Macroeconomic Dynamics, **16** (Supplement 3), 2012, 299–311. Printed in the United States of America.
doi:10.1017/S1365100510000878

ARTICLES

EXISTENCE AND UNIQUENESS OF STEADY-STATE EQUILIBRIUM IN A GENERALIZED OVERLAPPING GENERATIONS MODEL

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Galor and Ryder [*Journal of Economic Theory* 49 (1989), 360–375] establish conditions for the existence of equilibrium in a Diamond-type overlapping-generations (OLG) model. Although theoretically appealing, these conditions are implicit and not convenient to apply. This paper provides explicit and easily applied conditions for the existence and uniqueness of steady-state equilibrium, with which one only needs to check the first derivatives of the production and utility functions and their interactions, with no need to solve the optimization problem. Our theorems on the existence and uniqueness of steady-state equilibrium can be applied to a larger class of OLG models that do not require second-order differentiability of the production and utility functions. We present examples to show how to check the existence and uniqueness of equilibrium.

Keywords: Existence, Uniqueness, Equilibrium, Generalized Overlapping-Generations Model

1. INTRODUCTION

The overlapping-generations (OLG) model introduced by Allais (1947) and Samuelson (1958) and extended by Diamond (1965) has been widely used in economic analyses, in fields such as public economics, monetary economics, economic growth, development economics, and international economics. However, the existence and uniqueness of equilibrium has been an issue haunting the users of the OLG model. Most prior studies have either provided illustrative numerical examples to show the existence of equilibrium or simply assumed that equilibrium

The authors thank two anonymous referees, the editor, and an associate editor for their helpful comments and suggestions. The authors also thank Oded Galor for his encouragement. Part of the research was completed during Jinlu Li's visit to the School of Economics at Peking University. Address correspondence to: Jinlu Li, Department of Mathematics, Shawnee State University, Portsmouth, OH 45662, USA; e-mail: jli@shawnee.edu.

 1365-1005/11

exists in the OLG model. This paper will develop a new approach to existence and uniqueness and provide easily checkable conditions for the existence and uniqueness of nontrivial steady-state equilibrium in an OLG model.

Galor and Ryder (1988, 1989, 1991) advanced the literature by examining the conditions for the existence of equilibrium for the Diamond-type OLG model with productive capital. They demonstrated that the Inada conditions, along with well-behaved preferences, are not sufficient to warrant the existence of a nontrivial equilibrium in an OLG economy with production. They introduced the strengthened Inada condition and provided sufficient conditions for the existence of nontrivial steady-state equilibrium, which are requirements on production function and savings function and the interaction of these functions. Studies extending Galor and Ryder's work (1989) have usually imposed more restrictive conditions. For example, Konishi and Perera-Tallo (1997) imposed the condition that the labor share of output does not go to zero when the capital stock goes to zero, called nonvanishing labor share (which is stronger than Galor and Ryder's strengthened Inada condition), whereas on the consumption side they imposed the condition that the Engel curve must be concave near the origin.[1]

This paper introduces a new approach to the existence and uniqueness of steady-state equilibrium in OLG models. We generalize the existing OLG model by relaxing the assumption of second-order differentiability of production and utility functions. Thus, our existence theorems can be applied to a broader range of preferences and technologies, specifically, to models with production and utility functions not second-order differentiable. Our theorems require neither the strengthened Inada condition as in Galor and Ryder (1989), nor the traditional Inada condition as in McCallum (1983).

Despite the generalization, we are able to provide explicit and easily applicable conditions for the existence and uniqueness of equilibrium in the OLG model. Our conditions involve only first derivatives of production and utility functions.[2] One only needs to check the derivatives of the production and utility functions and their interactions, without solving for the optimization problem. We provide examples to illustrate how to use our theorems.

2. THE GENERALIZED MODEL

The model is a generalized Diamond-type OLG model with productive capital. The economy produces one good, which can be either consumed or invested. Individuals live for two periods, working and saving in the first period, and being retired and consuming savings and accrued interest in the second period. Individuals are identical within and across generations.

Let L_t be the number of workers in period t. The population grows at a rate n, i.e., $L_{t+1} = (1+n)L_t$. Let K_t be capital in period t. The production function exhibits constant returns to scale in both factors. The output, in period t, is $Y_t = F(L_t, K_t) = L_t f(k_t)$, and output per worker is $y_t = f(k_t)$, where $k_t = K_t/L_t$ is the capital–labor ratio, and $y_t = Y_t/L_t$ is the ratio of output to labor.

https://doi.org/10.1017/S1365100510000878 Published online by Cambridge University Press

Factor markets are perfectly competitive, so that the rate of return to each factor is equal to its marginal product; i.e., $r_t = f'(k_t)$, $w_t = f(k_t) - k_t f'(k_t)$, where r_t is the rate of return on capital in period t and w_t is the rate of return to labor, or the wage rate.

In this paper, the production function f is defined on $[0, \infty)$ and is assumed to be continuously differentiable of degree one, strictly increasing, and strictly concave.

Individuals born at time t have an intertemporal utility function, $u(c_t^t, c_{t+1}^t)$. The utility function is continuous over nonnegative first- and second-period consumption. In this paper, we always assume that the utility function is continuously differentiable of degree one on R_{++}^2 and is strictly increasing in c_t^t for any fixed level of $c_{t+1}^t > 0$, strictly increasing in c_{t+1}^t for any fixed level of $c_t^t > 0$, and strictly quasiconcave. More precisely, the utility function, u, satisfies the following conditions:

$$u_1\left(c_t^t, c_{t+1}^t\right) > 0, \quad u_2\left(c_t^t, c_{t+1}^t\right) > 0, \quad \forall \left(c_t^t, c_{t+1}^t\right) \gg 0,$$

$$\lim_{c_t \to 0} u_1\left(c_t^t, c_{t+1}^t\right) = \infty, \quad \forall c_{t+1}^t > 0,$$

$$\lim_{c_{t+1} \to 0} u_2\left(c_t^t, c_{t+1}^t\right) = \infty, \quad \forall c_t^t > 0,$$

$$u_{11}\left(c_t^t, c_{t+1}^t\right) [< 0, \quad \text{and} \quad u_{22}\left(c_t^t, c_{t+1}^t\right) < 0.$$

Let s_t be savings for an individual. For any given utility function, substituting the two-period budget constraints into the utility function, we obtain a constrained utility function for an individual,

$$u\left(c_t^t, c_{t+1}^t\right) = u[w_t - s_t, (1 + r_{t+1} - \delta)s_t],$$

where δ is the depreciation rate. For any fixed $w_t > 0$ and $r_{t+1} > 0$, we define the optimal (utility-maximizing) savings function in period t of an agent born in period t as

$$s_t = s(w_{t,} r_{t+1}) = \operatorname*{arg\,max}_{0 \le s_t \le w_t} u[w_t - s_t, (1 + r_{t+1} - \delta)s_t]. \quad \textbf{(1)}$$

Under continuity, monotonicity, and the strict quasiconcavity of preferences for the utility function u, it is known that the savings function $s(w_{t,} r_{t+1})$ is well-defined, single-valued and continuous.

The market clearing condition, demand for capital equal to supply of capital (savings), is as follows:

$$k_{t+1} = \frac{s[f(k_t) - k_t f'(k_t), f'(k_{t+1})]}{1+n}$$
$$= \frac{\operatorname*{arg\,max}_{0 \le s_t \le f(k_t) - k_t f'(k_t)} u[f(k_t) - k_t f'(k_t) - s_t, (1 + f'(k_{t+1}) - \delta)s_t]}{1+n}. \quad \textbf{(2)}$$

In the steady-state equilibrium of the generalized OLG economy, all the endogenous variables are time-invariant, and thus, the above dynamics of the system can

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be represented by the following equation:

$$k = \frac{s[f(k) - kf'(k),\ f'(k)]}{1+n}$$
$$= \frac{\underset{0 \le s \le f(k) - k_t f'(k)}{\arg\max} \quad u[f(k) - kf'(k) - s, (1 + f'(k) - \delta)s]}{1+n}. \qquad (3)$$

In contrast to prior studies, throughout this paper we assume that the production and utility functions have continuous differentials of only first degree on R^2_+. The relaxation of the second derivatives assumption allows us to consider a much broader range of production technologies and preferences.

3. SOME PRELIMINARIES

For a given production function f and a given population growth rate n satisfying $-1 < n$, we define three functions:

$$\phi(k) = s(f(k) - kf'(k),\ f'(k))$$
$$= \arg\max u[f(k) - kf'(k) - s, (1 + f'(k) - \delta)s], \quad \text{for all } k > 0, \qquad (4)$$

$$P(f, n) = \{k > 0\colon f(k) - kf'(k) > (1+n)k\}, \qquad (5)$$

$$\xi(k) = \frac{u_1[f(k) - kf'(k) - (1+n)k, (1+n)k(1 + f'(k) - \delta)]}{u_2[f(k) - kf'(k) - (1+n)k, (1+n)k(1 + f'(k) - \delta)]},$$
$$\text{for all } k \in P(f, n), \qquad (6)$$

where $\phi(k)$ is the savings function, $P(f, n)$ is a set of capital–labor ratios k that satisfy $f(k) - kf'(k) > (1+n)k$ (i.e., the wage rate is greater than investment (savings), or the first-period consumption is positive), and $\xi(k)$ stands for the marginal rate of substitution between first- and second-period consumption. Note that second-period consumption is positive, i.e., $(1+n)k(1 + f'(k) - \delta) > 0$, for all $k > 0$, and the domain of function $\xi(k)$ is $P(f, n)$.

We now present three lemmas in order to establish a theorem of existence.

LEMMA 1. *If the utility function u satisfies conditions (8), (9), and (10), then*

(i) for any $c > 0$, we have

$$\limsup_{(c^1,c^2)\to(c,0)} u_1(c^1, c^2) < \infty \quad \textit{and} \quad \limsup_{(c^1,c^2)\to(0,c)} u_2(c^1, c^2) < \infty; \qquad (7)$$

(ii) for an yc > 0, we have

$$\lim_{(c^1,c^2)\to(c,0)} u_2(c^1, c^2) = \infty \quad \lim_{(c^1,c^2)\to(0,c)} u_1(c^1, c^2) = \infty; \qquad (8)$$

(iii) for any given value $w > 0$, $r > 0$, we have

$$0 < s(w, r) < w; \qquad (9)$$

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i.e., for any given constrained utility function, we can obtain an interior solution, not a corner solution.

Proof. See Online Appendix 1 (http://journals.cambridge.org/MDY). ■

The following theorem describes the equilibrium conditions in general for this economy.

PROPOSITION 1. *Suppose that the utility function u satisfies conditions (8), (9), and (10), and the production function f and f' satisfy conditions (4)–(6), then for any given population growth rate n satisfying $-1 < n$, and for any given depreciation rate δ satisfying $0 \leq \delta \leq 1$, $\bar{k}$ is a nontrivial steady-state equilibrium of the OLG economy if $\bar{k} \in P(f, n)$ and satisfies*

$$\xi(\bar{k}) = 1 + f'(\bar{k}) - \delta. \tag{10}$$

Proof. See Online Appendix 2 for a straightforward proof. ■

Remark. If the production function *f* satisfies the condition

$$f(k) - kf'(k) < (1+n)k, \quad \text{for all } k > 0, \text{ that is, } P(f, n) = \emptyset,$$

then for any utility function *u* that satisfies conditions (8)–(10) [or conditions (8) and (9)], the only steady-state equilibrium of the OLG economy is characterized by zero production and consumption (a trivial equilibrium).

The result given in the above remark is well known in the literature. In our generalized model, it can be proved using Proposition 1.

4. PROPERTIES OF THE UTILITY AND PRODUCTION FUNCTIONS

This section discusses some properties of the utility and production functions that will be utilized in the next section to establish the existence of nontrivial steady-state equilibrium. In order to find a nontrivial steady-state equilibrium, we need to find an equilibrium *k* that satisfies the condition in Proposition 1 (i.e., equation (10)). Traditionally, to determine the range of possible equilibria, an upper bound to the attainable capital $\tilde{k}$ is first figured out, so that the possible nontrivial steady-state equilibrium will be in (0, $\tilde{k}$), as in Galor and Ryder (1989). We will find another upper bound that will narrow the range of possible equilibrium values of the capital–labor ratio. The following proposition describes the upper bound of the capita–labor ratio, which is essential in this section.[3]

PROPOSITION 2. *If the production function f, defined on $[0, \infty)$, satisfies conditions (4)–(6), then there exists $\hat{k} > 0$ such that*

$$f(k) - kf'(k) < (1+n)k, \quad \textit{for all } k > \hat{k}. \tag{11}$$

Moreover, if $\hat{k}$ is taken to be the inferior limit of k satisfying (11), then $\hat{k}$ is an upper bound of the possible equilibrium capital–labor ratio.

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To derive the conditions for the existence of a nontrivial steady-state equilibrium, we need to discuss the characteristics of utility functions further. The next proposition describes the behavior of the utility function at the boundary.

LEMMA 2. *If the utility function u satisfies conditions (8), (9), and (10), then for any given* $c > 0$*, we have*

$$\lim_{(c^1,c^2)\to(0,c)} \frac{u_1(c^1,c^2)}{u_2(c^1,c^2)} = \infty, \tag{12}$$

$$\lim_{(c^1,c^2)\to(c,0)} \frac{u_1(c^1,c^2)}{u_2(c^1,c^2)} = 0. \tag{13}$$

Furthermore, if there exists $\hat{k} \in \overline{P(f,n)}$ *such that* $f(\hat{k}) - \hat{k} f'(\hat{k}) = (1+n)\hat{k}$*, then we have*

$$\lim_{k\to\widehat{k},k\in P(f,n)} \xi(k) = \infty. \tag{14}$$

Proof. The first two equalities of this proposition follow immediately from parts (i) and (ii) of Lemma 1. The last equality can be derived from the first equality using the condition $f(\hat{k}) - \hat{k} f'(\hat{k}) - (1+n)\hat{k} = 0$. ■

5. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF NONTRIVIAL STEADY-STATE EQUILIBRIUM

We are now ready to identify some conditions on the utility and production functions, which ensure that the generalized OLG model has at least one nontrivial steady-state equilibrium.

THEOREM 1. *Assume that the utility function u satisfies conditions (8), (9), and (10), and the production function f satisfies conditions (4)–(6). If the production function f and the utility function u satisfy the conditions*

(a) $P(f,n) \neq \emptyset$*,*
(b) There exists $k' \in \overline{P(f,n)}$ *such that*

$$\liminf_{k\to k',k\in P(f,n)} \frac{\xi(k)}{f'(k)} < 1, \tag{15}$$

then for a given population growth rate n satisfying $-1 < n$*, and for any given depreciation rate* δ *satisfying* $0 \le \delta \le 1$*, the OLG economy has at least one nontrivial steady-state equilibrium.*

Proof. From conditions (a) and (b), we can find a point $k_0 \in P(f,n)$, which is very near to point $\hat{k} \in \overline{P(f,n)}$, such that

$$\xi(k_0) < f'(k_0). \tag{16}$$

From the definition of $P(f,n)$, we have

$$f(k_0) - k_0 f'(k_0) > (1+n)k_0.$$

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Because the production function f and its derivative function f' satisfy conditions (4)–(7), applying Lemma 4, we see that the set $\{k > k_0 : f(k) - kf'(k) < (1+n)k\}$ is not empty. Hence if we let

$$k_1 = \inf\{k > k_0 \colon f(k) - kf'(k) < (1+n)k\},$$

by using the continuity of f and f', we get $k_0 < k_1$, $[k_0, k_1) \subset P(f, n)$, and

$$f(k_1) - k_1 f'(k_1) = (1+n)k_1. \tag{17}$$

From the continuity and the conditions in (8), (9), and (10) on the utility function, the function $\xi(k)$ is well-defined and continuous on the interval $[k_0, k_1)$. For any given rate of capital depreciation δ satisfying $0 \le \delta \le 1$, inequality (16) implies

$$\xi(k_0) < 1 + f'(k_0) - \delta. \tag{18}$$

On the other hand, from conditions (4)–(6) and applying (17) and Lemma 1, we get

$$\lim_{k \to k_1^-} \xi(k) = \infty. \tag{19}$$

Because $\lim_{k \to k_1} f'(k) = f'(k_1) < \infty$, applying (19), there exists $k_2 \in (k_0, k_1)$ which is near to k_1, such that

$$\xi(k_2) > 1 + f'(k_2) - \delta. \tag{20}$$

Combining (18) and (20), from the continuity of the functions $\xi(k)$, $f'(k)$ and applying the Intermediate Value Theorem, there exists at least one point $\bar{k} \in (k_0, k_1)$ such that

$$\xi(\bar{k}) = 1 + f'(\bar{k}) - \delta.$$

From Proposition 1, we see that $\bar{k}$ is a nontrivial steady-state equilibrium. ■

Remark. If $\lim_{k \to \widehat{k}} f'(k) < \infty$, then condition (b) in Theorem 1 can be replaced by the following stronger condition:

(b′) there exists $\widehat{k} \in \overline{P(f,n)}$ such that $\liminf\limits_{k \to \widehat{k}, k \in P(f,n)} \xi(k) < \lim\limits_{k \to \widehat{k}} f'(k)$. **(15′)**

In case $\liminf_{k \to \widehat{k}, k \in P(f,n)} \xi(k) = \infty$ and $\lim_{k \to \widehat{k}} f'(k) = \infty$, we can consider the limit of their ratios and apply the condition in (15). But the condition in (15′) is not applicable in this case.

The properties and behavior of the production function f and its derivative function f' near zero have drawn a lot of attention in the literature. On the basis of Theorem 1, we can easily obtain the following corollary as a special case where k' is 0.

COROLLARY 1. *Assume that the utility function u satisfies conditions (8), (9), and (10), and the production function f satisfies conditions (4)–(6). If the production function f and the utility function u satisfy the conditions*

(a) $0 \in \overline{P(f,n)}$ *(implying* $P(f,n) \neq \emptyset$*),*
(b) $\liminf_{k\to 0, k\in P(f,n)} \xi(k)/f'(k) < 1$,

then for a given population growth rate n satisfying $-1 < n$*, and for any given depreciation rate* δ *satisfying* $0 \leq \delta \leq 1$*, the OLG economy has at least one nontrivial steady-state equilibrium.*

Prior studies have been based on strong assumptions on the production function. The Inada condition on production function f is defined by

$$\lim_{k\to 0} f'(k) = \infty. \quad (\mathbf{21})$$

Galor and Ryder (1989) developed a stronger condition than the Inada condition on the production function, called the strengthened Inada condition,

$$\lim_{k\to 0}(-kf''(k)) > 1 + n. \quad (\mathbf{22})$$

Using this condition, they provided significant results about the existence of equilibrium in the OLG models. Note that this condition is not used in this paper, and the existence of $f''(k)$ is not required in this paper. Theorem 1 and Corollary 1 depend on neither the strengthened Inada condition nor the Inada condition. With the Inada condition, we can obtain the following corollary based on Corollary 1.

COROLLARY 2. *Assume that the utility function u satisfies conditions (8), (9), and (10), and the production function f satisfies conditions (4)–(6) and the Inada condition. If the production function f and the utility function u satisfy the conditions*

(a) $0 \in \overline{P(f,n)}$ *[which implies that* $P(f,n) \neq \emptyset$*],*
(b) $\liminf_{k\to 0, k\in P(f,n)} \xi(k) < \infty$,

then for a given population growth rate n satisfying $-1 < n$*, and for any given depreciation rate* δ *satisfying* $0 \leq \delta \leq 1$*, the OLG economy has at least one nontrivial steady-state equilibrium.*

Proof. Combining condition (b) of this corollary and the Inada condition, we obtain

$$\liminf_{k\to 0, k\in P(f,n)} \frac{\xi(k)}{f'(k)} = 0.$$

Then this corollary follows immediately from Theorem 1. ■

The Inada condition does not ensure that $P(f,n) \neq \emptyset$. The following lemma shows that Galor and Ryder's condition ensures that $P(f,n) \neq \emptyset$.

LEMMA 3. *If the production function f and its derivative function* f' *satisfy conditions (4)–(6) and if* f'' *exists on* $(0,\infty)$ *satisfying* $\lim_{k\to 0}(-kf''(k)) > 1+n$*, then there exists* $\kappa_0 > 0$ *such that* $f(k) - kf'(k) > (1+n)k$*, for all* $0 < k < \kappa_0$*. Therefore,* $P(f,n) \neq \emptyset$*.*

Proof. Under conditions (4)–(7), it is known that

$$f(k) - kf'(k) > 0, \quad \text{for all } k > 0. \tag{23}$$

Using $f(0) \geq 0$ and (23), we get

$$\lim_{k\to 0}(f(k) - kf'(k)) = a \geq 0. \tag{24}$$

So if we define the value of $f(k) - kf'(k)$ to be a when $k = 0$, then $f(k) - kf'(k)$ is a continuous function on $[0, \infty)$. From Galor and Ryder's condition (22), there exists $\kappa_0 > 0$ such that $-kf''(k) > 1 + n$, for all $0 < k < \kappa_0$. Noting that

$$\frac{d(f(k) - kf'(k))}{dk} = -kf''(k),$$

and applying the Mean Value Theorem for function $f(k) - kf'(k)$, for all $0 < k < \kappa_0$, we have

$$\begin{aligned} f(k) - kf'(k) &= -k_1 f''(k_1)k + a \text{ (for some } 0 < k_1 < k) \\ &\geq -k_1 f''(k_1)k \\ &> (1+n)k. \end{aligned}$$

■

Applying Corollary 2 and Lemma 5, we obtain the following corollary.

COROLLARY 3. *Assume that the utility function u satisfies conditions (8), (9), and (10), and the production function f satisfies conditions (4)–(6). If the production function f is continuously differentiable of degree 2 and satisfies Galor and Ryder's strengthened Inada condition in (22), and the utility function u satisfies condition (b) in Corollary 2, i.e.,*

$$\lim_{k\to 0}(-kf''(k)) > 1 + n \quad \text{and} \quad \liminf_{k\to 0, k\in P(f,n)} \xi(k) < \infty,$$

then for a given population growth rate n that satisfies $-1 < n$, *and for any given depreciation rate* δ *that satisfies* $0 \leq \delta \leq 1$, *the OLG economy has at least one nontrivial steady-state equilibrium.*

Remarks.

(1) In Corollary 3 above, the first condition is the strengthened Inada condition introduced by Galor and Ryder (1989). The second condition is developed by us.
(2) The result in Corollary 3 is more general than the result in Proposition 2 in Konishi and Perera-Tallo (1997). The first condition of Proposition 2 in their paper is exactly the same as the first condition of Corollary 3. But the second condition in their

paper is

$$\limsup_{k\to 0} \xi(k) < \infty,$$

which is stronger or more restrictive than the second condition of Corollary 3. Hence, Proposition 2 in Konishi and Perera-Tallo (1997) is a special case of Corollary 3.

We now provide two examples to show how to use our theory to check the existence of equilibrium. The first example is simple, with a typical utility and a production function, whereas the second one is complicated, with both production and utility functions not being second-order differentiable.

Example 1
Assume that the utility function is $u(c_t^t, c_{t+1}^t) = c_t^t c_{t+1}^t$, where $c_t^t = f(k_t) - k_t f'(k_t) - k_{t+1}$ and $c_{t+1}^t = k_{t+1}[1 + f'(k_{t+1})]$, based on the budget constraints. In the steady state, $u(c_1, c_2) = c_1 c_2$, where $c_1 = f(k) - kf'(k) - k$ and $c_2 = k[1 + f'(k)]$. Assume that the steady-state production function is $f(k) = 2k^{0.5}$, which implies that $f'(k) = k^{-0.5}$. Assume, for simplicity, that $n = 0$ and $\delta = 0$.

Now let us examine the set $P(f, n)$, which includes all k that enable nonnegative first-period consumption:

$$\begin{aligned} P(f,n) &= \{k > 0\colon f(k) - kf'(k) > (1+n)k\} \\ &= \{k > 0\colon 2k^{0.5} - k^{0.5} - k > 0\} \\ &= \{k > 0\colon \sqrt{k} < 1\} \\ &= \{k > 0\colon k < 1\} \\ &= (0, 1). \end{aligned}$$

It is clear that $P(f, n) \neq \emptyset$, which implies that the closure $\overline{P(f,n)} = [0, 1] \neq \emptyset$. Therefore, condition (a) in Theorem 1 is met. By taking the partial derivates of the utility function u, we obtain the marginal rate of substitution between the first and second periods of consumption, $\xi(k)$, as follows:

$$\xi(k) = \frac{u_1}{u_2} = \frac{c_2}{c_1} = \frac{k[1 + f'(k)]}{f(k) - kf'(k) - k} \quad \text{for all } k \in P(f,n).$$

The ratio of $\xi(k)$ to the marginal product of capital $f'(k)$ is

$$\frac{\xi(k)}{f'(k)} = \frac{k[1 + f'(k)]}{(f(k) - kf'(k) - k)f'(k)} = \frac{k + k^{0.5}}{(2k^{0.5} - k^{0.5} - k)k^{-0.5}} = \frac{k + k^{0.5}}{1 - k^{0.5}}.$$

Condition (b) in Theorem 1 requires that there exist $k' \in \overline{P(f,n)}$ such that

$$\liminf_{k\to \widehat{k}, k\in P(f,n)} \frac{\xi(k)}{f'(k)} < 1.$$

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It is easy to see that, in this example, if $k < -1 + \sqrt{2}$, then $\xi(k)/f'(k) < 1$. Hence, any number $\hat{k} \in (0, -1 + \sqrt{2})$ will meet condition (b) in Theorem 1. Based on Theorem 1, there exists at least one steady-state equilibrium in this economy.

Setting the marginal rate of substitution equal to the slope the intertemporal budget constraint, we can solve for the equilibrium easily:

$$\xi(k) = \frac{k[1 + f'(k)]}{f(k) - kf'(k) - k} = \frac{k + k^{0.5}}{2k^{0.5} - k^{0.5} - k}$$
$$= \frac{k + k^{0.5}}{k^{0.5} - k} = 1 + f'(k) = 1 + k^{-0.5}.$$

Thus, $k = \frac{13-4\sqrt{3}}{16}$ is a unique nontrivial equilibrium.

Example 2
Define a production function and a utility function as follows:

$$f(k) = \begin{cases} 3(2k - k \ln k), & k \in (0, 1] \\ 6k^{1/2}, & k \in [1, \infty). \end{cases}$$

$$u(c^1, c^2) = \begin{cases} (c^1 c^2)^{\frac{1}{3}}, & if (c^1, c^2) \in R_+ \times [0, 1] \\ \frac{2}{3}(c^1)^{\frac{1}{3}}(c^2)^{\frac{1}{2}} + \frac{1}{3}(c^1)^{\frac{1}{3}}, & if (c^1, c^2) \in R_+ \times [1, \infty). \end{cases}$$

Both the production function and the utility function are continuously differentiable of degree one but not two. With $n = 0$ and $\delta = 0$, the OLG economy has a unique nontrivial steady-state equilibrium, approximately $\bar{k} = 2.82296$.[4]

6. UNIQUENESS OF NONTRIVIAL STEADY-STATE EQUILIBRIUM

In this section, we provide a uniqueness theorem for nontrivial steady-state equilibrium for the generalized OLG model.

THEOREM 2. *Assume the utility functional u satisfies conditions (8)–(10) and the production function f satisfies the conditions (4)–(6). If the utility function u and the production function f satisfy the following conditions:*

(a) there exists $\hat{k} > 0$ such that $P(f, n) = (0, \hat{k})$;
(b) $\liminf_{k \to 0} \xi(k)/f'(k) < 1$;
(c) Function $\xi - f'$ is strictly increasing on $(0, \hat{k})$,

then for the given population growth rate n with $-1 < n$, and for any given depreciation rate δ with $0 \leq \delta \leq 1$, the OLG economy has a unique nontrivial steady-state equilibrium.

Proof. For all $k \in (0, \hat{k})$, we define $\gamma(k) = \xi(k) - (1 + f'(k) - \delta)$. From Theorem 1, we know that $\bar{k}$ is a nontrivial steady-state equilibrium of the OLG economy if and only if $\bar{k} \in P(f, n)$ and

$$\gamma(\bar{k}) = \xi(\bar{k}) - (1 + f'(\bar{k}) - \delta) = 0.$$

From condition (c) of this theorem, the function g is a strictly increasing function on $(0, \hat{k})$. Noting that the function $\gamma(k)$ is defined on $P(f, n) = (0, \hat{k})$, the equation $\gamma(k) = 0$ has at most one solution; that is, the OLG economy has at most one nontrivial steady-state equilibrium. From Corollary 2, conditions (a) and (b) of this theorem ensure that the OLG economy has at least one nontrivial steady-state equilibrium. Of course, all the nontrivial steady-state equilibria must be in $P(f, n) = (0, \hat{k})$ ■

Remarks on Theorems 1 and 2.

(1) In the current model, given the production and utility functions and the set $P(f, n)$, the function $\xi(k)$ can be derived explicitly.
(2) The conditions in Theorems 1 and 2 can be explicitly checked.
(3) The production function in Corollary 1 of Galor and Ryder (1989), which is considered a major implication of Propositions 4 and 5 in their paper, is required to satisfy the extended Inada condition. Lemma 5 in our paper shows that the extended Inada condition is a stronger or more restrictive condition than our condition $P(f, n) \neq \emptyset$. Thus, the conditions on production function in Theorems 1 and 2 are weaker (less restrictive) than Galor and Ryder's conditions in Propositions 4 and 5 of Galor and Ryder (1989).
(4) In Propositions 4 and 5 of Galor and Ryder (1989), the conditions involve the saving function and its partial derivatives, and are more difficult to check.

7. CONCLUDING REMARKS

This paper has provided easily applicable theorems to determine the existence and uniqueness of nontrivial steady-state equilibrium in an OLG model. Our conditions for the existence and uniqueness are explicit and less restrictive. Based on our theorem, one need only check the first derivatives of the production and utility functions and their interactions to determine whether an OLG model has a nontrivial equilibrium and whether the equilibrium is unique, with no need to solve the optimization problem.

Our theorems can be applied to OLG models with a broader range of production and utility functions, which are only first-order differentiable. Although most economists assume well-behaved utility and production functions, real world production and utility functions may be more complicated. Advanced economic analyses in the future may need to be based on more realistic utility and production functions. Our theorems are more general. Numerous models that violate the conditions required by the earlier studies can still have nontrivial steady-state equilibria based on our conditions.

NOTES

1. Following Galor and Ryder's (1989) work, many other studies emerged. Wang (1993) showed the conditions for the existence of equilibrium in an OLG model with production and uncertainty. Galor (1992) developed a two-sector OLG model with the utility and production functions being twice continuously differentiable and characterized the dynamic system globally. He established sufficient conditions for the existence of nontrivial steady-state equilibrium in a two-sector OLG model. Li and Lin (2008) provided easily checkable conditions for the existence and uniqueness of equilibrium in the two-sector OLG model.

2. The assumption concerning the existence of unique (nontrivial) steady-state equilibrium is restrictive in prior studies, including Ihori (1978), Tirole (1985), and Weil (1987), where the third derivatives of utility and production functions are involved. The condition imposed by Galor and Ryder (1989) involves second derivatives.

3. This proposition can be shown by a result proved by Boldrin (1992), and Jones and Manuelli (1992). See Online Appendix 3 for a direct proof.

4. A detailed proof is provided in Online Appendix 4.

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