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REGULARITY AND STABILITY OF EQUILIBRIA IN AN OVERLAPPING GENERATIONS GROWTH MODEL

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In an exogenous-growth economy with overlapping generations, the Cobb–Douglas production, any positive life-cycle productivity, and time-separable constant elasticity of substitution (CES) utility, we analyze local stability of a balanced growth equilibrium (BGE) with respect to changes in consumption endowments, which could be interpreted as a transfer policy. We show that generically, in the space of parameters, equilibria around a BGE are locally unique and are locally differentiable functions of endowments, with derivatives given by kernels. Furthermore, those equilibria are stable in the sense that the effects of temporary changes decay exponentially toward $\pm\infty$.

Keywords: Regularity of Infinite Economies, Policy Evaluation, Overlapping Generations, Exogenous Growth

1. INTRODUCTION

1.1. Motivation and Some Related Literature

Whether the task is to analyze a pension reform, a change in social security system, or an environmental project with a longer than human life, it is impossible to conduct the analysis without a model that distinguishes between generations and does not rely too heavily on an assumption that individual lives are infinite. Policy analysis in overlapping generations models brings new insights that are lacking in static or representative agents' models [as is also stressed in De La Croix and Michel (2002); Erosa and Gervais (2002)], largely due to the built-in agents'

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heterogeneity: the savings-consumption trade-off varies over the life cycle and in an overlapping generation model agents of all ages are present at any point in time. The basic variant of such models was introduced in Allais (1947) and Samuelson (1958); see De La Croix and Michel (2002); Kotlikoff (2002) for an overview. Working with overlapping generations models has a cost however, since they are prone to multiplicity (possibly continua) of equilibria [Kehoe and Levine (1985); Geanakoplos and Polemarchakis (1991)], even in the presence of capital accumulation [Muller and Woodford (1988)], therefore potentially invalidating "comparative dynamics" exercises.

Our main goal is to provide a first step to enable "comparative dynamics" in an overlapping generations model with production (and capital accumulation). We show that *it is possible* to compute the reaction of a competitive equilibrium to changes in policy when time is taken as a real line, as opposed to the models where indeterminacy results are established—there time proceeds in discrete jumps and is being truncated at 0. A particular variant of such a model comes back to Cass and Yaari (1967), and has experienced some recent development (d'Albis and Augeraud-Véron 2007, 2009). Probably, the closest contribution (based on the defined objectives) is by Burke (1990), who considers a discrete-time, but "eternal" economy and establishes determinacy of equilibria, in contrast to the previous literature, Kehoe and Levine (1985). Burke observes that only in a world where time is not truncated one can model truly perfect foresight. Fully anticipated policies in such a model can lead to a predictable "smooth" change in equilibrium. More positive results in this respect appear in Demichelis and Polemarchakis (2007) demonstrating that indeterminacy [as in Kehoe and Levine (1985)] disappears in an exchange model (with no production) when time is extended infinitely far into the past and the gap in time between transactions tends to zero.[1]

This is the main reason we work with a continuous time (taken as $\mathbf{R}$) overlapping generations model to perform a classical (à la Debreu) comparative statics exercise asking when one should expect a smooth response of equilibrium variables to a "small" change in endowment perturbations, interpreted as a policy (transfers of consumption goods across individuals, including a net transfer into the economy). In contrast to the existing literature on overlapping generations, we do not seek to describe the equilibrium system as a dynamic system (with a finite memory); rather, we view equilibrium variables as elements of a vector space, very much like we do in a finite economy. The advantage is the ability to analyze policy changes enacted over some *interval of time*, as opposed to just one-period changes, "impulses," traditionally dealt with in dynamic systems, and to identify the speed of convergence of the perturbed equilibrium to the status-quo (which should eventually yield differentiability of welfare). However, this requires some ground work, i.e., describing the spaces hosting the policy changes and the resulting equilibrium responses, done in Mertens and Rubinchik (2014), as well as characterizing equilibria of the model, done in Mertens and Rubinchik (2013).

The extension of the classical analysis of regularity (as in Debreu, 1970) for infinite economies based on Sard's theorem (Sard, 1942) has been focused on models with the finite number of consumers. Using Smale's extension (Smale, 1965) might impose too many restrictions for the overlapping generations (OG) models where consumption goods are indexed by time, real line, and so is the date of birth of each consumer. Furthermore, the argument suggested in Chichilnisky and Zhou (1998) and Covarrubias (2010) who use such an extension is based on a decomposition property of Fredholm maps, see e.g., Lang, 1969, IX, Sect. 2, Theorem 6. One of the elements in the decomposition should be invertible (with the other being finite dimensional) and verifying invertibility of a map in infinite dimensional spaces, in general, is a nontrivial task, which one might hope to circumvent by appealing to indirect arguments, such as transversality theorems in the first place. Indeed, it is the invertibility of the derivative of the equilibrium map with respect to "endogenous variables" that is needed in order to apply the appropriate implicit function theorem (IFT) which, in turn, yields regularity.

Shannon and Zame (2002) work with another extension of Sard's theorem (Shannon, 2006) that does not require the equilibrium map to be as smooth as in the classical variants of the theorem; rather, they impose just the Lipschitz continuity property. As a result, generic determinacy is achieved for a wider class of preferences (of consumers in an exchange economy). However, the analysis applies only to an economy with a finite number of consumers; besides, only Pareto equilibria are analyzed. The latter, recall, is a substantial restriction in OG economies wherein first welfare theorem does not apply.

Instead of extending Sard's theorem, we use Wiener's theorem on the spectrum of convolution operators to assure generic invertibility of the derivative of the equilibrium map required by the IFT.

Moreover, we use a single IFT to establish both regularity and stability of balanced growth equilibria: Solutions to the equilibrium fixed point equation are found to be smooth functions of endowment changes and converge exponentially back to the baseline equilibrium for a generic set of parameters describing an OG model. The new approach should help to identify a tractable way to verify regularity for a wide class of infinite economies.[2]

The most exciting part of the new approach is the ability to analytically evaluate the first-order (approximated) response of the equilibrium variables to the policy change, i.e., we provide a way to calculate it. We work with a fully parameterized classical model that admits balanced growth: time-separable CES utility, constant-returns-to-scale Cobb–Douglas production, exogenous growth, and a linear capital depreciation. The analysis is broken into very small steps and each is generalized to the extent possible within the scope of a single paper. All the tools presented here can be used for a wide range of other models except for our proof of genericity that relies on a specific "trick": yet even that might shed light on a way to generalize.

Let us stress that the approach we offer is in no way competing with a numerical one; rather, it is complementary to it. Trying to find an equilibrium corresponding to a change in policy might be futile in the absence of any regularity results, especially with indeterminacy lurking. But even when regularity is established, constructing a good algorithm to find a new equilibrium is a challenge, since the map that has to be at its fixed point in equilibrium is not necessarily a contraction, and so starting with a guess for, say, a capital path in the vicinity of an equilibrium and simply applying the map recursively might lead away from the equilibrium. On the other hand, some of the steps in the approach we suggest can potentially be done using numerical methods.

We start in Section 2 by briefly describing the overlapping generations model from Mertens and Rubinchik (2013). Borrowed from there also is the characterization of its equilibria, interior with respect to irreversibility constraints,[3] presented in Section 3. Then, follows Section 4, which contains the outline for the rest of the paper along with the related definitions. The proofs that are not in the text and intermediary formal statements with their proofs are in the online appendix.

2. THE MODEL

For the analysis, we use the general variant of the model in Mertens and Rubinchik (2013) with the Cobb–Douglas production and the characterization of the competitive equilibrium there.

The life span of any individual born at $x \in \mathbf{R}$ is $[0, 1]$.

$$U(\hat{c}_x) = \int_0^1 e^{-\beta s} u(\hat{c}_{x,s}) ds, \text{ with } u(z) = \frac{z^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}, \text{ for } \sigma \neq 1, ^4$$

is his life-time utility defined over the set of individual consumption plans $\hat{c}_{x,s}$, $\overline{\mathbf{R}}_+$-valued Lebesgue-measurable functions of age s for every x.

An individual derives income from renting labor and receiving transfers.[5] Efficiency of labor varies with age according to a nonnull integrable function $\zeta_s \geq 0$ over the life-span, $[0, 1]$, and zero elsewhere. His time sells for $\int_0^1 w_{x+s} \zeta_s ds$, where x is his birth date, and w_t is the (per unit efficiency) wage rate at time t. His initial endowment of consumption goods is $\omega_{x,s}$ at age s, ω is locally integrable.[6] So, with the Arrow–Debreu price of consumption goods denoted by p^C, the individual's lifetime wealth is the value of his endowment (of consumption goods and of leisure)

$$M_x \stackrel{\text{def}}{=} \int_0^1 (p^C_{x+s} \omega_{x,s} + w_{x+s} \zeta_s) ds,$$

provided that integral is well defined.[7] In the baseline (status-quo) equilibrium, the endowment is null.

The instantaneous production set is a subset of $\mathbf{R}^5$ describing feasible transformations of contracted productive labor L_t, capital K_t, investment I_t, consumption C_t, and an intermediate good Y_t called "output," produced using a Cobb–Douglas technology

$$Y_t = A K_t^{\alpha} L_t^{1-\alpha}, \ 0 < \alpha < 1, A > 0.$$

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Aggregate total *productive labor* available at t is

$$L_t = N_0 e^{\gamma t} \int_{t-1}^{t} \zeta_{t-x} e^{\nu x} dx = N_0 e^{(\gamma+\nu)t} \int_0^1 \zeta_s e^{-\nu s} ds,$$

where $N_x dx \stackrel{\text{def}}{=} N_0 e^{\nu x} dx$ ($N_0 > 0$) individuals are born in $[x, x + dx]$, $\forall x \in \mathbf{R}$, ν is the rate of the population growth, and γ is the per-capita productivity growth. Aggregate capital evolves according to the differential equation[8] $K_t' = I_t - \delta K_t$ with depreciation factor $\delta > 0$ and is subject to the following assumption.

Assumption 1 (Initial Condition). For any feasible K, $e^{\delta t} K_t$ converges exponentially to 0 at $-\infty$.

The production set is any closed convex cone with free-disposal, containing the graph of the production function and the activities of transforming output into consumption and investment, and contained in the closed convex cone spanned by the production function, free-disposal, and two-way transformations of output into consumption and investment.

Perfectly competitive firms have finite lives, for profits to be well defined. Price of output, p_t^Y, is assumed to be finite. Labor L_t is bought from individuals at price w_t, and capital K_t is rented from investment firms at rate r_t. Aggregate consumption at time t C_t is sold at price p_t^C to individuals, and I_t at price p_t^I to investment firms.[9]

Notation 1. $\eta = (\gamma + \nu)(1 - \sigma) + \beta\sigma$, $R = \gamma + \nu + \delta$. Intensive variables: $k_t = \frac{K_t}{L_t}$, $y_t = \frac{Y_t}{L_t}$, $i_t = \frac{I_t}{L_t}$, $c_t = \frac{C_t}{L_t}$, $\varphi_s = \frac{e^{-\nu s}\zeta_s}{\int_0^1 e^{-\nu u}\zeta_u du}$, $E_{t,s} = \frac{N_{t-s}\omega_{t-s,s}}{L_t}$,[10] $\Omega_t = \int_0^1 E_{t,s} ds$.

3. THE EQUILIBRIUM SYSTEM

3.1. The Policy Space and the Induced Equilibria

Competitive equilibrium is defined in the classical way. Its characterization, derived in Mertens and Rubinchik (2013), Corollaries 11–12, implies that, first, it is a solution of the equilibrium system, or a fixed point of the map defined below as Υ, and, second, it satisfies inequalities, which assure that the solution yields nonnegative consumption and does not violate (possible) irreversibility constraints imposed by technology (hence $0 < i_t < y_t$ a.e.). The characterization is summarized below in Proposition 1 with the corresponding references to the previous results.

Here, we focus on two types of equilibria. The first is the status-quo, prevailing in the absence of a "policy intervention," so that $E = 0$ (cf. Notation 2). We assume it is a *balanced growth equilibrium* (BGE), i.e., an equilibrium in which capital grows exponentially. The second is a (possibly nonstationary) equilibrium that emerges as a result of a perturbation of consumption endowments, or transfers, E, which we will refer to as a *perturbed equilibrium*.

Recall, our task is to represent an equilibrium (in the vicinity of the status-quo) as a smooth map (ϖ) from the space of exogenous policy variable, E, into the

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spaces of equilibrium quantities and prices (in case there are multiple equilibria, ϖ picks one). Naturally, equilibrium conditions as well as assumptions imposed on the exogenous variable, E, determine the spaces where the equilibrium variables belong. The transfers, E, specify the "amount" of consumption good given to (or taken from) a person who is of age s at time t; hence it, itself, is a map $\mathbf{R} \times [0, 1] \to \mathbf{R}$. Our very first take on this problem (Mertens and Rubinchik, 2009) was written assuming E is uniformly bounded, which we found restrictive. To allow for a richer set of transfer policies and yet work with tractable spaces, we derived and compiled the relevant properties of the so-called "amalgams" in Mertens and Rubinchik (2014). Here is the definition adapted for the analysis of this model.

DEFINITION 1 (amalgams). *i. Given a relatively compact measurable subset H of $\boldsymbol{R}^n$ with nonempty interior,*

$$L_{p,q} \stackrel{\text{def}}{=} \{f \text{ measurable} \colon \boldsymbol{R}^n \to \overline{\boldsymbol{R}} \mid \|f\|_{p,q} \stackrel{\text{def}}{=} \|x \mapsto \|\mathbb{1}_{x+H} f\|_q\|_p < \infty\},$$

mod null functions, for $1 \le p, q \le \infty$.[11]

ii. For $1 \le p \le \infty$, C_p is the subspace of continuous functions in $L_{p,\infty}$.

For example, for $h \colon \mathbf{R} \to \mathbf{R}$, let $H = [0, 1]$, then $\|h\|_{\infty,1} = \sup_x \int_{x-1}^{x} |h(t)| dt$. The policy variable E is defined on $\mathbf{R} \times [0, 1]$ and so $\|E\|_{\infty,1} = \sup_x \int_{x-1}^{x} \int_0^1 |E_{t,s}| ds dt$.

Assumption 2. $\|E\|_{\infty,1} < \infty$.

Note that the assumption does not require the volume of transfers to be either uniformly bounded or even totally summable over time.

3.2. Fixed Points of the Equilibrium Map, Υ, (Solutions) and Equilibria

Since the technology allows for irreversibility, the prices of consumption, investment, and output are not necessarily equal in an equilibrium; however the equilibria that will be characterized for our analysis here[12] are interior with respect to such constraints and so the physical good has the same price, p_t, in all its forms (consumption, intermediate, etc.) at any t.

The first part of the proposition shows that an equilibrium should be a solution to a fixed point (in k) of the map Υ defined there. The map is broken into 10 simple components each having a clear economic meaning. As a result, this necessary condition for equilibrium is described by a single equation, $\Upsilon(k, E) = k$.

PROPOSITION 1. *Given an endowment $E_{t,s} \in L_{\infty,1}$, define $\Upsilon \colon (k, E) \mapsto \tilde{k}$, from $C_\infty \times L_{\infty,1}$ to C_∞ as the composition of*

i. $k \mapsto y \colon y_t = A k_t^\alpha$,

ii. $k \mapsto \mathfrak{r} \colon \mathfrak{r}_t = R - \alpha A k_t^{\alpha-1}$ $(= R - \frac{\alpha y_t}{k_t}, = \gamma + \nu + \frac{p'_t}{p_t})$,

iii. $(\mathfrak{r}, E) \mapsto \mathcal{N}_1 \colon \mathcal{N}_{1,x} = \int_0^1 e^{\int_x^{x+s} \mathfrak{r}_t dt} E_{x+s,s} ds$,

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iv. $(y, \mathfrak{r}) \mapsto \mathcal{N}_2 \colon \mathcal{N}_{2,x} = \int_0^1 e^{\int_x^{x+s} \mathfrak{r}_t dt} \varphi_s y_{x+s} ds,$
v. $(\mathcal{N}_1, \mathcal{N}_2) \mapsto \mathcal{N} \colon \mathcal{N} = \mathcal{N}_1 + (1-\alpha)\mathcal{N}_2,\ \mathcal{N} \geq 0,$
vi. $\mathfrak{r} \mapsto \mathcal{D} \colon \mathcal{D}_x = \int_0^1 e^{-\eta s + (1-\sigma)\int_x^{x+s} \mathfrak{r}_t dt} ds,$
vii. $(\mathcal{N}, \mathcal{D}) \mapsto \mathcal{B} \colon \mathcal{B} = \frac{\mathcal{N}}{\mathcal{D}},\ \mathcal{B}_j = \frac{\mathcal{N}_j}{\mathcal{D}}\ (j = 1, 2),\ \mathcal{B} = \mathcal{B}_1 + (1-\alpha)\mathcal{B}_2,$
viii. $(\mathfrak{r}, \mathcal{B}) \mapsto c \colon c_t = \int_0^1 e^{-\eta u - \sigma \int_{t-u}^t \mathfrak{r}_s ds} \mathcal{B}_{t-u} du,$
xi. $(y, E, c) \mapsto i \colon i_t = y_t + \Omega_t - c_t,$
x. $i \mapsto \tilde{k} \colon \tilde{k}_t = e^{-Rt} \int_{-\infty}^t e^{Rs} i_s ds > 0.$

The prices then can be computed as follows:

$$p_t = p_0 e^{\int_0^t (\delta - R + \mathfrak{r}_s) ds}, \tag{1}$$

$$r_t = p_t (R - \mathfrak{r}_t), \tag{2}$$

$$w_t = (1-\alpha) e^{\gamma t} p_t. \tag{3}$$

The zeros of $F(k, E) \stackrel{\text{def}}{=} \Upsilon(k, E) - k$, *i.e., the fixed points of* Υ *(with implied values for* y, i, c, *etc.) characterize*

Interior Equilibria *all equilibria where* $0 < i_t < y_t$ *a.e., provided the fixed points satisfy* $0 < i_t < y_t$*;*
BGE *if* K_t *is exponential, all* BGE with $\omega = 0$.

Proof. The characterization of each of the equilibria is by Corollary 12.c,e in Mertens and Rubinchik (2013). k_t is uniformly bounded by Proposition 1.a in Mertens and Rubinchik (2013), $i_t \in L_{\infty,1}$ by Proposition 1.b in Mertens and Rubinchik (2013) and hence by equation x here, k is continuous, thus the range and the domain of Υ are as specified in the claim. ■

Note that the prices p_t, r_t, w_t are determined fully by the rest of the variables, in fact, it is enough to know the (adjusted) rate of change of prices, $\mathfrak{r}_t$, to determine the three. Moreover, none of the equilibrium variables determined by the conditions $i - x$ in Proposition 1 are affected by the three prices. Hence, we can easily drop the three prices from the list of the equilibrium variables to lighten the notation and the analysis.

By Theorem 3 in Mertens and Rubinchik (2013), the number of BGE is finite in this economy, and we denote by ϖ an equilibrium selection for a given E. So, first, at $E = 0$, ϖ has to return one of the BGE. More generally, we want to establish existence of *solutions*

$$(k, y, c, \mathfrak{r}, i) \in P \stackrel{\text{def}}{=} C_\infty^4 \times L_{\infty,1},$$

of the *equation system* $i - x$ in Proposition 1, together with $F(k, E) = 0$, as functions of $E \in L_{\infty,1}(\mathbf{R}^2)$ (in a neighborhood of 0). By "equation system" we mean all equations there, but excluding the inequality $\mathcal{N} \geq 0$ part v.

Furthermore, for a *solution* to be an *equilibrium,* it has to satisfy, in addition, the inequality $\mathcal{N} \geq 0$, thus assuring that individual consumption is nonnegative,[13] and, for all interior equilibria, the constraints $0 < i_t < y_t$, implying that capital is positive as required by part x. For the BGE with zero transfers, this condition is satisfied, cf. Mertens and Rubinchik, 2013, Corollary 12.e.

4. THE ROADMAP

4.1. The Notions of Regularity, Stability, and Genericity

We will say that a BGE is *regular* if in some open $\|\cdot\|_{\infty,1}$ neighborhood of $E = 0$, there is a Fréchet–differentiable map from an endowment E to the corresponding perturbed equilibrium. It is *stable* if each perturbed equilibrium converges exponentially to the status-quo path.

Our task is to show regularity and stability of equilibria for a generic economy in the following sense.

DEFINITION 2. *The* parameter space*, or the space of economies, is*

$$\wp = \{(R, \alpha, \eta, \sigma, \varphi(ds)) \mid (R, \sigma) \in \boldsymbol{R}^2_{++},\ \ \alpha \in]0, 1[,\ \ \varphi(ds) \in \Delta([0, 1])\},$$

with the weak-topology on* $\Delta([0, 1])$*, the probabilities on* $[0, 1]$.

DEFINITION 3. *A subset of* $\wp$ *is* negligible *if its section for any fixed probability distribution* $\varphi(ds)$ *in* $\Delta([0, 1])$ *has the Lebesgue measure* 0.

A subset is generic *if its complement is contained in a countable union of closed negligible sets.*

4.2. The Building Blocks of the Unified Approach

Our task is to establish regularity and stability with one (implicit function) theorem.

Heuristically, since any equilibrium has to be a solution to the fixed point problem

$$F(k, E) \stackrel{\text{def}}{=} \Upsilon(k, E) - k = 0,$$

if such a solution is an equilibrium, the derivative with respect to E of, say, equilibrium capital path, has to be determined by an IFT:

$$\varpi'_k = -\left(\frac{\partial F}{\partial k}\right)^{-1} \circ \frac{\partial F}{\partial E}. \qquad \textbf{(1)}$$

Now our task is two-fold: First, prove that F is sufficiently smooth, and second, show that $\frac{\partial F}{\partial k}$ is generically invertible.

To assure both regularity and stability, we extend the notion of smoothness used in the IFT. We want to assure that the implicit function ϖ is smooth in a family (or a range) of spaces. Importantly, the neighborhood where this implicit function exists should be the same for the whole family. The family is indexed by a couple: an open interval, $\Lambda \subset \mathbf{R}$, containing zero, and a set $[1, \infty]$. The second component provides flexibility to control, using $p \in [1, \infty]$, the "global" behavior

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of the policy (changes), cf. ft. 11, its role will be clear once the main theorem is stated. The first component will be shown to indicate the range of the speeds of exponential convergence, $\lambda \in \Lambda$, of the perturbed equilibrium to the status quo, thus allowing us to analyze stability.

Recall that the equilibrium variables belong either to $L_{\infty,1}$ or to $C_\infty \subset L_\infty$. Since the endowment perturbation, E is defined on a subset of $\mathbf{R}^2$ (time and age), we will take $\mathbf{R}^n$ as domain for all. Using the definition of an amalgam above, we say that a function f defined on $\mathbf{R}^n$ belongs to $L^\lambda_{p,q}$ for some $\lambda \in \mathbf{R}$, if and only if $\phi_\lambda(f)$ is in $L_{p,q}$, where $\phi_\lambda(f(t)) = e^{\langle \boldsymbol{\lambda}, t\rangle} f(t)$ and $\boldsymbol{\lambda} = (\lambda, 0, \ldots, 0) \in \mathbf{R}^n$ so that only the first component of $t \in \mathbf{R}^n$ is multiplied by the exponential. The definition for $C^\lambda_p \subset L^\lambda_{p,\infty}$ is similar, thus covering all the cases we need. The formal definitions of these spaces and the resulting Banach families are in online Appendix A.1.

The definition of smoothness (S^1) that we develop requires existence of a Gâteaux derivative and this derivative to be locally Lipschitz in all the spaces of the Banach family, with the common Lipschitz constant, cf. Definition 2, in online Appendix A.1. Although using the definition directly in order to verify the S^1 property might be tedious, we offer an array of sufficient conditions, basic "building blocks" (Mertens and Rubinchik, 2014, Proposition 4,5), which are easier to apply. This is the approach used here: Lemma 16(ii) in online Appendix C showing that Υ is S^1 is an example of such construction.

The IFT for families of Banach spaces based on a general definition of S^1 is formulated and proved in Mertens and Rubinchik (2014), Theorem 3. To apply it, as we mentioned before, not only the smoothness of the equilibrium map $F(k, E) = \Upsilon(k, E) - k$ has to be verified, but also the existence of an *inverse* of its derivative, $\frac{\partial F}{\partial k}$.

For that, in online Appendix A.2, we establish Proposition 1, which is based on the theorem of Wiener (1932) (also known as Wiener's lemma), which implies that the spectrum of a convolution operator (on a Banach space) can be computed using its Fourier transform. How does it help us?

First, recall that a complex number $z \in \mathbf{C}$ is in the *spectrum* of an element A of a Banach algebra if $A - zI$ is not invertible, where I is the unit element of the algebra [cf. e.g., Lang, 1969, IV, Section 2, p. 68]. Heuristically, again, notice that $\frac{\partial F}{\partial k}$ is $\frac{\partial \Upsilon}{\partial k}$ minus the identity (derivative of k with respect to itself). To apply the Wiener theorem, we have to first show that the derivative $\frac{\partial \Upsilon}{\partial k}$ is given by a convolution kernel, i.e., when applied to a perturbation, δk (from the space of continuous real-valued functions with sup norm, where k belongs), can be written as a convolution of some real-valued function $\mathbf{k}^k_k$ and the perturbation, δk. Then, we can apply a variant of the theorem of Wiener to explicitly calculate the spectrum of that operator and check whether $z = 1$ belongs to it, i.e., whether the spectrum contains a point whose real part is one and whose imaginary component is zero. If the answer is yes, then $\frac{\partial F}{\partial k}$ is not invertible, and otherwise it is. In addition, the same tool provides a way to calculate the inverse of $\frac{\partial F}{\partial k}$, when it exists.

We take this idea a little further and ask: "What is the range of real numbers λ around zero for which the spectrum of the operator with kernel $\phi_\lambda(\mathbf{k}_k^k): t \mapsto e^{\lambda t}\mathbf{k}_k^k(t)$ does not contain unity?" The explicit answer is given in Corollary 7 (online Appendix E), determining the endpoints of the interval Λ that was mentioned before, and the same range gives us the bounds on exponential convergence of the equilibrium back to the status quo, as will be shown later.

Now, it is clear that we have a plan: (1) prove the equilibrium system is smooth and calculate the derivative of the key map, Υ, with respect to the endogenous variable, capital k, and the endogenous variation in endowments (transfers), E; (2) find BGE, evaluate the derivative there and show that $\frac{\partial \Upsilon}{\partial k}$ at a BGE is given by a convolution kernel; (3) calculate the spectrum of the kernel and show that generically (in parameters) it does not contain unity, calculate interval Λ as described above, calculate the inverse of $\frac{\partial F}{\partial k}$, and show it is the same for a family of spaces, indexed by Λ and $[1, \infty]$; (4) use the IFT to calculate the response of the solution [a fixed point of $\Upsilon(\cdot, E)$] to the change in transfers for a generic economy, thus establishing the regularity; (5) use the boundaries of the interval Λ to determine the speed of exponential convergence of the solutions to the baseline, thus establishing stability; (6) show when regularity and stability hold also for equilibria (solutions that satisfy equilibrium inequality constraints).

With this roadmap in hand, let us start the work.

5. SMOOTHNESS OF THE EQUILIBRIUM SYSTEM

The first step in applying the IFT is to verify the smoothness of the system of equations that the equilibrium has to satisfy. Recall, the system in our case reduces to a single equation, $\Upsilon(k, E) = k$, where Υ is defined as a composition of several maps. The existence of the derivatives and their computation is relegated to online Appendix C. In Lemma 16 there, we construct the derivative $\frac{\partial \Upsilon}{\partial k}$ step by step, showing smoothness of each intermediate map in Proposition 1 using the elementary properties developed in online Appendix B.2, and then apply the S^1 property of the composition (Mertens and Rubinchik, 2014, Proposition 5). The key elements of the calculation are represented in Figure 1. The figure reveals several economic insights. First, as one would probably expect, a change in policy (δE), can have effects beyond the time of its enactment: Its direct effect is on aggregate consumption (through the income effect) and on investment (through material balance), both of which affect the path of capital accumulation, thereby translating into the change in the output and in the net interest rate ($\mathfrak{r}$), both of which, in turn, affect consumption and investment, thus creating the indirect effect of the policy. The presence of the indirect effect is due to two (possibly related reasons): capital accumulation and consumption smoothing.

There is no uncertainty in the model, so all the changes are fully anticipated. Thus, policy change can have an effect both prior and after its enactment. This brings us to the second insight.

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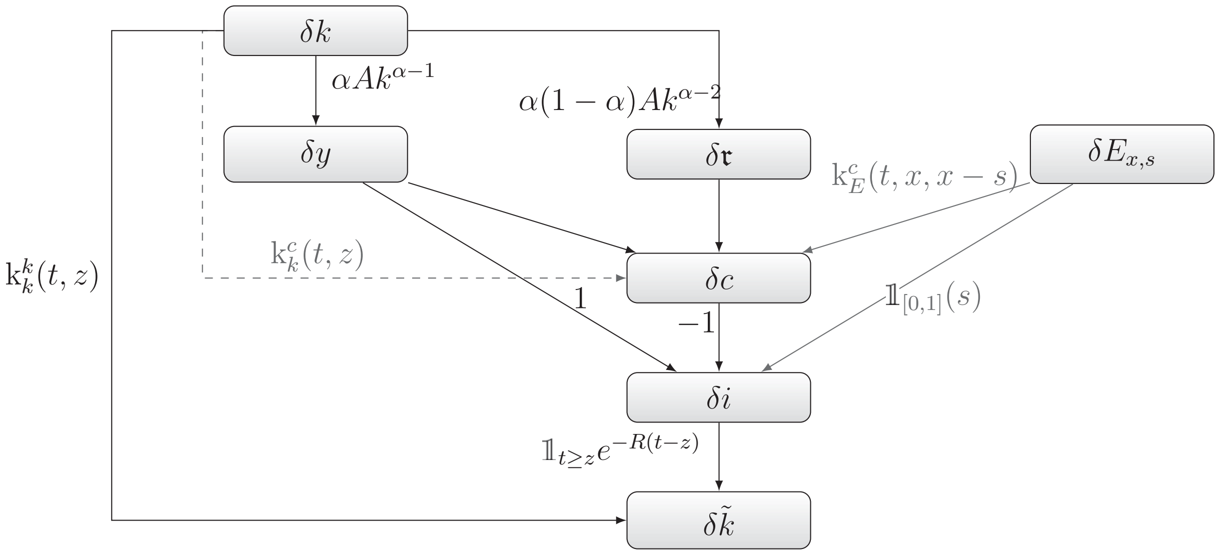

FIGURE 1. A schematic representation of the derivative of the equilibrium map. Exogenous, policy induced, or direct effects are represented by the two arrows on the right and the rest are indirect (equilibrium) effects. The effect of change in capital, δk, on output (y) and net interest ($\mathfrak{r}$) is immediate, so the derivative is "simple", e.g., $\delta y_t = \alpha A k_t^{\alpha-1}\delta k_t$ and the same is true about effects of aggregate consumption and output on investment, the effects are both immediate and one to one. The effect of an *aggregate* endowment change on investment is immediate too. The rest of the effects are "smoothed out" and are given by kernels, e.g., $\delta c_t = \iint \mathrm{k}^c_E(t,x,x-s)\delta E_{x,s}ds\,dx + \int \mathrm{k}^c_k(t,z)\delta k_z dz$.

Some of the effects are instantaneous, as is explained in the caption, whereas others are spread out in time. In the latter case, the effect (the corresponding derivative) is given by a kernel, denoted by k, representing components of the partial derivatives of the equilibrium variables (appearing in the superscript) with respect to other variables (appearing in the subscript).

Roughly, if the derivative of, e.g., aggregate (normalized) consumption, c, with respect to capital, k, is given by a kernel $\mathrm{k}_k^c(t, z)$, then the variation in consumption at time t that is due to the change in capital (the dotted arrow in the graph), corresponding to the joint effect of the wage income, $(1-\alpha)y$, and (the rate of change in) prices, $\mathfrak{r}$, is $\int \mathrm{k}_k^c(t, z)\delta k(z)dz$. So, to calculate the full effect of capital path variation on the change in consumption at a given time t, one has to aggregate over all the related "times" indexed by z. This, again, brings us to the need to model time as a real line, rather than truncating it at some arbitrary point "0": It is up to equilibrium forces to determine how long in advance the policy's impact will be present, any artificial truncation potentially sweeps away anticipation of the policy change by the forward-looking agents.

The interpretation of the kernel is simple: $\mathrm{k}_k^c(t, z)$ is the impulse response of consumption at time t, to a "shock" to capital at time z. At this stage, the derivatives reflect only partial effects, without taking into account the fixed point feature of the general equilibrium. To describe the latter, we will appeal to the IFT, for application of which the kernel k_k^k of the derivative $\frac{\partial \Upsilon}{\partial k}$ will be of particular interest.

We now proceed to calculating the baseline equilibria and evaluating the derivatives there.

6. THE BASELINE EQUILIBRIA, BGE AND $\frac{\partial \Upsilon}{\partial k}$ AT BGE

6.1. BGE and the Equilibrium Graph

Here, we characterize BGE without transfers, or baseline equilibria. The characterization is based on Corollary 13 and Remark 24 in Mertens and Rubinchik (2013), but it can also be viewed (and proven as) a corollary to Proposition 1.

Notation 6.1. Let $\Phi(z) \stackrel{\text{def}}{=} \frac{e^z-1}{z}$, $\mathcal{F}(\mathfrak{r}) \stackrel{\text{def}}{=} \frac{\Phi(-\mathfrak{r}\sigma-\eta)}{\Phi(\mathfrak{r}(1-\sigma)-\eta)}$.

COROLLARY 1. *The set of* BGE is the set of constant solutions $\mathfrak{r}$ of the system

$$\mathfrak{r}\Big(\frac{1-\mathcal{F}(\mathfrak{r})\int e^{\mathfrak{r}s}\varphi(ds)}{\mathfrak{r}}(R-\mathfrak{r})-\frac{\alpha}{1-\alpha}\Big)=0, \tag{2}$$

with the rest of the (constant) quantities determined by

i. $k=\left[\frac{R-\mathfrak{r}}{A\alpha}\right]^{\frac{1}{\alpha-1}}$,
ii. $y=Ak^{\alpha}$,
iii. $i=Rk$,
iv. $\mathcal{N}=(1-\alpha)y\int_0^1 e^{\mathfrak{r}s}\varphi_s ds$,

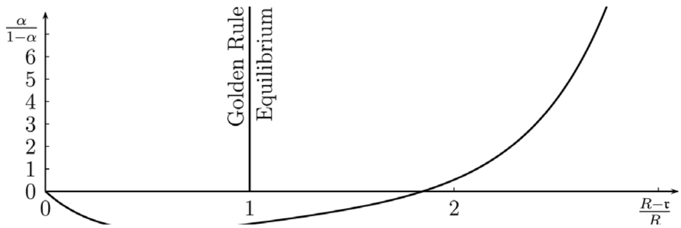

FIGURE 2. $R = 11, \sigma = .5, \eta = 2, a = .2, b = .75$. Two equilibria $\forall\alpha$.

v. $c = \mathcal{F}(\mathfrak{r})\mathcal{N}$,
vi. $\mathcal{D} = \Phi\big(\mathfrak{r}(1-\sigma)-\eta\big)$, $\mathcal{B} = \frac{\mathcal{N}}{\mathcal{D}}$.

Remark 1. Recall, the prices $p_t = p_0 e^{(\mathfrak{r}-\gamma-\nu)t}$, $w_t = (1-\alpha)e^{\gamma t}p_t$, $r_t = p_t(R-\mathfrak{r})$ are fully determined by $\mathfrak{r}$ and are omitted from the list of the equilibrium variables.

Remark 2. The number of BGE in this economy is finite by Theorem 3 in Mertens and Rubinchik (2013), since the production function $f(k) = Ak^{\alpha}$ satisfies Assumption 5 there and is analytic.

Whether the number of BGE is odd or even depends on the relative magnitudes of three parameters: First, the minimal working age, i.e., the youngest age at which individual productivity becomes strictly positive, second, the minimal tax/transfer age, i.e., the earliest age when the individual gets the transfers, and, third, individual preferences parameter, σ, cf. Mertens and Rubinchik, 2013, Theorem 3. Clearly, if the number of equilibria is even, a BGE cannot be unique.

Recall, we are interested in the set of parameters for which *any* BGE is regular and stable. For that it is convenient to work with an equilibrium graph.

DEFINITION 4. *Let $\mathfrak{G}$ be the cross product of the parameter space, $\wp$, and $P = C^4_\infty \times L_{\infty,1}$, containing the equilibrium variables: $(k, y, c, \mathfrak{r}, i)$. The* equilibrium graph *(restricting attention to* BGE*) is the subset G of $\mathfrak{G}$ composed of all points satisfying Corollary 1.*

To illustrate, we borrow (with slight modifications) from Mertens and Rubinchik (2013) the projection of such graphs into the space of a single parameter α and an equilibrium variable, $1 - \mathfrak{r}/R$. The graph in these figures contains combinations of α and $\mathfrak{r}$ that constitute a BGE (while the rest of the parameters are fixed). So, for example, to find all BGE for $\alpha = \frac{1}{3}$, one can draw a horizontal line at the level of $\frac{\alpha}{1-\alpha} = \frac{1}{2}$, and then each of its intersections with the graph will correspond to a BGE.

Figures 2–5 show the BGE of economies with $\varphi(s) = \frac{1}{b-a}\mathbb{1}_{[a,b]}(s)$ and reasonable parameters (time unit being 1 lifetime).

To distinguish "bad" or *exceptional* points in the graph where equilibrium is not regular, we proceed by evaluating the derivative of the equilibrium system at any baseline equilibrium, BGE.

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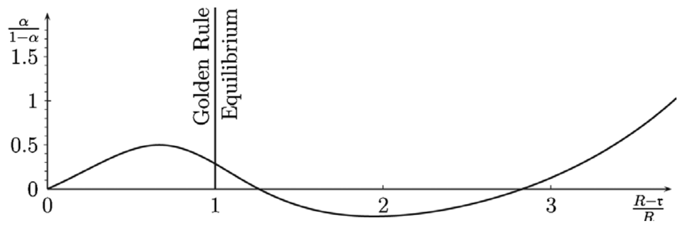

FIGURE 3. $R = 11$, $\sigma = .25$, $\eta = 2$, $a = .135$, $b = .5$. Two to four equilibria.

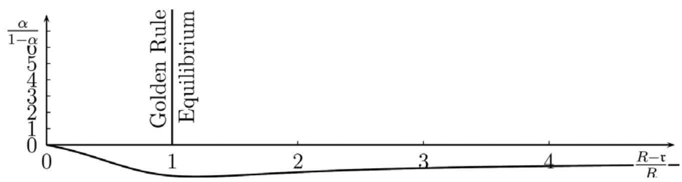

FIGURE 4. $R = 10$, $\sigma = .25$, $\eta = 2.5$, $a = .25$, $b = .75$. One equilibrium $\forall\alpha$.

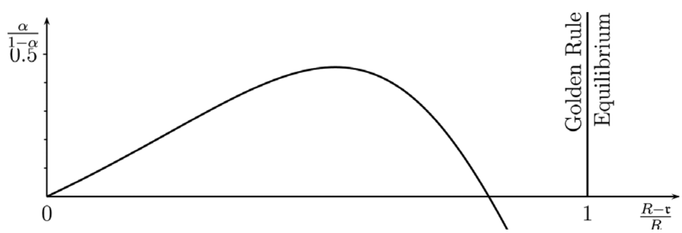

FIGURE 5. $R = 15$, $\sigma = .24$, $\eta = 1.9$, $a = .24$, $b = .55$. One or three equilibria.

6.2. $\partial\Upsilon/\partial k$ at a BGE

The objective here, according to our plan in Section 4.2, is to first, verify that the derivative is given by a *convolution kernel*, i.e., the impulse response depends only on the distance in time from the shock. For example, the kernel of the derivative of consumption with respect to capital becomes $\mathbf{k}_k^c : \mathbf{R} \to \mathbf{R}$, the function of only a single variable, i.e., $\mathrm{k}_k^c(t, z) = \mathbf{k}_k^c(t - z)$. In this case, it is easy to illustrate its interpretation as an impulse response. Note that $\int \mathbf{k}_k^c(t - x)\epsilon_z(x)dx = \mathbf{k}_k^c(t - z)$, where $\epsilon_z(x)$ is a unit mass at $x = z$.[14] So, the response of consumption δc at time t to a (fully anticipated) shock $\delta k(x) = \epsilon_z(x)$, which "happens" at time z, is $\mathbf{k}_k^c(t - z)$, which depends only on the difference (in time) between t and z.[15]

Second, we calculate the Fourier transform[16] of the kernel, which will then be used to determine its spectrum.

The formal result, Lemma 17, is in online Appendix D. Here, again, we resort to a diagram, Figure 6, to summarize the results. In particular, observe that the effect of an investment "shock" on capital decays exponentially at a fixed rate R.

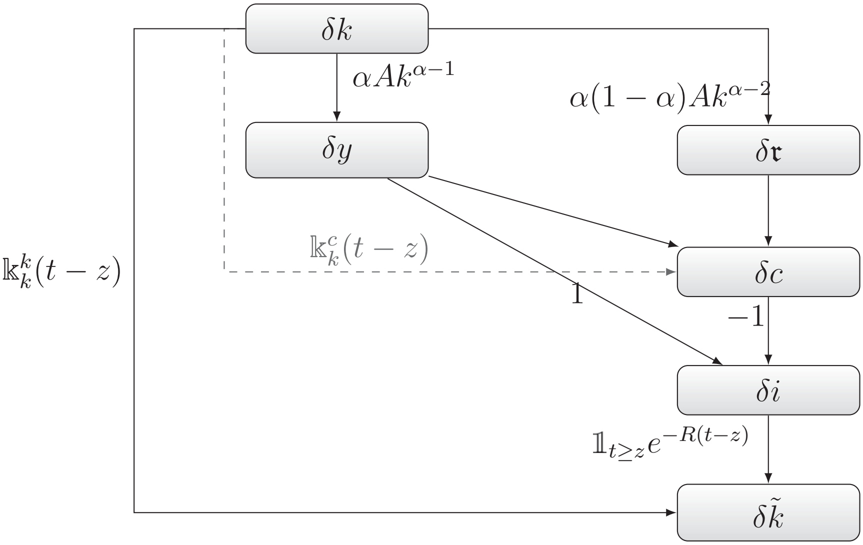

FIGURE 6. The derivative of the equilibrium map with respect to endogenous k, $\partial\Upsilon/\partial k$ at a BGE. Here, all the kernels become convolution kernels $\mathbf{k}$, so the magnitude of the effect depends only on the distance in time from the perturbation. By Lemma 17 in online Appendix D, the convolution kernel $\mathbf{k}_k^k$ of $\partial\Upsilon/\partial k$ is integrable. Its Fourier transform is $\widehat{\mathbf{k}}_k^k(\omega) = \frac{1}{R-\mathrm{i}\omega}(R - \mathfrak{r} - \widehat{\mathbf{k}}_k^c(\omega))$.

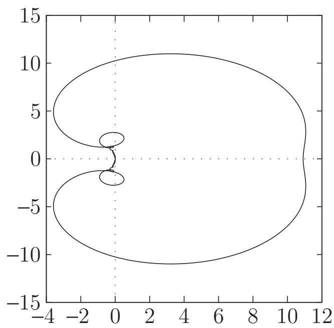

FIGURE 7. BGE of Figure 1 with $\frac{R-\mathfrak{r}}{R} = 2$.

This will be used later to determine the ranges of exponential convergence of the perturbed equilibrium.

Recall, $\partial F/\partial k$ equals $\partial\Upsilon/\partial k$ minus the identity, $\partial F/\partial k$ is not invertible if and only if unity is in the spectrum of $\partial\Upsilon/\partial k$. Next, we show that unity is not in the spectrum, which holds generically in the space of parameters of the model.

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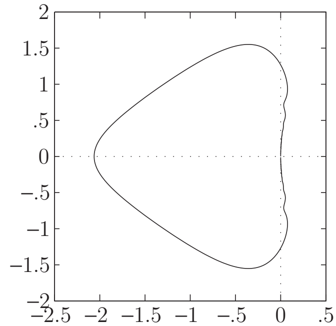

FIGURE 8. GRE of Figure 1 with $\alpha = .3$.

7. GENERIC INVERTIBILITY

As follows from Section 4.2, now the task of identifying points on the equilibrium graph where the derivative is noninvertible is reduced to finding the points where the Fourier transform of convolution kernel $\widehat{\mathbf{k}}_k^k$ of the derivative $\partial\Upsilon/\partial k$ at a BGE returns 1 for some parameter $\omega \in [0, \infty]$. The range of a Fourier transform is a subset of complex numbers, **C**. We are interested in "bad" points for which the *real part* of the transform, $\Re(\widehat{\mathbf{k}}_k^k)$, is unity, whereas the *imaginary part*, $\Im(\widehat{\mathbf{k}}_k^k)$, is zero for some ω.

7.1. Illustrating Spectra of $\frac{\partial\Upsilon}{\partial k}$ at a BGE

Figures 7–14 show the spectra of the derivative of the equilibrium map, or the range of $\widehat{\mathbf{k}}_k^k$, for the four example economies as in Figure 2–5 where the life-time efficiency of labor is zero between 0 and a, unity between a and b and zero thereafter, i.e., $\varphi(s) = \frac{1}{b-a}\mathbb{1}_{[a,b]}(s)$.

First, note that spectrum is a line (not an "area") and even if it contains $(1, 0)$ in the complex plane, it *looks like* a "rare" occasion: A slight change in parameters should shift the line away from the problematic $(1, 0)$. This last claim, of course, has to be proven and will be established in Proposition 2.

As an aside, one could also notice that not all the spectra are in a unit circle; hence, one should not expect Υ to be a contraction in k in general, thus making it difficult to find an equilibrium numerically (cf. the discussion in Section 1).

7.2. $\frac{\partial F}{\partial k}$ is Generically Invertible at a BGE

Recall that one of the parameters of the economy is a life-cycle productivity measure φ, and so a negligible set of parameters (cf. Definition 3) is defined for a fixed φ. The proof of the genericity claim (Proposition 2) is based on a well-known property of analytic functions on **C**: Their zeros (points for which the function returns zero) are isolated, i.e., there is an open neighborhood of any such point

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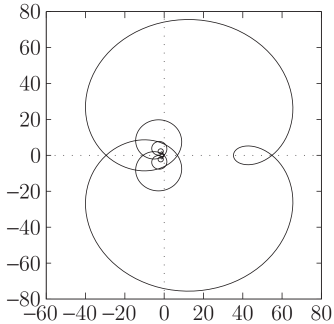

FIGURE 9. BGE of Figure 2 with $\frac{R-\tau}{R} = 3$.

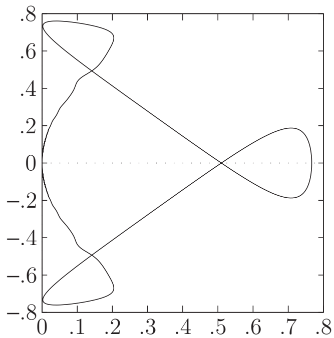

FIGURE 10. GRE of Figure 2 with $\alpha = .3$.

that has no other zeros. An extension of this property for $\mathbf{C}^n$ case is in Lemma 21 in online Appendix I and this is indeed all we need for our case. Here, we state the next result, its proof is in online Appendix E.

PROPOSITION 2. *The set $\mathcal{G} \subseteq \wp$ where* 1 *is not a value of $\widehat{\mathbf{k}}_k^k$ for any* BGE is generic.

Remark 3. The proof of Proposition 2 involves showing that negligibility is preserved when going from the equilibrium graph to the parameter space. This problem is reduced to the (trivial) one-dimensional version of a statement that a C^1 map from $\mathbf{R}^n$ to $\mathbf{R}^n$ preserves negligibility (or, more generally, replacing $\mathbf{R}^n$ above by a n-dimensional manifolds with boundary). Such a statement might be provable directly from Sard's theorem and the IFT. That might be the right tool to handle the above problem in general.

Remark 4. On the other hand, our technique to prove genericity relied on the fact that the discrete set of FT parameters ω that make the imaginary part of $\widehat{\mathbf{k}}_k^k$

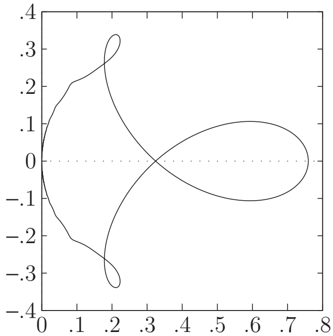

FIGURE 11. BGE of Figure 2 with $\frac{R-\mathfrak{r}}{R} = \frac{1}{2}$.

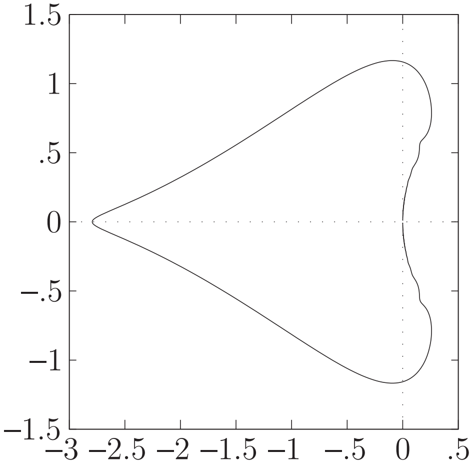

FIGURE 12. GRE of Figure 3 with $\alpha = .3$.

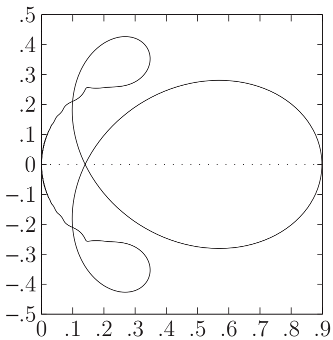

FIGURE 13. BGE of Figure 4 with $\frac{R-\mathfrak{r}}{R} = \frac{1}{2}$.

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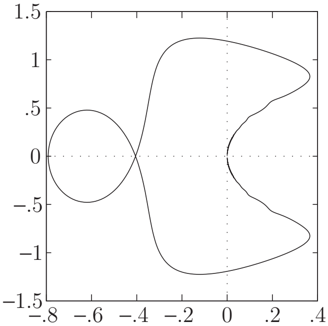

FIGURE 14. GRE of Figure 4 with $\alpha = .3$.

zero did not depend on one of the parameters (R), whereas the second condition (that the real part of $\widehat{\mathbf{k}}_k^k$ is unity) was satisfied for a finite set of ω for any given R. This is where we relied on the specification of the model.

8. SPEED OF CONVERGENCE

8.1. Establishing the Speed of Convergence

We assume here that we are dealing with generic equilibria and investigate the speed of convergence to 0 of the kernel of $(\frac{\partial F}{\partial k})^{-1}$, which will later be seen to be also the speed of convergence of perturbed equilibria back to the original equilibrium, cf. Remark 7.

Corollary 7 (of Proposition 1) from Appendix E defines the interval Λ: It can be computed from the primitives of the model for a given choice of a BGE. We want the kernel $\mathbf{k}_k^k$ to be invertible not only in L_1, but also in L_1^λ. So, the task then is to determine interval Λ such that $e^{\lambda t}\mathbf{k}_k^k(t)$ is invertible for $\lambda \in \Lambda$. We know by Proposition 2 that for a generic point on the equilibrium graph, $\lambda = 0$ is in that set. Corollary 7 establishes that the set is an interval (including zero). The upper bound of Λ is the highest λ for which $e^{\lambda t}\mathbf{k}_k^k(t)$ is *not* invertible, i.e., when its Fourier transform returns unity. But the Fourier transform of $e^{\lambda t}\mathbf{k}_k^k(t)$ is a Laplace transform ($\mathcal{L}$) of the original kernel, $\mathbf{k}_k^k$. So, to rephrase, in order to find the upper bound of Λ, we have to find the root $z = \lambda + i\omega \in \mathbf{C}$ of the equation $(\mathcal{L}\mathbf{k}_k^k)(z) = 1$ with the lowest positive real part λ, and to find the lower bound of Λ, we need the highest negative such λ. In addition, of course, the transform has to exist, i.e., return finite values for any parameter λ in the range. The convolution kernel $\mathbf{k}_k^k$ is itself a convolution of several components (as is evident from Figure 6), one of which is $\mathbb{1}_{t\geq z}e^{-R(t-z)}$. Thus, its Laplace transform is the product of the Laplace transforms of these components. Clearly, if the real part of z is higher than R, the Laplace transform of $\mathbb{1}_{t\geq z}e^{-R(t-z)}$ diverges. Therefore, the interval Λ has to be reduced to $]-\infty, R[$ even before the computation of the roots mentioned

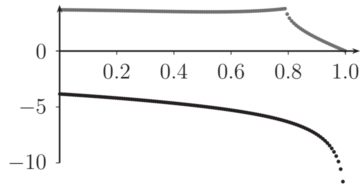

FIGURE 15. λ_+ and λ_-; GRE of Figure 2.

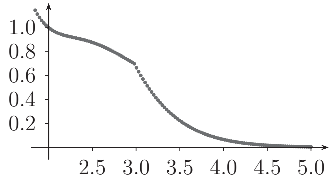

FIGURE 16. λ_+ and λ_-; BGE of Figure 2.

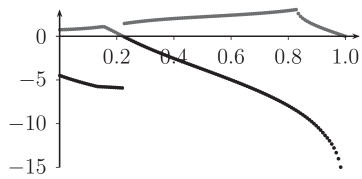

FIGURE 17. λ_+ and λ_-; GRE of Figure 3.

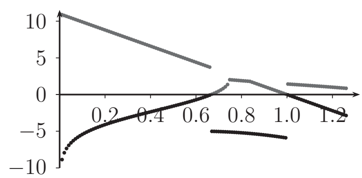

FIGURE 18. λ_+ and λ_-; BGE of Figure 3.

above. The results of the computations for the example economies are illustrated in Figures 15–22.

Corollary 8 in online Appendix E establishes that the same inverse $g - \mathbf{1}$ of the fixed point map that we previously calculated for $\lambda = 0$ is also valid for the whole family of spaces indexed by λ in the interval Λ.

Next, by Corollary 9, in online Appendix E, the "end points" of the interval Λ indicate the speed of exponential convergence of the operator g and, as will follow from the main result, these are the speeds of convergence of the perturbed equilibria toward the original equilibrium at $-\infty$ and $+\infty$.

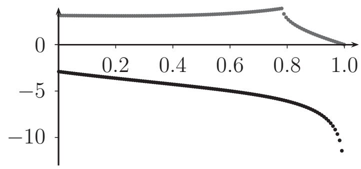

FIGURE 19. λ_+ and λ_-; GRE of Figure 4.

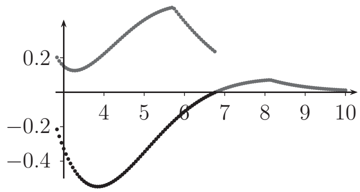

FIGURE 20. λ_+ and λ_-; BGE of Figure 3.

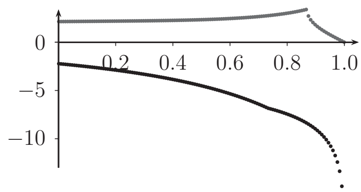

FIGURE 21. λ_+ and λ_-; GRE of Figure 5.

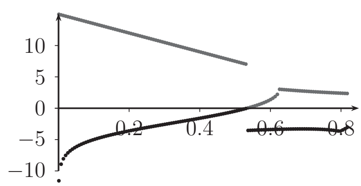

FIGURE 22. λ_+ and λ_-; BGE of Figure 5.

8.2. Illustrating the Speed of Convergence

Figures 15–22 illustrate, for example the economies of Section 7.1, the rates of convergence λ_- (below the horizontal axis) and λ_+ (above the horizontal axis) as a function of α for the GRE and of $x = 1 - \frac{\mathfrak{r}}{R}$ $(= \frac{\alpha Y}{I})$ for the other BGE. Figures 18 and 20 refer to the low and high intervals of x in Figure 3 for which BGE exists.

It is possible to show by a direct computation that $z = \mathfrak{r}$ is always a root for BGE;[17] it is the straight line passing through $(0, R)$ and $(1, 0)$, and segments of it are visible, e.g., in Figures 18 and 22. Another segment of it is λ_- after the critical

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point in Figure 20, which does not appear since being < -60 it would fall far off the page, same for the whole of λ_- in Figure 16.

Critical points (combinations of parameters where the invertibility fails) correspond to the intersections with the horizontal axis. All but one of them are "trivial," in the sense that they correspond to real roots, which indicate the local extrema of the corresponding BGE curves in Figures 2–5, and the intersection with the GRE line, as well as $\alpha = 1$. The one exception is the critical point in Figure 20 ($x = 6.768475$, $z = \pm 8.07776\mathrm{i}$).

9. FIRST RESULTS

At this stage, we could have already stated our main result establishing regularity and stability of solutions, Lemma 18 (in online Appendix F), which certainly can be viewed as such. This lemma rests on the IFT formulated for the Banach families (Mertens and Rubinchik, 2014, Theorem 3) and it consists of several statements.

One of them is the standard regularity property assuring that there is a neighborhood B of zero in the space of endowment perturbations E such that there is an implicit function ϖ from B into the neighborhood of each *solution* to the equilibrium fixed point problem, cf. Section 3.2.

Moreover, for any $\varepsilon > 0$, there is a smaller ball B_ε around zero and a compact interval $\Lambda^\varepsilon \subseteq \Lambda$ with $\max\{\min\{0, \lambda_- + \varepsilon\}, \frac{-1}{\varepsilon}\}$ and $(\lambda_+ - \varepsilon)^+$ in its interior such that on B_ε and for the whole family of spaces indexed by $\lambda \in \Lambda^\varepsilon$, the implicit function ϖ is smooth and differentiable with the derivative that has a Lipschitz property. In addition, the derivative of F with respect to capital, k, is invertible when evaluated at point $[E, \varpi_k(E)]$ for any E in the ball, where ϖ_k is the "capital component" of the function ϖ.

Remark 5. We could as well have viewed our fixed point map Υ, e.g., as a map from i to $\tilde{\imath}$, rather than from k to $\tilde{k}$; basically everything still goes through in the same way. However, then one obtains a weaker "local uniqueness" result in Lemma 18 in online Appendix F: It would then refer to a δ' neighborhood of ϖ_i and that would be in $L_{\infty,1}$. The map $i \to k$ is continuous and injective, but the inverse is not at all continuous, so our present result is definitely sharper (and simpler).

Next step is to present the results in a simpler form: with easier metrics and using standard differentiability concepts. In addition, we prefer to formulate the results independently of the fixed-point map used,[18] a.o., to get correct bounds for each variable for its own sake—i.e., "to cover our tracks."

In order to accomplish this last objective, we will show that the derivatives (ϖ') are mostly given by properly behaving kernels so that the statement of the main result (based on the IFT) is independent of the fixed-point map used (on i, on k,...), apart, of course, from the proper specification of the space where the underlying variable (i, on k,...) lives, cf. Remark 9.

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Remark 6. In the language of dynamic systems, the expression of ϖ' by kernels $\mathrm{k}(x; t, t-s)$ is equivalent to the full system of impulse responses for all possible (small) impulses: $\mathrm{k}(x; t, t-s)$, as a function of x, is the response to an impulse (in E) at (t, s). At this stage, the system of impulse responses incorporates the full force of the equilibrium effects, as opposed to the partial effects illustrated in Figure 1.

10. THE KERNEL REPRESENTATION OF THE DERIVATIVE

The kernel representation of the derivative of the implicit function ϖ, done in online Appendix G (Lemmata 19 and 20), combines several results established so far. The "nice" properties of the derivative of Υ (the map that defines the equilibrium fixed point) with respect to endogenous k and exogenous E (Lemma 16 in online Appendix C) along with the existence of the inverse of $\frac{\partial F}{\partial k}$ in a neighborhood of the equilibrium point (Lemma 18 in online Appendix F) yield the "nice" properties of the "building blocks" of ϖ' by Lemma 2 in online Appendix B.1. These "building blocks" are the inverses of the kernels described in Lemma 16. Importantly, this calculation gives the analytic form of the full equilibrium response to the exogenous changes in transfer policy, E.

Lemma 19 from online Appendix G.1 shows that our results on invertibility are quite sharp: either the spectrum of the kernel of $\frac{\partial \Upsilon}{\partial k}$ at a BGE contains 1, i.e., for some ω, $e^{i\omega t}$, and hence the two-dimensional space of linear combinations of $\cos(\omega t)$ and $\sin(\omega t)$, solves the linearization of the fixed point problem (so, $\frac{\partial F}{\partial k}$ is not even injective on C_∞, the basic space for k), or there exists a full neighborhood of 0, both for E and for λ, where $\frac{\partial F}{\partial k}$ is invertible, with the same (cf. Appendix H) inverse $\mathrm{h} - \mathbf{1}$, in all the above operator spaces.

Lemma 20 in online Appendix G.2 is devoted to identifying discontinuous part of each kernel. This simplifies the numerical problem of finding the inverses to that of computing their continuous parts (chiefly that of Lemma 19 from Appendix G, the others are just a matter of integration),[19] which are everywhere well defined, thus turning the problem into a "well-posed problem." Otherwise, kernels would just be maps to equivalence classes of measurable functions: quite unrealistic to compute.

Moreover, this lemma has an interesting economic interpretation. It shows that the investment is going to absorb part of a "shock" of the endowment (E) directly: Its derivative is a sum of a kernel operator and a "spike," whereas the rest of the variables are responding more smoothly to the change in E with at most one discontinuity in the corresponding kernel.[20]

In addition, it becomes possible to translate the smoothness of the implicit function ϖ (inherited from the smoothness of Υ by the IFT) into the properties of the kernels composing the derivative of ϖ. This, in turn, allows one to formulate the main results in terms of a simpler norm (for the kernels) that expresses directly the exponential convergence aspect, cf. notation 11.1.

11. MAIN RESULTS

11.1. Regularity and Stability of Solutions

Here, we finally present our main results, applying Lemma 18 in online Appendix F that is based on the IFT to a solution of the equilibrium system and derive the smoothness properties of each solution ϖ as a function of the exogenous E defined in a neighborhood B of zero ($E = 0$). For most of what follows, we write $\varpi(E)$ to indicate this dependence. Recall that $\varpi(E)$ itself is an array of functions $(k, c, y, \mathfrak{r}, i)$, where the first four are continuous and bounded and hence will be referred to as C_∞ components, whereas $i \in L_{\infty,1}$. To refer to a particular component, we are going to use a subscript, e.g., $(\varpi_k(E))(x)$ denotes the value of capital (k) at time x of a solution [in the neighborhood of BGE $\varpi(0)$] corresponding to endowment path E. The derivative, ϖ', too, depends on E, as in Lemma 18. The derivative itself, recall, maps δE, an element of $L_{\infty,1}$, to the variation in the solution, i.e., a profile $(\delta k, \delta c, \delta y, \delta\mathfrak{r}, \delta i)$. So, e.g., $((\varpi_k'(E))(\delta E))(x)$ denotes the change (variation) in capital δk at time x [from the initial level $(\varpi_k(E))(x)$] as a result of a perturbation δE of endowments from their initial level E.

Recall, again that the set $\mathcal{G} \subseteq \wp$ is, as in Proposition 2, a generic subset of parameters where $\partial F/\partial k$ is invertible. Convolution ($\star$) of two measures is defined in the usual fashion, cf. online Appendix A.2, Definition 4. The following notation is used in the statement of the theorem.

Notation 11.1. $\psi_{\lambda,\lambda'}(z) = e^{\lambda z} + e^{\lambda' z}$, and for a compact interval $J \subset \Lambda =]-\infty, R[$, $\psi_J = \psi_{\min J, \max J}$. For a kernel $\mu(x, dt)$, $\|\mu\|_J^{CC} = \sup_x \int \psi_J(x - t)|\mu(x, dt)|$, $(= \sup_x(\psi_J \star |\mu(x, \cdot)|)(x))$. For a kernel $\mathrm{k}(x; t, s)$, let $\|\mathrm{k}\|_J^{LC} = \sup_x \mathit{ess\ sup}_{s,t} \psi_J(x - t)|\mathrm{k}(x; t, s)|$.

$\lambda_-^\varepsilon = \min \Lambda^\varepsilon$, and $\lambda_+^\varepsilon = \max \Lambda^\varepsilon$. $\psi_\varepsilon = \psi_{\Lambda^\varepsilon}$, for a kernel $\mu(x, dt)$, $\|\mu\|_\varepsilon = \|\mu\|_{\Lambda^\varepsilon}^{cc}$, and for $\mathrm{h}(x, t)$, $\|\mathrm{h}\|_\varepsilon = \|\mathrm{h}(x, t)dt\|_\varepsilon$.

The first statement (i) of the theorem is a standard implication of an IFT, only this time it is formulated for the "classical" Fréchet-C^1 differentiability concept.

The second part (1), which also follows from the IFT, is a basis for stability. Here, the hard work of defining families of spaces indexed by $\lambda \in \Lambda^\varepsilon$ pays off: The implications of the IFT hold for all the spaces in the family, so the response to the change in endowments diminishes exponentially with time in both directions (past and future). Moreover, the ground work done when analyzing Υ becomes useful here: We proved that the equilibrium map is smooth, which also includes a Lipschitz property. The inverse of the derivative $\frac{\partial F}{\partial k}$ also inherited this property and so does the derivative of the implicit function with respect to the endowment. It follows that the derivative is very smooth itself, thus making it a "good" first-order approximation to the function around the chosen BGE, as is shown in part (iia). Also, as a result, we get proximity of two arbitrary equilibria emerging under two different endowments that in the neighborhood of zero, part (iic).

THEOREM 1. i. *Everywhere in* $\mathcal{G}$, $\exists\delta>0$ *and an* $\|\cdot\|_{\infty,1}$*-open ball B s.t., for any* BGE $\varpi(0)$, $\forall E\in B$, there is a unique solution $\varpi(E)\in P$ with $\|\varpi_k(E)-\varpi_k(0)\|\leq\delta$, and s.t. ϖ is Fréchet-C^1 on B.

ii. *Furthermore,* $\forall\varepsilon>0$ *there is an* $\|\cdot\|_{\infty,1}$*-open ball* $B_\varepsilon\subseteq B$ *s.t.,* $\forall$BGE *there is a compact interval* $\Lambda^\varepsilon\subseteq\Lambda$ *with* $\max\{\min\{0,\lambda_-+\varepsilon\},\frac{-1}{\varepsilon}\}$ *and* $(\lambda_+-\varepsilon)^+$ *in its interior, s.t., on* B_ε*:*

(*a*) $E\mapsto\varpi_q'(E)$ *is Lipschitz from* $\|\cdot\|_{\infty,1}$ *to* $\|\cdot\|^{\mathrm{LC}}_{\Lambda^\varepsilon}$ *for any component* $q\in\{k,y,c,\mathfrak{r},i\}$ *of* ϖ.

(*b*) ϖ *is differentiable in the strong sense: For any component* $q\in\{k,y,c,\mathfrak{r},i\}$ *of* ϖ *there is* $L\in\boldsymbol{R}$: $\forall x$,

$$|(\varpi_q(E+\delta E))(x)-(\varpi_q(E))(x)-((\varpi_q'(E))(\delta E))(x)|$$
$$\leq L\|\delta E\|_{\infty,1}\Big[\frac{1}{\psi_\varepsilon}\star\int|\delta E(\cdot,s)|ds\Big](x).$$

(*c*) $\exists L:\forall\lambda\in\Lambda^\varepsilon,\forall p,\ \forall E_1,E_2\in B_\varepsilon$,

$$\|\varpi_q(E_1)-\varpi_q(E_2)\|^\lambda_p\leq L\|E_1-E_2\|^\lambda_p,$$

for any component $q\in\{k,y,c,\mathfrak{r},i\}$ *of* ϖ.

Moreover, the bound can be tightened: with k^q_E *the kernel of the derivative of any component* $q\in\{k,y,c,\mathfrak{r},i\}$ *of* ϖ, $K^q\overset{\text{def}}{=}\sup_{B_\varepsilon}\|\mathrm{k}^q_E\|^{\mathrm{LC}}_{\Lambda^\varepsilon}$ *and* $\delta E=E_1-E_2$, $\forall x$,

$$|(\varpi_m(E_1))(x)-(\varpi_m(E_2))(x)|$$
$$\leq K^m\Big[\frac{1}{\psi_\varepsilon}\star\int|\delta E(\cdot,s)|ds\Big](x),\quad m\in\{k,y,c,\mathfrak{r}\},$$

and for the investment, i-th, coordinate,

$$|(\varpi_i(E_1))(x)-(\varpi_i(E_2))(x)-\int\delta E_{x,s}ds|$$
$$\leq K^i\left[\frac{1}{\psi_\varepsilon}\star\int|\delta E(\cdot,s)|ds\right](x).$$

(*d*) *For* $\lambda\in\Lambda^\varepsilon$, $\delta E\mapsto\varpi_q(E+\delta E)-\varpi_q(E)$ *is sequentially continuous from* $(L^\lambda_1,\sigma(L^\lambda_1,L^{-\lambda}_\infty))$ *to* C^λ_1 *for any* C_∞*-component* $q\in\{k,y,c,\mathfrak{r}\}$ *of* ϖ *and to* $(L^\lambda_1,\sigma(L^\lambda_1,L^{-\lambda}_\infty))$ *for the i-th component.*

Remark 7. Whereas points iia–iib express the regularity aspect (i.e., C^1) in a sharper way, points iic–iid express a very strong form of stability, or "no hysteresis": that the effects of a perturbation decay exponentially at rates λ_+ and λ_- at $+\infty$ and $-\infty$.

But, it is point iia that is the basic one, since point iib follows from it and points iic and iid follow from iib: regularity and stability are a single theorem!

Remark 8. (iid) is a "weak" analog of Lemma 18.iii in online Appendix F. Observe that even with $\lambda=0$ and at $E=0$, next result cannot extend to $L_{p,1}$ instead of L_1: shifting δE yields then a sequence converging weakly to 0, while the corresponding equilibria are also obtained by shifting, hence cannot converge to 0 in C_p.

Proof. (i): By Lemma 18 in online Appendix F and (Mertens and Rubinchik, 2014, Proposition 3.v).

(iia): For the C_∞-components use Lemma 20 in online Appendix G.2 and Proposition 4 from Appendix B.4. The result for i follows too, since the non-kernel part is constant in E and the kernel represents a linear operator from $L_{\infty,1}(\mathbf{R}^2)$ to C_∞ (being the difference of the kernels of ϖ_y' and ϖ_c').

(iib): Denote by k_E the kernel of the derivative, $\varpi_q(E)$, of a C_∞-component q of the solution[21] evaluated at E, each such kernel is well defined by Lemma 20 in online Appendix G.2. Let

$$g_z \stackrel{\text{def}}{=} (\varpi_q(E + z\delta E))(x) - (\varpi_q(E))(x) - z\iint \mathrm{k}_E(x; t, t-s)\delta E_{t,s} ds dt,$$

then, since by point i, ϖ is Fréchet and hence Gateaux differentiable,

$$g_z' = \iint [\mathrm{k}_{E+z\delta E}(x; t, t-s) - \mathrm{k}_E(x; t, t-s)]\delta E_{t,s} ds dt.$$

Hence, $|g_z' - g_0'| \le \iint |\mathrm{k}_{E+z\delta E}(x; t, s) - \mathrm{k}_E(x; t, s)||\delta E_{t,s}| ds dt$. By point iia and Proposition 4 from online Appendix B.4, $\exists L$ s.t. $\forall x$, $|\mathrm{k}_{E+\delta E}(x; t, u) - \mathrm{k}_E(x; t, u)| \le L\|\delta E\|_{\infty,1} \frac{1}{\psi_\varepsilon(x-t)}$ a.e. Hence, $|g_z'| \le Lz\|\delta E\|_{\infty,1}\left[\frac{1}{\psi_\varepsilon} \star \int |\delta E(\cdot, s)| ds\right](x)$. Since $\frac{1}{\psi_\varepsilon} \in L_{1,\infty}$, convolution with it $\in \mathbb{L}(L_{\infty,1}, C_\infty)$ by Theorem 1.xiii in Mertens and Rubinchik (2014); hence, the bound is finite for any finite z. Integrate now over $z \in [0, 1]$.

The result for the i-th component then follows, exactly as in the proof of point (iia).

(iic): The first claim is from Lemma 18.iii in online Appendix F. For the pointwise bound, replace k_E^q by its upper bound and δE by $|\delta E|$ in the derivative and integrate on the segment joining E_1 and E_2.

(iid): B_ε is weakly closed in L_1^λ by Lemma 15 in online Appendix B.5. For the C_∞-components, (iic) implies, by the "λ-transposition" (Lemma 1 in Appendix B) of Corollary 6 in Mertens and Rubinchik (2014), that, with $E_2 = E$, $E_1 = E + \delta E$, and δE in a weakly compact set in L_1^λ, the $|\varpi(E+\delta E) - \varpi(E)|$ are majorized by a compact family in C_1^λ. Conclude by Lemma 1 in Mertens and Rubinchik (2014) and Lemma 15 in online Appendix B.5. The result for i follows by Lemma 20 in online Appendix G.2. ■

11.2. From Solutions to Equilibria

For the results of the main theorem to apply to equilibria as well, the solution emerging as a result of a perturbation of policy (δE) has to also satisfy the interiority constraints, i.e., individual consumption should be positive and the two inequalities $0 < i_t < y_t$ should hold. Hence, in addition to the assumptions of the previous theorem, one has to assure that in the neighborhood where the theorem holds, first, the life-time transfers are not too negative for almost all individuals so that $\mathcal{N}_x$, the normalized individual income, is positive,[22] and, second, the aggregate transfer is uniformly bounded, and hence the last two inequalities hold.

THEOREM 2. $\exists\delta_1 > 0$ *such that,* $\forall$ BGE*,* $\mathcal{N}$ *of the equilibrium in the neighborhood of a given* BGE*,* $\{E \in B \mid ess\ sup_x \int E^-_{x+s,s} ds \leq \delta_1\}$*, is uniformly bounded away from* 0 *on that neighborhood.*

δ can be chosen as in Theorem 1 and $\exists\delta_0 > 0$*, s.t., on the set where* $\|\int E_{\cdot,s} ds\|_\infty \leq \delta_0$ *and* $ess\ sup_x \int E^-_{x+s,s} ds \leq \delta_1$*,* $\forall$BGE*,* $\varpi(E)$ *is the unique equilibrium of the E-perturbed economy s.t.* $\|\varpi_k(E) - \varpi_k(0)\| \leq \delta$ *and* $\|\varpi_c(E) - \varpi_c(0)\| \leq \delta$.

Proof. By Proposition 1. iii-v, for sufficiently small δ_1, $\mathcal{N}_x \geq \varepsilon$, since $\varpi_{\mathfrak{r}}$ is bounded and ϖ_k, thus ϖ_y, bounded away from 0.

To show that $\varpi(E)$ is an equilibrium, we need, by Proposition 1, that $\mathcal{N} \geq 0$, and that, if not in the basic model, $0 < i_t < y_t$ a.e., since $\inf k_t > 0$ by Corollary 10 in online Appendix F.

ϖ is continuous by Theorem 1, so the theorem remains true for any smaller δ, by adjusting B. By choosing δ, sufficiently small $\varpi_y(E) - \varpi_y(0)$ and $\varpi_c(E) - \varpi_c(0)$ will be uniformly small; hence $i_t = y_t + \int E_{t,s} ds - c_t$ implies that $\|\varpi_i(E) - \varpi_i(0)\|$ will be uniformly arbitrarily small if $\|\int E_{\cdot,s} ds\|_\infty \leq \delta_0$ for sufficiently small δ_0; in particular, $0 < i_t < y_t$ will hold: $\varpi(E)$ is an equilibrium.

It remains to show the uniqueness part. Since the equality $i_t = y_t + \int E_{t,x} ds - c_t$ holds for any equilibrium, the same argument as above shows that, for δ and δ_0 sufficiently small the inequality $0 < i_t < y_t$ will hold; Proposition 1 implies then that the whole equation system must hold, so the equilibrium must be $\varpi(E)$, by the uniqueness statement in Theorem 1. And for the basic model, the result follows from $k_t > 0$ (Corollary 10 in online Appendix F). ■

Remark 9. Just for $p = \infty$, since Λ^ε can be taken as a compact interval approximating Λ as close as desired from inside (so with 0 in interior), the theorem implies a very strong form of stability (i.e., *uniform* convergence to the status-quo), toward both $-\infty$ and $+\infty$, "at any exponential rate in Λ."

11.3. Technical Remarks (On the Choice of the Underlying Spaces)

Remark 10. The usual (strong) definition of a function being C^k on a set is that the function has a C^k extension to some open set containing the given set. Using this definition is the only way to distinguish the adequate topologies (for describing continuity, derivatives, etc.) from the region where the implicit function exists. Our results are exactly of this sort, the "solutions" of Theorem 1 being the required extension.

This approach uses the fact that demand extends naturally and smoothly for $M < 0$. Nothing is specific to this example there: We mentioned in Mertens and Rubinchik (2012) that homogeneity of utility w.r.t. to consumption goods was essential for balanced growth, and this is indeed a pure implication of homogeneity. If labor does not enter the utility function, as here, homogeneity implies directly that demand is positively homogeneous of degree 1 in total income, M.

Remark 11. Conditions for regularity of the BGE's w.r.t. variations in the parameters are trivial: It suffices that when restricting all functions in Υ in Proposition 1 to be constants, at each BGE $\frac{d\tilde{k}}{dk} \neq 1$, i.e., equivalently $\widehat{\mathbf{k}}_k^k(0) \neq 1$. In particular, on the generic set $\mathcal{G}$, regularity w.r.t. variations in the parameters also holds.

COROLLARY 2. *On the set of E's described in Theorem 2, the* k, y, $\mathfrak{r}$ *components of* ϖ *have values in a* $\tau(C_\infty, M)$ *compact set. If the ess sup condition in Theorem 2 were strengthened to ess sup*$_x$ $\int |E_{x+s,s}| ds \leq \delta_0$, *the whole map* ϖ *in Theorem 1 would become compact valued on this domain.*

Proof. As seen above, ϖ_i are bounded in L_∞, so, with the weak*-topology, they live in a compact metric space. On that space, first k, then y, $\mathfrak{r}$, then the others are, as in the proof of Lemma 15 in online Appendix B.5, continuous functions of i. If, furthermore, the $\int |E_{x+s,s}| ds$ are bounded in L_∞, this implies that $\mathcal{N}_{1,x}$ are also thus bounded, and one can then give a similar argument for the remaining components using in addition the relative compactness of the $\mathcal{N}_{1,x}$. ∎

Remark 12. The second case in the corollary suggests reinterpreting $E_{x+s,s} ds$ as a measure $\mu_x(ds)$ on [0, 1] (or a single measure on $\mathbf{R} \times [0, 1]$), with the weak topology of the dual of $L_1^{C([0,1])}(\mathbf{R})$. The second constraint identifies a ball in this space, compact and metrisable in this duality; and the first constraint defines then a closed subset, inducing the $\sigma(L_\infty, L_1)$ topology on $\int E_{\cdot,s} ds$. The equilibrium equations obviously still make sense with such endowments, since they imply prices are even C^1, and this would make ϖ a continuous function on a compact metric space.

12. CONCLUSIONS

We have demonstrated that "comparative statics" around a BGE in an overlapping generations model in continuous time with respect to a fully anticipated transfer is feasible. Although tedious, such work has a high payoff: The IFT assures not only the existence of the one-to-one map between the "policy parameter" (transfer of endowments) and the resulting equilibrium in the neighborhood of a chosen BGE, but also the Lipschitz property of its derivative, hence assuring the first-order approximation of that map given by the derivative is "of high quality."

Perhaps the most surprising result is the (uniform over Λ) Lipschitz property of the derivatives, Theorem 1. iia, yielding exponential convergence of two non-BGE equilibria in the neighborhood of a BGE to each other (Theorem 1. iic), i.e., the stability result in its strong form.

NOTES

1. In addition, there is also recent work analyzing equilibria in continuous-time overlapping generations models; see d'Albis and Augeraud-Véron (2007, 2009) based on the seminal contributions of Yaari (1964) and Cass and Yaari (1967).

2. To stress, our choice of taking time as $\mathbf{R}$ to index birth- and transaction dates was not dictated by the choice of tools; rather, the main concern was the indeterminacy results mentioned above. One can use Wiener's theorem for "discrete time" models, in fact, one of the textbook examples of its use is for the space of infinite summable sequences; see (Kolmogorov and Fomin, 1989, p. 603, Appendix "Banach Algebras" by V. Tikhomirov).

3. With three potentially different physical goods, output, consumption, investment, while we assume that output can always be transformed into investment and consumption, the opposite transformation might or might not be feasible.

4. u is extended by continuity to $[0, +\infty]$.

5. The profits will be null due to constant-returns-to-scale production, defined below.

6. A function is locally integrable on a topological space if every point has a measurable neighborhood where the function is integrable.

7. Since for any consumption bundle $\hat{c}$ any equivalent function (coinciding with $\hat{c}$ a.e.) has the same utility and the same budget, we will think of it as an equivalence class of $\overline{\mathbf{R}}_{+}$-valued measurable functions. The same applies to all flows, Y_t, I_t, C_t, labor- and capital services, and to their prices p^Y, p^I, p^C, w, and r. On the other hand, capital is a stock, so it and its price p_t are defined pointwise, and no measurability restriction has a reason to apply. By the usual convention in measure theory, define any product of prices and quantities as 0 in the case of a product $0 \times \infty$ or $\infty \times 0$. This allows us to think of either prices or quantities as measures.

8. In its integral form the equation is $K_t = e^{-\delta(t-t_0)} K_{t_0} + \int_{t_0}^{t} e^{-\delta(t-s)} I_s ds$, where the integral is a Lebesgue integral [without loss of generality in this model, cf. Mertens and Rubinchik 2013, Corollary 2].

9. For the detailed description of the technology available to investment firms and the appropriate definition of their profits, see Mertens and Rubinchik, 2013, Section 2.2.3.

10. $E_{t,s}$ is the normalized (per unit of productive labor at time t) endowment at time t of all individuals of age s, and so Ω_t denotes the aggregate normalized endowment at time t.

11. Note that the existence of the inner norm, $\|\mathbb{1}_{x+H} f\|_q$, limits only the "local" behavior (on H shifted by x) of the function f, whereas the map from x to the inner norm shifts the "locality" and thus the existence of the outer norm restricts the global behavior of the function.

12. For an extended characterization, see Mertens and Rubinchik (2013).

13. $\mathcal{B}$ has the same sign as the equilibrium consumption of an individual at age u, born at $t-u$ [cf. Mertens and Rubinchik, 2013, Comment 21], which is also the sign of $\mathcal{N}$ by part (vii), since $\mathcal{D} \geq 0$, which follows from part (vi).

14. $\int \epsilon_z(x)dx = 1$ and $\epsilon_z(x) = 0$ for all $x \neq z$.

15. Note that $\mathbf{k}_k^c$ as above is also a continuous-time counterpart of the classical impulse response function used in time-series analysis, e.g., Hamilton, 1994, p. 318, equations 11.4.1,11.4.2.

16. For an integrable function g, its *Fourier transform* (FT) is $\widehat{g}(\omega) = \int e^{i\omega t} g(t)dt$, a complex-valued function defined on $\mathbf{R}$.

17. Recall, the relevant equation is $(\mathcal{L}\mathbf{k}_k^k)(z) = 1$, where $z = \lambda + i\omega \in \mathbf{C}$, cf. the preceding subsection.

18. For a given equilibrium equation system: Indeed, the system could be represented, instead of a map from k to $\tilde{k}$, as a map from i to $\tilde{i}$, for example.

19. Re-expressing equation (2) in Lemma 19.ii (which defines h) with the continuous part S of h gives $S(x, z) = \int S(x, t)\mathrm{k}(t, z)dt$ plus a known continuous function; one can solve that equation numerically. The $\|\cdot\|_{\Lambda^\varepsilon}^{\mathrm{LC}}$ bound of Lemma 19.i should prove very useful for the numerical analysis of this (e.g., local density of a grid); however, to investigate appropriate truncations, it seems "stability" results for the kernels themselves would be needed. That is beyond our scope here.

20. Interestingly, it appears to be consistent with the empirical findings, Lee et al. (2011), if one takes the oil price change as a proxy for the endowment shock.

21. So that $q \in \{k, y, c, \mathfrak{r}\}$.

22. $\mathcal{N}_x$ has the same sign as consumption of an individual born at x (cf. Section 3.2).

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