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UNIFYING TIME-TO-BUILD THEORY

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Discrete-time and continuous-time models with time to build do not possess the same dimension. In this paper we address the dimensionality issue by revising the way time to build is traditionally measured when transforming the economic dynamics from discrete to continuous time. We propose our new procedure as an alternative approach which allows for a unifying theory of the two classes of economic growth models. Interestingly enough, discrete-time models admit a change in stability for a value of the time-to-build parameter where their continuous counterpart admits an Hopf bifurcation.

Keywords: Discrete and Continuous-Time, Time to Build, Mixed Functional Differential Equations

1. INTRODUCTION

The dimensionality of a continuous-time neoclassical growth model switches from finite to infinite as soon as capital takes time to become productive. In fact, the presence of the time-to-build parameter, $J > 0$, pins down a system of mixed functional differential equations in capital and consumption, whose associated characteristic equation is transcendental in the complex field; the number of its roots is therefore infinite.

In contrast, in its discrete-time version, the time-to-build assumption simply increases the order of the dynamical system describing the economy by a factor of $2J$, with J an integer multiple of the chosen calendar time unit, and thus it remains finite. In particular, the characteristic equation associated with a neoclassical growth model [as in Kydland and Prescott (1982), among others] is algebraic of corresponding degree.

Our objective is to propose a procedure that preserves the time-to-build structure of capital in the passage from discrete to continuous time and, in doing so, explains the change in dimensionality. For this purpose, we need to be careful in assigning the time-to-build parameter its appropriate measure. In fact, with regard to the problem of the nature of time in neoclassical growth models, we maintain a sort

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of agnostic point of view, thus not entering the debate on the "proper" dimension that the time variable must be given in any particular economic setting. Our focus remains, instead, the study of the morphological differences that the model's dynamics may exhibit when written by choosing a continuous vs. a discrete time approach. In this respect the transformation issue—namely the problem of deriving the "correct" continuous counterpart of an assigned discrete-time dynamics, and vice versa—is particularly relevant to us, and will deserve specific attention further on.

Since the early contributions on the relation between discrete- and continuous-time models [see May (1970), Foley (1975), and Turnovsky (1977), among others], the conventional procedure for moving from discrete to continuous time has been to rescale all the key equations describing the economy along any subinterval $[t, t+h]$ and let the period length, h , tend to zero; then the discrete-time capital equation $K_{t+1} = f(K_t)$ is usually rewritten as

$$\Delta_h(K_t) = K_{t+h} - K_t = h\Delta_1(K_{t+1}) = h(f(K_t) - K_t),$$

and computing the limit of the ratio $\Delta_h(K_t)/h$ for $h \to 0$, one obtains

$$\dot{K}(t) = f(K(t)) - K(t).$$

Thus, the continuous-time version has the same dimensionality as its discrete-time counterpart. However, the two economies, as described by the two versions of the model, are de facto very different: capital becomes productive immediately in continuous time, whereas it took one period (for example, one quarter) in discrete time. This difference emerges as a consequence of the way the time to build is measured. In the explained procedure it coincides with the *period length*, or using a more updated terminology, the time-to-build parameter is measured in terms of *frequency of decision making* [see Anagnostopoulos and Giannitsaru (2013)].[1] Then the time-to-build structure of capital is lost as the length of the period goes to zero, or equivalently the frequency of decision making goes to infinity. On the other hand, there is no economic reason that an increase in the frequency of decisions should, all else equal, change the time-to-build structure of capital: rephrasing the well-known example of Böhm-Bawerk (1889), a ship cannot be built in a day even if the agents make their decisions continuously.[2]

In this paper, we try to resolve this question, and at the same time to clarify the dimensionality issue, by proposing an alternative way of measuring the time-to-build parameter in terms of period length units.[3] This is done in Section 2, whereas we study, in Section 3, how this new measure may lead to differences in the qualitative and quantitative dynamic behavior of the economy when we compare the continuous- with the discrete-time case. The main qualitative difference emerges when the time to build is equal to one period length; in fact, the former shows damping output fluctuations in front of monotonic convergence of the latter. On the other hand, the two dynamics are very similar as soon as the time-to-build parameter is an integer multiple of period length h by a factor greater than one.

Interestingly enough, we show that the discrete-time model changes its stability properties when its continuous counterpart has a Hopf bifurcation.[4]

Finally, at the end of Section 3, we also show that the amplitude of the damping fluctuations in output can be significantly larger in the continuous-time version than in its discrete-time counterpart. This last result is one more step in understanding the importance of the time-to-build assumption as a source of output fluctuations [e.g., Wen (1998)].

2. DESCRIPTION OF THE ECONOMY

2.1. Calendar Time, Market Period and Time-to-Build

Let us consider an infinite-horizon economy where the investment and consumption decisions of a representative agent are taken along a discrete sequence of dates,

$$t, t+h, t+2h, t+3h, \ldots = \{t+sh\}_{s\in\mathbf{N}},$$

where $t \in \mathbf{R}$ is the fixed initial time and the *period length,* h, represents the minimum time interval separating two different and not simultaneous economic decisions. We assume h to be an integer submultiple of a chosen calendar time unit $\mathcal{T}$; that is,

$$h = \frac{1}{m} \qquad \text{in } \mathcal{T} \text{ units},$$

with $m \in \mathbf{N}$ an integer.

Now we introduce time to build as a basic feature of the economy, and admit any investment to yield new capital after a fixed time delay, whose length is equal to J when measured in calendar time units. By assuming the time-to-build delay, J, to be a multiple of the period length h, we may write

$$\alpha \cdot h = J,$$

with $\alpha \in \mathbf{N}$ an integer number. Combining the two expressions, we may easily observe that

$$J = \frac{\alpha}{m} \qquad \text{in } \mathcal{T} \text{ units},$$

with $\alpha/m \in \mathbf{Q}$ a rational number. We remark that the integer parameter α, whose value represents the measure of J in terms of period units h, will be particularly relevant later when we will show that the dynamics describing the discrete-time version of our model adds up to a difference-equation system of order 2α.

As an example, choose the calendar time unit $\mathcal{T} = 1$ year, the period length h spanning two months, and a time-to-build value J of two and one-half years. With reference to the symbols above one easily computes $h = \frac{1}{6}\mathcal{T}$, $m = 6$, $J = 15h = 15/6\mathcal{T} = 2.5\mathcal{T}$.

In this context, the decisions' frequency increases as the period length h decreases. As h becomes smaller and smaller and converges to zero, we eventually

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arrive to a limit representation of the model, where the decisions are taken continuously. We call this limit case the *continuous-time version* of the model. It is also worth noting that, for any given J, the integer ratio $\alpha = J/h$ will diverge to infinity, because any investment will yield new capital after an infinite number of periods, whenever their length h tends to zero.[5]

2.2. From Discrete to Continuous Time

The capital accumulation equation of a neoclassical growth model with time to build and with a constant-returns-to-scale Cobb–Douglas production function is

$$K_{t+J+sh} = A_h K_{t+sh}^a L_{t+sh}^{1-a} + (1-\delta_h) K_{t+J+sh-h} - C_{t+sh}, \tag{1}$$

where the depreciation rate of capital has been rescaled, as well as the technology parameter, the latter to capture the production Y_{t+sh} within the interval $(t+sh, t+(s+1)h)$. The dimension of the equation (1) is mJ, which corresponds to the number of economic decisions that can be taken between t and $t+J$.

To complete the description of the economy, we also assumed, as in Hansen (1985), an instantaneous utility function in consumption and labor that allows for indivisible labor supply. After the intertemporal preference discount factor is rescaled, the social planner problem for this economy can be written as

$$\max \sum_{s=0}^{\infty} \beta_h^s (\log C_{t+sh} - BL_{t+sh}), \tag{2}$$

subject to the capital accumulation equation (1), given the (constant) initial conditions of capital

$$K_i = \bar{K}, \qquad \text{with} \qquad i = t, t+h, t+2h, \ldots, t+\alpha h = t+J.$$

From the Lagrangian, the first-order conditions with respect to C_{t+sh}, L_{t+sh}, and K_{t+sh+J} are

$$\beta_h^s \frac{1}{C_{t+sh}} = \mu_{t+sh}, \tag{3}$$

$$\beta_h^s B = \mu_{t+sh}(1-a) A_h K_{t+sh}^a L_{t+sh}^{-a}, \tag{4}$$

$$\mu_{t+sh} = \mu_{t+sh+J}\, a A_h K_{t+sh}^{a-1} L_{t+sh}^{1-a} + \mu_{t+sh+h}(1-\delta_h), \tag{5}$$

where μ_{t+sh} is the Lagrangian multiplier. These first-order conditions hold for any $s \in \mathbf{N}$ and any $t \in \mathbf{R}^+ \cup \{0\}$. Moreover, the following intratemporal substitution condition between consumption and leisure and the Euler equation can be derived

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easily from (3), (4), and (5):

$$BC_{t+sh} = (1-a)A_h K_{t+sh}^{a} L_{t+sh}^{-a}, \tag{6}$$

$$\frac{C_{t+sh+h}}{C_{t+sh}} = \beta_h \left[\beta_h^{\frac{J}{h}-1} \frac{C_{t+sh+h}}{C_{t+J+sh}} a A_h K_{t+sh}^{a-1} L_{t+sh}^{1-a} + 1 - \delta_h \right]. \tag{7}$$

It is worth noting that the dimension of this discrete-time model is always finite and equal to 2α.

DEFINITION 1. *An optimal equilibrium path of the economy is any sequence* $\{C_{t+sh}\}_{s=1}^{\infty}$, $\{L_{t+sh}\}_{s=1}^{\infty}$, *and* $\{K_{t+sh+J}\}_{s=1}^{\infty}$ *that satisfies equations (1), (6), and (7) plus the initial history of capital and the transversality condition* $\lim_{t+sh\to\infty} \mu_{t+sh} K_{t+sh} = 0$ *for any choice of* $s \in \mathbf{N}$ *and any* $t \in \mathbf{R}^{+} \cup \{0\}$.

Finally, in order to deduce a continuous-time counterpart from the previous discrete-time dynamical model, we study equations (1), (6), and (7) as the parameter h shrinks to zero.

PROPOSITION 1. *Assume the flow variables to be constant within a period and a linear rescaling of the parameters. Then the continuous-time counterpart of equations (1), (6), and (7) is*

$$\dot{K}(t+J) = AK(t)^{a} L(t)^{1-a} - \delta K(t+J) - C(t), \tag{8}$$

$$BC(t) = (1-a)AK(t)^{a} L(t)^{-a}, \tag{9}$$

$$\frac{\dot{C}(t)}{C(t)} = \left[\frac{C(t)}{C(t+J)} e^{-\rho J} a AK(t)^{a-1} L(t)^{1-a} - \delta - \rho \right]. \tag{10}$$

Proof. See the Appendix. ■

Remark 1. The procedure used to move to continuous time is theoretically consistent, because equation (10) coincides with the Euler equation we would have obtained by

$$\max \int_0^{\infty} [\log C(t) - BL(t)] e^{-\rho t} dt,$$

subject to the capital accumulation equation (8).

Remark 2. The dimension of the model in continuous time is always infinite, as also appears clear from equations (8), (9), and (10), which form a system of mixed functional differential equations, whereas the dimension of the same model in discrete time is always finite and equal to 2α. This difference in dimensionality is not puzzling any more and is consistent with the procedure used to move from

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discrete to continuous time. In fact, for a given J, the dimension of the discrete-time model is $2\alpha = 2\frac{J}{h} \to +\infty$, which goes to infinity as $h \to 0$.

Remark 3. All the results obtained in this section can easily be extended to a more general framework with a neoclassical utility function and a neoclassical production function.

Remark 4. The model setup, formed by expressions (1) and (2), contains all the specifications of time to build generally used in the literature under the assumptions of Proposition 1; in fact,

(a) If $h = 1$, the model describes an economy with gestation lags in production as in Kydland and Prescott (1982);
(b) If $h \to 0$, the model describes an economy with gestation lags as defined by Kalecki (1935) and also recently used in growth models [e.g., Asea and Zak (1999) and Bambi et al. (2008)].

3. TRANSITIONAL DYNAMICS

The transitional dynamics of the discrete-time model is studied under the assumptions of Proposition 1. Our local analysis starts with a linearization around the unique steady state (C^*, K^*) with

$$C^* = \frac{A(1-a)}{B}\left[\frac{a\beta_h^{\frac{J}{h}}(1+\rho h)}{\rho+\delta}\right]^{\frac{a}{1-a}} \quad \text{and}$$

$$K^* = \frac{\left[a\beta_h^{\frac{J}{h}}(1+\rho h)\right]C^*}{A(\rho+\delta) - \delta\left[a\beta_h^{\frac{J}{h}}(1+\rho h)\right]}$$

of the system of equations (1), (6), and (7). After some boring algebra, we obtain

$$\frac{\tilde{k}_{t+sh} - \tilde{k}_{t+sh-h}}{h} = \frac{\delta+\rho}{a}\beta_h^{1-\frac{J}{h}}\tilde{k}_{t+sh-J} - \delta\tilde{k}_{t+sh-h} + \left[\frac{\rho+\delta}{a^2}\beta_h^{-\frac{J}{h}} - \delta\right]\tilde{c}_{t+sh-J}, \tag{11}$$

$$\frac{\tilde{c}_{t+sh+h} - \tilde{c}_{t+sh}}{h} = \frac{\rho+\delta}{1-\delta h}\tilde{c}_{t+sh} - \frac{\rho+\delta}{a(1-\delta h)}\tilde{c}_{t+sh+J}, \tag{12}$$

where $\tilde{x}$ indicates the deviation of the variable $x = \log X$ with $X = C,\ K$ from its steady state value x^*. The characteristic equation associated with this system is the product of the characteristic equation of the homogeneous part of (11)

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and (12),

$$\underbrace{\left[\frac{\varpi^{\alpha}-\varpi^{\alpha-1}}{h}-\frac{\delta+\rho}{a}\beta_h^{1-\alpha}+\delta\varpi^{\alpha-1}\right]}_{=\Delta_k^h(\varpi)}$$

$$\cdot\underbrace{\left[\frac{\varpi-1}{h}+\frac{\rho+\delta}{a(1-\delta h)}\varpi^{\alpha}-\frac{\rho+\delta}{1-\delta h}\varpi\right]}_{=\Delta_c^h(\varpi)}=0, \qquad \textbf{(13)}$$

where ϖ indicates a generic root of the polynomial $\Delta_k^h(\varpi)\cdot\Delta_c^h(\varpi)$ whose order is $2\alpha = 2mJ$.[6] If half of the roots of this polynomial are outside the unit circle, then the Blanchard–Kahn [Blanchard and Kahn (1980)] condition for a unique equilibrium path is respected. However this condition can be verified only numerically when the polynomial degree is sufficiently high.[7] For example, when $h = 1$ quarter and the other parameters' values are chosen as in Kydland and Prescott's quarterly calibration ($a = 0.36$, $\delta = 0.025$, and $\rho = 0.01$ to yield a return on capital of 1.625% when $J = 4$), we find, for an initial 1% deviation of initial capital from the steady state, saddlepath stability with consumption converging monotonically while capital converges by damping fluctuations.

In contrast, the local dynamics of the economy can be *analytically* derived in continuous time. By linearizing the system (8), (9), and (10) around the steady state, we obtain

$$\dot{\tilde{k}}(t)=\frac{\delta+\rho}{a}e^{\rho J}\tilde{k}(t-J)-\delta\tilde{k}(t)-\left[\frac{\delta+\rho}{a^2}e^{\rho J}-\delta\right]\tilde{c}(t-J), \qquad \textbf{(14)}$$

$$\dot{\tilde{c}}(t)=(\delta+\rho)\tilde{c}(t)-\frac{\delta+\rho}{a}\tilde{c}(t+J), \qquad \textbf{(15)}$$

whose characteristic equation is

$$\underbrace{\left[\lambda-\frac{\delta+\rho}{a}e^{\rho J}e^{-\lambda J}+\delta\right]}_{=\Delta_k(\lambda)}\cdot\underbrace{\left[z+\frac{\rho+\delta}{a}e^{zJ}-(\rho+\delta)\right]}_{=\Delta_c(z)}=0, \qquad \textbf{(16)}$$

where λ and z indicate generic roots of the transcendental equations $\Delta_k(\lambda)$ and $\Delta_c(z)$, respectively.[8] It is worth noting that the system (14), (15) and its characteristic equation, (16), are the continuous-time counterparts of the system (11), (12) and its characteristic equation, (13), which might well be obtained alternatively by computing the limit of the latter as $h \to 0$. The change in dimensionality is now clear because the 2α-degree polynomial becomes the transcendental equation (16), having infinite roots as $h \to 0$.

From an application of the D-subdivision method [see, for example, Bambi (2008)], we can learn a lot about the spectrum of eigenvalues of $\Delta_k(\lambda) = 0$ and $\Delta_c(z) = 0$. In particular, under the assumption that $J \in (0, \bar{J})$, with $\bar{J}$ the unique

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real root of $J = \frac{3a\pi}{2(\delta+\rho)} e^{-\rho J}$, the former has one positive real root, $\lambda_{\hat{v}}$, and infinitely many complex conjugate roots, $\lambda_v = x_v + i y_v$, with negative real part, whereas the latter has one negative real root, $\hat{z}$, and infinitely many complex conjugate roots with positive real part. Taking into account these results, the following proposition can be proved:

PROPOSITION 2. *The equilibrium of the economy is locally determinate when $J \in (0, \bar{J})$. Moreover, the optimal path of consumption converges monotonically, whereas the optimal path of capital converges by damping fluctuations. Their functional forms are, respectively,*

$$\tilde{c}(t) = c_0 e^{\hat{z}t} \qquad with \;\; c_0 = (\hat{z} - \lambda_{\hat{v}}) h'(\lambda_{\hat{v}}) n_{\hat{v}}, \tag{17}$$

$$\tilde{k}(t) = \left(n_{\hat{v}} - 2 \sum_{v=1}^{+\infty} c_0 \Phi_{1,v} \right) e^{\hat{z}t} + 2 \sum_{v=1}^{+\infty} [(\varsigma_v + c_0 \Phi_{1,v}) \cos y_v t - (\omega_v + c_0 \Phi_{2,v}) \sin y_v t] e^{x_v t}, \tag{18}$$

where $n_v = \varsigma_v + i\omega_v$ indicates the eigenvector associated with the eigenvalue λ_v with $v \neq \hat{v}$, whereas $\Phi_{1,v}$ and $\Phi_{2,v}$ are constants. Moreover, persistent endogenous fluctuations raise when $J = \bar{J}$ through a Hopf bifurcation.[9]

Proof. See the Appendix. ■

Comparing these results with those in the discrete framework with $m = 1$, we have also verified numerically that the change in stability in discrete time arrives at J^0, with $J^0 =]\bar{J}[$, where $]x[$ indicates the smallest integer not less than or equal to x.[10] It is also worth noting that for $J = 2$ the discrete-time model always has damping "improper oscillations" in capital because of the presence of two negative and less than one real roots,[11] and for $J = 1$ monotonic convergence. This discrepancy with the continuous-time case is due to the fact that the fluctuations arrive at a frequency less than 1 and thus cannot be captured by the discrete-time model when $m = 1$. More generally, monotonic convergence in discrete time emerges as soon as $J = h$. These results on stability have been verified for different parameterizations and are summarized in Table 1.

Finally, we have studied whether a reduction of the number of decisions, m, within a calendar-time unit, assumed to be a quarter, may positively affect the amplitude of the output fluctuations under the same calibration of the parameters and initial condition of capital as before. To this purpose, we have first calculated the output deviations from the steady state value for different choices of m at some specific calender dates x. More precisely, we have computed $\tilde{y}_m(x)$ with $m = 1, 20, 100, 300$ and $x = J, J+1, J+2, J+3, 2J$ with $J = 4$ quarters. The last choice has been made to focus on the time interval where the highest peak takes place. Thus we have found the percentage deviations $D(m, x) = \frac{\tilde{y}_m(x) - \tilde{y}_1(x)}{y_1(x)} \cdot 100$

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TABLE 1. Taxonomy of the dynamics

	Discrete time (period length h)	Continuous time ($J \neq h \to 0$)	Continuous time ($J = h \to 0$)
	Local stability		
No output fluctuations	$J = h$	never	always
Damping output fluctuations	$h < J < J^0$	$0 < J < \bar{J}$	never
Persistent output fluctuations[a]	$J = J^0 =]\bar{J}[$	$J = \bar{J}$	never
	Local instability		
Explosive output fluctuations	$J > J^0 = h$	$J > \bar{J}$	never

[a]The condition indicates the change from local stability to local instability, which in continuous time arrives through a Hopf bifurcation.

for the values of m and x previously indicated and reported them in Figure 1. What emerges is that increasing the number of operations within a calendar time unit leads to higher output volatility. More precisely, the amplitude percentage difference may be up to 6% higher when we compare the discrete-time model with $m = 1$ with the continuous-time model with $m = \infty$.[12] Intuitively, this result emerges because the agents can increasingly smooth their consumption path as the number of decisions within a calendar-time unit increases, and by their doing so, the output will result more and more volatile.

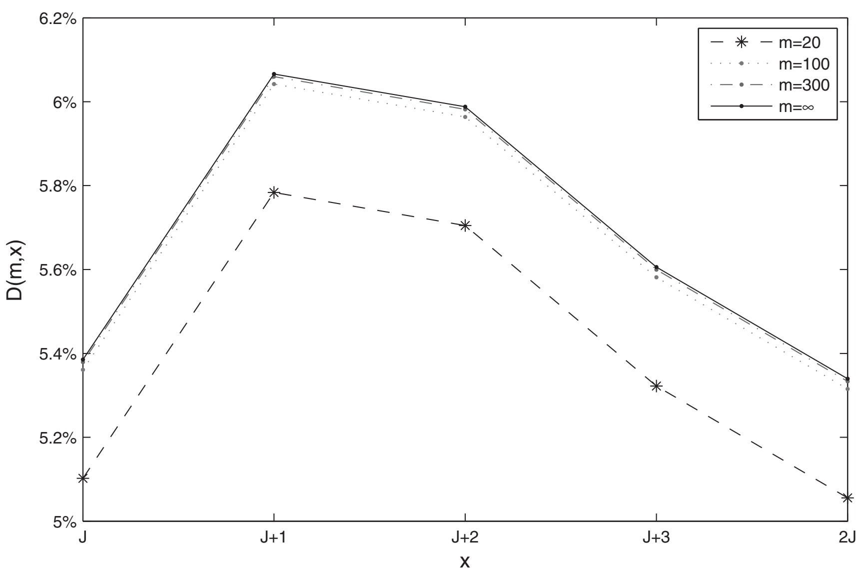

FIGURE 1. Number of economic decisions, m, per unit of calendar time and its effect at different dates, x, on the output volatility, $D(m, x)$.

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NOTES

1. To some extent, we share with these authors the distinction between calendar time and frequency of decision making, but the "objective" of our research is substantially different. In particular, it refers to models with time to build, which are explicitly excluded by their analysis.

2. See also Licandro and Puch (2006) on this point.

3. A similar measurement issue was pointed out by Burmeister and Turnovsky (1976) in their attempt to find the continuous-time version of a discrete-time model with adaptive expectation.

4. The proposed procedure for moving correctly from discrete to continuous time can easilybe extended to vintage capital models [e.g., Boucekkine et al. (1997, 2005)].

5. Regarding this convergence–divergence procedure, we stress that it will not depend on the specific form given to the converging sequence $\{h_k\}$.

6. The characteristic equation (13) was obtained by searching for a solution of the form $\tilde{c}_{t+sh} = \tilde{c}_0 \varpi^s$ and $\tilde{k}_{t+sh} = \varpi^{s+\alpha} \tilde{k}_{-\alpha}$.

7. For example, the geometrical method proposed by Grandmont et al. (1998) for investigating the stability of a dynamical system can be applied in our case only when $mJ \leq 1$.

8. The distinction between λ and z is not essential but it will become useful in the proof of Proposition 2 in order to find the equilibrium path of consumption and capital.

9. The Hopf bifurcation emerges because at $J = \bar{J}$ the real part of two conjugate complex roots changes its sign.

10. See also Bosi and Ragot (2010) on the preservation of stability properties and bifurcations in the case of discretization of continuous-time models.

11. The term "improper oscillations" is also used in Gandolfo (1996, p. 18) to distinguish this type of oscillations from trigonometric oscillations.

12. The continuous-time capital path (18) is approximated by computing the first 2,000 roots around the imaginary axis using the Lambert function [see for more details Jarlebring and Damm (2007)].

13. It is worth noting that the parameters δ, ρ, and β and their period length correspondents δ_h, ρ_h, and β_h are related by the conditions $\delta_h = 1-(1-\delta)^h$, $\rho_h = (1+\rho)^h - 1$, and $\beta_h = \beta^h$. Hence, $h\delta$ and $h\rho$ are, respectively, the first-order approximations of the functions $\delta_h(\delta)$ and $\rho_h(\rho)$ in a neighborhood of zero, for any given h.

REFERENCES

Anagnostopoulos, A. and C. Giannitsarou (2013) Indeterminacy and period length under balanced budget rules. *Macroeconomic Dynamics* 17, 898–919.

Asea, P. and P. Zak (1999) Time-to-build and cycles. *Journal of Economic Dynamics and Control* 23, 1155–1175.

Bambi, M. (2008) Endogenous growth and time-to-build: The AK case. *Journal of Economic Dynamics and Control* 32, 1015–1040.

Bambi, M., G. Fabbri, and F. Gozzi (2012) Optimal policy and consumption smoothing effects in the time-to-build AK model. *Economic Theory* 50(3), 635–669.

Bellman, R. and K. Cooke (1963) *Differential–Difference Equations*. New York: Academic Press.

Böhm-Bawerk, E. (1889) Positive theory of capital. In *Capital and Interest*. Libertarian Press.

Bosi, S. and L. Ragot (2010) Time, Bifurcations and Economic Applications. Thema working paper 2010–01, Université de Cergy, Pontoise, France.

Boucekkine, R., M. Germain, and O. Licandro (1997) Replacement echoes in the vintage capital growth model. *Journal of Economic Theory* 74, 333–348.

Boucekkine, R., O. Licandro, L. Puch, and F. del Rio (2005) Vintage capital and the dynamics of the AK model. *Journal of Economic Theory* 120, 39–72.

Blanchard, O.J. and C.M. Kahn (1980) The solution of linear difference models under rational expectations. *Econometrica* 48(5), 1305–1311.

Burmeister, E. and S. Turnovsky (1976) The specification of adaptive expectations in continuous-time dynamic economic models. *Econometrica* 44(5), 879–905.

Foley, D.K. (1975) On two specifications of asset equilibrium in macroeconomic models. *Journal of Political Economy* 81(11), 305–324.

Gandolfo, G. (1996) *Economic Dynamics*. Springer.

Grandmont, J.-M., P. Pintus, and R. de Vilder (1998) Capital–labour substitution and competitive nonlinear endogenous business cycles. *Journal of Economic Theory* 80, 14–59.

Hansen, G. (1985), Indivisible labor and the business cycle. *Journal of Monetary Economics* 16, 309–325.

Jarlebring, E. and T. Damm (2007) The Lambert *W* function and the spectrum of some multidimensional time-delay system. *Automatica* 43, 2124–2128.

Kalecki, M. (1935) A macroeconomic theory of the business cycle. *Econometrica* 3, 327–344.

Kydland, F.E. and E.C. Prescott (1982) Time-to-build and aggregate fluctuations. *Econometrica* 50(6), 1345–1370.

Licandro, O. and L. Puch (2006), Is Discrete-Time a Good Representation of Continuous-Time? Mimeo, EUI.

May, J. (1970) Period analysis and continuous analysis in Patinkin's macroeconomic model. *Journal of Economic Theory* 2, 1–9.

Turnovsky, S.J. (1977) On the formulation of continuous-time macroeconomic models with asset accumulation. *International Economic Review* 18(1), 1–28.

Wen, Yi (1998) Investment cycles. *Journal of Economic Dynamics and Control* 7, 1139–1165.

Winkler, R. (2011) A note on the optimal control of stocks accumulating with a delay. *Macroeconomic Dynamics* 15, 565–578.

APPENDIX

Proof of Proposition 1. First, the flow variable consumption is rescaled by assuming a constant rate of change through each unit of period length h:

$$C_{t+sh} = \int_{t+sh}^{t+(s+1)h} C(t+sh, v)dv = hC(t+sh).$$

Following the terminology used in Turnovsky (1977), $C(t+sh, v)$ is the planned (finite) consumption rate at time instant v, chosen on date $t+sh$, over the interval $[t+sh, t+(s+1)h]$; $C(t+sh)$ represents its corresponding average value; and C_{t+sh} indicates the cumulated consumption of the representative agent over the interval $[t+sh, t+(s+1)h]$.

Second, the instantaneous rate of production is assumed to be constant over any calendar time unitary interval. Hence the technological coefficients A_h and A—which appear in the functions measuring, respectively, the economy's cumulated output along any period length h and the economy's cumulated output along a calendar unitary interval—will be related by the simple linear condition $A_h = hA$.

Finally, a *linear* rescaling of the other parameters leads to $\delta_h = h\delta$ and $\beta_h = \frac{1}{1+\rho h}$.[13] Once equations (1), (6), and (7) are rewritten, taking these assumptions into account, their continuous counterparts, namely (8), (9), and (10), can be easily obtained as h shrinks to zero. ■

Proof of Proposition 2. The spectrum of roots of the system is given by the union of the spectrum of roots of $h_k(\lambda)$ and $h_c(z)$. Because the system is recursively solvable, it is possible to write any solution of (15) using the finite Laplace transform method developed in Belmann and Cooke (1963, pp. 197–205):

$$\tilde{c}(t) = \lim_{\ell\to\infty} \sum_{z_m \in C_\ell} a_m e^{z_m t}, \tag{A.1}$$

where the contour C_ℓ, with $\ell = 1, 2, \ldots$, cuts the negative and positive imaginary axis to the points $-y_\ell$ and y_ℓ, respectively. Moreover, a_m is the residue of the root z_m and depends on the boundary (initial) condition of consumption. The general convergent solution of consumption can be obtained by ruling out all the roots with positive real part, imposing $a_m = 0\ \forall m \neq \hat{m}$, and then rewriting (A.1) as

$$\tilde{c}(t) = a_{\hat{m}} e^{\hat{z}t}, \tag{A.2}$$

with $\hat{z}$ the negative real root of $h_c(z)$. On the other hand, the spectrum of $h_k(\lambda)$ is characterized by one positive real root and all the other complex conjugates, with negative real part when $J \in \left(0, \bar{J}\right)$. The general solution of capital [Belmann and Cooke (1963); Bambi (2008)] is

$$\tilde{k}(t) = \sum_{v}^{\infty} n_v e^{\lambda_v t} - \int_0^t c(s) \sum_{v}^{\infty} \frac{e^{\lambda_v (t-s)}}{h'_{\lambda_v}} ds \tag{A.3}$$

$$= \sum_{v}^{\infty} n_v e^{\lambda_v t} - \int_0^t a_{\hat{m}} e^{\hat{z}s} \sum_{v}^{\infty} \frac{e^{\lambda_v (t-s)}}{h'_{\lambda_v}} ds \tag{A.4}$$

$$= \sum_{v}^{\infty} (n_v + P_v) e^{\lambda_v t} - \sum_{v}^{\infty} P_v e^{\hat{z}t}, \tag{A.5}$$

where n_v is the residue of the root λ_v, and $P_v = \frac{a_{\hat{m}}}{(\lambda_v - \hat{z}) h'(\lambda_v)}$. Moreover, capital convergence requires that $a_{\hat{m}} = c_0$, with

$$c_0 = (\hat{z} - \lambda_{\hat{v}}) h'(\lambda_{\hat{v}}) n_{\hat{v}}, \tag{A.6}$$

which is the unique condition to rule out the divergent solution induced by $\lambda_{\hat{v}}$ and to make the transversality conditions hold. Then the optimal path of consumption is monotonic.

On the other hand, (18) can be obtained by applying several properties of complex numbers to (A.5). After $a_{\hat{m}} = c_0$ is imposed, the solution for capital is written

$$\tilde{k}(t) = \sum_{v \neq \hat{v}}^{\infty} (n_v + P_v) e^{\lambda_v t} - \sum_{v}^{\infty} P_v e^{\hat{z}t}, \tag{A.7}$$

where, as shown in Bellman and Cooke (1963), the residues are

$$n_v = \frac{\bar{k} + \bar{k}(\lambda_v + \delta) \int_{-J}^{0} e^{-\lambda_v s} ds}{1 + J \Lambda e^{-\lambda_v J}} = \frac{\bar{k}\left[1 + (\lambda_v + \delta)\left(-\frac{1}{\lambda_v}\right)(1 - e^{\lambda_v J})\right]}{1 + J \Lambda e^{-\lambda_v J}}, \tag{A.8}$$

with $\tilde{k}(t) = \bar{k} \in \mathbf{R}^+$ in $t \in [-J, 0]$ and $\Lambda = \frac{\delta + \rho}{a} e^{\rho J}$.

Now letting $\varsigma_v = \mathrm{Re}(n_v)$, $\omega_v = \mathrm{Im}(n_v)$, $\lambda = x + iy$, and $\bar{\lambda}$ its conjugate, we rewrite

$$\sum_{v} P_v = -n_{\hat{v}} + c_0 \sum_{v \neq \hat{v}}^{\infty} \left(\frac{1}{(\tilde{z} - \lambda_v) h'(\lambda_v)} + \frac{1}{(\tilde{z} - \bar{\lambda}_v) h'(\bar{\lambda}_v)} \right), \tag{A.9}$$

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which, after some algebra and taking into account the relation $\bar{\lambda}e^{-\bar{\lambda}J}+\lambda e^{-\lambda J} = 2(x\cos yJ + y\sin yJ)e^{-xJ}$, becomes

$$\sum_{v} P_v = -n_{\hat{v}} + 2c_0 \sum_{v\neq\hat{v}}^{\infty} \Phi_{1,v}, \qquad \textbf{(A.10)}$$

with

$$\Phi_{1,v} = \frac{(\hat{z}-x_v)(1+J\Gamma e^{-x_v J}\cos y_v J) - J\Gamma y_v e^{-x_v J}\sin y_v J}{(\hat{z}^2 - 2\tilde{z}x_v + x_v^2 + y_v^2)(1+J^2\Gamma^2 e^{-2Jx_v} + 2J\Gamma e^{-x_v J}\cos y_v J)}. \qquad \textbf{(A.11)}$$

Now let us study the second piece of the solution of capital, namely

$$\sum_{v\neq\hat{v}} (n_v + P_v)e^{\lambda_v t} = \sum_{v\neq\hat{v}} \left(n_v e^{\lambda_v t} + \bar{n}_v e^{\bar{\lambda}_v t}\right) + \sum_{v\neq\hat{v}} \left(P_v e^{\lambda_v t} + \bar{P}_v e^{\bar{\lambda}_v t}\right) \qquad \textbf{(A.12)}$$

$$= 2\sum_{v\neq\hat{v}} (\varsigma_v \cos y_v t + \omega_v \sin y_v t)\, e^{-x_v t} + \sum_{v\neq\hat{v}} \left(P_v e^{\lambda_v t} + \bar{P}_v e^{\bar{\lambda}_v t}\right), \qquad \textbf{(A.13)}$$

which, after some algebra and taking into account the trigonometric relations $\cos(a+b) = \cos a\cos b - \sin a \sin b$ and $\sin(a+b) = \sin a\cos b + \cos a \sin b$, becomes

$$\sum_{v\neq\hat{v}} (n_v + P_v)e^{\lambda_v t} = 2\sum_{v\neq\hat{v}} [(\varsigma_v + c_0\Phi_{1,v})\cos y_v t - (\omega_v + c_0\Phi_{2,v})\sin y_v t]e^{x_v t}, \qquad \textbf{(A.14)}$$

where

$$\Phi_{2,v} = \frac{y_v + J\Gamma e^{-x_v J}\,[(\hat{z}-x_v)\sin y_v J + y_v \cos y_v J]}{(\hat{z}^2 - 2\tilde{z}x_v + x_v^2 + y_v^2)(1+J^2\Gamma^2 e^{-2Jx_v} + 2J\Gamma e^{-x_v J}\cos y_v J)}. \qquad \textbf{(A.15)}$$

■

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