = 1,2,\ldots,L$, $A \subseteq \{1,2,\ldots,L\}$*, and* $s_k \geq 0$, $k \in A$*,*

$$\beta_m + \sum_{k\in A} \delta_k(s_k)\gamma_{km} \leq \beta'_m + \sum_{k\in A} \delta'_k(s_k)\gamma'_{km}. \quad \textbf{(13)}$$

(ii) For every $k = 1,2,\ldots,L$, $A \subseteq \{1,2,\ldots,L\}$*, and* $s_k \geq 0$*,*

$$\delta_k(s_k)\left(\gamma_k + \sum_{m\in A} \gamma_{km}\right) \geq \delta'_k(s_k)\left(\gamma'_k + \sum_{m\in A} \gamma'_{km}\right). \quad \textbf{(14)}$$

PROOF: We will show that conditions (1) and (2) imply (13) and (14) for special choices of $\mathbf{x}$, $\mathbf{y}$ and increasing sets Γ.

We introduce the following notation. For $\mathbf{z} \in \mathbf{Z}_+^L$ set $[\geq\mathbf{z}] = \{\mathbf{w}:\mathbf{w} \geq \mathbf{z}\}$ and $[\leq\mathbf{z}] = \{\mathbf{w}:\mathbf{w} \leq \mathbf{z}\}$. Then, $[\geq\mathbf{z}]$ and $[\leq\mathbf{z}]^c$ are increasing. Note also that the union and the intersection of increasing sets are increasing sets.

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For proving (13) for a fixed $m = 1,2,\dots,L$, we can assume without loss that $A \subseteq \{1,2,\dots,L\}\backslash\{m\}$, since $\gamma_{mm} = 0$. Fix m, A, and $s_k \geq 0$, $k \in A$. Take

$$\mathbf{x} = \mathbf{y} = \sum_{h\in A} s_h \mathbf{e}_h, \qquad \Gamma = [\geq \mathbf{x} + \mathbf{e}_m] \cup \bigcup_{j\in A} [\geq \mathbf{x} - \mathbf{e}_j + \mathbf{e}_m]. \tag{15}$$

Then,

$$\begin{aligned}\Gamma(\mathbf{x}) = \Gamma(\mathbf{y}) &= \{i : \mathbf{x} + \mathbf{e}_i \geq \mathbf{x} - \mathbf{e}_j + \mathbf{e}_m, \text{ for some } j \in A \text{ or } \mathbf{x} + \mathbf{e}_i \geq \mathbf{x} + \mathbf{e}_m\} \\ &= \{i : \mathbf{e}_i + \mathbf{e}_j \geq \mathbf{e}_m, \text{ for some } j \in A \text{ or } \mathbf{e}_i \geq \mathbf{e}_m\} = \{m\},\end{aligned} \tag{16}$$

since $\mathbf{e}_j \neq \mathbf{e}_m$ for $j \in A$ (because we assumed that $A \subseteq \{1,2,\dots,L\}\backslash\{m\}$).

Similarly, for every $k \in \{1,2,\dots,L\}\backslash\{m\}$, we obtain

$$\Gamma(\mathbf{x} - \mathbf{e}_k) = \Gamma(\mathbf{y} - \mathbf{e}_k) = \begin{cases} \{m\} & \text{if } k \in A \\ \varnothing & \text{otherwise.} \end{cases} \tag{17}$$

Taking into account (16) and (17), (5) and (6) give

$$\sum_{\mathbf{z}\in\Gamma} q(\mathbf{x},\mathbf{z}) = \beta_m + \sum_{k\in A} \delta_k(s_k)\gamma_{km}, \qquad \sum_{\mathbf{z}\in\Gamma} q'(\mathbf{y},\mathbf{z}) = \beta'_m + \sum_{k\in A} \delta'_k(s_k)\gamma'_{km}$$

and (1) implies (13).

For proving (14) for a fixed $k = 1,2,\dots,L$, we can assume without loss that $A \subseteq \{1,2,\dots,L\}\backslash\{k\}$, since $\gamma_{mm} = 0$. Fix k, A, and $s_k \geq 0$. Take

$$\mathbf{x} = \mathbf{y} = s_k \mathbf{e}_k, \qquad \Gamma = [\leq \mathbf{x} - \mathbf{e}_k]^c \cap \bigcap_{j\in A} [\leq \mathbf{x} - \mathbf{e}_k + \mathbf{e}_j]^c. \tag{18}$$

Then, $\Gamma^c = [\leq \mathbf{x} - \mathbf{e}_k] \cup \bigcup_{j\in A} [\leq \mathbf{x} - \mathbf{e}_k + \mathbf{e}_j]$ and, we can easily see, as in the previous case, that

$$\Gamma^c(\mathbf{x}) = \Gamma^c(\mathbf{y}) = \{k\}, \tag{19}$$

and for every $m \in \{1,2,\dots,L\}\backslash\{k\}$, we obtain

$$\Gamma^c(\mathbf{x} + \mathbf{e}_m) = \Gamma^c(\mathbf{y} + \mathbf{e}_m) = \begin{cases} \{k\} & \text{if } m \in A \\ \varnothing & \text{otherwise.} \end{cases} \tag{20}$$

Taking into account (19) and (20), (2), (8), and (9) imply (14). ■

We will now state as corollaries two results that have been reported in the literature. Their original derivations use certain coupling constructions.

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Corollary 1 (Lopez et al. [5] conditions): *Consider two L-station Jackson networks with Markov semigroups P and P′ and parameters* $(\beta_m, \delta_k(\cdot), \gamma_{km}, \gamma_k)$ *and* $(\beta'_m, \delta'_k(\cdot), \gamma'_{km}, \gamma'_k)$*, respectively, with increasing service rates at each station. Conditions (13) and (14) are necessary and sufficient for* $P \leq_{\text{st}} P'$*.*

Proof: The necessity of conditions (13) and (14) has been established in Theorem 2. Condition (13) together with the assumption of the increasing service rates at each station imply condition (3). Therefore, in this case, (13) and (14) are also sufficient. ■

Corollary 2 (Lindvall [3] conditions): *Consider two L-station Jackson networks with Markov semigroups P and P′ and parameters* $(\beta_m, \delta_k(\cdot), \gamma_{km}, \gamma_k)$ *and* $(\beta'_m, \delta'_k(\cdot), \gamma'_{km}, \gamma'_k)$*, respectively, with identical constant service rates at each station and identical transition probabilities* $(\delta_k = \delta'_k, \gamma_{km} = \gamma'_{km}, \gamma_k = \gamma'_k)$*. Then,* $P \leq_{\text{st}} P'$ *if and only if* $\beta_m \leq \beta'_m$ *for every m.*

Consider now two Jackson networks with constant service rates and parameters $(\beta_m, \delta_k, \gamma_{km}, \gamma_k)$ and $(\beta'_m, \delta'_k, \gamma'_{km}, \gamma'_k)$. Lindvall [4] observed that if we modify the conditions of Corollary 2 into the more general $\delta_k \geq \delta'_k$, $\gamma_{km} = \gamma'_{km}, \gamma_k = \gamma'_k$ and $\beta_m \leq \beta'_m$, the two networks may not be comparable. In fact, it is immediate to see that Theorems 1 and 2 imply the following result in this case.

Corollary 3: *Consider two L-station Jackson networks with Markov semigroups P and P′ and parameters* $(\beta_m, \delta_k(\cdot), \gamma_{km}, \gamma_k)$ *and* $(\beta'_m, \delta'_k(\cdot), \gamma'_{km}, \gamma'_k)$*, respectively, with constant service rates at each station. Suppose that* $\delta_k \geq \delta'_k$*,* $\gamma_{km} = \gamma'_{km}$*, and* $\gamma_k = \gamma'_k$*. Then,* $P \leq_{\text{st}} P'$ *if and only if for every* $m = 1, 2, \ldots, L$ *and* $A \subseteq \{1, 2, \ldots, L\}$

$$\beta_m + \sum_{k \in A} \delta_k \gamma_{km} \leq \beta'_m + \sum_{k \in A} \delta'_k \gamma'_{km}.$$

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