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Infinitely many solutions for three classes of self-similar equations with p-Laplace operator: Gelfand, Joseph–Lundgren and MEMS problems

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We study global solution curves and prove the existence of infinitely many positive solutions for three classes of self-similar equations with p-Laplace operator. In the $p = 2$ case these are well-known problems involving the Gelfand equation, the equation modelling electrostatic micro-electromechanical systems (MEMS), and a polynomial nonlinearity. We extend the classical results of Joseph and Lundgren to the case in which $p \neq 2$, and we generalize the main result of Guo and Wei on the equation modelling MEMS.

Keywords: parametrization of the global solution curves; infinitely many solutions

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1. Introduction

We consider radial solutions on a ball in $\mathbb{R}^n$ for three special classes of equations involving the p-Laplace operator, those that are self-similar under scaling. We now explain our approach for one of the classes, that involving the p-Laplace version of the equation that arises in the modelling of electrostatic micro-electromechanical systems (MEMS),

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda \frac{|x|^{\alpha}}{(1-u)^{q}} = 0 \quad \text{for } |x| < 1,\ u = 0 \text{ when } |x| = 1, \tag{1.1}$$

where $p > 1$, $\alpha > 0$, $q > 0$, $u = u(x)$, $x \in \mathbb{R}^n$, $n \geqslant 1$ (see [6, 7, 14]). Here λ is a positive parameter. We are looking for solutions satisfying $0 < u < 1$. Radial solutions of this equation satisfy

$$\begin{gathered} \varphi(u')' + \frac{n-1}{r}\varphi(u') + \lambda \frac{r^{\alpha}}{(1-u)^{q}} = 0 \quad \text{for } 0 < r < 1, \\ u'(0) = u(1) = 0, \qquad 0 < u(r) < 1, \end{gathered} \tag{1.2}$$

with $\varphi(v) = v|v|^{p-2}$. It is easy to see that $u'(r) < 0$ for all $0 < r < 1$, which implies that the value of $u(0)$ gives the maximum value (or the L^{∞}-norm) of our solution. Moreover, $u(0)$ is a *global parameter*, i.e. it uniquely identifies the solution pair $(\lambda, u(r))$; see, for example, [10]. It follows that a two-dimensional curve in the

$(\lambda, u(0))$ plane completely describes the solution set of (1.2). The self-similarity of this equation allows one to parametrize the global solution curve, using the solution of a single initial-value problem:

$$\varphi(w')' + \frac{n-1}{t}\varphi(w') = \frac{t^\alpha}{w^q}, \quad w(0) = 1, \ w'(0) = 0. \tag{1.3}$$

Its solution $w(t)$ is a positive and increasing function, which can be easily computed numerically. Following Pelesko [14], we show that the global solution curve of (1.2) is given by

$$(\lambda, u(0)) = \left(\frac{t^{\alpha+p}}{w^{p+q-1}(t)}, 1 - \frac{1}{w(t)} \right),$$

parametrized by $t \in (0, \infty)$. In particular,

$$\lambda = \lambda(t) = t^{\alpha+p}/w^{p+q-1}(t) \quad \text{and} \quad \lambda'(t) = t^{\alpha+p-1}w^{-p-q}[(\alpha+p)w - t(p+q-1)w'],$$

so that the solution curve travels to the right (respectively, left) in the $(\lambda, u(0))$ plane if $(\alpha+p)w - t(p+q-1)w' > 0$ (respectively, $(\alpha+p)w - t(p+q-1)w' < 0$). This makes us interested in the roots of the function $(\alpha+p)w - t(p+q-1)w'$. If we set this function to zero,

$$(\alpha+p)w - t(p+q-1)w' = 0,$$

then the general solution of this equation is

$$w(t) = ct^\beta, \quad \beta = \frac{\alpha+p}{p+q-1}.$$

Quite remarkably, if we choose the constant

$$c = c_0 = \left[\frac{1}{\beta^{p-1}[(p-1)(\beta-1)+n-1]} \right]^{1/(p+q-1)},$$

then

$$w_0(t) = c_0 t^\beta$$

also solves the equation in (1.3), along with $w(t)$. We show that $w(t)$ tends to $w_0(t)$ as $t \to \infty$, and the solution curve of (1.2) makes infinitely many turns if and only if $w(t)$ and $w_0(t)$ intersect infinitely many times. We give a sharp condition for that to happen, thus generalizing the main result of Guo and Wei [7] to the case in which $p \neq 2$ (with a simpler proof). In [11] we called $w(t)$ *the generating solution*, and $w_0(t)$ *the guiding solution*.

We apply a similar approach to a class of equations with polynomial $f(r, u)$ generalizing the well-known results of Joseph and Lundgren [9], and to the p-Laplace version of the generalized Gelfand equation, where we easily recover the corresponding result of Jacobsen and Schmitt [8].

For each of the three classes of equations we show that along the solution curves (as $u(0) \to \infty$) the solutions tend to a singular solution (for which $u(r) \to \infty$ or $u'(r) \to \infty$ as $r \to 0$). Moreover, one can calculate the singular solutions explicitly, which is truly a remarkable feature of self-similar equations. Singular solutions were studied previously by many authors, including Budd and Norbury [2], Merle and Peletier [13] and Flores [5].

2. Parametrization of the solution curves

We begin with the p-Laplace version of the generalized Gelfand equation

$$\varphi(u')' + \frac{n-1}{r}\varphi(u') + \lambda r^{\alpha} e^{u} = 0 \quad \text{for } 0 < r < 1,\ u'(0) = 0,\ u(1) = 0, \tag{2.1}$$

where $\varphi(v) = v|v|^{p-2}$, $p > 1$. Observe that $\varphi(sv) = s^{p-1}\varphi(v)$ for any constant $s > 0$. Assume that $u(0) = a > 0$. We set $u = w + a$, $t = br$. The constants a and b are assumed to satisfy

$$\lambda = b^{\alpha+p} e^{-a}.$$

Then (2.1) becomes

$$\varphi(w')' + \frac{n-1}{t}\varphi(w') + t^{\alpha} e^{w} = 0, \quad w(0) = 0,\ w'(0) = 0. \tag{2.2}$$

The solution of this problem $w(t)$, which is a negative and decreasing function, is defined for all $t > 0$, and it may be easily computed numerically. (Write this equation as $[t^{n-1}\varphi(w')]' = -t^{n+\alpha-1}e^{w} < 0$, conclude that $t^{n-1}\varphi(w') < 0$, and then $w'(t) < 0$ for all t.) We have

$$0 = u(1) = a + w(b),$$

so that $a = -w(b)$, and then $\lambda = b^{\alpha+p}e^{w(b)}$. The solution curve for (2.1) is

$$(\lambda, u(0)) = (b^{\alpha+p}e^{w(b)}, -w(b)),$$

parametrized by $b \in (0, \infty)$. The solution of (2.1) at b is $u(r) = w(br) - w(b)$. It will be convenient to write the solution curve as

$$(\lambda, u(0)) = (t^{\alpha+p}e^{w(t)}, -w(t)), \tag{2.3}$$

parametrized by $t \in (0, \infty)$, and $w(t)$ is the solution of (2.2). The solution of (2.1) at the parameter value t is $u(r) = w(tr) - w(t)$.

We next consider the problem

$$\begin{gathered}\varphi(u')' + \frac{n-1}{r}\varphi(u') + \lambda\frac{r^{\alpha}}{(1-u)^{q}} = 0 \quad \text{for } 0 < r < 1, \\ u'(0) = u(1) = 0, \qquad 0 < u(r) < 1,\end{gathered} \tag{2.4}$$

which arises in the modelling of MEMS; see [6,7,14]. Here λ is a positive parameter, $q > 0$ and $\alpha > 0$ are constants, and as before $\varphi(v) = v|v|^{p-2}$, $p > 1$. Any solution $u(r)$ of (2.4) is a positive and decreasing function (by the maximum principle), so that $u(0)$ gives its maximum value. Our goal is to compute the solution curve $(\lambda, u(0))$. Let $1 - u = v$. Then $v(r)$ satisfies

$$\varphi(v')' + \frac{n-1}{r}\varphi(v') = \lambda\frac{r^{\alpha}}{v^{q}} \quad \text{for } 0 < r < 1,\ v'(0) = 0,\ v(1) = 1. \tag{2.5}$$

Assume that $v(0) = a$. We scale as follows: $v(r) = aw(r)$ and $t = br$. The constants a and b are assumed to satisfy

$$\lambda = a^{p+q-1}b^{\alpha+p}. \tag{2.6}$$

Then (2.5) becomes

$$\varphi(w')' + \frac{n-1}{t}\varphi(w') = \frac{t^\alpha}{w^q}, \quad w(0) = 1,\ w'(0) = 0. \tag{2.7}$$

The solution of this problem is a positive increasing function that is defined for all $t > 0$. We have

$$1 = v(1) = aw(b),$$

and so $a = 1/w(b)$, and then $\lambda = b^{\alpha+p}/w^{p+q-1}(b)$. The solution curve $(\lambda, u(0))$ is

$$\left(\frac{b^{\alpha+p}}{w^{p+q-1}(b)}, 1 - \frac{1}{w(b)}\right),$$

parametrized by $b \in (0, \infty)$. It will be convenient to write the solution curve in the form

$$(\lambda, u(0)) = \left(\frac{t^{\alpha+p}}{w^{p+q-1}(t)}, 1 - \frac{1}{w(t)}\right), \tag{2.8}$$

parametrized by $t \in (0, \infty)$. In the $p = 2$ case, this parametrization was first derived by Pelesko [14] and was then used in [6]. The solution of (2.4) at t is $u(r) = 1 - w(tr)/w(t)$.

Finally, we consider the problem (with the constants $p > 1$, $q > 1$, $\alpha \geqslant 0$)

$$\begin{gathered}\varphi(u')' + \frac{n-1}{r}\varphi(u') + \lambda r^\alpha (1+u)^q = 0 \quad \text{for } 0 < r < 1, \\ u'(0) = u(1) = 0,\end{gathered} \tag{2.9}$$

which was analysed for the case in which $p = 2$ and $\alpha = 0$ by Joseph and Lundgren [9]. If we set $1 + u = v$, then $v(r)$ satisfies

$$\varphi(v')' + \frac{n-1}{r}\varphi(v') + \lambda r^\alpha v^q = 0, \quad v'(0) = 0,\ v(1) = 1. \tag{2.10}$$

Assuming that $v(0) = a$, we scale as follows: $v(r) = aw(r)$ and $t = br$. The constants a and b are assumed to satisfy

$$\lambda = \frac{b^{p+\alpha}}{a^{q-p+1}}. \tag{2.11}$$

Then (2.10) becomes

$$\varphi(w')' + \frac{n-1}{t}\varphi(w') + t^\alpha w^q = 0, \quad w(0) = 1,\ w'(0) = 0. \tag{2.12}$$

The solution of (2.12) satisfies $w'(t) < 0$, so long as $w(t) > 0$ (the function $t^{n-1}\varphi(w'(t))$ is zero at $t = 0$, and its derivative is negative). It follows that either there is a t_0 such that $w(t_0) = 0$ and $w(t) > 0$ on $(0, t_0)$, or $w(t) > 0$ on $(0, \infty)$ and $\lim_{t\to\infty} w(t) = a \geqslant 0$. It is easy to see that $a = 0$ in the second case. Indeed, assuming that $a > 0$, we have $[t^{n-1}\varphi(w')]' \leqslant -a^q t^{n+\alpha-1}$, and integrating we conclude that $w(t) \leqslant 1 - ct^\gamma$, with some $c > 0$ and $\gamma = (\alpha+p)/(p-1) > 0$, contradicting that $w(t) > 0$ on $(0, \infty)$.

LEMMA 2.1. *Assume that*

$$q > \frac{np - n + p + p\alpha}{n - p}. \tag{2.13}$$

Then $w(t) > 0$ *and* $w'(t) < 0$ *on* $(0, \infty)$, *with* $\lim_{t\to\infty} w(t) = 0$.

Proof. In view of the above remarks, we need to exclude the possibility that $w(t_0) = 0$ and $w(t) > 0$ on $(0, t_0)$. Recall that for the equation

$$\varphi(w')' + \frac{n-1}{t}\varphi(w') + f(t, w) = 0$$

the Pohozhaev function

$$P(t) = t^n[(p-1)\varphi(w')w' + pF(t, w)] + (n-p)t^{n-1}\varphi(w')w$$

is easily seen to satisfy

$$P'(t) = t^{n-1}[npF(t, w) - (n-p)wf(t, w) + ptF_t(t, w)],$$

where $F(t, w) = \int_0^w f(t, z)\,\mathrm{d}z$ (see, for example, [10, p. 136]). Here

$$P'(t) = t^{n-1+\alpha}\left[\frac{np}{q+1} - (n-p) + \frac{p\alpha}{q+1}\right]w^{q+1} < 0.$$

Since $P(0) = 0$ and $P(t_0) > 0$, we have a contradiction. □

As before, we have

$$1 = v(1) = aw(b),$$

and so $a = 1/w(b)$, and then $\lambda = b^{p+\alpha}w^{q-p+1}(b)$. Under condition (2.13), the solution curve $(\lambda, u(0))$ is

$$\left(b^{p+\alpha}w^{q-p+1}(b), \frac{1}{w(b)} - 1\right),$$

parametrized by $b \in (0, \infty)$. The solution at b is $u(r) = w(br)/w(b) - 1$. It will be convenient to write the solution curve in the form

$$(\lambda, u(0)) = \left(t^{p+\alpha}w^{q-p+1}(t), \frac{1}{w(t)} - 1\right), \tag{2.14}$$

parametrized by $t \in (0, \infty)$. The solution of (2.9) at t is $u(r) = w(tr)/w(t) - 1$.

3. The equation modelling MEMS

We consider problem (2.4), whose solution curve is given by (2.8), where $w(t)$ is the solution of (2.7). We have $\lambda(t) = t^{\alpha+p}/w^{p+q-1}(t)$, where $w(t)$ is the solution of (2.7), and so

$$\lambda'(t) = t^{\alpha+p-1}w^{-p-q}[(\alpha+p)w - t(p+q-1)w'].$$

We are interested in the roots of the function $(\alpha + p)w - t(p + q - 1)w'$. If we set this function to zero,

$$(\alpha + p)w - t(p + q - 1)w' = 0,$$

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then the general solution of this equation is

$$w(t) = ct^{\beta}, \quad \beta = \frac{\alpha + p}{p + q - 1}.$$

Quite remarkably, if we choose the constant

$$c = c_0 = \left[\frac{1}{\beta^{p-1}[(p-1)(\beta-1)+n-1]}\right]^{1/(p+q-1)},$$

under the condition that

$$(p-1)(\beta-1)+n-1 > 0, \tag{3.1}$$

then

$$w_0(t) = c_0 t^{\beta}$$

also solves the equation in (2.7), along with $w(t)$. We shall show that $w(t)$, the solution of initial-value problem (2.7), tends to $w_0(t)$ as $t \to \infty$, and the issue turns out to be whether $w(t)$ and $w_0(t)$ cross infinitely many times as $t \to \infty$.

LEMMA 3.1. *Assume that $w(t)$ and $w_0(t)$ intersect infinitely many times. Then the solution curve of (2.4) makes infinitely many turns.*

Proof. Assuming that $w(t)$ and $w_0(t)$ intersect infinitely many times, let $\{t_n\}$ denote the points of intersection. At the $\{t_n\}$s, $w(t)$ and $w_0(t)$ have different slopes (by uniqueness for initial-value problems). Since $(\alpha+p)w_0(t_n)-t_n(p+q-1)w_0'(t_n) = 0$, it follows that $(\alpha+p)w(t_n) - t_n(p+q-1)w'(t_n) < 0$ (respectively, $(\alpha+p)w(t_n) - t_n(p+q-1)w'(t_n) > 0$) if $w(t)$ intersects $w_0(t)$ from below (respectively, above) at t_n. Hence, on any interval (t_n, t_{n+1}) there is a point t_0 where $(\alpha+p)w(t_0)-t_0(p+q-1)w'(t_0) = 0$, i.e. $\lambda'(t_0) = 0$, and t_0 gives a critical point. Since $\lambda'(t_n)$ and $\lambda'(t_{n+1})$ have different signs, the solution curve changes its direction over (t_n, t_{n+1}). □

We shall need Sturm–Picone's comparison theorem, which is well known (see, for example, [12, p. 5]).

LEMMA 3.2. *Let $u(t)$ and $v(t)$ respectively be classical solutions of*

$$(a(t)u')' + b(t)u = 0 \tag{3.2}$$

and

$$(a_1(t)v')' + b_1(t)v = 0. \tag{3.3}$$

Assume that the given differentiable functions $a(t)$, $a_1(t)$ and continuous functions $b(t)$ and $b_1(t)$ satisfy

$$b_1(t) \geqslant b(t) \quad \text{and} \quad 0 < a_1(t) \leqslant a(t) \quad \text{for } t \geqslant t_0 > 0. \tag{3.4}$$

In the case in which $a_1(t) = a(t)$ and $b_1(t) = b(t)$ for all t, assume additionally that $u(t)$ and $v(t)$ are not constant multiples of one another. Then, for $t \geqslant t_0$, $v(t)$ has a root between any two consecutive roots of $u(t)$.

Lemma 3.3. *Consider the equation*

$$(a_0(t)(1+f(t))v')' + \frac{n-1}{t}a_0(t)(1+f(t))v' + b_0(t)(1+g(t))v = 0, \tag{3.5}$$

with given differentiable functions $a_0(t) > 0$ *and* $f(t)$*, and continuous functions* $b_0(t) > 0$ *and* $g(t)$*. Assume that* $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0$ *and that there is an* $\varepsilon > 0$ *such that any solution of*

$$(a_0(t)(1+\varepsilon)v')' + \frac{n-1}{t}a_0(t)(1+\varepsilon)v' + b_0(t)(1-\varepsilon)v = 0 \tag{3.6}$$

has infinitely many roots. Then any solution of (3.5) has infinitely many roots.

Proof. We rewrite (3.5) in the form (3.2), with $a(t) = t^{n-1}a_0(t)(1+f(t))$ and $b(t) = t^{n-1}b_0(t)(1+g(t))$, and we rewrite (3.6) in the form (3.3), with $a_1(t) = t^{n-1}a_0(t)(1+\varepsilon)$ and $b_1(t) = t^{n-1}b_0(t)(1-\varepsilon)$. For large t the inequalities in (3.4) hold and lemma 3.2 applies. □

The linearized equation for (2.7) is

$$(\varphi'(w')z')' + \frac{n-1}{t}\varphi'(w')z' = -qt^{\alpha}w^{-q-1}z.$$

At the solution $w = w_0(t)$, this becomes

$$(a_0(t)z')' + \frac{n-1}{t}a_0(t)z' + b_0(t)z = 0, \tag{3.7}$$

with

$$a_0(t) = \varphi'(w_0') = (p-1)c_0^{p-2}\beta^{p-2}t^{(p-2)(\beta-1)}$$

and

$$b_0(t) = qt^{\alpha}w_0^{-q-1} = qc_0^{-q-1}t^{\alpha-\beta(q+1)}.$$

One simplifies (3.7) to read

$$z'' + \frac{[(p-2)(\beta-1)+n-1]}{t}z' + \frac{q\beta[(p-1)(\beta-1)+n-1]}{(p-1)t^2}z = 0,$$

which is an Euler equation! The roots of its characteristic equation,

$$r(r-1) + [(p-2)(\beta-1)+n-1]r + \frac{q\beta[(p-1)(\beta-1)+n-1]}{p-1} = 0,$$

are complex valued provided that

$$[(p-2)(\beta-1)+n-2]^2 < \frac{4q\beta[(p-1)(\beta-1)+n-1]}{p-1}.$$

We write this inequality in the form

$$A\beta^2 + B\beta - C > 0, \tag{3.8}$$

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with $A = 4(p-1)q - (p-1)(p-2)^2$, $B = 4q(n-p) - 2(p-1)(p-2)(n-p)$ and $C = (p-1)(n-p)^2$. We shall have $A > 0$ provided that

$$4q - (p-2)^2 > 0. \tag{3.9}$$

For (3.8) to hold, we need $\beta = (\alpha+p)/(p+q-1)$ to be greater than the larger root of this quadratic, i.e. $\beta > (-B + \sqrt{B^2+4AC})/2A$ (assuming that (3.9) holds), which gives

$$\frac{\alpha+p}{p+q-1} > \frac{(p-n)(2q-p^2+3p-2) + 2|n-p|\sqrt{q(p+q-1)}}{(p-1)[4q-(p-2)^2]}. \tag{3.10}$$

THEOREM 3.4. *Assume that* $q > 0$, $p > 1$, *with*

$$(p-1)(\beta-1) + n - 1 > \beta, \tag{3.11}$$

and that conditions (3.9) and (3.10) hold. Then the solution curve of

$$\begin{gathered}\varphi(u')' + \frac{n-1}{r}\varphi(u') + \lambda\frac{r^\alpha}{(1-u)^q} = 0 \quad \text{for } 0 < r < 1, \\ u'(0) = u(1) = 0, \qquad 0 < u(r) < 1,\end{gathered} \tag{3.12}$$

makes infinitely many turns. Moreover, along this curve (as $u(0) \to \infty$*),* $\lambda \to \lambda_0 = 1/c_0^{q-1} = \beta^{p-1}[(p-1)(\beta-1)+n-1]$*, and* $u(r)$ *tends to* $1 - r^\beta$ *for* $r \neq 0$*, which is a solution of the equation in (3.12).*

Proof. In view of lemma 3.1, we need to show that $w(t)$ and $w_0(t)$ intersect infinitely many times. Let $P(t) = w(t) - w_0(t)$. Then $P(t)$ satisfies

$$(a(t)P')' + \frac{n-1}{t}a(t)P' + b(t)P = 0, \tag{3.13}$$

where

$$a(t) = \int_0^1 \varphi'(sw'(t) + (1-s)w_0'(t))\,\mathrm{d}s, \tag{3.14}$$

$$b(t) = qt^\alpha \int_0^1 \frac{1}{[sw(t)+(1-s)w_0(t)]^{q+1}}\,\mathrm{d}s. \tag{3.15}$$

We claim that it is impossible for $P(t)$ to keep the same sign over some infinite interval (t_0, ∞) while tending to a constant as $t \to \infty$. Assuming the contrary, write

$$\begin{aligned}a(t) &= (p-1)(w_0')^{p-2}\int_0^1 \left|s\frac{w'(t)}{w_0'(t)} + (1-s)\right|^{p-2} \mathrm{d}s = a_0(t)(1+o(1)),\\ b(t) &= qt^\alpha \frac{1}{w_0^{q+1}}\int_0^1 \frac{1}{[s(w(t)/w_0(t)) + (1-s)]^{q+1}}\,\mathrm{d}s = b_0(t)(1+o(1))\end{aligned}$$

as $t \to \infty$. (Observe that $w(t)/w_0(t) \to 1$, since $P(t)$ tends to a constant, and $w'(t)/w_0'(t) \to 1$, by L'Hospital's rule, as $t \to \infty$.) Since Euler's equation (3.7) has infinitely many roots on (t_0, ∞), we conclude by lemma 3.3 that $P(t)$ must vanish on that interval too, which is a contradiction.

Next we show that if $P(t_0) = 0$, then $P(t)$ remains bounded for all $t > t_0$. Assume that $P'(t_0) < 0$; the case in which $P'(t_0) > 0$ is similar. Then $P(t) < 0$ for $t > t_0$, with $t - t_0$ small. From (3.13), $t^{n-1}a(t)P'(t)$ is increasing for $t > t_0$ so that

$$P'(t) > -\frac{a_0}{a(t)t^{n-1}} \quad \text{for } t > t_0 \text{ (with } a_0 = -t_0^{n-1}a(t_0)P'(t_0) > 0).$$

Since solutions of the linear equation (3.13) cannot go to infinity over a bounded interval, we may assume that t_0 is large, and then, by the above, $a(t) \sim a_0(t) \sim a_1 t^{(p-2)(\beta-1)}$ for $t > t_0$ and some $a_1 > 0$. It follows that, for some $a_2 > 0$,

$$P'(t) > -\frac{a_2}{t^{n-1+(p-2)(\beta-1)}} = -\frac{a_2}{t^{1+\varepsilon}} \quad \text{for } t > t_0, \tag{3.16}$$

with $\varepsilon = n - 2 + (p-2)(\beta-1) > 0$, in view of (3.11). Integrating over (t_0, t) and using that $n \geqslant 3$, we conclude the boundedness of $P(t)$, so long as $P(t) < 0$. If another root of $P(t)$ is encountered, we repeat the argument. Hence, $P(t)$ remains bounded for all $t > t_0$.

From (3.13) we see that $P(t)$ cannot have points of positive minimum or points of negative maximum. We claim that if $P(t)$ has one root, it has infinitely many roots. Indeed, assume that $P(t_1) = 0$, and say $P'(t_1) > 0$. For $t > t_1$, $P(t)$ remains bounded but cannot tend to a constant. Hence, $P(t)$ will have to turn back and become decreasing, but it cannot have a positive local minimum or tend to a constant. Hence, $P(t_2) = 0$ at some $t_2 > t_1$, and so on.

We have $P(0) = 1$, so that $(t^{n-1}a(t)P'(t))' < 0$ for small $t > 0$. The function $q(t) \equiv t^{n-1}a(t)P'(t)$ satisfies $q(0) = 0$ and $q'(t) < 0$, and so $q(t) < 0$. It follows that $P'(t) < 0$ for small $t > 0$. Since $P(t)$ cannot turn around or tend to a constant, we conclude the existence of the first root t_1 of $P(t)$, thereby implying the existence of infinitely many roots.

We show next that $w(t) \to w_0(t)$ as $t \to \infty$. Let t_k and t_{k+1} be two consecutive roots of $P(t)$ and let $P'(t_k) < 0$ so that $P(t) < 0$ on (t_k, t_{k+1}). Let τ_k be the unique minimum point of $P(t)$ on (t_k, t_{k+1}). For negative $P(t)$ we have inequality (3.16) with t_k in place of t_0. Integrating this inequality over (t_k, τ_k), we get

$$P(\tau_k) > \bar{c}(\tau_k^{-\varepsilon} - t_k^{-\varepsilon}) \quad \text{(with some } \bar{c} > 0),$$

which implies that $|P(\tau_k)| \to 0$ as $k \to \infty$. The case in which $P'(t_k) > 0$ is similar, so that $w(t) \to w_0(t)$ along the solution curve. Since $u(r) = 1 - w(tr)/w(t)$, it follows that along the solution curve $u(r)$ tends to $1 - w_0(tr)/w_0(t) = 1 - r^\beta$, while $\lambda(t)$ tends to $1/c_0^{q-1}$. □

Observe that in the case in which $\beta \in (0,1)$, the limiting solution $1 - r^\beta$ is *singular* because $u'(0)$ is not defined. Notice also that condition (3.11) implies (3.1). Finally, observe that in the case in which $\beta \in (0,1)$, condition (3.11) implies that $n \geqslant 2$. Indeed, we can rewrite (3.11) as $n > 2\beta + p(1-\beta)$, which is a point between $p > 1$ and 2.

One special case in which this theorem applies is the following. Assume that $n \geqslant p$ so that (3.10) becomes

$$\frac{\alpha + p}{p + q - 1} > (n - p)\frac{2\sqrt{q(p+q-1)} + p^2 - 3p + 2 - 2q}{(p-1)[4q - (p-2)^2]}.$$

Then (3.10) holds, provided that

$$\left.\begin{aligned} 2\sqrt{q(p+q-1)}+p^2-3p+2-2q>0,\\ 4q>(p-2)^2,\\ p\leqslant n<p+\frac{(\alpha+p)(p-1)[4q-(p-2)^2]}{(p+q-1)(2\sqrt{q(p+q-1)}+p^2-3p+2-2q)}. \end{aligned}\right\} \tag{3.17}$$

Observe that the third inequality ($n \geqslant p$) implies that condition (3.1) holds, and the second inequality is just (3.9). Hence, the three inequalities in (3.17) imply the theorem. In the $p = 2$ case, the first and the second inequalities hold automatically, while the third gives Guo and Wei's condition [7].

4. The generalized Joseph–Lundgren problem

We now study problem (2.9). Its solution curve is represented by (2.14), under condition (2.13), where $w(t)$ is the solution of (2.12). In particular, $\lambda(t) = t^{p+\alpha}w^{q-p+1}(t)$, and we wish to know how many times this function changes the direction of monotonicity for $t \in (0, \infty)$. (Here $w(t)$ is the generating solution of (2.12).) Compute

$$\lambda'(t) = t^{p+\alpha-1}w^{q-p}(t)[(p+\alpha)w(t) + (q-p+1)tw'(t)],$$

so that we are interested in the roots of the function $(p+\alpha)w + (q-p+1)tw'$. If we set this function to zero,

$$(p+\alpha)w + (q-p+1)tw' = 0,$$

then the general solution of this equation is $w(t) = at^{-\beta}$, with $\beta = (p+\alpha)/(q-p+1)$. If we choose the constant a as

$$a = a_0 = [(n-p)\beta^{p-1} - (p-1)\beta^p]^{1/(q-p+1)},$$

then $w_0(t) = a_0t^{-\beta}$ is the guiding solution of (2.12) (we have $(n-p)\beta^{p-1} - (p-1)\beta^p > 0$, under condition (2.13), if $n > p$).

Lemma 4.1. *Assume that $w(t)$ and $w_0(t)$ intersect infinitely many times. Then the solution curve of (2.9) makes infinitely many turns.*

Proof. Indeed, assuming that $w(t)$ and $w_0(t)$ intersect infinitely many times, let $\{t_n\}$ denote their points of intersection. At the $\{t_n\}$s, $w(t)$ and $w_0(t)$ have different slopes (by uniqueness for initial-value problems). Since $(p+\alpha)w_0(t_n)+(q-p+1)t_nw_0'(t_n) = 0$, it follows that $(p+\alpha)w(t_n)+(q-p+1)t_nw'(t_n) > 0$ (respectively, $(p+\alpha)w(t_n)+(q-p+1)t_nw'(t_n) < 0$) if $w(t)$ intersects $w_0(t)$ from below (respectively, above) at t_n. Hence, on any interval (t_n, t_{n+1}) there is a point t_0 where $(p+\alpha)w(t_0) + (q-p+1)t_0w'(t_0) = 0$, i.e. $\lambda'(t_0) = 0$ and t_0 is a critical point. Since $\lambda'(t_n)$ and $\lambda'(t_{n+1})$ have different signs, the solution curve changes its direction over (t_n, t_{n+1}). □

The linearized equation for (2.12) is

$$(\varphi'(w')z')' + \frac{n-1}{t}\varphi'(w')z' + qt^\alpha w^{q-1}z = 0.$$

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At the solution $w = w_0(t)$, this becomes

$$(a_0(t)z')' + \frac{n-1}{t}a_0(t)z' + b_0(t)z = 0 \tag{4.1}$$

with $a_0(t) = \varphi'(w_0')$ and $b_0(t) = qt^\alpha w_0^{q-1}$. One simplifies (4.1) to Euler's equation

$$z'' + \frac{-(\beta+1)(p-2)+n-1}{t}z' + \frac{qa_0^{q-p+1}}{(p-1)\beta^{p-2}t^2}z = 0. \tag{4.2}$$

Let us first consider the case in which $p = 2$, $\alpha = 0$ and $n > 2$. Then $\beta = 2/(q-1)$, $a_0 = [\beta(n-\beta-2)]^{1/(q-1)}$, and (4.2) becomes

$$t^2 z'' + (n-1)tz' + q\beta(n-\beta-2)z = 0.$$

Its characteristic equation

$$r(r-1) + (n-1)r + q\beta(n-\beta-2) = 0$$

has the roots

$$r = \frac{-(n-2) \pm \sqrt{(n-2)^2 - 4q\beta(n-\beta-2)}}{2}.$$

These roots are complex if

$$(n-2)^2 - 4q\beta(n-2) + 4q\beta^2 < 0.$$

On the left we have a quadratic in $n-2$ with two positive roots. The largest value of $n-2$ for which this inequality holds corresponds to the larger root of this quadratic, i.e.

$$n - 2 < \frac{4q}{q-1} + 4\sqrt{\frac{q}{q-1}}. \tag{4.3}$$

We shall show that infinitely many solutions occur if (4.3) holds and

$$q > \frac{n+2}{n-2}. \tag{4.4}$$

(The last condition ensures that the generating solution $w(t)$ is defined for all $t > 0$, by lemma 2.1.) In terms of n, conditions (4.3) and (4.4) imply that

$$\frac{2+2q}{q-1} < n < 2 + \frac{4q}{q-1} + 4\sqrt{\frac{q}{q-1}}, \tag{4.5}$$

which is the condition from [9] (and implies that $n > 2$). Thus we shall recover the following classical theorem of Joseph and Lundgren [9].

Theorem 4.2. *Assume that conditions (4.3) and (4.4) hold (or (4.5) holds). Then the solution curve of (2.9) makes infinitely many turns. Moreover, along this curve (as $u(0) \to \infty$), $\lambda \to \lambda_0 = a_0^{q-1}$ and $u(r)$ tends to $r^{-\beta} - 1$ for $r \neq 0$, which is a singular solution of the equation in (2.9).*

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We shall give a proof of a more general result below.

For general p and α, the characteristic equation for (4.2) is

$$r(r-1)+Ar+B=0, \tag{4.6}$$

with $A=-\beta(p-2)+n-p+1$ and $B=(q(n-p)/(p-1))\beta-q\beta^2$. The roots of (4.6),

$$r=\frac{-(A-1)\pm\sqrt{(A-1)^2-4B}}{2},$$

are complex provided that

$$(A-1)^2-4B<0,$$

which simplifies to

$$(n-p)^2-\theta(n-p)+\gamma<0, \tag{4.7}$$

with

$$\theta=2\beta(p-2)+\frac{4q\beta}{p-1}, \qquad \gamma=(p-2)^2\beta^2+4q\beta^2. \tag{4.8}$$

On the left in (4.7) we have a quadratic in $n-p$ with two positive roots. The largest value of $n-p$ for which the inequality (4.7) holds corresponds to the larger root of this quadratic, i.e.

$$n-p<\frac{\theta+\sqrt{\theta^2-4\gamma}}{2}. \tag{4.9}$$

We shall show that infinitely many solutions occur if conditions (2.13) and (4.9) hold. In terms of n, conditions (2.13) and (4.9) imply that

$$\frac{pq+p+p\alpha}{q-p+1}<n<p+\frac{\theta+\sqrt{\theta^2-4\gamma}}{2}. \tag{4.10}$$

The first inequality in (4.10) implies that

$$(\beta+1)(p-2)<n-2, \tag{4.11}$$

which in turn gives that $n>p$.

The critical exponent in (4.9) was computed earlier by Cabré and Sanchón in the context of semi-stable and extremal solutions of p-Laplace equations in [3], where the authors considered equations on general domains and more general $f(u)$; see also [1, 4].

THEOREM 4.3. *Assume that* $\lim_{t\to\infty}(w(t)/w_0(t))=1$ *(in the* $p=2$ *case, this follows by [1, lemma 2.2]). Assume also that conditions (2.13) and (4.9) hold (or (4.10) holds). Then the solution curve of*

$$\varphi(u')'+\frac{n-1}{r}\varphi(u')+\lambda r^{\alpha}(1+u)^q=0 \quad \text{for } 0<r<1, \tag{4.12}$$
$$u'(0)=u(1)=0,$$

makes infinitely many turns. Moreover, along this curve (as $u(0)\to\infty$*),* $\lambda\to\lambda_0=a_0^{q-1}$ *and* $u(r)$ *tends to* $r^{-\beta}-1$ *for* $r\neq 0$*, which is a singular solution of the equation in (4.12).*

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Proof. In view of lemma 4.1, we need to show that $w(t)$ and $w_0(t)$ intersect infinitely many times, and they tend to each other as $t \to \infty$. Let $P(t) = w(t) - w_0(t)$. Then $P(t)$ satisfies

$$(a(t)P')' + \frac{n-1}{t} a(t)P' + b(t)P = 0, \tag{4.13}$$

where

$$a(t) = \int_0^1 \varphi'(sw'(t) + (1-s)w_0'(t))\, \mathrm{d}s, \tag{4.14}$$

$$b(t) = qt^\alpha \int_0^1 [sw(t) + (1-s)w_0(t)]^{q-1}\, \mathrm{d}s. \tag{4.15}$$

We claim that it is impossible for $P(t)$ to keep the same sign over some infinite interval (t_0, ∞). Assuming the contrary, write ($a_0(t)$ and $b_0(t)$ were defined in (4.1))

$$a(t) = (p-1)(-w_0')^{p-2} \int_0^1 \left| s\frac{w'(t)}{w_0'(t)} + (1-s) \right|^{p-2} \mathrm{d}s = a_0(t)(1 + o(1)),$$

$$b(t) = qt^\alpha w_0^{q-1} \int_0^1 \left[s\frac{w(t)}{w_0(t)} + (1-s) \right]^{q-1} \mathrm{d}s = b_0(t)(1 + o(1))$$

as $t \to \infty$. We have $w(t)/w_0(t) \to 1$, and then $w'(t)/w_0'(t) \to 1$, by L'Hospital's rule, as $t \to \infty$. Since Euler's equation (3.7) has infinitely many solutions on (t_k, ∞), we conclude by lemma 3.3 that $P(t)$ must vanish on that interval too, which is a contradiction. It follows that $P(t)$ has infinitely many roots, which implies that $w(t)$ and $w_0(t)$ have infinitely many points of intersection, and hence the solution curve makes infinitely many turns.

Since $u(r) = w(tr)/w(t) - 1$, it follows that along the solution curve $u(r)$ tends to $w_0(tr)/w_0(t) - 1 = r^{-\beta} - 1$ for $r \neq 0$. □

5. The generalized Gelfand problem

We now use the representation (2.3) for the solution curve of (2.1). In particular, $\lambda(t) = t^{\alpha+p}\mathrm{e}^{w(t)}$, where $w(t)$ is the generating solution of (2.2), and the issue is how many times this function changes its direction of monotonicity for $t \in (0, \infty)$. Compute

$$\lambda'(t) = t\mathrm{e}^w(\alpha + p + tw'),$$

so that we are interested in the roots of the function $\alpha + p + tw'$. If we set this function to zero,

$$\alpha + p + tw' = 0,$$

then the solution of this equation is of course $w(t) = a - (\alpha + p)\ln t$. Quite surprisingly, if we choose the constant $a = a_0 = \ln[(n-p)(\alpha+p)^{p-1}]$, assuming that $n > p$, then

$$w_0(t) = \ln[(n-p)(\alpha+p)^{p-1}] - (\alpha+p)\ln t$$

is a solution of the equation in (2.2)! We shall show that $w(t)$ (the solution of initial-value problem (2.2)) tends to $w_0(t)$ as $t \to \infty$, and give a condition for $w(t)$ and $w_0(t)$ to cross infinitely many times as $t \to \infty$.

Lemma 5.1. *Assume that $w(t)$ and $w_0(t)$ intersect infinitely many times. Then the solution curve of (2.1) makes infinitely many turns.*

Proof. Indeed, assuming that $w(t)$ and $w_0(t)$ intersect infinitely many times, let $\{t_n\}$ denote the points of intersection. At the $\{t_n\}$s, $w(t)$ and $w_0(t)$ have different slopes (by uniqueness for initial-value problems). Since $\alpha + p + t_n w_0'(t_n) = 0$, it follows that $\alpha+p+t_n w'(t_n) > 0$ (respectively, $\alpha+p+t_n w'(t_n) < 0$) if $w(t)$ intersects $w_0(t)$ from below (respectively, above) at t_n. Hence, on any interval (t_n, t_{n+1}) there is a point t_0 where $\alpha + p + t_0 w'(t_0) = 0$, i.e. $\lambda'(t_0) = 0$, and t_0 is a critical point. Since $\lambda'(t_n)$ and $\lambda'(t_{n+1})$ have different signs, the solution curve changes its direction over (t_n, t_{n+1}). □

The linearized equation for (2.2) is

$$(\varphi'(w')z')' + \frac{n-1}{t}\varphi'(w')z' + t^{\alpha}\mathrm{e}^{w}z = 0.$$

At the solution $w = w_0(t)$ this becomes

$$(a_0(t)z')' + \frac{n-1}{t}a_0(t)z' + b_0(t)z = 0, \tag{5.1}$$

with

$$a_0(t) = \varphi'(w_0') = \frac{(p-1)(p+\alpha)^{p-2}}{t^{p-2}} \quad \text{and} \quad b_0(t) = t^{\alpha}\mathrm{e}^{w_0} = \frac{(n-p)(p+\alpha)^{p-1}}{t^{p}}.$$

Simplifying (5.1) gives

$$(p-1)t^2 z'' + (p-1)(n-p+1)tz' + (n-p)(p+\alpha)z = 0,$$

which is Euler's equation! Its characteristic equation

$$(p-1)r(r-1) + (p-1)(n-p+1)r + (n-p)(p+\alpha) = 0$$

has the roots

$$r = \frac{-(p-1)(n-p) \pm \sqrt{(p-1)(n-p)[(p-1)(n-p) - 4(p+\alpha)]}}{2(p-1)}.$$

The roots are complex if $n - p > 0$ and the quantity in the square brackets is negative (the opposite inequalities lead to a vacuous condition), i.e. when

$$p < n < \frac{p^2 + 3p + 4\alpha}{p-1}. \tag{5.2}$$

We now easily recover the following result of Jacobsen and Schmitt [8], which was a generalization of the famous theorem of Joseph and Lundgren [9].

Theorem 5.2. *Assume that condition (5.2) holds. Then the solution curve of*

$$\varphi(u')' + \frac{n-1}{r}\varphi(u') + \lambda r^{\alpha}\mathrm{e}^{u} = 0 \quad \textit{for } 0 < r < 1,\ u'(0) = 0,\ u(1) = 0, \tag{5.3}$$

makes infinitely many turns. Moreover, along this curve (as $u(0) \to \infty$), $\lambda \to \mathrm{e}^{a_0} = (n-p)(p+\alpha)^{p-1}$ and $u(r)$ tends to $-(p+\alpha)\ln r$ for $r \neq 0$, which is a singular solution of the equation in (5.3).

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Proof. We follow the proof of the theorem 3.4. In view of lemma 5.1, we need to show that $w(t)$ and $w_0(t)$ intersect infinitely many times. Let $P(t) = w(t) - w_0(t)$. Then $P(t)$ satisfies

$$(a(t)P')' + \frac{n-1}{t}a(t)P' + b(t)P = 0, \tag{5.4}$$

where

$$a(t) = \int_0^1 \varphi'(sw'(t) + (1-s)w_0'(t))\,\mathrm{d}s, \tag{5.5}$$

$$b(t) = t^\alpha \int_0^1 \mathrm{e}^{sw(t)+(1-s)w_0(t)}\,\mathrm{d}s. \tag{5.6}$$

Compared with the proof of the theorem 3.4, we have a complication here: in the case in which $P(t)$ tends to a constant p_0 as $t \to \infty$, we cannot conclude that $b(t) = b_0(t)(1 + o(1))$, unless $p_0 = 0$.

We claim that it is impossible for $P(t)$ to keep the same sign over some infinite interval (t_0, ∞) while tending to a constant $p_0 \neq 0$ as $t \to \infty$. Assume, on the contrary, that $P(t) > 0$ on (t_0, ∞) and $\lim_{t\to\infty} P(t) = p_0 > 0$. We may assume that

$$P(t) > \tfrac{1}{2}p_0 > 0 \quad \text{on } (t_1, \infty) \text{ with some } t_1 > t_0. \tag{5.7}$$

Write (5.4) as

$$(t^{n-1}a(t)P')' = -t^{n-1}b(t)P. \tag{5.8}$$

As before,

$$a(t) = a_0(t)(1 + f(t)) \quad \text{with } f(t) \to 0 \text{ as } t \to \infty. \tag{5.9}$$

Writing $b(t) = t^\alpha \mathrm{e}^{w_0(t)} \int_0^1 \mathrm{e}^{sP(t)}\,\mathrm{d}s$, we see that

$$b(t) = b_0(t)(p_1 + g(t)) \tag{5.10}$$

with $p_1 = \int_0^1 \mathrm{e}^{sp_0}\,\mathrm{d}s > 1$, and $g(t) \to 0$ as $t \to \infty$. By (5.7), (5.8) and (5.10),

$$(t^{n-1}a(t)P')' < -c_1 t^{n-p-1} \quad \text{on } (t_1, \infty)$$

for some constant $c_1 > 0$. Integrating this inequality over (t_1, t), we get

$$t^{n-1}a(t)P' < c_2 - c_3 t^{n-p} \quad \text{on } (t_1, \infty) \tag{5.11}$$

for some constants $c_2 > 0$ and $c_3 > 0$ (using that $n > p$). By (5.9),

$$a(t) > c_4 t^{-p+2} \quad \text{on } (t_2, \infty)$$

for some constants $c_4 > 0$ and $t_2 > t_1$. Using this in (5.11), we have

$$P' < \frac{c_2}{c_4} t^{-n+p-1} - \frac{c_3}{c_4} t^{-1} \quad \text{on } (t_2, \infty).$$

Integrating this over (t_2, t) and using that $n > p$,

$$P(t) < c_5 + \frac{c_2}{c_4(-n+p)} t^{-n+p} - \frac{c_3}{c_4} \ln t < c_5 - \frac{c_3}{c_4} \ln t$$

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for some constant $c_5 > 0$. Hence, $P(t)$ has to vanish at some $t > t_2$, contradicting the assumption that $P(t) > 0$ on (t_0, ∞). This proves that $p_0 = 0$. We conclude that $p_1 = 1$ in (5.10), and the rest of the proof is similar to that of theorem 3.4. □

If $p = 2$ and $\alpha = 0$, condition (5.2) becomes $2 < n < 10$, the classical condition of Joseph and Lundgren [9].

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