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CIRCULARITY IN SOUNDNESS AND COMPLETENESS

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The strings, my lord, are false.

Shakespeare, The Tragedy of Julius Caesar

Abstract. We raise an issue of circularity in the argument for the completeness of first-order logic. An analysis of the problem sheds light on the development of mathematics, and suggests other possible directions for foundational research.

§1. Introduction. In a recent discussion I took part in with a number of philosophers engaged in investigations into the philosophy of mathematics over dinner one evening, the question arose whether the argument for the Soundness Theorem of mathematical logic is circular or not.[1] This was fairly rapidly dispatched by a number of people present, but I was surprised by the reactions I received when I expressed concerns that the Completeness Theorem might have circularity issues. My colleagues obviously thought this a much weaker position to hold. This article attempts to express those doubts and why they might matter.

§2. The Soundness Theorem. The notation $\Sigma \vdash \sigma$ will be used to express the assertion that there is a formal proof of the statement σ (typically a string of symbols from a finite alphabet) from assumptions in Σ (typically a set of strings from the same finite alphabet), where the proof obeys some formally defined and precise rules for classical first-order logic. The conclusion $\Sigma \vDash \sigma$ expresses that every interpretation making Σ true also makes σ true, where usually a particular restricted family of mathematical interpretations is under consideration, and 'true' means in terms of Tarski's definition of Truth. The Soundness Theorem is the familiar theorem stating

$$\Sigma \vdash \sigma \quad \text{implies} \quad \Sigma \vDash \sigma$$

for all such Σ, σ.

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[1]Throughout this article I am interested in classical first-order logic, and I use Soundness and Completeness, capitalized, to denote the standard theorems for this logic. In contrast, soundness and completeness, without capitals, will denote the desirable properties of formal systems that first-order logic or any other system of logic may have. It is possible that similar points I make might be valid for other logics such as intuitionistic, though this has not been investigated.

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It seems that the argument for this is circular, since knowledge of semantics and whether or not $\Sigma \vDash \sigma$ must be founded in some system of argumentation, which, if it is the system of $\vdash$ itself, quickly leads to infinite regress:

$$\begin{array}{rcl} \Sigma \vdash \sigma & \text{implies} & \vdash `\Sigma \vDash \sigma' \\ \vdash `\Sigma \vDash \sigma' & \text{implies} & \vdash `\vDash `\Sigma \vDash \sigma'' \\ \vdash `\vDash `\Sigma \vDash \sigma'' & \text{implies} & \vdash `\vDash `\vDash `\Sigma \vDash \sigma''' \end{array}$$

and so on. This is not unlike the point in Lewis Carroll's *What the Tortoise Said to Achilles* [2] and whilst not incorrect is not usually helpful.

The main issue in this article is with Completeness, but the matter of soundness of mathematical theories is one that has received a considerable amount of attention. Taken in the form given above, the Soundness Theorem requires a metatheory that is able to express details of a formal system and its notion of proof, and as usual we must have some confidence that the theory described by the metatheory is the one we want to study. The proof of Soundness goes by induction on the length of proof, so this must be available. Most importantly we also need a (semantic) interpretation of the theory and a notion of semantics for it. In the case when for example our theory Σ is first-order Peano Arithmetic (PA) the metatheory must have access to a model or interpretation of PA and access to a notion of truth over this. It might be, but need not be, the standard natural numbers $\mathbb{N}$ with the usual addition and multiplication operations.

As just described, it is preferable to regard the Soundness Theorem as a 'relative consistency result' and see it as stating that provability in the formal system is sound relative to the system required to set up the notion of semantic entailment, $\vDash$. A certain amount of induction and recursion is required for this argument: most obviously enough induction and recursion must be available in the metatheory to carry out the syntactical operations required and to perform the induction on the length of formal derivation. Less obviously, Tarski's definition of truth is a recursive definition, where the recursion over formulas is necessary to enable a single $\vDash$ to apply to all formulas of arbitrary quantifier complexity in the language. To be sure, Tarski's definition can be given to any finite stage without induction. Thus without induction on quantifier complexity we can define truth for Σ_n formulas for any fixed n, and prove in PA the consistency of $I\Sigma_n$ (the subtheory of PA formed by restricting the induction axiom to Σ_n formulas [10]).

There is a possibility that certain applications of Soundness may indeed be circular in a different way to the Lewis Carroll type of problem. In view of the implicit induction required for Tarskian semantics, there would be a problem should Soundness be used as a means to justify a theory of induction. It would seem to be a highly worthwhile project to identify the *inductive content* (as opposed to the semantic content) of Tarski's definition. Note that both the semantic and inductive aspects of Tarski's definition are allied to the well-known increase in complexity as one looks at semantics for an increasing number of quantifier blocks—an increase in complexity that seems unavoidable and intractable (in the technical sense of the word) even when one restricts interpretations to finite domains.

In the form given above, and even with a reasonably strong metatheory accepted, it seems to me that the Soundness Theorem on its own does not give credence to the commonly held assertion that if one accepts the consistency of PA (for example) then one is entitled to accept the consistency of PA $+$ Con(PA). Even if one knows Con(PA) through knowledge of an interpretation of PA it does not follow that that interpretation is (or is known to be) ω-standard, and so it does not follow that it satisfies the arithmetized sentence Con(PA). The precise details depend on our metatheory itself.

To take this further and to infer soundness of a system such as PA or its extensions by semi-formal or informal means—and this is surely one of the main aims of such foundational studies—is more difficult still. In an excellent article, Dummett [3] discusses the issue of soundness thoroughly. He concludes that there may be an idealist (or constructivist) view by which progress can be made. An alternative but not necessarily contradictory position is that by examination and checking of a great deal of cases, and also perhaps by measurement in the physical world and grounding ones intuitions on these measurements, one might conclude (in the sense perhaps of Popper's Scientific Discovery [14]) that the best and most useful explanation is indeed the existence of an interpretation of PA satisfying the reasonable mathematical properties we expect.

Thus the notion of semantic entailment itself might be given informally or else might be given in some more formal or semi-formal sense. As it happens, mathematicians are, rightly or wrongly, typically very comfortable with a semantic notion of Truth based on mathematical structures. The process presented by the proof of the Soundness Theorem (and this process is a straightforward recursion on the length of a proof) can be mimicked in any such situation, and its validity checked by other informal means. Indeed this is often done in mathematical practice, and as such provides supporting evidence of the correctness of the theorem.

It should be added that it would seem that, from a mathematical perspective, the Soundness Theorem is not of much practical use, since the existence of a formal proof of σ from Σ may be more compelling than the manipulation of this proof into semantic arguments why σ should follow from Σ. Put another way, Soundness is actually comparatively weak as a piece of mathematics. Nevertheless, despite the apparent limitations of Soundness when seen in this way, it remains one of the most important and useful results in mathematical logic. Given statements Σ and σ it may be difficult to see why there is no formal argument of σ from Σ from the point of view of direct arguments with syntax, even though this is the subject matter of the assertion '$\Sigma \nvdash \sigma$' and the search for such an argument is the natural and direct approach. But instead, constructing an interpretation of Σ in which σ fails, and invoking the contrapositive of the Soundness Theorem is commonly the more successful approach. For example, although there are now methods of ordinal analysis of proofs that allow us to show in a direct way from arguing about proofs that PA $\nvdash$ PH, where PA is first-order Peano

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arithmetic and PH is the Paris–Harrington statement [13], the original argument for 'PA $\nvdash$ PH' was conceived in a semantic way, and the original proof was an indirect one via Soundness.[2] Thus there remains a puzzle relating to Soundness that, although it is rather weak mathematically and presents little extra information, it is in practice rather useful.

§3. The Completeness Theorem. It is my contention that the converse result, the Completeness Theorem

$$\Sigma \vDash \sigma \quad \text{implies} \quad \Sigma \vdash \sigma,$$

is much more problematic in its correct interpretation, where as before $\vdash$ is provability in normal first-order logic.

It is not a problem to set up the notions of first-order provability and semantic entailment via Tarski's definition of Truth in one's favorite metatheory—set theory, say—and prove the above result in that system. Moreover, since set theory itself is not at the outset intended to be a metamathematical theory, but rather a theory of mathematical objects, the proof of Completeness in set theory gives genuine new information. The issue is, as it was for Soundness, whether some of the familiar foundational consequences normally drawn from Completeness are reasonable. First and foremost of these is the question whether the Completeness Theorem supports the assertion,

First-order logic is complete.

By this I mean that it is complete for ordinary mathematics as is commonly taught in a typical pure mathematics department in a University—in the U.K. or U.S.A. perhaps. Mathematicians in such departments do their work with whatever means are available to get the results they need. In particular, the law of excluded middle is always assumed where needed, as are nonconstructive arguments using the Axiom of Choice. Additional assumptions, such as the continuum hypothesis (or its negation) are very rarely assumed. In some sense there is (in mathematics departments) a consensus that mathematics is precisely what is described by ZFC.

My focus is not the usual one of concern that first-order logic is too strong (and some other logic, possibly a constructive logic based on intuitionistic logic should be preferred) but rather on whether first-order logic is *strong enough* for practical mathematics, and whether there are not in fact other new logical principles that do not follow from first order logic that should be accepted.

The argument for first-order logic in 'ordinary' mathematics is rather compelling. If I am a mathematician working in, say, group theory, interested in properties of groups that can be expressed in the language of first order logic, I am comfortable with the idea that I need to look for theorems (with

[2]It takes us off topic slightly, but it might be argued that the statement PH itself arose from semantic considerations, so that the Paris–Harrington Theorem may not have even been discovered at the time except by semantic means.

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proofs of some kind) or counter-examples. As mentioned, most mathematicians are perfectly happy with the idea of counter-examples, i.e., structures such as groups and a notion of semantics attached to them, corresponding to Tarski's definition of Truth. The Completeness Theorem states that if I cannot find a counter-example then there is a proof of the relevant theorem, not just in some general informal mathematical terms, but in the formal first-order language following the formal rules. If there are no such proofs it even provides a means for constructing a counter-example. Thus for most mathematicians one can reduce one's activity to finding counter-examples or finding first-order proofs.[3]

It is this conclusion that is normally drawn from Completeness, and it is precisely this argument that I suggest is circular. The issue is important because, by the reduction to first order logic just provided, mathematicians in other disciplines (such as in set theory, or some areas of real analysis where there are problematic questions such as the continuum problem, or more generally what subsets of the reals are available, and whether they are all measurable, etc.) need only look for *first-order* principles that settle these questions, and there is the expectation that this is a helpful reduction. But it may be that this is based on erroneous reasoning.

My counter-argument here is based on the fact that the Completeness Theorem is not proved in the form given above, but rather in its contrapositive form,

$$\Sigma \nvdash \perp \quad \text{implies} \quad \Sigma \nvDash \perp.$$

In other words, one proves (using set theory or whatever foundational system one has as the metatheory) that if a set of sentences Σ is *consistent*, i.e. $\Sigma \nvdash \perp$, then *there is a structure or interpretation of* Σ, that is $\Sigma \nvDash \perp$ holds.[4]

There are two problems. The first is a mathematical one that complicates the issue but must be given careful examination, and is that the interpretation of Σ is not given constructively but typically requires a weak form of the Axiom of Choice (more precisely, the Boolean Prime Ideal Theorem, BPIT, or a statement equivalent to this) to do its work. The second is more pertinent and is the observation that the metatheory (some flavor of set theory or arithmetic, including the BPIT) begs the question of what interpretations or structures should be available, by presenting them all in advance. If one did not know the rules for first-order logic one would be able to discover them by working through the proof of Completeness and at each step, when some formal proof is required, writing down the simplest formally Sound rule that justifies the step in question.

Thus the proof of the Completeness Theorem simply fits some logic to the (known, or implied) class of interpretations or structures given by the metatheory, and checks that the rules of that logic are adequate for this class

[3]To be sure there are important ways of 'speeding up' first-order logic to ensure that proofs are quicker to find, shorter to write down and/or clearer to understand, but the logic with these extra speed-up principles added is still equivalent to first-order logic.

[4]I am assuming of course that the target logic has standard rules for reductio ad absurdum and for the false statement $\perp$.

of interpretations. Our metatheory is typically a set theory or arithmetic with BPIT in some form, though with additional assumptions on our language and axioms Σ may be taken to be weaker than this—see below. This metatheory is of course based on first-order logic (though the first-order nature of this logic is sometimes hidden by appealing to multi-sorted languages such as that of second-order arithmetic as if some higher order logic is being employed). Our conclusion that first-order logic is complete for ordinary mathematics is then seen to be based on a circularity, that first-order logic is complete for the class of interpretations in a possibly artificially devised class based on a metatheory based on first-order logic.

To illustrate the problem, suppose a mathematician rejected the idea of uncountable structures and insisted that all countable structures should have a recursive (i.e. computable) presentation.[5] Such a mathematician would have a reasonable and self-consistent notion of semantic entailment, $\Sigma \vDash_{\mathrm{Rec}} \sigma$, meaning that σ holds in all recursively presented models of Σ. He would presumably reject ZFC as a metatheory, as it leads to unacceptable consequences such as the existence of nonrecursive interpretations, but the logic based on $\Sigma \vDash_{\mathrm{Rec}} \sigma$ is not unreasonable. For example it is consistent, i.e. $\Sigma \nvDash_{\mathrm{Rec}} \perp$ for a great number of Σ including of course the empty set, since there are recursive models available. This mathematician could accept all rules for first-order logic as being sound, since the Soundness Theorem still holds in the form $\Sigma \vdash \sigma$ implies $\Sigma \vDash_{\mathrm{Rec}} \sigma$, where $\vdash$ means for the rules of first-order logic. He would not however accept that first-order logic is complete, since for example Gödel's Second Incompleteness Theorem $\mathrm{PA} \nvdash \mathrm{Con}(\mathrm{PA})$ for PA in first-order logic can be given adequate syntactic arguments, but $\mathrm{PA} \vDash_{\mathrm{Rec}} \mathrm{Con}(\mathrm{PA})$ since no model of $\mathrm{PA} + \neg\mathrm{Con}(\mathrm{PA})$ is recursive by Tennenbaum's result [20].[6]

Recursive models have been a theme of mathematical logic since at least the 1950s, and a number of results aim at highlighting the distinctions between $\vDash$ and $\vDash_{\mathrm{Rec}}$. For a discussion that brings many of these ideas together and starts to explore the model theory of $\vDash_{\mathrm{Rec}}$ in a more systematic way, see Stolboushkin [19]. There are a number of interesting technical questions related to the entailment notion $\vDash_{\mathrm{Rec}}$ and I will return to this later.

As an even more extreme example, but one that is certainly of use in computer science, finite model theory is the model theory of finite structures with associated semantics $\vDash_{\mathrm{Fin}}$. For this theory we even have $\mathrm{PA} \vDash_{\mathrm{Fin}} \perp$. I do not hold that finite models, nor even recursive models, are the only interesting ones, and certainly I would not myself restrict the mathematical universe in such a way: the examples are given to illustrate the consistency of the view that first-order logic is not complete and thus demonstrate the

[5] I.e. should have domain $\mathbb{N}$ with all predicates and functions recursive.

[6] This argument can easily be adapted to show that, working in one of the usual metatheories such as ZFC or a fragment of second-order arithmetic, there is no recursive system of proof $\vdash_{\mathrm{Rec}}$ that is Sound and Complete for $\vDash_{\mathrm{Rec}}$. One needs only modify the proof of the Gödel Incompleteness Theorem for a Sound and recursive $\vdash_{\mathrm{Rec}}$ to show that $\mathrm{PA} \nvdash \mathrm{Con}_{\mathrm{Rec}}(\mathrm{PA})$ where $\mathrm{Con}_{\mathrm{Rec}}$ means in terms of $\vdash_{\mathrm{Rec}}$. But of course $\mathrm{PA} \vDash_{\mathrm{Rec}} \mathrm{Con}_{\mathrm{Rec}}(\mathrm{PA})$ holds for the same reason.

inadequacy of the argument purporting to show that it is. Nevertheless, the collection of recursively presented interpretations and its semantics is very natural. It might conceivably be the case that some flavor of mathematics with a different class of interpretations for our logical languages is more appropriate for scientific work in understanding our physical universe, and in particular then some stronger logical principles should be adopted. But I have no sensible suggestions in this regard.

Looking at the argument given above, what we see is that it is more accurate to see the Completeness Theorem as having presented mathematicians with a self-consistent class of examples and counter-examples, rather than to see it as justifying a system of proof. It seems highly unlikely that mathematicians before 1930 (with the possible exception of Cantor) had any clear conception of what sets or structures or interpretations might be admissible to mathematics, and therefore the more accurate view of the historical development of the subject is that first-order logic and the Completeness Theorem in particular and the set theory needed to prove it in its full generality were developed alongside each other, and together they provide mathematicians with a clear conception of what might or might not constitute a mathematical interpretation or structure. Furthermore, without a thorough investigation into the foundational issues of the axioms of set theory and the Axiom of Choice particularly, general techniques for transfinite induction and recursion, Zorn's lemma and other nonconstructive methods may never have been accepted into the mathematical mainstream.

§4. **Towards an analysis of completeness.** It is well known that the Completeness Theorem for propositional logic in finitely many propositional letters,

$$\Sigma \nvdash \perp \quad \text{implies} \quad \text{there is a valuation making each } \tau \in \Sigma \text{ true}$$

(where Σ is a finite set of propositional formulas) is provable directly by constructive means. Perhaps the most straightforward argument uses tableaux. If $\Sigma \nvdash \perp$ then there is a completed tableau from Σ in which every formula is subdivided into its atomic subformulas, and this is possible because Σ is finite and so there are only finitely many subformulas to consider in the propositional logic. Then one of the paths yields an appropriate valuation, or else the whole tableau can be converted to a proof of $\perp$ in Σ. The same argument also constructively shows

$$\text{no valuation making each } \tau \in \Sigma \text{ true makes } \sigma \text{ true} \quad \text{implies} \quad \Sigma \vdash \neg\sigma,$$

for any completed tableau for Σ, σ has all its paths closed, and this tableau can be constructively converted to a proof showing $\Sigma \vdash \neg\sigma$. Hilbert [9], Post [15], and Bernays [1] all had proofs of the completeness of propositional logic in this sense. See particularly Zach [21, p. 340] who traces Hilbert's formulation of completeness from

> 'whether the axioms suffice to prove all "facts" of the theory in question' (1905)

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to

> '[whether] we always obtain an inconsistent system of axioms by adding a formula which is so far not derivable,' (1917–18)[7]

statements that are clearly 'absolute', i.e., not relative to some particular choice of metatheory. In particular, Hilbert does not (and nor would we expect it of him in 1905) suggest that the collection of "facts" to be captured is dependent on a choice of position or metatheory from which one views those "facts".[8]

There is an issue of nonconstructivity in the Completeness Theorem for propositional logic with infinitely many propositional letters, however. Our mathematician who accepts only recursive objects will believe that propositional logic is not complete. To see this one needs only recall that there are recursive trees of infinite $0, 1$-branching trees with no recursive infinite path. It is then easy to use such a tree to set up a consistent recursive set of propositional formulas Σ with no recursive valuation. More generally, the status of the Completeness Theorem for propositional logic (in the sense of Reverse Mathematics) is that it is equivalent to WKL_0 over RCA_0, where RCA_0 is the base system of Reverse Mathematics, the acronym standing for 'Recursive Comprehension Axiom', and WKL_0 is the system obtained by adding to RCA_0 an axiom for Weak König's Lemma—see e.g., Simpson [18].

Gödel's 1930 proof [8] of the Completeness Theorem for first-order logic follows Skolem's technique of using what are today called Skolem functions to reduce formulas to a simple Prenex Form, and then apply what is essentially Completeness for propositional logic in its form with infinitely many letters. The more familiar Henkin argument proceeds in a similar way, using a family of (Henkin-)constants in place of Skolem functions. In the case when the set of sentences Σ one wishes to provide an interpretation for is presented in a recursive way suitable for investigation in a metatheory based on arithmetic, the system WKL_0 based on the König tree argument suffices to prove Completeness. In more general cases (for example when there is no arithmetization or Gödel numbering of syntax) the BPIT is required. I should also mention that the issue of nonconstructivity for Completeness in WKL_0 is no longer one concerning the Axiom of Choice: merely that non-recursive sets are required.

To take the discussion further we will need to be more precise as to what the issues are and what reasonable starting assumptions could be. We will look at logical statements in various languages or signatures, and to prevent the

[7] Zach's translations.

[8] The improved statement of 1917–18 follows Hilbert's thinking of the time and the development of his Programme, but is slightly misleading here. My understanding of Hilbert's proof of the completeness of propositional logic is that a statement $\phi(p, q, \dots)$ in propositional logic should be interpreted as a universal, $\forall p, q, \dots \phi(p, q, \dots)$, with $p, q, \dots$ thought of as ranging over the two truth values, and this only allows Hilbert to give a proof of what we would now call Completeness for propositional logic in the case of finitely many propositional letters. However, with propositions interpreted in this way, the new formulation of completeness for propositional logic is correct.

discussion becoming too broad I continue to restrict these logical statements to the syntax and scope of first-order logic, and allow that truth for these logical statements is defined in a way very similar to (if not exactly the same as) Tarski's definition of Truth. I take it that there is a potential collection of sets, models, and interpretations for these logical statements. I want to be a little more flexible about these: they may come from a Platonistic account of mathematics, and thus correspond to 'existing abstract objects'; they may come from a physical description of the universe (or part of it) and thus correspond to 'existing physical objects'; or they might be virtual or ideal objects posited by a (formal or informal) theory. The universe of interpretations and our arguments about them will be referred to as the *metatheory*. This, together with the definition of Truth, gives rise to a notion of semantic entailment $\vDash$, called *true semantic entailment*. The problem is to identify the nature of $\vDash$. We do have a system of proof for first-order logic, $\vdash_{\mathrm{FO}}$, and the statement that first-order logic is complete is

$$\Sigma \vDash \sigma \quad \text{implies} \quad \Sigma \vdash_{\mathrm{FO}} \sigma.$$

The Completeness Theorem (formulated in ZFC) is the statement

$$\Sigma \vDash_{\mathrm{ZFC}} \sigma \quad \text{implies} \quad \Sigma \vdash_{\mathrm{FO}} \sigma$$

of ZFC, where I have placed a subscript on $\vDash$ to indicate that this notion of semantic entailment is the one internal to ZFC and may not correspond to true semantic entailment. Other weaker theories (such as ACA_0, WKL_0, PA) also prove versions of the Completeness Theorem, sometimes restricted to arithmetized theories and sets of sentences.

At this point, it is worth looking briefly at Kreisel's squeezing argument [11]. As summarized by Field [5, Chapter 2], it involves two separate notions of 'validity' or semantic entailment: an intuitive notion which we can identify with our true semantic entailment $\vDash$ and a formalized notion $\vDash_{\mathrm{ZFC}}$. The difficulty is to bring these into focus and at least show they are extensionally identical,

$$\Sigma \vDash_{\mathrm{ZFC}} \sigma \;\Leftrightarrow\; \Sigma \vDash \sigma.$$

The point is that we may use the Completeness Theorem in the form '$\Sigma \vDash_{\mathrm{ZFC}} \sigma$ implies $\Sigma \vdash_{\mathrm{FO}} \sigma$', and the Intuitive Soundness Theorem in the form '$\Sigma \vdash_{\mathrm{FO}} \sigma$ implies $\Sigma \vDash \sigma$' (both of which being available and justifiable) to obtain the full equivalence of $\vDash_{\mathrm{ZFC}}$ and $\vDash$ from the direction

$$\Sigma \vDash \sigma \;\Rightarrow\; \Sigma \vDash_{\mathrm{ZFC}} \sigma. \tag{1}$$

Although there may be intuitive arguments for (1) from what is now traditional mathematical practice, this implication is precisely the one that is being disputed and investigated in this article.

The model theory and proof theory of the suggested metatheories, ZFC, ACA_0, WKL_0, PA, etc., should and does play and important role in exploring and clarifying the foundational questions here, and it is quite impossible to survey all the many positive contributions made in this direction in a short article. I will point to two examples that highlight the nature of some of these contributions as well as raising questions of their own.

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My first example concerns whether there is a Whitehead group[9] which is not free [17]. This is the Whitehead problem, and I heard about it first (as an undergraduate) from an algebraic topologist who had been working on an equivalent problem for quite a few days without realizing the connections or the contribution due to Shelah. Specifically, Shelah showed that no such groups exist in models of $\mathrm{ZFC} + V = L$ but the existence of such groups follows from Martin's axiom plus the negation of CH.

The question this (and many other similar results in set theory) raises, is what results like this say for our notion of true semantic entailment? We tend to think of $\models$ and mathematical truth in absolute terms. Is this reasonable? Results of the type just given are not enough in themselves to allow any such groups to be allowable structures in our universe, and even though this is a question that is often raised as part of general mathematical research experts in Foundational Studies do not (currently) offer any further guidance other than the observation that additional means going beyond ZFC will be required to solve the problem. Many important contributions from set theory are often similarly complicated and potentially confusing to the non-expert. Most working mathematicians (including my topologist friend) therefore duck such issues, in this case by accepting that 'the Whitehead problem is independent of the axioms of mathematics[10] .' This seems a great pity, and a preferable state of affairs would be to present the matter in a way that is palatable to the nonlogician and in such a way that she or he can and might want to contribute to the problems instead of avoiding them.

In terms of our true semantics and what structures should play a part in it, set theoretic research, dealing with questions of absoluteness for example, is highly relevant. There are questions such as: which interpretations internal to some model of set theory can be regarded as interpretations in their own right?

My second example is that of the metamathematics of WKL_0 itself. When one looks at minimal theories required for the Completeness Theorem one can restrict attention to first-order theories which are given in a recursive way via some arithmetical notions such as Gödel numbering. I will assume that these notions of Gödel numbering and the particular presentation of the non-logical axioms are clear, unambiguous, and correct for the intended application, though of course one could explore these ideas further and no doubt find exceptional cases which are problematic. As we have seen, when such a first-order theory is so-described and is known or assumed to be consistent, then models of this theory can be given, working in the metatheory WKL_0.

Various question then arise. In particular, what is the constructive content of the Completeness Theorem as proved from WKL_0? This is precisely the 'unwinding' programme of Kreisel in the particular case of a specialization

[9]The details are unimportant for this discussion, but an abelian group A is Whitehead if whenever B is abelian and $f : B \to A$ is a surjective group homomorphism with $\ker f \cong \mathbb{Z}$, there is $g : A \to B$ with $fg = \mathrm{id}_A$.

[10]By which is meant ZFC, of course.

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of the Completeness Theorem—see Feferman [4]. Quite a lot is known about this, including the initial work by Scott on what are now called Scott sets [16] (which are the second-order part of ω-models of WKL_0), and their rediscovery and application by Friedman [6] to models of arithmetic. (See Simpson [18], especially Chapter IV, for aspects of this in second-order arithmetic and Kaye [10], especially Chapter 13, for applications to first-order arithmetic.)

It may help to observe that in the proof (in WKL_0, say, or by a direct tree argument) of the Completeness Theorem for a recursive theory, when presented carefully, only one recursive tree needs to be traversed to find an infinite path in it. By additional coding, one may present the proof of Completeness for all recursive first order theories using a single infinite path from a particular recursive binary tree encoding all such theories.[11] A recursive binary tree that has no recursive infinite path necessarily has several infinite paths.[12] It is precisely this that makes the WKL_0 axioms analogous to Choice, though they are of course provable directly. What might our recursive model theorist make of this?

If he could be persuaded to accept countable models with domain $\mathbb{N}$ and finitely many relations in some Scott set $\mathscr{X}$, and this is a rather large 'if' since there are always nonrecursive models in such an $\mathscr{X}$, he would no doubt be willing to accept a model $(\mathbb{N}, R)$ where R is recursive relative to one of the models in $\mathscr{X}$. In other words, his collection of models is $\mathscr{X}$ itself.

The theory WKL_0 and its models, the Scott sets, are precisely the universes where Completeness and Soundness hold. Thus if $\mathscr{X}$ is a Scott set and $\vDash_{\mathscr{X}}$ is semantic entailment relative to models in $\mathscr{X}$ we have, for each recursive Σ, the set Σ is defined by a Δ^0_1 formula of WKL_0 and

$$\Sigma \vDash_{\mathscr{X}} \sigma \ \Leftrightarrow\ (\mathbb{N}, \mathscr{X}) \vDash \text{`}\Sigma \vDash \sigma\text{'} \ \Leftrightarrow\ (\mathbb{N}, \mathscr{X}) \vDash \text{`}\Sigma \vdash \sigma\text{'} \ \Leftrightarrow\ \Sigma \vdash \sigma.$$

The first bi-implication here is a restatement of $\vDash_{\mathscr{X}}$, the second is by Completeness and Soundness in WKL_0, and the third follows from the fact that $(\mathbb{N}, \mathscr{X})$ is an ω-model, so a proof in this model has a standard Gödel-number; then (as observed by Gödel) the usual syntactic notions of Gödel-number, formula and proof, etc., are all described by primitive recursive functions and so mean the same thing in $(\mathbb{N}, \mathscr{X})$ as in the metatheory (i.e., the notion of first-order proof is absolute) hence the third bi-implication above holds. This is the key observation by Scott and the central reason for the importance of Scott sets: that they do indeed describe precisely the constructive content of the Completeness Theorem.

[11] The related result due to Scott that Scott sets are completion closed [10, Theorem 13.3] explains the idea.

[12] This is because if the tree is recursive and has a unique infinite path then this path can be found by the algorithm which operates as follows. At node n, the algorithm simultaneously searches the tree below both daughter nodes to check which of them is finite. One is finite, by our assumption that there is precisely one infinite path through the tree, and so can be traversed completely, so in one case the algorithm returns the answer. When this happens, the algorithm takes the non-finite daughter node and repeats the process.

This analysis may not be wholly satisfactory to our recursive mathematician, however. He might be sufficiently uncomfortable with the need for nonrecursive sets to ask for further justification. Given a recursive Σ, he asks is there some *canonical model* that interprets Σ? And here our answer will disappoint.

Given a nonrecursive set $A \subseteq \mathbb{N}$ there is always a Scott set $\mathscr{X}$ such that $A \notin \mathscr{X}$ [10, Lemma 11.2] (see also Simpson [18, Chapter VIII]). Thus semantic entailment for recursive models, $\vDash_{\mathrm{Rec}}$ has some claim to being the *weakest* logic for which every model is canonical in some sense. It would be of particular interest to explore these directions further and explore the relative merits of $\vDash_{\mathrm{Rec}}$ and $\vDash_{\mathscr{X}}$ for Scott sets $\mathscr{X}$ in more detail.

Before I leave WKL_0 there is one other remarkable result concerning it of foundational interest that should be mentioned, and that is the Harrington–Friedman result that WKL_0 and RCA_0 share the same Π^1_1 consequences [18, Chapter IX]. This result can be restated and proved using techniques from PA too: The theories WKL_0, RCA_0 and their first-order counterpart $\mathrm{I}\Sigma_1$ [10, Section 7.1 and Chapter 10] all have the same first-order consequences, and are all Π^0_2 conservative over primitive recursive arithmetic, PRA. It is arguable that PRA is a reasonable base for mathematics (i.e. for finitistic methods, in the sense of Hilbert).[13] The conservation result relating WKL_0 to PRA can be proved by direct proof-theoretic means in PRA itself: one can show (in PRA) that if p is a proof from WKL_0 of σ where σ is Π^0_2, then there is a way of manipulating this proof p into a proof of σ from PRA itself. In the special case where σ is an absurdity $\perp$, this gives an equiconsistency result $\mathrm{Con}(\mathrm{PRA}) \Rightarrow \mathrm{Con}(\mathrm{WKL}_0)$ in PRA. This is one of the very few places where Hilbert's programme can be realized successfully, and this in itself gives good grounds to accept WKL_0 as consistent and perhaps appropriate. If we are able in the future to justify completeness of first-order logic based on the proof of Completeness in WKL_0 and certain intuitive properties of the true semantic entailment relation, it seems to me that these properties of WKL_0 will most likely be highly relevant.

Of course all the discussion here has been based on the notion of true semantic entailment and the class of structures we admit in the universe. The other aspect of circularity in the inference of completeness from the Completeness Theorem is the logic that we use in our metatheory. Unpicking this is likely to be a much more delicate task: our informal arguments in the metatheory are (rightly or wrongly) based on conditioning derived from mathematical practice, and from first-order logic in particular, so justifying first-order logic by logical arguments in the metatheory is fraught with difficulty. The theories we have been examining (WKL_0, RCA_0, $\mathrm{I}\Sigma_1$, and PRA) are all based on first-order logic and the conclusion we wish to draw is an absolute one concerning first-order logic. Whether this is a problem

[13]But there really is an issue here: PRA provides functions that grow at rates corresponding to the individual levels of the Ackermann function, and so certainly defines functions that are intractable in the sense of complexity theory and much more besides. It is not *a priori* obvious that our intuitions for such functions are sufficient to guarantee the consistency and relevance of PRA.

will depend on one's views of how mathematics is ultimately grounded, and quite possibly also what mathematics is for. Nevertheless, even if one were to accept usual practice in such informal arguments, one should accept that there is an assumption here, or an argument to be made, and that others with different starting points may have a different opinion.

I close this section with a brief mention of another well-known result 'characterizing' first-order logic, and offer its analysis (in the context of the discussion here) as a worthy programme of study. Lindström's Theorem [12] is (rightly) regarded as a highlight of mathematical logic characterizing first-order logic. Essentially, it says that first-order logic is unique amongst abstract logics as being the only one enjoying certain basic properties, chief amongst them being countable compactness and the downward Löwenheim Theorem. Lindström's Theorem 'suffers' (and I do not mean this in any pejorative sense) the same issues when it is regarded as a 'justification' for first-order logic. The processes used in the proof of Lindström's Theorem are very much in the realm of first order logic with a class of allowed models (implicitly) given. And it is not so surprising, perhaps, that the theorem says that this class of models corresponds to first-order logic itself in some strong way. Once again, the theorem can, and does, give significant foundational insight, through its mixture of internal and external arguments, including in particular (externally) its view of all interpretations and the true semantic entailment, and (internally) its application of back-and-forth systems and isomorphisms.

§5. Conclusions. First-order logic, with its Completeness and Soundness Theorems, is a remarkable achievement. It is certainly true that the arguments about it, Completeness and Soundness in particular, show it to be robust and natural and deserving of our attention. It is not the case that the Completeness of first-order logic shows it to be necessarily strong enough for any particular application, however, except where the metatheory can be taken to be a particular flavor of set theory or arithmetic which proves the theorem. Thus, some of the claims for first-order logic (including many I myself have made in the classroom) have to be treated with considerable care, to say the least.

As a foundational device, first-order logic (alongside other logics) provides a framework for discussing our foundational issues in mathematics, and again it is invaluable in this respect. But it may be wise not to allow the Completeness and Soundness Theorems for first order logic to mislead us into thinking that first-order logic necessarily captures usual mathematical practice in every respect. In Section 4 particularly I have suggested further avenues for investigation into just these matters.

Having said this, the circularity of the argument that takes us from Completeness for first-order logic to the completeness of first-order logic has its beneficial side too. Whatever structures one might have originally accepted in the mathematical universe, for example whether one originally might have accepted the existence of nonstandard models of arithmetic which are necessarily nonrecursive, techniques from the proof of the Completeness Theorem and more generally the use of set theory as a standard all-encompassing theory for mathematics force us to extend what might have been a rather limited mathematical view, to examination of structures built by nonconstructive means. First-order logic, even if it was not the 'original intended logic of mathematics' (if that phrase means anything at all), becomes the practical logic of modern mathematics through its interrelations with set theory. Those interrelations, I hope to have convinced the reader, beg questions, in a way that should properly be considered as a species of *circulus in probando*. But the resulting shift in emphasis that arises is highly positive, and asks how one can profitably use the new structures and tools at our disposal.

These issues also shed some light on the development of mathematical thinking in the twentieth century, and the acceptance of set theory as a suitable framework for mathematics, as new existence principles became available and more widely accepted. Hand in hand with this was the need to understand these principles and their limitations, and first-order logic played an intimate role in both aspects of this development, as did set theory in the development of first-order logic.

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