Proceedings of the Royal Society of Edinburgh, **147A**, 743–761, *2017*
DOI:10.1017/S0308210516000287

The algebra of symmetric analytic functions on L_∞

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(MS received 17 November 2015; accepted 21 December 2015)

We consider the algebra of holomorphic functions on L_∞ that are *symmetric*, i.e. that are invariant under composition of the variable with any measure-preserving bijection of $[0, 1]$. Its spectrum is identified with the collection of scalar sequences $\{\xi_n\}_{n=1}^\infty$ such that $\{\sqrt[n]{|\xi_n|}\}_{n=1}^\infty$ is bounded and turns to be separable. All this follows from our main result that the subalgebra of *symmetric* polynomials on L_∞ has a natural algebraic basis.

Keywords: polynomials and analytic functions on Banach spaces; measure preserving; symmetric polynomials; spectra of algebras

2010 *Mathematics subject classification:* Primary 46J15; 46J20
Secondary 46E15; 46E25; 46G20

1. Introduction

In [14] Nemirovski and Semenov considered functions on ℓ_p spaces invariant under the permutation of variables and their approximation by the same kind of polynomials. They used the term *symmetric* for such functions and it appears that this was the first time that symmetric functions of an infinite number of variables were dealt with. Some of their results were generalized by González *et al*. [10] to real separable rearrangement-invariant function spaces. In [3] Alencar *et al*. studied in detail the algebra of functions that are symmetric in the ball algebra $A(B_{\ell_p})$ and described its spectrum. Chernega *et al*. [7–9] deal with the analogous situation for the space $\mathcal{H}_b(\ell_p)$ of analytic functions of bounded type, including the study of convolution operators on the algebra of symmetric functions.

Let L_∞ be the *complex* Banach space of all Lebesgue measurable essentially bounded complex-valued functions x on $[0, 1]$ with norm

$$\|x\|_\infty = \operatorname*{ess\,sup}_{t\in[0,1]} |x(t)|.$$

Let Ξ be the set of all measurable bijections of $[0, 1]$ that preserve the measure. A function $F\colon L_\infty \to \mathbb{C}$ is called Ξ-*symmetric* (or just *symmetric* when the context is clear) if, for every $x \in L_\infty$ and for every $\sigma \in \Xi$,

$$F(x \circ \sigma) = F(x).$$

For every $n \in \mathbb{N} \cup \{0\}$ we define $R_n \colon L_\infty \to \mathbb{C}$ by

$$R_n(x) = \int_0^1 x^n(t)\,\mathrm{d}t.$$

We call the functions R_n the *elementary symmetric polynomials*.

Our main result states that $(R_n)_{n=0}^{\infty}$ is an algebraic basis for the space of continuous symmetric polynomials P on L_∞, that is, for each given such P, there is a (unique) polynomial q in finitely many variables such that

$$P(x) = q(R_1(x), \dots, R_m(x)) \quad \text{for every } x \in L_\infty.$$

It has been shown (see [10] in the real case and [6] in the complex case) that the above equality holds for functions x in the closed linear span of the characteristic functions of the dyadic intervals.

The Ξ-symmetric continuous polynomials on L_p-spaces, $1 \leqslant p < \infty$, have been investigated in [6,10,14]; the article [6] studies analytic functions invariant under the action of some group of operators from a more general point of view. The present paper also fits in this setting.

2. Measurable automorphisms

A measure space is a triple $(\Omega, \mathcal{F}, \nu)$, where Ω is a set, $\mathcal{F}$ is a σ algebra of its subsets and $\nu \colon \mathcal{F} \to [0, +\infty)$ is a measure. An *isomorphism* between two measure spaces $(\Omega_1, \mathcal{F}_1, \nu_1)$ and $(\Omega_2, \mathcal{F}_2, \nu_2)$ is an invertible map $f \colon \Omega_1 \to \Omega_2$ such that f and f^{-1} are both measurable and measure-preserving maps. In the case $(\Omega_1, \mathcal{F}_1, \nu_1) = (\Omega_2, \mathcal{F}_2, \nu_2)$ the mapping f is called a *measurable automorphism*. Two measure spaces $(\Omega_1, \mathcal{F}_1, \nu_1)$ and $(\Omega_2, \mathcal{F}_2, \nu_2)$ are called *isomorphic modulo zero* if there exist null sets $M \subset \Omega_1$ and $N \subset \Omega_2$ such that measure spaces $\Omega_1 \setminus M$ and $\Omega_2 \setminus N$ are isomorphic [15].

In this paper we shall only consider the Lebesgue measure on $[0,1]$. Clearly, Ξ is the set of all measurable automorphisms of $[0,1]$.

The following simple proposition shows that a measurable automorphism of $[0,1]$ can be defined up to a null set.

Proposition 2.1. *Let f be an isomorphism modulo zero of $[0,1]$. Then there exists a measurable automorphism of $[0,1]$ that coincides with f almost everywhere.*

Proof. There exist null sets M and N such that

$$f \colon [0,1] \setminus M \to [0,1] \setminus N$$

is an isomorphism. Let $\mathcal{K}$ be the Cantor set. Let

$$C_1 = \mathcal{K} \cap ([0,1] \setminus M) \quad \text{and} \quad C_2 = \mathcal{K} \cap ([0,1] \setminus N).$$

The sets

$$U = \mathcal{K} \cup f^{-1}(C_2) \cup M \quad \text{and} \quad V = \mathcal{K} \cup f(C_1) \cup N$$

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are both null sets of the cardinality of the continuum. Let h be the bijection between U and V. Clearly,

$$g(t) = \begin{cases} h(t) & \text{if } t \in U, \\ f(t) & \text{if } t \in [0,1] \setminus U, \end{cases}$$

is a bijection, and therefore it is a measurable automorphism of $[0,1]$. □

For every $E \subset [0,1]$ let

$$\mathbf{1}_E(t) = \begin{cases} 1 & \text{if } t \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.2. *Let $E_1, \ldots, E_N \subset [0,1]$ be measurable sets such that $\mu(E_j \cap E_k) = 0$ if $j \neq k$. Then there exist $\sigma_{E_1,\ldots,E_N} \in \Xi$ such that*

$$\mathbf{1}_{E_m} = \mathbf{1}_{[b_{m-1}, b_m]} \circ \sigma_{E_1,\ldots,E_N}$$

for every $m \in \{1, \ldots, N\}$ almost everywhere on $[0,1]$, where $b_k = \sum_{j=1}^{k} \mu(E_j)$ for $k \in \{1, \ldots, N\}$ and $b_0 = 0$.

Proof. Without loss of generality we can assume that the E_m are disjoint. Let $E_{N+1} = [0,1] \setminus \bigcup_{m=1}^{N} E_m$ and $b_{N+1} = 1$. By [15, § 2, nos 1–4], every measurable subset $E \subset [0,1]$ is isomorphic modulo zero to an interval of length $\mu(E)$. Thus, every E_m is isomorphic modulo zero to $[b_m, b_{m+1}]$. Let f_m be the proper isomorphisms. Let

$$\sigma_{E_1,\ldots,E_N}(t) = f_m(t)$$

if $t \in E_m$, $m \in \{1, \ldots, N+1\}$. Evidently, $\sigma_{E_1,\ldots,E_N}$ satisfies the stated conditions. □

3. Symmetric functions on L_∞

Theorem 3.1. *For every sequence*

$$\xi = \{\xi_n\}_{n=1}^{\infty} \subset \mathbb{C}$$

such that the sequence $\{\sqrt[n]{|\xi_n|}\}_{n=1}^{\infty}$ is bounded, there exists $x_\xi \in L_\infty$ such that $R_n(x_\xi) = \xi_n$ for every $n \in \mathbb{N}$.

Proof. Let ε_k be the kth Rademacher function, i.e.

$$\varepsilon_k(t) = \operatorname{sgn} \sin 2^k \pi t.$$

It is known (see [11, ch. 3] or [2, p. 162]) that the series $\sum_{k=1}^{\infty} \varepsilon_k(t) u_k$ converges almost everywhere on $[0,1]$ if and only if $\sum_{k=1}^{\infty} |u_k|^2$ converges. Consequently, the series

$$\sum_{k=1}^{\infty} \frac{\varepsilon_k(t)}{k+1}$$

converges almost everywhere on $[0,1]$. For every $n \in \mathbb{N}$ we define

$$p_n(t) = \exp\left(\frac{\mathrm{i}\pi}{2n} \sum_{k=1}^{\infty} \frac{\varepsilon_k(t)}{k+1}\right).$$

Clearly, $|p_n(t)| = 1$ almost everywhere. Therefore, $p_n \in L_\infty$ and $\|p_n\|_\infty = 1$.

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Next, we provide some lemmas we need to complete the proof.

LEMMA 3.2. *For every $m, n \in \mathbb{N}$ we have*

$$R_m(p_n) = \prod_{k=1}^{\infty} \cos\left(\frac{\pi}{2}\frac{m}{n(k+1)}\right).$$

Proof of lemma 3.2. By definition,

$$R_m(p_n) = \int_0^1 (p_n(t))^m \,\mathrm{d}t.$$

The sequence of functions $\{p_n^{(l)}\}_{l=1}^{\infty} \subset L_\infty$ given by

$$p_n^{(l)}(t) = \exp\left(\frac{\mathrm{i}\pi}{2n}\sum_{k=1}^{l}\frac{\varepsilon_k(t)}{k+1}\right)$$

converges almost everywhere to p_n. By the dominated convergence theorem,

$$\int_0^1 (p_n(t))^m \,\mathrm{d}t = \lim_{l\to\infty}\int_0^1 (p_n^{(l)}(t))^m \,\mathrm{d}t.$$

Note that

$$\begin{aligned}
\int_0^1 (p_n^{(l)}(t))^m \,\mathrm{d}t &= \int_0^1 \exp\left(\frac{\mathrm{i}\pi m}{2n}\sum_{k=1}^{l}\frac{\varepsilon_k(t)}{k+1}\right)\mathrm{d}t \\
&= \int_0^1 \exp\left(\frac{\mathrm{i}\pi m}{2n}\frac{\varepsilon_1(t)}{2}\right)\exp\left(\frac{\mathrm{i}\pi m}{2n}\sum_{k=2}^{l}\frac{\varepsilon_k(t)}{k+1}\right)\mathrm{d}t \\
&= \exp\left(\frac{\mathrm{i}\pi m}{2n}\frac{1}{2}\right)\int_{[0,1/2]}\exp\left(\frac{\mathrm{i}\pi m}{2n}\sum_{k=2}^{l}\frac{\varepsilon_k(t)}{k+1}\right)\mathrm{d}t \\
&\quad + \exp\left(\frac{\mathrm{i}\pi m}{2n}\frac{(-1)}{2}\right)\int_{[1/2,1]}\exp\left(\frac{\mathrm{i}\pi m}{2n}\sum_{k=2}^{l}\frac{\varepsilon_k(t)}{k+1}\right)\mathrm{d}t \\
&= 2\cos\left(\frac{\pi m}{2n}\frac{1}{2}\right)\int_{[0,1/2]}\exp\left(\frac{\mathrm{i}\pi m}{2n}\sum_{k=2}^{l}\frac{\varepsilon_k(t)}{k+1}\right)\mathrm{d}t \\
&= 4\cos\left(\frac{\pi m}{2n}\frac{1}{2}\right)\cos\left(\frac{\pi m}{2n}\frac{1}{3}\right)\int_{[0,1/4]}\exp\left(\frac{\mathrm{i}\pi m}{2n}\sum_{k=3}^{l}\frac{\varepsilon_k(t)}{k+1}\right)\mathrm{d}t \\
&= 2^l\prod_{k=1}^{l}\cos\left(\frac{\pi m}{2n}\frac{1}{k+1}\right)\int_{[0,1/2^l]}\mathrm{d}t \\
&= \prod_{k=1}^{l}\cos\left(\frac{\pi m}{2n}\frac{1}{k+1}\right).
\end{aligned}$$

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Therefore,

$$R_m(p_n) = \lim_{l\to\infty} \prod_{k=1}^{l} \cos\left(\frac{\pi m}{2n}\frac{1}{k+1}\right) = \prod_{k=1}^{\infty} \cos\left(\frac{\pi m}{2n}\frac{1}{k+1}\right).$$

□

Let $\alpha_k = \exp(2\pi \mathrm{i} k/n)$ for $k = 1, 2, \dots, n$. We define the function $y_n \colon [0,1] \to \mathbb{C}$ in the following way: For $t \in [(k-1)/n, k/n)$, where $k \in \{1, \dots, n\}$, we set

$$y_n(t) = \alpha_k p_n(nt - k + 1).$$

Note that $y_n \in L_\infty$ and $\|y_n\|_\infty = 1$.

LEMMA 3.3. *For $m, n \in \mathbb{N}$, we have*

$$R_m(y_n) = \begin{cases} M & \text{if } m = n, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$M = \prod_{k=1}^{\infty} \cos\left(\frac{\pi}{2}\frac{1}{k+1}\right).$$

Proof. Note that

$$\begin{aligned} R_m(y_n) &= \int_0^1 (y_n(t))^m \,\mathrm{d}t = \sum_{k=1}^{n} \int_{[(k-1)/n, k/n)} (\alpha_k p_n(nt - k + 1))^m \,\mathrm{d}t \\ &= \left(\frac{1}{n}\sum_{k=1}^{n} \alpha_k^m\right) \int_0^1 (p_n(t))^m \,\mathrm{d}t \\ &= \left(\frac{1}{n}\sum_{k=1}^{n} \alpha_k^m\right) R_m(p_n). \end{aligned}$$

Since

$$\frac{1}{n}\sum_{k=1}^{n} \alpha_k^m = \begin{cases} 1 & \text{if } m \text{ is a multiple of } n, \\ 0 & \text{otherwise,} \end{cases}$$

it follows that $R_m(y_n) = 0$ if m is not a multiple of n. However, if m is a multiple of n, i.e. $m = k_0 n$ for some $k_0 \in \mathbb{N}$, we have

$$R_m(y_n) = R_m(p_n) = \prod_{k=1}^{\infty} \cos\left(\frac{\pi}{2}\frac{m}{n(k+1)}\right) = \prod_{k=1}^{\infty} \cos\left(\frac{\pi}{2}\frac{k_0}{k+1}\right).$$

If $m \neq n$, then $k_0 > 1$ and one of the factors is equal to $\cos\frac{1}{2}\pi$, and therefore $R_m(y_n) = 0$. In the $m = n$ case we have

$$R_n(y_n) = \prod_{k=1}^{\infty} \cos\left(\frac{\pi}{2}\frac{1}{k+1}\right) = M.$$

□

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We now continue with the proof of theorem 4.3. We set

$$x_n(t) = \frac{1}{\sqrt[n]{M}} y_n(t).$$

Taking into account that $\|y_n\|_\infty = 1$ and $0 < M < 1$, we have

$$\|x_n\|_\infty = \frac{1}{\sqrt[n]{M}} \|y_n\|_\infty = \frac{1}{\sqrt[n]{M}} \leqslant \frac{1}{M}.$$

From lemma 3.3 it follows that

$$R_m(x_n) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Now we construct x_ξ. For $t \in [(2^{n-1} - 1)/2^{n-1}, (2^n - 1)/2^n)$, where $n \in \mathbb{N}$, we set

$$x_\xi(t) = 2\sqrt[n]{\xi_n} x_n(2^n t - 2^n + 2).$$

Since the sequence $\{\sqrt[n]{|\xi_n|}\}_{n=1}^{\infty}$ is bounded, there exists $a > 0$ such that $|\xi_n| \leqslant a^n$ for every $n \in \mathbb{N}$. Note that

$$\|x_\xi\|_\infty \leqslant \sup_{n \in \mathbb{N}} 2|\sqrt[n]{\xi_n}| \|x_n\|_\infty \leqslant \frac{2a}{M}.$$

Thus, $x_\xi \in L_\infty$. For $m \in \mathbb{N}$ we have

$$\begin{aligned} R_m(x_\xi) &= \int_0^1 (x_\xi(t))^m \,\mathrm{d}t \\ &= \sum_{n=1}^{\infty} (2\sqrt[n]{\xi_n})^m \int_{[(2^{n-1}-1)/2^{n-1}, (2^n-1)/2^n)} (x_n(2^n t - 2^n + 2))^m \,\mathrm{d}t \\ &= \sum_{n=1}^{\infty} (2\sqrt[n]{\xi_n})^m \frac{1}{2^n} \int_0^1 (x_n(t))^m \,\mathrm{d}t = (2\sqrt[m]{\xi_m})^m \frac{1}{2^m} \\ &= \xi_m. \end{aligned}$$

□

4. Homogeneous symmetric continuous polynomials on L_∞

A mapping $P\colon X \to Y$, where X and Y are Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively, is called an n-homogeneous polynomial if there exists an n-linear symmetric mapping $A_P\colon X^n \to Y$ such that

$$P(x) = A_P(x, \overset{n}{\ldots}, x) \quad \text{for every } x \in X.$$

Here 'symmetric' means that $A_P(x_{\tau(1)}, \ldots, x_{\tau(n)}) = A_P(x_1, \ldots, x_n)$ for every permutation $\tau\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$.

It is known (see, for example, [13, theorem 1.10]) that A_P can be recovered from P by means of the so-called polarization formula:

$$A_P(x_1, \ldots, x_n) = \frac{1}{n! 2^n} \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \varepsilon_1 \cdots \varepsilon_n P(\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n). \tag{4.1}$$

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We shall use the polynomial formula (see [13, theorem 1.8])

$$P(x_1+\cdots+x_k)=\sum_{n_1+\cdots+n_k=n}\frac{n!}{n_1!\cdots n_k!}A_P(x_1,\overset{n_1}{\ldots},x_1,x_2,\overset{n_2}{\ldots},x_2,\ldots,x_k,\overset{n_k}{\ldots},x_k) \tag{4.2}$$

and its corollary, the binomial formula (see [13, corollary 1.9])

$$P(x+y)=\sum_{m=0}^{n}\binom{n}{m}A_P(x,\overset{n}{\ldots},x,y,\overset{m}{\ldots},y). \tag{4.3}$$

It is known that an n-homogeneous polynomial $P\colon X\to Y$ is continuous if and only if

$$\|P\|=\sup_{\|x\|_X\leqslant 1}\|P(x)\|_Y<+\infty.$$

Similarly, an n-linear mapping $A\colon X^n\to Y$ is continuous if and only if

$$\|A\|=\sup_{\|x_1\|_X\leqslant 1,\ldots,\|x_n\|_X\leqslant 1}\|A(x_1,\ldots,x_n)\|_Y<+\infty.$$

Clearly, if A is continuous, then

$$\|A(x_1,\ldots,x_n)\|_Y\leqslant\|A\|\|x_1\|_X,\ldots,\|x_n\|_X \tag{4.4}$$

for every $x_1,\ldots,x_n\in X$.

We restrict our attention to scalar-valued $\varXi$-symmetric n-homogeneous continuous polynomials on L_∞. In this section we shall prove that every such a polynomial can be represented as an algebraic combination of the elementary symmetric polynomials R_n. First, we prove some auxiliary results.

Remark 4.1. For every symmetric k-homogeneous polynomial $Q\colon L_\infty\to\mathbb{C}$ and functions $\sigma\in\varXi, x_1,\ldots,x_k\in L_\infty$ we have

$$A_Q(x_1\circ\sigma,\ldots,x_k\circ\sigma)=A_Q(x_1,\ldots,x_k).$$

Proof. By the polarization formula (4.1) and by the symmetry of Q,

$$\begin{aligned}A_Q(x_1\circ\sigma,&\ldots,x_k\circ\sigma)\\&=\frac{1}{k!2^k}\sum_{\varepsilon_1,\ldots,\varepsilon_k=\pm1}\varepsilon_1\cdots\varepsilon_kQ(\varepsilon_1(x_1\circ\sigma)+\cdots+\varepsilon_k(x_k\circ\sigma))\\&=\frac{1}{k!2^k}\sum_{\varepsilon_1,\ldots,\varepsilon_k=\pm1}\varepsilon_1\cdots\varepsilon_kQ((\varepsilon_1x_1+\cdots+\varepsilon_kx_k)\circ\sigma)\\&=\frac{1}{k!2^k}\sum_{\varepsilon_1,\ldots,\varepsilon_k=\pm1}\varepsilon_1\cdots\varepsilon_kQ(\varepsilon_1x_1+\cdots+\varepsilon_kx_k)=A_Q(x_1,\ldots,x_k).\end{aligned}$$

□

Let us prove that the coefficients of an algebraic combination of elementary symmetric polynomials can be recovered from the values of the combination.

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Proposition 4.2. *For every $N \in \mathbb{N}$ there exist functions $x_j \in L_\infty$ and constants $u^{(j)}_{k_1,\ldots,k_N} \in \mathbb{C}$, where $j \in \{1,\ldots,(N+1)^N\}$ and $k_1,\ldots,k_N \in \{0,\ldots,N\}$, such that for every functional $G\colon L_\infty \to \mathbb{C}$ of the form*

$$G(x) = \sum_{k_1=0}^{N} \cdots \sum_{k_N=0}^{N} \alpha_{k_1,\ldots,k_N} R_1^{k_1}(x) \cdots R_N^{k_N}(x),$$

where $\alpha_{k_1,\ldots,k_N} \in \mathbb{C}$, the following equality holds:

$$\alpha_{k_1,\ldots,k_N} = \sum_{j=1}^{(N+1)^N} u^{(j)}_{k_1,\ldots,k_N} G(x_j).$$

Proof. For every $j \in \{1,\ldots,(N+1)^N\}$, by theorem 3.1, there exists $x_j \in L_\infty$ such that $R_m(x_j) = j^{(N+1)^{m-1}}$ for $1 \leqslant m \leqslant (N+1)^N$ and $R_m(x_j) = 0$ for $m > (N+1)^N$. Then

$$\sum_{k_1=0}^{N} \cdots \sum_{k_N=0}^{N} \alpha_{k_1,\ldots,k_N} j^{k_1+k_2(N+1)+k_3(N+1)^2+\cdots+k_N(N+1)^{N-1}} = G(x_j), \quad j \in \{1,\ldots,(N+1)^N\}. \tag{4.5}$$

It is easy to check that the expression $k_1+k_2(N+1)+k_3(N+1)^2+\cdots+k_N(N+1)^{N-1}$ takes all the values from 0 to $(N+1)^N - 1$.

Thus, the determinant of the system of linear equations (4.5) is, up to permutation, a Vandermonde determinant, which is not equal to zero. Therefore, there exist constants $u^{(j)}_{k_1,\ldots,k_N} \in \mathbb{C}$, where $j \in \{1,\ldots,(N+1)^N\}$ and $k_1,\ldots,k_N \in \{0,\ldots,N\}$, such that

$$\alpha_{k_1,\ldots,k_N} = \sum_{j=1}^{(N+1)^N} u^{(j)}_{k_1,\ldots,k_N} G(x_j).$$

□

Theorem 4.3. *Every symmetric continuous n-homogeneous polynomial $P\colon L_\infty \to \mathbb{C}$ can be uniquely represented as*

$$P(x) = \sum_{k_1+2k_2+\cdots+nk_n=n} \alpha_{k_1,\ldots,k_n} R_1^{k_1}(x) \cdots R_n^{k_n}(x),$$

where $k_1,\ldots,k_n \in \mathbb{N} \cup \{0\}$ and $\alpha_{k_1,\ldots,k_n} \in \mathbb{C}$. In other words, $\{R_n\}$ forms an algebraic basis in the algebra of symmetric continuous polynomials on L_∞.

Proof. Once the existence of the coefficients is proved, uniqueness follows from proposition 4.2.

For the existence, we proceed by induction on n. In the $n = 1$ case the polynomial P is a symmetric continuous linear functional. Let $g\colon [0,1] \to \mathbb{C}$,

$$g(t) = P(\mathbf{1}_{[0,t]}).$$

By the symmetry and the linearity of P,

$$\begin{aligned} g(t_1+t_2) &= P(\mathbf{1}_{[0,t_1+t_2]}) \\ &= P(\mathbf{1}_{[0,t_1]} + \mathbf{1}_{[t_1,t_1+t_2]}) \\ &= P(\mathbf{1}_{[0,t_1]}) + P(\mathbf{1}_{[t_1,t_1+t_2]}) \\ &= P(\mathbf{1}_{[0,t_1]}) + P(\mathbf{1}_{[0,t_2]}) = g(t_1) + g(t_2), \end{aligned} \tag{4.6}$$

where $t_1, t_2 \in [0,1]$ such that $t_1 + t_2 \leqslant 1$. Thus, g is additive, that is, it satisfies the Cauchy functional equation. By the continuity of P we also have the boundedness of g. It is well known that every bounded additive function on $[0,1]$ is necessarily linear (see, for example, [1, ch. 2]. Thus, g is linear, and therefore $g(t) = tg(1)$ for every $t \in [0,1]$.

Hence,

$$P(\mathbf{1}_{[0,t]}) = tP(\mathbf{1}_{[0,1]}). \tag{4.7}$$

Let E be a measurable subset of $[0,1]$. Applying proposition 2.2 to $N = 1$ and $E_1 = E$, there exists $\sigma_E \in \varXi$ such that $\mathbf{1}_E = \mathbf{1}_{[0,\mu(E)]} \circ \sigma_E$ almost everywhere on $[0,1]$. By the symmetry of P and by (4.7),

$$P(\mathbf{1}_E) = P(\mathbf{1}_{[0,\mu(E)]}) = \mu(E)P(\mathbf{1}_{[0,1]}).$$

For the simple measurable function $x = \sum_{j=1}^{J} h_j \mathbf{1}_{E_j}$, where $h_j \in \mathbb{C}$ and $E_j \subset [0,1]$, by the linearity of P,

$$P(x) = \sum_{j=1}^{J} h_j P(\mathbf{1}_{E_j}) = P(\mathbf{1}_{[0,1]}) \sum_{j=1}^{J} h_j \mu(E_j) = P(\mathbf{1}_{[0,1]}) R_1(x).$$

Since the set of simple measurable functions is dense in L_∞, the continuity of P leads to

$$P(x) = P(\mathbf{1}_{[0,1]}) R_1(x)$$

for every $x \in L_\infty$. This completes the proof for the $n = 1$ case.

Assume the statement of the theorem holds for every $m \in \{1, \dots, n-1\}$. We prove it for n. To do this we provide several lemmas.

LEMMA 4.4. *Let $1 \leqslant m < n$ and $[a,b] \subset [0,1]$ and let $y_1, \dots, y_{n-m} \in L_\infty$ be such that the restrictions of $y_1, \dots, y_{n-m}$ to $[a,b]$ are constant. Then there exists a constant $C_1(m,a,b) > 0$ such that, for every measurable subset E of $[a,b]$,*

$$|A_P(\mathbf{1}_E, \overset{m}{\dots}, \mathbf{1}_E, y_1, \dots, y_{n-m})| \leqslant \mu(E)\|y_1\|_\infty \cdots \|y_{n-m}\|_\infty C_1(m,a,b).$$

Proof. For every $x \in L_\infty$ let

$$\hat{x}(t) = \begin{cases} x\left(\dfrac{t-a}{b-a}\right) & \text{if } t \in [a,b], \\ 0 & \text{if } t \in [0,1] \setminus [a,b]. \end{cases}$$

Consider the m-homogeneous polynomial $L(x) = A_P(\hat{x}, \overset{m}{\dots}, \hat{x}, y_1, \dots, y_{n-m})$.

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For every $\sigma \in \varXi$ let

$$\tilde{\sigma}(t) = \begin{cases} a + (b-a)\sigma\left(\dfrac{t-a}{b-a}\right) & \text{if } t \in [a,b], \\ t & \text{if } t \in [0,1] \setminus [a,b]. \end{cases}$$

Clearly, $\tilde{\sigma} \in \varXi$. We claim that $\widehat{x \circ \sigma} = \hat{x} \circ \tilde{\sigma}$. Indeed, if $t \in [a,b]$, then

$$\widehat{x \circ \sigma}(t) = (x \circ \sigma)\left(\frac{t-a}{b-a}\right)$$

and

$$\begin{aligned} (\hat{x} \circ \tilde{\sigma})(t) &= \hat{x}(\tilde{\sigma}(t)) \\ &= \hat{x}\left(a + (b-a)\sigma\left(\frac{t-a}{b-a}\right)\right) \\ &= x\left(a + (b-a)\sigma\left(\frac{t-a}{b-a}\right) - a\right)\left(\frac{1}{b-a}\right) \\ &= (x \circ \sigma)\left(\frac{t-a}{b-a}\right), \end{aligned}$$

while if $t \in [0,1] \setminus [a,b]$, then $\widehat{x \circ \sigma}(t) = 0$, and so $(\hat{x} \circ \tilde{\sigma})(t) = \hat{x}(\tilde{\sigma}(t)) = \hat{x}(t) = 0$. Thus, the claim is verified, and hence

$$L(x \circ \sigma) = A_P(\widehat{x \circ \sigma}, \overset{m}{\ldots}, \widehat{x \circ \sigma}, y_1, \ldots, y_{n-m}) = A_P(\hat{x} \circ \tilde{\sigma}, \overset{m}{\ldots}, \hat{x} \circ \tilde{\sigma}, y_1, \ldots, y_{n-m}).$$

Since y_j are constant on $[a,b]$, it follows that $y_j \circ \tilde{\sigma} = y_j$ for $j \in \{1, \ldots, n-m\}$. Therefore,

$$A_P(\hat{x} \circ \tilde{\sigma}, \overset{m}{\ldots}, \hat{x} \circ \tilde{\sigma}, y_1, \ldots, y_{n-m}) = A_P(\hat{x} \circ \tilde{\sigma}, \overset{m}{\ldots}, \hat{x} \circ \tilde{\sigma}, y_1 \circ \tilde{\sigma}, \ldots, y_{n-m} \circ \tilde{\sigma}).$$

By remark 4.1,

$$A_P(\hat{x} \circ \tilde{\sigma}, \overset{m}{\ldots}, \hat{x} \circ \tilde{\sigma}, y_1 \circ \tilde{\sigma}, \ldots, y_{n-m} \circ \tilde{\sigma}) = A_P(\hat{x}, \overset{m}{\ldots}, \hat{x}, y_1, \ldots, y_{n-m}) = L(x).$$

Thus, $L(x \circ \sigma) = L(x)$ for every $\sigma \in \varXi$. The continuity of A_P implies that of L. Hence, L is a continuous symmetric m-homogeneous polynomial on L_∞. By the induction hypothesis, L can be represented as

$$L(x) = \sum_{k_1 + 2k_2 + \cdots + mk_m = m} \alpha^{([a,b])}_{k_1,\ldots,k_m}(y_1, \ldots, y_{n-m}) R_1^{k_1}(x) \cdots R_m^{k_m}(x),$$

where the coefficients depend on $[a,b]$ and $y_1, \ldots, y_{n-m}$.

By proposition 4.2, there exist functions $x_j \in L_\infty$ and constants $u^{(j)}_{k_1,\ldots,k_m} \in \mathbb{C}$, where $j \in \{1, \ldots, (m+1)^m\}$ and $k_1, \ldots, k_m \in \{0, \ldots, m\}$, such that

$$\alpha^{([a,b])}_{k_1,\ldots,k_m}(y_1, \ldots, y_{n-m}) = \sum_{j=1}^{(m+1)^m} u^{(j)}_{k_1,\ldots,k_m} L(x_j).$$

Thus,

$$\alpha^{([a,b])}_{k_1,\dots,k_m}(y_1,\dots,y_{n-m}) = \sum_{j=1}^{(m+1)^m} u^{(j)}_{k_1,\dots,k_m} A_P(\widehat{x_j},\overset{m}{\dots},\widehat{x_j},y_1,\dots,y_{n-m}).$$

This implies that $\alpha^{([a,b])}_{k_1,\dots,k_m}\colon (L_\infty)^{n-m}\to\mathbb{C}$ is a continuous $(n-m)$-linear mapping. Therefore, by (4.4),

$$|\alpha^{([a,b])}_{k_1,\dots,k_m}(y_1,\dots,y_{n-m})| \leqslant \|\alpha^{([a,b])}_{k_1,\dots,k_m}\|\|y_1\|_\infty\cdots\|y_{n-m}\|_\infty.$$

Let

$$\hat{E} = \left\{\frac{t-a}{b-a}\colon t\in E\right\}.$$

Since $E\subset[a,b]$, it follows that $\hat{E}\subset[0,1]$. Since

$$\widehat{\mathbf{1}_{\hat{E}}} = \mathbf{1}_E,$$

it follows that

$$\begin{aligned} A_P(\mathbf{1}_E,\overset{m}{\dots},\mathbf{1}_E,y_1,\dots,y_{n-m}) &\\ &= A_P(\widehat{\mathbf{1}_{\hat{E}}},\overset{m}{\dots},\widehat{\mathbf{1}_{\hat{E}}},y_1,\dots,y_{n-m}) = L(\mathbf{1}_{\hat{E}})\\ &= \sum_{k_1+2k_2+\cdots+mk_m=m} \alpha^{([a,b])}_{k_1,\dots,k_m}(y_1,\dots,y_{n-m})R_1^{k_1}(\mathbf{1}_{\hat{E}})\cdots R_m^{k_m}(\mathbf{1}_{\hat{E}})\\ &= \sum_{k_1+2k_2+\cdots+mk_m=m} \alpha^{([a,b])}_{k_1,\dots,k_m}(y_1,\dots,y_{n-m})\mu(\hat{E})^{k_1+k_2+\cdots+k_m}. \end{aligned}$$

Taking into account that

$$\mu(\hat{E})^{k_1+k_2+\cdots+k_m} \leqslant \mu(\hat{E}) = \frac{1}{b-a}\mu(E),$$

we have

$$\begin{aligned} |A_P(\mathbf{1}_E,\overset{m}{\dots},\mathbf{1}_E,y_1,\dots,y_{n-m})| &\\ &\leqslant \frac{1}{b-a}\mu(E)\|y_1\|_\infty\cdots\|y_{n-m}\|_\infty \sum_{k_1+2k_2+\cdots+mk_m=m}\|\alpha^{([a,b])}_{k_1,\dots,k_m}\|. \end{aligned}$$

Defining

$$C_1(m,a,b) = \frac{1}{b-a}\sum_{k_1+2k_2+\cdots+mk_m=m}\|\alpha^{([a,b])}_{k_1,\dots,k_m}\|,$$

we obtain the result. □

LEMMA 4.5. *Let $1\leqslant m<n$ and $[a,b],[c_1,d_1],[c_2,d_2]\subset[0,1]$ be such that $\mu([a,b]\cap[c_1,d_1])=0$, $\mu([a,b]\cap[c_2,d_2])=0$, $\mu([c_1,d_1]\cap[c_2,d_2])=0$ and $\mu([c_1,d_1]\cup[c_2,d_2])\leqslant\mu([a,b])$. Let $y_1,\dots,y_{n-m}\in L_\infty$ be such that the restrictions of $y_1,\dots,y_{n-m}$ to $[a,b]\cup[c_1,d_1]\cup[c_2,d_2]$ are constant. Then there exists a constant $C(m,a,b)>0$ such that*

$$\begin{aligned} |A_P(\mathbf{1}_{[c_1,d_1]\cup[c_2,d_2]},\overset{m}{\dots},\mathbf{1}_{[c_1,d_1]\cup[c_2,d_2]},y_1,\dots,y_{n-m})| &\\ &\leqslant \mu([c_1,d_1]\cup[c_2,d_2])\|y_1\|_\infty\cdots\|y_{n-m}\|_\infty C(m,a,b). \end{aligned}$$

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Proof. Let

$$\sigma_1(t) = \begin{cases} a + t - c_1 & \text{if } t \in (c_1, d_1), \\ c_1 + t - a & \text{if } t \in (a, a + d_1 - c_1), \\ a + d_1 - c_1 + t - c_2 & \text{if } t \in (c_2, d_2), \\ c_2 + t - (a + d_1 - c_1) & \text{if } t \in (a + d_1 - c_1, a + d_1 - c_1 + d_2 - c_2), \\ t & \text{otherwise.} \end{cases}$$

Evidently, $\sigma_1 \in \varXi$.

By proposition 4.1,

$$A_P(\mathbf{1}_{[c_1,d_1]\cup[c_2,d_2]}, \overset{m}{\ldots}, \mathbf{1}_{[c_1,d_1]\cup[c_2,d_2]}, y_1, \ldots, y_{n-m}) \\ = A_P(\mathbf{1}_{[c_1,d_1]\cup[c_2,d_2]} \circ \sigma_1, \overset{m}{\ldots}, \mathbf{1}_{[c_1,d_1]\cup[c_2,d_2]} \circ \sigma_1, y_1 \circ \sigma_1, \ldots, y_{n-m} \circ \sigma_1).$$

Since $\mathbf{1}_{[c_1,d_1]\cup[c_2,d_2]} \circ \sigma_1 = \mathbf{1}_{[a,a+d_1-c_1+d_2-c_2]}$ and $y_j \circ \sigma_1 = y_j$ for every $j \in \{1, \ldots, n-m\}$, it follows that

$$A_P(\mathbf{1}_{[c_1,d_1]\cup[c_2,d_2]}, \overset{m}{\ldots}, \mathbf{1}_{[c_1,d_1]\cup[c_2,d_2]}, y_1, \ldots, y_{n-m}) \\ = A_P(\mathbf{1}_{[a,a+d_1-c_1+d_2-c_2]}, \overset{m}{\ldots}, \mathbf{1}_{[a,a+d_1-c_1+d_2-c_2]}, y_1, \ldots, y_{n-m}).$$

By lemma 4.4, there exists a constant $C_1(m, a, b) > 0$ such that

$$|A_P(\mathbf{1}_{[a,a+d_1-c_1+d_2-c_2]}, \overset{m}{\ldots}, \mathbf{1}_{[a,a+d_1-c_1+d_2-c_2]}, y_1, \ldots, y_{n-m})| \\ \leqslant \mu([a, a + d_1 - c_1 + d_2 - c_2]) \|y_1\|_\infty \cdots \|y_{n-m}\|_\infty C_1(m, a, b).$$

We set $C(m, a, b) = C_1(m, a, b)$. This completes the proof of the lemma. □

LEMMA 4.6. *There exists a sequence* $\{s_k\}_{k=1}^\infty \subset [0, 1]$ *such that* $\lim_{k\to\infty} s_k = 0$ *and, for every sequence* $\{r_k\}_{k=1}^\infty$ *such that* $0 \leqslant r_k \leqslant s_k$,

$$|P(\mathbf{1}_{[0,r_k]})| \leqslant \frac{1}{k}(\|P\| + 1).$$

Proof of lemma 4.6. We set $s_1 = 1$. By the continuity of P,

$$|P(\mathbf{1}_{[0,r_1]})| \leqslant \|P\| < \|P\| + 1$$

for every $0 \leqslant r_1 \leqslant s_1$.

For $k \geqslant 2$ let $t \geqslant 0$ be such that $kt < \frac{1}{2}$. Since $\mathbf{1}_{[0,kt]} = \sum_{j=1}^k \mathbf{1}_{[(j-1)t,jt]}$, by the polynomial formula (4.2),

$$P(\mathbf{1}_{[0,kt]}) = \sum_{n_1+n_2+\cdots+n_k=n} \frac{n!}{n_1!n_2!\cdots n_k!} \\ \times A_P(\mathbf{1}_{[0,t]}, \overset{m}{\ldots}, \mathbf{1}_{[0,t]}, \mathbf{1}_{[t,2t]}, \overset{n_2}{\ldots}, \mathbf{1}_{[t,2t]}, \ldots, \mathbf{1}_{[(k-1)t,kt]}, \overset{n_k}{\ldots}, \mathbf{1}_{[(k-1)t,kt]}),$$

where $n_1, n_2, \ldots, n_k \in \mathbb{N} \cup \{0\}$. For every multi-index $(n_1, n_2, \ldots, n_k)$ let

$$l = l_{(n_1,n_2,\ldots,n_k)} \in \{1, \ldots, k\}$$

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be the position of the first non-zero element in $(n_1, n_2, \dots, n_k)$. The sum of the addends for which $n_l = n$ is equal to $kP(\mathbf{1}_{[0,t]})$ due to the symmetry of P. Therefore,

$$kP(\mathbf{1}_{[0,t]}) = P(\mathbf{1}_{[0,kt]}) - \sum_{\substack{n_1+n_2+\cdots+n_k=n\\ n_l<n}} \frac{n!}{n_1!n_2!\cdots n_k!} \times A_P(\mathbf{1}_{[0,t]}, \overset{n_1}{\dots}, \mathbf{1}_{[0,t]}, \mathbf{1}_{[t,2t]}, \overset{n_2}{\dots}, \mathbf{1}_{[t,2t]}, \dots, \mathbf{1}_{[(k-1)t,kt]}, \overset{n_k}{\dots}, \mathbf{1}_{[(k-1)t,kt]}).$$

If $n_l < n$, then by lemma 4.5, in which we set $m = n_l$, $[a, b] = [\frac{1}{2}, 1]$, $[c_1, d_1] = [(l-1)t, lt]$ and $c_2 = d_2$,

$$|A_P(\mathbf{1}_{[0,t]}, \overset{n_1}{\dots}, \mathbf{1}_{[0,t]}, \mathbf{1}_{[t,2t]}, \overset{n_2}{\dots}, \mathbf{1}_{[t,2t]}, \dots, \mathbf{1}_{[(k-1)t,kt]}, \overset{n_k}{\dots}, \mathbf{1}_{[(k-1)t,kt]})| \leqslant tC(n_l, \tfrac{1}{2}, 1).$$

Taking into account that $|P(\mathbf{1}_{[0,kt]})| \leqslant \|P\|$, we have

$$|kP(\mathbf{1}_{[0,t]})| \leqslant \|P\| + t \sum_{\substack{n_1+n_2+\cdots+n_k=n\\ n_l<n}} \frac{n!}{n_1!n_2!\cdots n_k!} C(n_l, \tfrac{1}{2}, 1).$$

We set

$$s_k = \min\left\{(2k+1)^{-1}, \left(\sum_{\substack{n_1+n_2+\cdots+n_k=n\\ n_l<n}} \frac{n!}{n_1!n_2!\cdots n_k!} C(n_l, \tfrac{1}{2}, 1)\right)^{-1}\right\}.$$

Then, for $0 \leqslant t \leqslant s_k$,

$$k|P(\mathbf{1}_{[0,t]})| \leqslant \|P\| + 1.$$

Thus,

$$|P(\mathbf{1}_{[0,r_k]})| \leqslant \frac{1}{k}(\|P\| + 1).$$

□

Lemma 4.7. *Let*

$$x = \sum_{j=1}^{N} h_j \mathbf{1}_{[a_j,b_j]},$$

where $h_j \in \mathbb{C}$ for $j \in \{1, \dots, N\}$ and $a_j < b_j \leqslant a_{j+1}$ for $j \in \{1, \dots, N-1\}$. Then, for every $l \in \{1, \dots, N\}$ and for all sequences

$$\{a_l^{(k)}\}_{k=1}^{\infty}, \qquad \{b_l^{(k)}\}_{k=1}^{\infty},$$

such that $a_l \leqslant a_l^{(k)} \leqslant a_l + \frac{1}{2}\min\{s_k, b_l - a_l\}$ and $b_l - \frac{1}{2}\min\{s_k, b_l - a_l\} \leqslant b_l^{(k)} \leqslant b_l$,

$$\lim_{k\to\infty} P(x^{(k)}) = P(x),$$

where

$$x^{(k)} = \sum_{j=1, j\neq l}^{N} h_j \mathbf{1}_{[a_j,b_j]} + h_l \mathbf{1}_{[a_l^{(k)}, b_l^{(k)}]}.$$

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Proof of lemma 4.7. Let $\delta^{(k)} = -h_l \mathbf{1}_{[a_l, a_l^{(k)}]} \cup [b_l^{(k)}, b_l]$. Then $x^{(k)} = x + \delta^{(k)}$. By the binomial formula (4.3),

$$P(x^{(k)}) = P(x) + P(\delta^{(k)}) + \sum_{m=1}^{n-1} \binom{n}{m} A_P(\delta^{(k)}, \overset{m}{\ldots}, \delta^{(k)}, x, \overset{n-m}{\ldots}, x).$$

Since P is a symmetric n-homogeneous polynomial, it follows that

$$P(\delta^{(k)}) = (-h_l)^n P(\mathbf{1}_{[a_l, a_l^{(k)}] \cup [b_l^{(k)}, b_l]}) = (-h_l)^n P(\mathbf{1}_{[0, a_l^{(k)} - a_l + b_l - b_l^{(k)}]}).$$

Since $a_l^{(k)} - a_l + b_l - b_l^{(k)} \leqslant s_k$, we have by lemma 4.6 that

$$|P(\mathbf{1}_{[0, a_l^{(k)} - a_l + b_l - b_l^{(k)}]})| \leqslant \frac{1}{k}(\|P\| + 1).$$

Thus,

$$|P(\delta^{(k)})| \leqslant \frac{1}{k}|h_l|^n(\|P\| + 1).$$

By lemma 4.5, where $[c_1, d_1] = [a_l, a_l^{(k)}]$, $[c_2, d_2] = [b_l^{(k)}, b_l]$, $[a, b] = [a_l, b_l]$ and $y_1 = \cdots = y_{n-m} = x$,

$$\begin{aligned} |A_P(\delta^{(k)}, \overset{m}{\ldots}, \delta^{(k)}, x, \overset{n-m}{\ldots}, x)| &= |h_l|^m |A_P(\mathbf{1}_{[a_l, a_l^{(k)}] \cup [b_l^{(k)}, b_l]}, \overset{m}{\ldots}, \mathbf{1}_{[a_l, a_l^{(k)}] \cup [b_l^{(k)}, b_l]}, x, \overset{n-m}{\ldots}, x)| \\ &\leqslant \mu([a_l, a_l^{(k)}] \cup [b_l^{(k)}, b_l]) |h_l|^m \|x\|_\infty^{n-m} C(m, a_l, b_l) \\ &\leqslant s_k |h_l|^m \|x\|_\infty^{n-m} C(m, a_l, b_l). \end{aligned}$$

Therefore,

$$|P(x^{(k)}) - P(x)| \leqslant \frac{1}{k}|h_l|^n(\|P\| + 1) + s_k \sum_{m=1}^{n-1} \binom{n}{m} |h_l|^m \|x\|_\infty^{n-m} C(m, a_l.b_l).$$

Recall that $\lim_{k\to\infty} s_k = 0$. Thus,

$$\lim_{k\to\infty} P(x^{(k)}) = P(x).$$

□

We now continue with the proof of theorem 4.3. For $M \in \mathbb{N}$ we set

$$G_M = \left\{ \sum_{j=1}^{2^M} d_j \mathbf{1}_{[(j-1)/2^M, j/2^M]} \colon d_j \in \mathbb{C} \right\} \quad \text{and} \quad G = \bigcup_{M=1}^{\infty} G_M.$$

Let

$$g_M(d_1, \ldots, d_{2^M}) = P\biggl(\sum_{j=1}^{2^M} d_j \mathbf{1}_{[(j-1)/2^M, j/2^M]} \biggr).$$

Clearly, g_M is a symmetric polynomial of degree n of 2^M scalar variables. It is well known (see, for example, [12, ch. XI, § 53]) that there exists a polynomial p_M such

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that

$$g_M(d_1,\dots,d_{2^M}) = p_M\left(\frac{1}{2^M}\sum_{j=1}^{2^M} d_j, \frac{1}{2^M}\sum_{j=1}^{2^M} d_j^2, \dots, \frac{1}{2^M}\sum_{j=1}^{2^M} d_j^n\right).$$

Hence, for every $x \in G_M$,

$$P(x) = p_M(R_1(x), R_2(x), \dots, R_n(x)).$$

That is,

$$P(x) = \sum_{k_1+2k_2+\cdots+nk_n=n} \alpha_{k_1,\dots,k_n} R_1^{k_1}(x) R_2^{k_2}(x)\cdots R_n^{k_n}(x). \tag{4.8}$$

Since $G_M \subset G_{M'}$ for $M \leqslant M'$, it follows that coefficients $\alpha_{k_1,\dots,k_n}$ do not depend on M. Thus, (4.8) holds for every $x \in G$.

Let

$$D = \left\{\frac{K}{2^M} \colon M \in \mathbb{N},\ K \in \{0,1,\dots,2^M\}\right\}.$$

and

$$x = \sum_{j=1}^N h_j \mathbf{1}_{[a_j,b_j]},$$

where $h_j \in \mathbb{C}$ for $j \in \{1,\dots,N\}$, $a_j < b_j \leqslant a_{j+1}$ for $j \in \{1,\dots,N-1\}$.

For every $l \in \{1,\dots,N\}$ we choose sequences $\{a_l^{(k)}\}_{k=1}^\infty, \{b_l^{(k)}\}_{k=1}^\infty \subset D$ such that

$$a_l \leqslant a_l^{(k)} \leqslant a_l + \tfrac{1}{2}\min\{s_k, b_l - a_l\},$$
$$b_l - \tfrac{1}{2}\min\{s_k, b_l - a_l\} \leqslant b_l^{(k)} \leqslant b_l.$$

Then, for every multi-index $\varkappa = (\varkappa_1, \varkappa_2, \dots, \varkappa_N) \in \mathbb{N}^N$, the function

$$x_\varkappa = \sum_{j=1}^N h_j \mathbf{1}_{\left[a_j^{(\varkappa_j)}, b_j^{(\varkappa_j)}\right]}$$

belongs to G. By using lemma 4.7 N times,

$$\begin{aligned} P(x) &= \lim_{\varkappa_1\to\infty}\lim_{\varkappa_2\to\infty}\cdots\lim_{\varkappa_N\to\infty} P(x_\varkappa) \\ &= \lim_{\varkappa_1\to\infty}\lim_{\varkappa_2\to\infty}\cdots\lim_{\varkappa_N\to\infty} \sum_{k_1+2k_2+\cdots+nk_n=n} \alpha_{k_1,\dots,k_n} R_1^{k_1}(x_\varkappa)\cdots R_n^{k_n}(x_\varkappa) \\ &= \sum_{k_1+2k_2+\cdots+nk_n=n} \alpha_{k_1,\dots,k_n} R_1^{k_1}(x)\cdots R_n^{k_n}(x). \end{aligned}$$

Now let

$$x = \sum_{j=1}^N h_j \mathbf{1}_{E_j},$$

where $E_1, E_2, \dots, E_N$ are disjoint measurable subsets of $[0,1]$. By proposition 2.2, there exists $\sigma = \sigma_{E_1,\dots,E_N} \in \varXi$ such that

$$\mathbf{1}_{E_m} = \mathbf{1}_{[\sum_{j=1}^{m-1}\mu(E_j), \sum_{j=1}^{m}\mu(E_j)]} \circ \sigma_{E_1,\dots,E_N}$$

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for every $m \in \{1, \ldots, N\}$ almost everywhere on $[0, 1]$. By the symmetry of P,

$$P(x) = P(x \circ \sigma^{-1}) = P\biggl(\sum_{j=1}^{N} h_j \mathbf{1}_{[\sum_{m=1}^{j-1} \mu(A_m), \sum_{m=1}^{j} \mu(A_m)]}\biggr)$$
$$= \sum_{k_1+2k_2+\cdots+nk_n=n} \alpha_{k_1,\ldots,k_n} R_1^{k_1}(x) \cdots R_n^{k_n}(x).$$

Hence, for every simple measurable function x, (4.8) holds. Since the set of simple measurable functions is dense in L_∞, the continuity of P yields that (4.8) holds for every $x \in L_\infty$. This completes the proof of theorem 4.3. □

The statement of theorem 4.3 for symmetric linear functionals is proved directly in [17].

Corollary 4.8. *Let $P_1, \ldots, P_m$ be symmetric polynomials on L_∞ such that*

$$\bigcap_{i=1}^{m} \operatorname{Ker}(P_i) = \emptyset.$$

Then there are symmetric polynomials $Q_1, \ldots, Q_m$ such that $\sum_{i=1}^{m} P_i Q_i = 1$.

Proof. Let $n = \max(\deg(P_i))$. Since (R_j) is an algebraic basis for the symmetric polynomials, there exist $p_i \in \mathcal{P}(\mathbb{C}^n)$ such that $P_i(x) = p_i(R_1(x), \ldots, R_n(x))$. If $p_i(\theta_1, \ldots, \theta_n) = 0$ for all $i = 1, \ldots, m$ for some point $(\theta_1, \ldots, \theta_n) \in \mathbb{C}^n$, then by theorem 3.1 there exists $x_\theta \in L_\infty$ such that $R_i(x_\theta) = \theta_i$ for all $i = 1, \ldots, m$. So x_θ would be a common zero of the P_i. Hence, the p_i have no common zeros, and thus by the Hilbert Nullstellensatz there are polynomials $q_1, \ldots, q_m \in \mathcal{P}(\mathbb{C}^n)$ such that $\sum_{i=1}^{m} p_i q_i = 1$. Set $Q_i = q_i(R_1, \ldots, R_n)$ to complete the proof. □

5. The space $H_{\mathrm{bs}}(L_\infty)$ and its spectrum

Let $H_{\mathrm{bs}}(L_\infty)$ be the Fréchet algebra of all entire symmetric functions $F\colon L_\infty \to \mathbb{C}$ which are bounded on bounded sets endowed with the topology of uniform convergence on bounded sets. Every such entire function can be described by its Taylor series of continuous homogeneous polynomials which in turn are symmetric as well. Therefore, by theorem 4.3, every $F \in H_{\mathrm{bs}}(L_\infty)$ can be represented as

$$F(x) = \sum_{n=0}^{\infty} \sum_{\substack{k_1+2k_2+\cdots+nk_n=n \\ k_j \geqslant 0}} \alpha_{k_1 k_2 \cdots k_n} R_1^{k_1}(x) R_2^{k_2}(x) \cdots R_n^{k_n}(x),$$

where $\alpha_{k_1 k_2 \cdots k_n} \in \mathbb{C}$ and the series converges uniformly on bounded sets.

Next, we point out some topological properties of functions in $H_{\mathrm{bs}}(L_\infty)$.

Proposition 5.1. *For every $F \in H_{\mathrm{bs}}(L_\infty)$ we have that*

(a) *its derivative mapping $\mathrm{d}F$ is weakly compact,*

(b) $\lim_j F(u_j) - F(v_j) = 0$ *for bounded sequences (u_j) and (v_j) such that $(u_j - v_j)$ is weakly null.*

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Proof. Let us first verify the statements for R_n. The multilinear form

$$(x_1,\dots,x_n)\in L_n\times\overset{n}{\cdots}\times L_n \xrightarrow{A} \int_0^1 x_1(t)\cdots x_n(t)\,\mathrm{d}t$$

is well defined and continuous. Recall that L_∞ embeds continuously into $L_p[0,1]$ for any $p>1$, since $|x|^p \leqslant \|x\|_\infty^p \chi_{[0,1]}$. Therefore, the polynomials

$$x\in L_\infty \overset{A_i}{\mapsto} A(x,\overset{i}{\dots},x,\cdot,\overset{n-i}{\cdots},\cdot)\in P(^{n-i}L_\infty)$$

factor through the reflexive space $L_n[0,1]$. Thus, they are weakly compact. In particular, the derivative mapping $\mathrm{d}R_n\colon L_\infty\to L_\infty^*$ given by $\mathrm{d}R_n(x)(y)=nA(x\dots,x,y)$ is weakly compact.

To check (b), recall that L_∞ has the Dunford–Pettis property. Hence, the weakly compact polynomials A_i are weakly sequentially continuous, that is,

$$\lim_j A_i(u_j-v_j,\overset{i}{\dots},u_j-v_j,\cdot,\overset{n-i}{\cdots},\cdot)=0\quad\text{for } i=1,\dots,n-1.$$

Now, use the binomial formula (4.3) to see that

$$\lim_j P(u_j)-P(v_j)=\lim_j P(u_j-v_j+v_j)-P(v_j)=0.$$

It is immediate that the derivative of the product of finitely many polynomials with weakly compact derivative (respectively, satisfying statement (b)) has weakly compact derivative (respectively, satisfies (b)). Hence, by theorem 4.3, every symmetric polynomial has a weakly compact derivative and satisfies (b).

Finally, for the Taylor series of F at 0, $F=\sum P_n$, that is made up of symmetric polynomials, we have that $\mathrm{d}F=\sum \mathrm{d}P_n$. Thus, we use [16, theorem 3.2] to deduce that $\mathrm{d}F$ is weakly compact. Moreover, (b) holds by suitably approximating F by symmetric polynomials. □

Note that there are (necessarily non-symmetric) polynomials on L_∞ whose derivatives are not weakly compact (see [5, remark 3.3(b)]) or do not satisfy (b) [4, example 2.4].

We denote by M_{bs} the spectrum of $H_{\mathrm{bs}}(L_\infty)$, that is, the set of all continuous complex-valued homomorphisms (characters) on $H_{\mathrm{bs}}(L_\infty)$. As usual, we shall consider M_{bs} endowed with the weak* topology, that is, the topology of convergence against functions in $H_{\mathrm{bs}}(L_\infty)$. For every character $\phi\in M_{\mathrm{bs}}$ we have

$$\phi(F)=\sum_{n=0}^{\infty}\sum_{\substack{k_1+2k_2+\cdots+nk_n=n\\ k_j\geqslant 0}}\alpha_{k_1k_2\cdots k_n}\phi(R_1)^{k_1}\phi(R_2)^{k_2}\cdots\phi(R_n)^{k_n}.$$

Thus, we see that ϕ is completely defined by its values on R_j ($j\in\mathbb{N}$). Hence, we can identify every $\phi\in M_{\mathrm{bs}}$ with the sequence $(\xi_1,\xi_2,\dots,\xi_n,\dots)$, where $\xi_j=\phi(R_j)$ ($j\in\mathbb{N}$).

Proposition 5.2. *Let $R(\phi)$ be the radius of $\phi\in M_{\mathrm{bs}}$. Then $|\xi_n|\leqslant R(\phi)^n$ for every $n\in\mathbb{N}$.*

https://doi.org/10.1017/S0308210516000287 Published online by Cambridge University Press

Proof. Let B denote the unit ball in L_∞. Since $|\phi(F)| \leqslant \|F\|_{R(\phi)B}$, and $\|R_n\|_B = 1$, we have $|\xi_n| = |\phi(R_n)| \leqslant \|R_n\|_{R(\phi)B} = R(\phi)^n$. □

Note that for $x \in L_\infty$ the point-evaluation functional δ_x defined by

$$\delta_x(F) = F(x), \quad F \in H_{\mathrm{bs}}(L_\infty),$$

belongs to M_{bs}. For every sequence $\{\xi_n\}_{n=1}^\infty \subset \mathbb{C}$ such that $\{\sqrt[n]{|\xi_n|}\}_{n=1}^\infty$ is bounded, there exists, by theorem 3.1, $x_\xi \in L_\infty$ such that $R_n(x_\xi) = \xi_n$ for every $n \in \mathbb{N}$. Hence, we have the following.

Corollary 5.3. *Every $\phi \in M_{\mathrm{bs}}$ is a point-evaluation functional.*

Corollary 5.4. *The set M_{bs} can be identified with the set $\Delta \subset \mathbb{C}^{\mathbb{N}}$ of all sequences $\{\xi_n\}_{n=1}^\infty \subset \mathbb{C}$ such that $\{\sqrt[n]{|\xi_n|}\}_{n=1}^\infty$ is bounded.*

Corollary 5.5. *The bounded subsets of M_{bs} are separable. Consequently, M_{bs} itself is separable.*

Proof. By the uniform boundedness principle the bounded subsets S of M_{bs} are equicontinuous and hence weak* relatively compact, and as a consequence the weak* topology coincides on them with the Hausdorff topology of the convergence against the symmetric polynomials.

Denote by Λ the identification mapping $\{\xi_n\}_{n=1}^\infty \in \Delta \to \phi \in M_{\mathrm{bs}}$. Endow $\Lambda^{-1}(S)$ with the induced product topology of $\mathbb{C}^{\mathbb{N}}$ that is a separable metrizable space. Then $\Lambda^{-1}(S)$ is a separable space. We shall deduce that S is separable just by checking that Λ is continuous on $\Lambda^{-1}(S)$. Indeed, let $(\{\xi_n^k\}_{n=1}^\infty)_k \in \Lambda^{-1}(S)$ be a convergent sequence to, say, $\{\mu_n\}_{n=1}^\infty \in \Lambda^{-1}(S)$. Then, for $\phi_k := \Lambda(\{\xi_n^k\})$ and $\omega := \Lambda(\{\mu_n\})$, we have

$$\lim_k \phi_k(R_n) = \lim_k \xi_n^k = \mu_n = \omega(R_n),$$

and also, according to theorem 4.3, $\lim_k \phi_k(P) = \omega(P)$ for all symmetric polynomials P. Thus, by the comment above, $\lim_k \phi_k = \omega$ for the weak* topology, as required.

To check that M_{bs} is separable, recall that there is a sequence $(L_i)_i$ of weak* compact subsets of M_{bs} such that $M_{\mathrm{bs}} = \bigcup_i L_i$, e.g. $L_i = \{\phi \in M_{\mathrm{bs}} \colon R(\varphi) \leqslant i\}$. Since for each of the L_i there is a countable dense subset D_i, it follows that $\bigcup_i D_i$ is a dense subset of M_{bs}. □

6. Functions of exponential type

From the general theory of entire functions of exponential type we have the following corollary.

Corollary 6.1. *The set M_{bs} can be identified with the set $\mathrm{Exp}(\mathbb{C})$ of all functions of exponential type on $\mathbb{C}$ vanishing at the origin by*

$$M_{\mathrm{bs}} \ni \phi \rightsquigarrow \sum_{n=1}^\infty \frac{\xi_n z^n}{n!} = \sum_{n=1}^\infty \frac{\phi(R_n) z^n}{n!} \in \mathrm{Exp}(\mathbb{C}).$$

The next theorem gives a new representation of functions of exponential type.

THEOREM 6.2. *Let $g\colon \mathbb{C} \to \mathbb{C}$ be a function of exponential type with $g(0) = 1$. Then there exists $x \in L_\infty$ such that*

$$g(z) = 1 + \sum_{n=1}^{\infty} \frac{R_n(x) z^n}{n!} = \int_0^1 \mathrm{e}^{zx(t)}\,\mathrm{d}t, \quad z \in \mathbb{C}.$$

Proof. The proof follows from corollaries 5.3 and 6.1 and direct calculations. □

Note that this representation is not unique because for example $R_n(\mathrm{e}^{2\pi \mathrm{i} t}) = 0$ for every n.

By [15], every Lebesgue–Rohlin space with continuous measure $(\Omega, \mathcal{F}, \nu)$ is isomorphic modulo zero to $[0, 1]$ with Lebesgue measure. Therefore, all of the results of this work are also valid for L_∞ on a Lebesgue–Rohlin space with continuous measure.

Acknowledgements

The work of P.G. is partly supported by Grant no. MINECO-MTM2014-053241-P, and that of T.V. and A.Z. is partly supported by MESU Grant no. 0115U002305.

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