*J. Aust. Math. Soc.* **111** (2021), 159–178
doi:10.1017/S144678871900048X

# PERTURBATION THEOREMS FOR FRACTIONAL CRITICAL EQUATIONS ON BOUNDED DOMAINS

**AZEB ALGHANEMI and HICHEM CHTIOUI**

(Received 28 February 2019; accepted 11 September 2019; first published online 9 March 2020)

Communicated by F. Cirstea

**Abstract**

We consider the fractional critical problem $A_s u = K(x)u^{(n+2s)/(n-2s)}, u > 0$ in $\Omega, u = 0$ on $\partial\Omega$, where $A_s, s \in (0, 1)$, is the fractional Laplace operator and $K$ is a given function on a bounded domain $\Omega$ of $\mathbb{R}^n, n \geq 2$. This is based on A. Bahri's theory of critical points at infinity in Bahri [*Critical Points at Infinity in Some Variational Problems*, Pitman Research Notes in Mathematics Series, 182 (Longman Scientific & Technical, Harlow, 1989)]. We prove Bahri's estimates in the fractional setting and we provide existence theorems for the problem when $K$ is close to 1.

2010 *Mathematics subject classification*: primary 35J60.

*Keywords and phrases*: fractional Laplace operator, critical equation.

## 1. Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^n, n \geq 2$, with smooth boundary $\partial\Omega$. Let $K : \overline{\Omega} \to \mathbb{R}$ be a given function. We look for solutions of the fractional partial differential equation (PDE)

$$\begin{cases} A_s u = K u^{(n+2s)/(n-2s)}, \\ \quad u > 0 \quad \text{in } \Omega, \\ \quad u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1-1}$$

Here $A_s, s \in (0, 1)$, denotes the fractional Laplace operator defined by using the spectrum of the Laplace operator $(-\Delta)$ in $\Omega$ with zero Dirichlet boundary condition.

In recent years extensive studies have been devoted to PDEs involving the fractional Laplacian $A_s$ due to its broad applications in many branches of sciences; see [14] and the references therein.

This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under Grant No. (KEP-PhD-41-130-38). The authors, therefore, acknowledge with thanks DSR technical and financial support.

After the seminal work of Caffarelli and Silvester [16] who developed a local interpretation of the problem in one more dimensions (see also [5, 18, 20] for similar extensions), many authors studied nonlinear problems involving the fractional Laplacian. For Riemannian manifolds without boundary, we refer the reader to [6–8, 11–13, 23]. For manifolds with boundary, only very few papers address problem (1-1). In [15, 18], the authors established existence results for the subcritical problems in the particular case $s = \frac{1}{2}$. In [19], Tan proved that (1-1) has no solution if $K = 1$ and $\Omega$ is a star-shaped domain. Later in [9], Abdelhedi, Chtioui and Hajaiej proved that (1-1) has a solution if $K = 1$ and $\Omega$ admits a nontrivial group of homology.

Our aim in this paper is to provide some conditions on $K$ and on the domain $\Omega$ to prove existence results for (1-1).

($A$) Assume that $(\partial K/\partial \nu)(x) < 0$, for all $x \in \partial\Omega$.

Here $\nu$ is the unit outward normal vector on $\partial\Omega$.

($nd$) Assume that $K$ is a $C^2$-positive function on $\overline{\Omega}$, having only nondegenerate critical points $y_0, \dots, y_\ell$ with

$$\Delta K(y_i) \neq 0, \ \forall i = 0, \dots, \ell \quad \text{if } n > 2 + 2s,$$

and

$$\frac{n-2s}{2} c_1 H(y_i, y_i) - \frac{n-2s}{n} \frac{c_2}{K(y_i)} \Delta K(y_i) \neq 0, \ \forall i = 0, \dots, \ell \quad \text{if } n = 2 + 2s,$$

where $c_1 = \int_{\mathbb{R}^n} (dz/(1+|z|^2)^{(n+2s)/2})$, $c_2 = (1/n) \int_{\mathbb{R}^n} |z|^2((|z|^2 - 1)/(1+|z|^2)^{n+1})\, dz$, and $H(.,.)$ is the regular part of the Green function associated to $A_s$. Let

$$\mathcal{K}^+ = \{y \in \Omega, \text{s.t.}, \nabla K(y) = 0 \text{ and } -\Delta K(y) > 0\} \quad \text{if } n > 2 + 2s,$$

$$\mathcal{K}^+ = \left\{y \in \Omega, \text{s.t. } \nabla K(y) = 0 \text{ and } \frac{n-2s}{2} c_1 H(y,y) - \frac{n-2s}{n} \frac{c_2}{K(y)} \Delta K(y) > 0\right\} \quad \text{if } n = 2 + 2s,$$

and

$$\mathcal{K}^+ = \{y \in \Omega, \text{s.t. } \nabla K(y) = 0\} \quad \text{if } n < 2 + 2s.$$

For any critical point $y$ of $K$, we denote by $\text{ind}(K, y)$ the Morse index of $K$ at $y$. The following theorem is the first main result of this paper.

THEOREM 1.1. *Let $K$ be a given function on a contractible bounded domain of $\mathbb{R}^n, n \geq 2$, satisfying conditions ($A$) and ($nd$). If there exists an integer $k_0$ such that*

(a) $n - \text{ind}(K, y) \neq k_0$, *for all $y \in \mathcal{K}^+$,*

(b)

$$\sum_{y \in \mathcal{K}^+, n - \text{ind}(K,y) \leq k_0 - 1} (-1)^{n - \text{ind}(K,y)} - 1 \neq 0,$$

*then (1-1) admits at least one solution provided that $K$ is close to 1.*

https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

Observe that for any integer $k_0 > \max\{n - \mathrm{ind}(K, y), y \in \mathcal{K}^+\}$, condition $(a)$ of Theorem 1.1 is trivially satisfied. For applications, we think that the following version is helpful:

THEOREM 1.2. *Let $\Omega$ be a contractible bounded domain of $\mathbb{R}^n, n \geq 2$, and let $K$ be a function satisfying conditions $(A)$ and $(nd)$. If*

$$\sum_{y \in \mathcal{K}^+} (-1)^{n-\mathrm{ind}(K,y)} - 1 \neq 0,$$

*then (1-1) admits at least one solution provided that $K$ is close to 1.*

The argument we use is able to extend the result of Theorem 1.2 to any bounded domain of $\mathbb{R}^n, n \geq 2$.

THEOREM 1.3. *Let $\Omega$ be a bounded domain of $\mathbb{R}^n, n \geq 2$. Under conditions $(A)$ and $(nd)$, if*

$$\sum_{y \in \mathcal{K}^+} (-1)^{n-\mathrm{ind}(K,y)} - \chi(\Omega) \neq 0,$$

*then (1-1) admits at least one solution provided that $K$ is close to 1. Here $\chi(\Omega)$ denotes the Euler–Poincaré characteristic of $\Omega$.*

Our method is based on Bahri's theory of critical points at infinity. For this theory, we refer to [1]. We will prove Bahri's estimates in the fractional framework and we use it to prove our existence results.

## 2. Estimates at infinity

**2.1. Variational framework.** Following [16, 18], we state the local equation associated to (1-1) on the half cylinder $C = \Omega \times [0, \infty)$. The celebrated fractional harmonic extension result of Caffarelli and Silvester [16] on $\mathbb{R}^n$ and Cabré and Tan [18] on bounded domains (see also [5, 20, 22]) says that any $u \in H_0^s(\Omega)$; for the fractional Sobolev space on $\Omega$, the problem

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla v) = 0 & \text{in } C, \\ v = 0 & \text{on } \partial_L C := \partial\Omega \times [0, \infty), \\ v = u & \text{on } \Omega \times \{0\}, \end{cases}$$

admits a unique solution denoted by $s - h(u)$ in the Sobolev space $H_{0L}^s(C)$ defined by the closure of

$$C_{0L}^\infty(C) := \{v \in C^\infty(\overline{C}), \text{s.t. } v = 0 \text{ on } \partial_L C\},$$

with respect to the norm

$$|v|^2 = \int_C t^{1-2s} |\nabla v|^2 \, dx \, dt.$$

It follows that $A_s$ is expressed as follows:

$$u \in H_0^s(\Omega) \mapsto A_s(u) = \partial_N^s (s - h(u))/_{\Omega \times \{0\}},$$

https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

where $N$ denotes the unit outward normal vector to $C$ on $\Omega \times \{0\}$ and

$$\partial_N^s(s-h(u))(x,0) = -c_s \lim_{t\to 0^+} t\frac{\partial(s-h(u))}{\partial t}(x,t).$$

Here $c_s := \Gamma(s)/2^{1-2s}\Gamma(1-s)$. In this way, problem (1-1) is equivalent to the local problem

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla v) = 0 & \text{in } C, \\ v > 0 & \text{in } C, \\ v = 0 & \text{on } \partial_L C, \\ \partial_N^s(v) = K(x) v^{(n+2s)/(n-2s)} & \text{on } \Omega\times\{0\}. \end{cases} \tag{2-1}$$

Therefore, $u$ is a solution of (1-1) if and only if $s-h(u)$ is a solution of (2-1).

Following [9], we state the Euler–Lagrange functional associated to (2-1). Let

$$\mathcal{H} = \{v \in H_{0L}^s(C), \text{s.t. } \mathrm{div}(t^{1-2s}\nabla v) = 0 \text{ in } C\}.$$

For any $v, w \in \mathcal{H}$, we denote

$$\langle v, w\rangle = c_s^{-1}\int_{\Omega\times\{0\}} \partial_N^s v(x,0) w(x,0)\,dx$$

and

$$|v|^2 = c_s^{-1}\int_{\Omega\times\{0\}} \partial_N^s v(x,0)v(x,0)\,dx = \int_C t^{1-2s}|\nabla v|^2\,dx\,dt.$$

Up to a multiplicative constant, $v$ is a solution of (2-1) in $\mathcal{H}$ if and only if $v$ is a critical point of

$$J(v) = \frac{|v|^2}{(\int_\Omega K(x)v(x,0)^{2n/(n-2s)}\,dx)^{(n-2s)/n}}, \quad v\in\Sigma^+,$$

where $\Sigma^+ = \{v\in\mathcal{H}, v\geq 0, |v| = c_s^{-1/2}\}$. Observe that the exponent $2n/(n-2s)$ corresponds to the critical exponent of the Sobolev trace embedding $\mathcal{H}\hookrightarrow L^q(\Omega)$. Since the critical Sobolev embedding is not compact, $J$ fails to satisfy the Palais–Smale condition. The following proposition characterizes all sequences failing the Palais–Smale condition. Let $\lambda > 0$ and $a\in\Omega$. We set

$$\delta_{(a,\lambda)}(x) = \beta_0\Big(\frac{\lambda}{1+\lambda^2|x-a|^2}\Big)^{(n-2s)/2}, \quad x\in\mathbb{R}^n.$$

Here $\beta_0$ is a fixed positive constant chosen so that $\widetilde{\delta}_{(a,\lambda)} := s - h(\delta_{(a,\lambda)})$ satisfies

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla\widetilde{\delta}_{(a,\lambda)}) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \partial_N^s\widetilde{\delta}_{(a,y)} = \widetilde{\delta}_{(a,y)}^{(n+2s)/(n-2s)} & \text{on } \mathbb{R}^n\times\{0\}. \end{cases}$$

Let $P\widetilde{\delta}_{(a,\lambda)}$ be the unique solution in $\mathcal{H}$ of

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla v) = 0 & \text{in } C, \\ v = 0 & \text{on } \partial_L C, \\ \partial_N^s v = \widetilde{\delta}_{(a,y)}^{(n+2s)/(n-2s)} & \text{on } \Omega\times\{0\}. \end{cases}$$

https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

Proposition 2.1 [10, 17]. *Assume that $J$ has no critical point in $\Sigma^+$. For any sequence $(u_k)_k$ in $\Sigma^+$ along which $J$ is bounded and its gradient goes to zero, there exist $p \in \mathbb{N}^*$, $(\varepsilon_k)_k \to 0$ in $\mathbb{R}_+$ and a subsequence $(u_{k_r})_r$ of $(u_k)_k$ such that $u_{k_r} \in V(p, \varepsilon_{k_r})$, for all $r \in \mathbb{N}$. Here*

$$V(p,\varepsilon) := \Big\{u \in \Sigma^+, \text{s.t. there exist } a_1,\dots,a_p \in \Omega, \text{there exist } \lambda_1,\dots,\lambda_p > \varepsilon^{-1}$$

$$\text{and } \alpha_1,\dots,\alpha_p > 0, \text{s.t. } \Big|u - \sum_{i=1}^{p} \alpha_i P\widetilde{\delta}_{(a_i,\lambda_i)}\Big| \langle \varepsilon, \lambda_i d(a_i, \partial\Omega) \rangle \varepsilon^{-1},$$

$$|\alpha_i^{4s/(n-2s)} K(a_i) J(u)^{n/(n-2s)} - 1| < \varepsilon, \text{ for all } i = 1,\dots,p$$

$$\text{and } \varepsilon_{ij} := \frac{1}{\Big(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j |a_i - a_j|^2\Big)^{(n-2s)/2}} < \varepsilon, \text{ for all } i \neq j\Big\}.$$

The parametrization of $V(p, \varepsilon)$ is as follows.

Proposition 2.2 [10]. *For any $u \in V(p, \varepsilon)$, the minimization problem*

$$\min\Big\{\Big|u - \sum_{i=1}^{p} \alpha_i P\widetilde{\delta}_{(a_i,\lambda_i)}\Big|, a_i \in \Omega, \lambda_i > 0, \alpha_i > 0, \forall i = 1,\dots,p\Big\}$$

*has a unique solution (up to permutation). Hence any $u \in V(p, \varepsilon)$ can be written as*

$$u = \sum_{i=1}^{p} \alpha_i P\widetilde{\delta}_{(a_i,\lambda_i)} + v,$$

*where $|v| < \varepsilon$ satisfies the condition*

$$(V_0) : \langle v, \varphi \rangle = 0 \quad \text{for } \varphi \in \Big\{P\widetilde{\delta}_{(a_i,\lambda_i)}, \frac{\partial P\widetilde{\delta}_{(a_i,\lambda_i)}}{\partial \lambda_i}, \frac{\partial P\widetilde{\delta}_{(a_i,\lambda_i)}}{\partial a_i}, i = 1,\dots,p\Big\}.$$

The next proposition deals with the expansion of $J$ in $V(p, \varepsilon)$. Its proof proceeds exactly as the one in [9, Proposition 1].

For any $x, y \in \Omega$ and $t \geq 0$, we denote

$$\widetilde{G}((x,t), y) = \frac{1}{\|(x-y,t)\|_{\mathbb{R}^{n+1}}^{n-2s}} - \widetilde{H}((x,t), y),$$

where $\widetilde{H}$ is the regular part of $\widetilde{G}$. It satisfies

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla \widetilde{H}(.,y)) = 0 & \text{in } C, \\ \widetilde{H}((x,t),y) = \dfrac{1}{\|(x-y,t)\|_{\mathbb{R}^{n+1}}^{n-2s}} & \text{on } \partial_L C, \\ \partial_N^s \widetilde{H}(.,y) = 0 & \text{on } \Omega \times \{0\}. \end{cases}$$

https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

Following [9, Lemma 3.4], we have the three estimates

$$P\widetilde{\delta}_{(a,\lambda)} = \widetilde{\delta}_{(a,\lambda)} - c_0 \frac{\widetilde{H}(.,a)}{\lambda^{(n-2s)/2}} + O\Big(\frac{1}{\lambda^{(n+2s)/2} d(a,\partial\Omega)^{n+2-2s}}\Big), \tag{2-2}$$

$$\lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial \lambda} = \lambda \frac{\partial \widetilde{\delta}_{(a,\lambda)}}{\partial \lambda} + \frac{n-2s}{2} c_0 \frac{\widetilde{H}(.,a)}{\lambda^{(n-2s)/2}} + O\Big(\frac{1}{\lambda^{(n+2s)/2} d(a,\partial\Omega)^{n+2-2s}}\Big), \tag{2-3}$$

$$\frac{1}{\lambda}\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial a} = \frac{1}{\lambda}\frac{\partial \widetilde{\delta}_{(a,\lambda)}}{\partial a} - c_0 \frac{\frac{\partial \widetilde{H}(.,a)}{\partial a}}{\lambda \lambda^{(n-2s)/2}} + O\Big(\frac{1}{\lambda\lambda^{(n+2s)/2} d(a,\partial\Omega)^{n+2-2s}}\Big).$$

We set

$$H(x,y) = \widetilde{H}((x,0),y), \forall x,y \in \Omega.$$

Proposition 2.3. *Let $p \geq 1$ and $\varepsilon > 0$ but small enough. For any $u = \sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda_i)} + v \in V(p,\varepsilon)$, we have the expansion*

$$\begin{aligned} J(u) = {} & \frac{S^{2s/n}\sum_{i=1}^p \alpha_i^2}{(\sum_{i=1}^p \alpha_i^{2n/(n-2s)} K(a_i))^{(n-2s)/n}} \Big\{1 + \frac{2w_{n-1}}{\sum_{i=1}^p K(a_i)} \\ & \times \Big[\sum_{i=1}^p \frac{1}{K(a_i)^{(n-2s)/2}} \Big(c_1 \frac{H(a_i,a_i)}{\lambda_i^{n-2s}} - c_2 \frac{\Delta K(a_i)}{K(a_i)\lambda_i^2}\Big) \\ & + \sum_{j\neq i} \frac{c_1}{(K(a_i)K(a_j))^{(n-2s)/4}} \Big(\frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{(n-2s)/2}} - \varepsilon_{ij}\Big)\Big] - f(v) + \frac{1}{S\sum_{i=1}^p \alpha_i^2} Q(v,v) \\ & + o\Big(\sum_{i=1}^p \frac{1}{(\lambda_i d(a_i,\partial\Omega))^{n-2s}} + \sum_{j\neq i}\varepsilon_{ij} + \sum_{i=1}^p \frac{1}{\lambda_i^2} + |v|^2\Big)\Big\}, \end{aligned}$$

*where*

$$Q(v,v) = |v|^2 - \frac{n+2s}{n-2s} \frac{\sum_{i=1}^p \alpha_i^2}{\sum_{i=1}^p \alpha_i^{2n/(n-2s)} K(a_i)} \int_\Omega K(x) \Big(\sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda_i)}\Big)^{4s/(n-2s)} v^2\,dx,$$

$$f(v) = \frac{2}{S\sum_{i=1}^p \alpha_i^{2n/(n-2s)} K(a_i)} \int_\Omega K(x)\Big(\sum_i \alpha_i P\widetilde{\delta}_{(a_i,\lambda_i)}\Big)^{(n+2s)/(n-2s)} v\,dx.$$

*$c_1 = \int_{\mathbb{R}^n} (dz/(1+|z|^2)^{(n+2s)/2})$, $c_2 = (1/n)\int_{\mathbb{R}^n} |z|^2((|z|^2-1)/(1+|z|^2)^{n+1})\,dz$, $S = \int_{\mathbb{R}^n}(dz/(1+|z|^2)^n)$ and $w_{n-1}$ is the volume of the unit sphere $\mathbb{S}^{n-1}$.*

Corollary 2.4. *If $\|K-1\|_{L^\infty(\overline{\Omega})}$ is small enough, then for any $u = \sum_{i=1}^p \alpha_i P\widetilde{\delta}_{(a_i,\lambda_i)} + v \in V(p,\varepsilon)$, we have*

$$J(u) = (pS)^{2s/n} + o(1) \quad \text{as } \varepsilon \to 0.$$

Let $\alpha_0 > 0$ such that $S^{2s/n} + \alpha_0 < (2S)^{2s/n}$ and let

$$J_{S^{2s/n}+\alpha_0} = \{u \in \Sigma^+, J(u) \leq S^{2s/n} + \alpha_0\}.$$

We point out that our next construction, to prove our existence theorems, lies in $J_{S^{2s/n}+\alpha_0}$. Therefore, our next analysis at infinity will be performed only in the set $V(1,\varepsilon)$.

https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

**2.2. Expansion in $V(1,\varepsilon)$.** In this subsection we provide the expansions of the gradient of $J$ at $\alpha\lambda(\partial P\widetilde{\delta}_{(a_i,\lambda_i)}/\partial\lambda)$ and $\alpha(1/\lambda)(\partial P\widetilde{\delta}_{(a_i,\lambda_i)}/\partial a)$, respectively.

PROPOSITION 2.5. *Assume that $K$ satisfies condition (nd). For any $u = \alpha P\widetilde{\delta}_{(a,\lambda)} \in V(1,\varepsilon)$, we have the expansion*

$$\Big\langle \partial J(u), \alpha\lambda\frac{\partial P\widetilde{\delta}_{a,\lambda}}{\partial\lambda}\Big\rangle = 2\alpha^2 J(u)\begin{cases} -\dfrac{n-2s}{2}c_1\dfrac{H(a,a)}{\lambda^{n-2s}} & \text{if } n<2+2s,\\ \dfrac{n-2s}{n}\dfrac{c_2}{K(a)}\dfrac{\Delta K(a)}{\lambda^2} & \text{if } n>2+2s,\\ \dfrac{n-2s}{n}\dfrac{c_2}{K(a)}\dfrac{\Delta K(a)}{\lambda^2}-\dfrac{n-2s}{2}c_1\dfrac{H(a,a)}{\lambda^{n-2s}} & \text{if } n=2+2s,\end{cases}$$
$$+ o\Big(\frac{1}{\lambda^2}+\frac{1}{(\lambda d(a,\partial\Omega))^{n-2s}}\Big).$$

PROOF OF PROPOSITION 2.5. Let $u = \alpha P\widetilde{\delta}_{(a,\lambda)} \in V(1,\varepsilon)$. Following [9], we have

$$\Big\langle \partial J(u), \alpha\lambda\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\Big\rangle = 2J(u)\Big[\alpha^2\Big\langle P\widetilde{\delta}_{(a,\lambda)}, \lambda\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\Big\rangle$$
$$-\alpha^{2n/(n-2s)}J(u)^{n/(n-2s)}\int_\Omega K(x)P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)}\lambda\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\,dx\Big].$$

LEMMA 2.6.

$$\Big\langle P\widetilde{\delta}_{(a,\lambda)}, \lambda\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\Big\rangle = \frac{n-2s}{2}c_1\frac{H(a,a)}{\lambda^{n-2s}} + o\Big(\frac{1}{(\lambda d(a,\partial\Omega))^{n-2s}}\Big).$$

PROOF.

$$\Big\langle P\widetilde{\delta}_{(a,\lambda)}, \lambda\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\Big\rangle$$
$$=\int_\Omega \widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)}\lambda\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\,dx$$
$$=\int_\Omega \widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)}\lambda\frac{\partial \widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\,dx + \int_\Omega \widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)}\Big(\lambda\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}-\lambda\frac{\partial \widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\Big)dx.$$

Using (2-3) and an expansion of the first order of $H(.,a)$ around $a$, we get

$$\int_\Omega \widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)}\Big(\lambda\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}-\lambda\frac{\partial \widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\Big)dx$$
$$=\frac{1}{\lambda^{(n-2s)/2}}\Big[-\frac{n-2s}{2}c_0 H(a,a)\int_{\mathbb{R}^n}\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)}\,dx$$
$$+O\Big(\int_{\mathbb{R}^n}|x-a|\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)}\,dx\Big)$$
$$+o\Big(\frac{1}{\lambda^{2s}d(a,\partial\Omega)^{n-2s+2}}\int_{\mathbb{R}^n}\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)}\,dx\Big)\Big]+O\Big(\frac{1}{\lambda^n}\Big).$$

https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

Using the fact that

$$\int_{\Omega} \widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \lambda \frac{\partial \widetilde{\delta}_{(a,\lambda)}}{\partial \lambda}\, dx = O\Big(\frac{1}{\lambda d(a,\partial\Omega)^n}\Big),$$

$$\int_{\mathbb{R}^n} \widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)}\, dx = \frac{1}{\lambda^{(n-2s)/2}} \int_{\mathbb{R}^n} \frac{dz}{(1+|z|^2)^{(n+2s)/2}}$$

and

$$\int_{\mathbb{R}^n} |x-a| \widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)}\, dx = O\Big(\frac{1}{\lambda^{(n-2s)/2+1}}\Big),$$

Lemma 2.6 follows. □

**Lemma 2.7.**

$$\begin{aligned}
&\alpha^{2n/(n-2s)} J(u)^{n/(n-2s)} \int_{\Omega} K(x) P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial \lambda}\, dx \\
&\quad = 2\alpha^2 \Big\langle P\widetilde{\delta}_{(a,\lambda)}, \lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial \lambda} \Big\rangle - \alpha^2 \frac{n-2s}{n} \frac{c_2}{K(a)} \frac{\Delta K(a)}{\lambda^2} + o\Big(\frac{1}{(\lambda d(a,\partial\Omega))^{n-2s}}\Big).
\end{aligned}$$

*Proof.* Let $\eta > 0$ be small enough.

$$\begin{aligned}
&\int_{\Omega} K(x) P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial \lambda}\, dx \\
&\quad = \int_{B(a,\eta)} K(x) P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial \lambda}\, dx + O\Big(\frac{1}{\lambda^n}\Big) \\
&\quad = K(a) \int_{B(a,\eta)} P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial \lambda}\, dx \\
&\qquad + \int_{B(a,\eta)} \nabla K(a)(x-a) P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial \lambda}\, dx \\
&\qquad + \frac{1}{2} \int_{B(a,\eta)} D^2K(a)(x-a, x-a) P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial \lambda}\, dx \\
&\qquad + o\Big(\int_{B(a,\eta)} |x-a|^2 P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial \lambda}\, dx\Big) + O\Big(\frac{1}{\lambda^n}\Big).
\end{aligned}$$

*Claim 1.*

$$I_1 := \int_{B(a,\eta)} P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial \lambda}\, dx = 2\Big\langle P\widetilde{\delta}_{(a,\lambda)}, \lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial \lambda} \Big\rangle + o\Big(\frac{1}{(\lambda d(a,\partial\Omega))^{n-2s}}\Big).$$

https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

Indeed,

$$\begin{aligned}
I_1 &:= \int_{B(a,\eta)} \widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\,dx \\
&\quad + \frac{n+2s}{n-2s} \int_{B(a,\eta)} \widetilde{\delta}_{(a,\lambda)}^{4s/(n-2s)} (P\widetilde{\delta}_{(a,\lambda)} - \widetilde{\delta}_{(a,\lambda)}) \lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\,dx \\
&\quad + O\Big(\int_\Omega |P\widetilde{\delta}_{(a,\lambda)} - \widetilde{\delta}_{(a,\lambda)}|^2 \widetilde{\delta}_{(a,\lambda)}^{4s/(n-2s)}\,dx\Big) \\
&= \int_\Omega \widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\,dx + \frac{n+2s}{n-2s} \int_\Omega \widetilde{\delta}_{(a,\lambda)}^{4s/(n-2s)} P\widetilde{\delta}_{(a,\lambda)} \lambda \frac{\partial \widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\,dx \\
&\quad - \frac{n+2s}{n-2s}\Big[\int_\Omega \widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \lambda \frac{\partial \widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\,dx \\
&\quad + \int_\Omega \widetilde{\delta}_{(a,\lambda)}^{4s/(n-2s)} (P\widetilde{\delta}_{(a,\lambda)} - \widetilde{\delta}_{(a,\lambda)}) \Big(\lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda} - \lambda \frac{\partial \widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\Big)\,dx\Big] \\
&\quad + O\Big(\int_\Omega |P\widetilde{\delta}_{(a,\lambda)} - \widetilde{\delta}_{(a,\lambda)}|^2 \widetilde{\delta}_{(a,\lambda)}^{4s/(n-2s)}\,dx\Big) + O\Big(\frac{1}{(\lambda d(a,\partial\Omega))^n}\Big) \\
&= 2\int_\Omega \widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\,dx + o\Big(\frac{1}{(\lambda d(a,\partial\Omega))^{n-2s}}\Big).
\end{aligned}$$

Hence Claim 1 follows.

*Claim 2.*

$$I_2 := \int_{B(a,\eta)} \nabla K(a)(x-a) P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\,dx = o\Big(\frac{1}{(\lambda d(a,\partial\Omega))^{n-2s}}\Big).$$

Indeed, using the estimates (2-2), (2-3) and (3-1) below, we have

$$\begin{aligned}
I_2 &= \int_{B(a,\eta)} \nabla K(a)(x-a) \widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \lambda \frac{\partial \widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\,dx \\
&\quad + O\Big(\frac{1}{\lambda^{(n-2s)/2} d(a,\partial\Omega)^{n-2s}} \int_{B(a,\eta)} |x-a| \widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)}\,dx\Big).
\end{aligned}$$

By a symmetry argument, we have

$$\int_{B(a,\eta)} \nabla K(a)(x-a) \widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \lambda \frac{\partial \widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\,dx = 0.$$

https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

Moreover,

$$\int_{B(a,\eta)} |x-a|\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)}\,dx$$
$$= \int_{B(a,\eta)} |x-a|\frac{\lambda^{(n+2s)/2}}{(1+\lambda^2|x-a|^2)^{(n+2s)/2}}\,dx$$
$$= \frac{1}{\lambda\lambda^{(n-2s)/2}}\int_{B(0,\lambda\eta)} \frac{|z|}{(1+|z|^2)^{(n+2s)/2}}\,dz = \begin{cases} O\Big(\dfrac{\log\lambda}{\lambda\lambda^{(n-2s)/2}}\Big) & \text{if } s=\dfrac{1}{2},\\ O\Big(\dfrac{1}{\lambda^{2s}\lambda^{(n-2s)/2}}\Big) & \text{if } s\neq\dfrac{1}{2}.\end{cases}$$

Hence Claim 2 follows.

*Claim 3.*

$$I_3 := \int_{B(a,\eta)} D^2K(a)(x-a,x-a)P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)}\lambda\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\,dx$$
$$= -\frac{n-2s}{2n}c_2\frac{\Delta K(a)}{\lambda^2}+o\Big(\frac{1}{\lambda^2}\Big).$$

Indeed, using the estimates (2-2), (2-3) and (3-1) below, we have

$$I_3 = \int_{B(a,\eta)} D^2K(a)(x-a,x-a)\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)}\lambda\frac{\partial\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\,dx$$
$$+O\Big(\frac{1}{\lambda^{(n-2s)/2}d(a,\partial\Omega)^{n-2s}}\int_{B(a,\eta)}|x-a|^2\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)}\,dx\Big).$$

Observe that

$$\int_{B(a,\eta)} D^2K(a)(x-a,x-a)\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)}\lambda\frac{\partial\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\,dx$$
$$= \frac{n-2s}{2}\sum_{1\le i,j\le n}\frac{\partial^2K(a)}{\partial x_i\partial x_j}\int_{B(a,\eta)}(x-a)_i(x-a)_j\frac{1-\lambda^2|x-a|^2}{(1+\lambda^2|x-a|^2)^{n+1}}\lambda^n\,dx.$$

Using the fact that for any $i\neq j$ we have

$$\int_{B(a,\eta)}(x-a)_i(x-a)_j\frac{1-\lambda^2|x-a|^2}{(1+\lambda^2|x-a|^2)^{n+1}}\lambda^n\,dx=0,$$

a change of variables $z=\lambda(x-a)$ yields

$$\int_{B(a,\eta)} D^2K(a)(x-a,x-a)\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)}\lambda\frac{\partial\widetilde{\delta}_{(a,\lambda)}}{\partial\lambda}\,dx$$
$$= \frac{n-2s}{2n}\frac{\Delta K(a)}{\lambda^2}\int_{\mathbb{R}^n}|z|^2\frac{1-|z|^2}{(1+|z|^2)^{n+1}}\,dz+O\Big(\frac{1}{\lambda^n}\Big).$$

Hence Claim 3 holds. Now using the fact that $\alpha^{4s/(n-2s)}J(u)^{n/(n-2s)}K(a)=1+o(1)$, Lemma 2.7 is valid. □

https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

Proposition 2.5 now follows from the estimates of Lemmas 2.6 and 2.7. □

PROPOSITION 2.8. *For any $u = \alpha P\widetilde{\delta}_{(a,\lambda)} \in V(1,\varepsilon)$, we have the expansion*

$$\Big\langle \partial J(u), \alpha \frac{1}{\lambda}\frac{\partial P\widetilde{\delta}_{a,\lambda}}{\partial a}\Big\rangle = -2\alpha^2 J(u)\bigg(c_3 \frac{\nabla K(a)}{K(a)\lambda}(1+o(1)) - c_1 \frac{\frac{\partial H(a,a)}{\partial a}}{\lambda^{n+1-2s}}\bigg) + O\Big(\frac{1}{\lambda^2}\Big) + o\Big(\frac{1}{(\lambda d(a,\partial\Omega))^{n+1-2s}}\Big),$$

*where* $c_3 = \int_{\mathbb{R}^n} |z|((|z|^2-1)/(1+|z|^2)^{n+1})\,dz$.

PROOF OF PROPOSITION 2.8. Let $u = \alpha P\widetilde{\delta}_{(a,\lambda)} \in V(1,\varepsilon)$. Then

$$\Big\langle \partial J(u), \alpha \frac{1}{\lambda}\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial a}\Big\rangle = 2J(u)\bigg[\alpha^2 \Big\langle P\widetilde{\delta}_{(a,\lambda)}, \frac{1}{\lambda}\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial a}\Big\rangle - \alpha^{2n/(n-2s)} J(u)^{n/(n-2s)} \int_\Omega K(x) P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \frac{1}{\lambda}\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial a}\,dx\bigg].$$

Arguing as in Lemmas 2.6 and 2.7, we have the following two estimates.

LEMMA 2.9.

$$\Big\langle P\widetilde{\delta}_{(a,\lambda)}, \frac{1}{\lambda}\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial a}\Big\rangle = -c_1 \frac{\frac{\partial H(a,a)}{\partial a}}{\lambda^{n+1-2s}} + o\Big(\frac{1}{(\lambda d(a,\partial\Omega))^{n+1-2s}}\Big).$$

LEMMA 2.10.

$$\alpha^{2n/(n-2s)} J(u)^{n/(n-2s)} \int_\Omega K(x) P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} \frac{1}{\lambda}\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial a}\,dx = 2\alpha^2 \Big\langle P\widetilde{\delta}_{(a,\lambda)}, \frac{1}{\lambda}\frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial a}\Big\rangle + \alpha^2 c_3 \frac{\nabla K(a)}{K(a)\lambda} + O\Big(\frac{1}{\lambda^2}\Big) + o\Big(\frac{1}{(\lambda d(a,\partial\Omega))^{n+1-2s}}\Big).$$

The proof of Proposition 2.8 follows from Lemmas 2.9 and 2.10. □

**2.3. On the $v$-part contribution.** Let $u = \alpha P\widetilde{\delta}_{(a,\lambda)} \in V(1,\varepsilon)$. By an expansion of $K$ around $a$, the quadratic form $Q$ defined in Proposition 2.3 is close to

$$|v|^2 - \frac{n+2s}{n-2s}\int_\Omega P\widetilde{\delta}_{(a,\lambda)}^{4s/(n-2s)} v^2\,dx.$$

Therefore it is positive definite; see [1]. Consequently, there exists an invertible positive operator such that $Q(v,v) = \langle Bv, v\rangle$ and the expansion of $J$ in $V(1,\varepsilon)$ given by Proposition 2.3 becomes

$$J(u) = \frac{S^{2s}}{(K(a))^{(n-2s)/2}}\bigg[1 + \frac{2w_{n-1}}{K(a)K(a)^{(n-2s)/2}}\Big(c_1 \frac{H(a,a)}{\lambda^{n-2s}} - c_2 \frac{\Delta K(a)}{K(a)\lambda^2}\Big) + \frac{1}{S\alpha^2}(\langle Bv, v\rangle - S\alpha^2 f(v))\bigg] + o\Big(\frac{1}{(\lambda d(a,\partial\Omega))^{n-2s}} + \frac{1}{\lambda^2} + |v|^2\Big).$$

https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

Observe that $J$ behaves on the $v$-variable as a function close to

$$\psi(v) := \langle Bv, v\rangle - S\alpha^2 f(v),$$

which is a coercive function. It follows that $\psi$ and therefore $v \mapsto J(\alpha P\widetilde{\delta}_{(a,\lambda)} + v)$ admit a unique minimum denoted by $\bar{v} = \bar{v}(\alpha, a, \lambda)$. The estimate of $|\bar{v}|$ is as follows.

Proposition 2.11. *For any* $u = \alpha P\widetilde{\delta}_{(a,\lambda)} \in V(1,\varepsilon)$, *we have*

$$|\bar{v}| \le M\Big(\frac{|\nabla K(a)|}{\lambda} + \frac{1}{\lambda^2}\Big) + M\begin{cases} \dfrac{1}{(\lambda d)^{(n+2s)/2}} & \text{if } n > 6s,\\ \dfrac{\log(\lambda d)^{2/3}}{(\lambda d)^{(n+2s)/2}} & \text{if } n = 6s,\\ \dfrac{1}{(\lambda d)^{n-2s}} & \text{if } n < 6s.\end{cases}$$

*Here* $d = d(a, \partial\Omega)$.

Proof. Since $\bar{v}$ minimizes $\psi$, we derive that

$$\psi'(\bar{v}) = B\bar{v} - S\alpha f = 0.$$

Using the fact that $B$ is invertible, we get

$$|\bar{v}| \le S\alpha\|B^{-1}f\| \le c\|f\|.$$

For any $v$ in the subspace $E_{\alpha a\lambda} = \{v \in \mathcal{H}, v \in (V_0)\}$, we have

$$\begin{aligned} f(v) &= \frac{2}{S\alpha^{2n/(n-2s)}K(a)}\int_\Omega K(x)P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} v\,dx\\ &= \frac{2}{S\alpha^{2n/(n-2s)}K(a)}\\ &\quad\times\Big(\int_{B(a,d)} K(x)P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} v\,dx + \int_{\Omega\setminus B(a,d)} K(x)P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} v\,dx\Big)\\ &:= \frac{2}{S\alpha^{2n/(n-2s)}K(a)}(I_1 + I_2).\end{aligned}$$

Using the Hölder inequality with $p = 2n/(n+2s)$ and $q = 2n/(n-2s)$, we have

$$\begin{aligned} |I_2| &\le c\Big(\int_{\Omega\setminus B(a,d)} P\widetilde{\delta}_{(a,\lambda)}^{2n/(n-2s)}\Big)^{(n+2s)/2n}\Big(\int_\Omega |v|^{2n/(n-2s)}\Big)^{(n-2s)/2n}\\ &\le c|v|\Big(\int_{\Omega\setminus B(a,d)} \widetilde{\delta}_{(a,\lambda)}^{2n/(n-2s)} + O\Big(\int_{\Omega\setminus B(a,d)} \widetilde{\delta}_{(a,\lambda)}^{4s/(n-2s)}|P\widetilde{\delta}_{(a,\lambda)} - \widetilde{\delta}_{(a,\lambda)}|\,dx\Big)\Big)^{(n+2s)/2n}\\ &\le c|v|\Big(\Big(\int_{\Omega\setminus B(a,d)} \widetilde{\delta}_{(a,\lambda)}^{2n/(n-2s)}\Big)^{(n+2s)/2n} + O\Big(\int_{\Omega\setminus B(a,d)} \widetilde{\delta}_{(a,\lambda)}^{4s/(n-2s)}|P\widetilde{\delta}_{(a,\lambda)} - \widetilde{\delta}_{(a,\lambda)}|\,dx\Big)\Big)\\ &\le c|v|\Big(\frac{1}{(\lambda d)^{(n+2s)/2}} + o\Big(\frac{1}{(\lambda d)^{(n+2s)/2}}\Big)\Big).\end{aligned}$$

https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

In order to compute $I_1$, we expand $K$ around $a$. We have

$$I_1 = \int_{B(a,d)} K(a) P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} v\, dx + \int_{B(a,d)} \nabla K(a)(x-a) P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} v\, dx + O\Big(\int_{B(a,d)} |x-a|^2 P\widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} |v|\, dx\Big) := K_1 + K_2 + K_3.$$

Using the fact that $v$ satisfies $(V_0)$, we have

$$\begin{aligned} |K_1| &\le c\Big(\int_{^cB(a,d)} \widetilde{\delta}_{(a,\lambda)}^{(n+2s)/(n-2s)} |v|\, dx + \int_{B(a,d)} \widetilde{\delta}_{(a,\lambda)}^{4s/(n-2s)} |P\widetilde{\delta}_{(a,\lambda)} - \widetilde{\delta}_{(a,\lambda)}||v|\, dx\Big) \\ &\le c|v|\Big(\frac{1}{(\lambda d)^{(n+2s)/2}} + \|P\widetilde{\delta}_{(a,\lambda)} - \widetilde{\delta}_{(a,\lambda)}\|_{L^\infty(\Omega)}\Big(\int_{B(a,d)} \widetilde{\delta}_{(a,\lambda)}^{8ns/(n^2-4s^2)}\, dx\Big)^{(n+2s)/2n}\Big). \end{aligned}$$

Observe that by (2-2) we have $\|P\widetilde{\delta}_{(a,\lambda)} - \widetilde{\delta}_{(a,\lambda)}\|_{L^\infty(\Omega)} \le c/(\lambda d)^{(n-2s)/2}$ and

$$\Big(\int_{B(a,d)} \widetilde{\delta}_{(a,\lambda)}^{8ns/(n^2-4s^2)}\, dx\Big)^{(n+2s)/2n} \le c \begin{cases} \dfrac{1}{(\lambda d)^{2s}} & \text{if } n > 6s, \\ \dfrac{\log(\lambda d)^{2/3}}{(\lambda d)^{n/3}} & \text{if } n = 6s, \\ \dfrac{1}{(\lambda d)^{(n-2s)/2}} & \text{if } n < 6s. \end{cases}$$

Thus

$$|K_1| = |v| \begin{cases} O\Big(\dfrac{1}{(\lambda d)^{(n+2s)/2}}\Big) & \text{if } n > 6s, \\ O\Big(\dfrac{\log(\lambda d)^{2/3}}{(\lambda d)^{2n/3}}\Big) & \text{if } n = 6s, \\ O\Big(\dfrac{1}{(\lambda d)^{n-2s}}\Big) & \text{if } n < 6s. \end{cases}$$

In the same way we have

$$|K_2| = |v| O\Big(\frac{|\nabla K(a)|}{\lambda}\Big) \quad \text{and} \quad |K_3| = |v| O\Big(\frac{1}{\lambda^2}\Big).$$

This completes the proof of Proposition 2.11. □

## 3. Concentration phenomenon in $V(1,\varepsilon)$

In this section we construct a decreasing pseudo-gradient for the functional $J$ in $V(1,\varepsilon)$ and use it to characterize the critical points at infinity of the problem of only one mass.

THEOREM 3.1. *Assume that $K$ satisfies conditions (A) and (nd). Then there exists a pseudo-gradient $W$ of $J$ such that for any $u = \alpha P\widetilde{\delta}_{(a,\lambda)} \in V(1,\varepsilon)$, we have:*

(i) $\langle \partial J(u), W(u)\rangle \le -c(|\nabla K(a)|/\lambda + 1/\lambda^2 + 1/(\lambda d)^{n+1-2s})$;

https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

(ii) $\langle \partial J(u+\bar v), W(u) + (\partial\bar v/\partial(\alpha, a, \lambda))(W(u))\rangle \le -c(|\nabla K(a)|/\lambda + 1/\lambda^2 + 1/(\lambda d)^{n+1-2s})$, *where $c$ is a fixed positive constant independent of $u$ and $\varepsilon$.*

(iii) *Furthermore, the distance $d(t) = d(a(t), \partial\Omega)$ increases if it is small enough.*

(iv) *Moreover, $W$ is a bounded vector field and the only region where the components $\lambda(t)$ are not bounded along the flow lines of $W$ are those where the concentration points $a(t)$ converge along the flow lines of $W$ to a critical point in $\mathcal{K}^+$.*

Proof. $\Omega$ being a regular bounded domain, there exists a small positive constant $d_0$ such that for any $a \in \Omega$ with $d(a, \partial\Omega) \le d_0$, there exists a unique $\bar a \in \partial\Omega$ satisfying $d(a, \partial\Omega) = \|a - \bar a\|$. Furthermore, under assumption $(A)$ we have $(\partial K/\partial\nu_a)(a) \le -c$. Here $\nu_a$ denotes the unit outward normal vector at $a$ of the boundary of $\Omega_a := \{x \in \Omega, d(x, \partial\Omega) \le d(a, \partial\Omega)\}$. Let

$$u = \alpha P\widetilde{\delta}_{(a,\lambda)} \in V(1, \varepsilon).$$

The construction of the required vector field $W(u)$ will be decomposed in two steps.

*Step 1.* Assume that $d(a, \partial\Omega) \le d_0$. In this case we claim that

$$H(a,a) \sim \frac{1}{(2d(a,\partial\Omega))^{n-2s}} \quad \text{and} \quad \frac{\partial H}{\partial \nu_a}(a,a) \sim \frac{2(n-2s)}{(2d(a,\partial\Omega))^{n-2s+1}} \quad \text{as } \varepsilon \text{ is small.} \tag{3-1}$$

Indeed, let $a' = 2\bar a - a$ ($a'$ is the symmetric point of $a$ with respect to $\partial\Omega$). By choosing $d_0$ small, $a' \notin \Omega$. For any $x, y \in \Omega$ and $t \ge 0$, we set

$$\varphi((x,t),y) = \widetilde{H}((x,t),y) - \frac{1}{\|(x-a',t)\|^{n-2s}}.$$

Since $a' \notin \Omega$, $\varphi$ satisfies

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla\varphi(.,y)) = 0 & \text{in } C, \\ \varphi((x,t),y) = \dfrac{1}{\|(x-y,t)\|^{n-2s}} - \dfrac{1}{\|(x-a',t)\|^{n-2s}} & \text{on } \partial\Omega, \\ \partial_N^s \varphi(.,y) = 0 & \text{on } \Omega\times\{0\}. \end{cases}$$

Using the maximum principle, and the fact that on $\partial\Omega$ we have

$$\left|\frac{1}{\|a-y\|^{n-2s}} - \frac{1}{\|a'-y\|^{n-2s}}\right| = o\left(\frac{1}{d(a,\partial\Omega)^{n-2s}}\right) \quad \text{for } d_0 \text{ small,}$$

we get

$$|\varphi((a,0),a)| = o\left(\frac{1}{d(a,\partial\Omega)^{n-2s}}\right).$$

The first estimate of (3-1) follows. In the same way we prove the second estimate of (3-1). We set

$$\dot a = -\frac{\nu_a}{\lambda}. \tag{3-2}$$

https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

We move $a$ along the differential equation (3-2). $u$ satisfies

$$\dot{u} = W_1(u) \quad \text{where } W_1(u) = -\alpha \frac{1}{\lambda} \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial a} \nu_a.$$

The expansion of Proposition 2.8 yields

$$\langle \partial J(u), W_1(u) \rangle = 2\alpha^2 J(u) \Big( c_3 \frac{\frac{\partial K}{\partial \nu_a}(a)}{K(a)\lambda} - c_1 \frac{\frac{\partial H(a,a)}{\partial \nu_a}}{\lambda^{n+1-2s}} \Big) + O\Big(\frac{1}{\lambda^2}\Big) + o\Big(\frac{1}{(\lambda d(a,\partial\Omega))^{n-2s+1}}\Big).$$

Using assumption $(A)$ and estimate (3-1), we get

$$\begin{aligned}\langle \partial J(u), W_1(u) \rangle &\leq -c\Big(\frac{1}{\lambda} + \frac{1}{(\lambda d(a,\partial\Omega))^{n-2s+1}}\Big) \\ &\leq -c\Big(\frac{|\nabla K(a)|}{\lambda} + \frac{1}{\lambda^2} + \frac{1}{(\lambda d(a,\partial\Omega))^{n-2s+1}}\Big).\end{aligned}$$

Condition (i) of Theorem 3.1 is then satisfied. Observe that in this region $W_1$ has no action on the variable $\lambda$ and moves the concentration point $a$ inward of $\Omega$.

*Step 2.* Assume that $d(a,\partial\Omega) \geq d_0/2$. In this case we move the concentration point $a$, producing the equation

$$\dot{a} = \frac{1}{\lambda} \frac{\nabla K(a)}{|\nabla K(a)|},$$

if $a$ is far from the critical points of $K$. If $a$ is near a critical point of $K$, we move $\lambda$, resulting in the equation

$$\dot{\lambda} = \operatorname{sign}(\chi_a)\lambda,$$

where

$$\operatorname{sign}(\chi_a) = \begin{cases} 1 & \text{if } n < 2+2s, \\ \operatorname{sign}(-\Delta K(a)) & \text{if } n > 2+2s, \\ \operatorname{sign}\Big(\frac{n-2s}{2} c_1 H(a,a) - \frac{n-2s}{n} c_2 \frac{\Delta K(a)}{K(a)}\Big) & \text{if } n = 2+2s. \end{cases}$$

Since $K$ satisfies condition $(nd)$, there exist fixed positive constants $\eta_0$ and $\rho_0$ such that for any $a \in \Omega$,

$$|\nabla K(a)| \leq 2\eta_0 \Rightarrow \exists y \in \mathcal{K} \text{ s.t. } a \in B(y,\rho_0) \quad \text{and} \quad \operatorname{sign}(\chi_a) = \operatorname{sign}(\chi_y).$$

Define

$$\begin{aligned} \varphi : \mathbb{R} &\to \mathbb{R} \\ t &\mapsto \begin{cases} 1 & \text{if } |t| \leq \dfrac{\eta_0}{\varepsilon^\gamma}, \\ 0 & \text{if } |t| \geq \dfrac{2\eta_0}{\varepsilon^\gamma}, \end{cases} \end{aligned}$$

where $\gamma := \min(1, n-2s)$. Observe that if $n \geq 3$, the $\gamma = 1$ We set

$$W_2(u) = \varphi(\lambda^\gamma |\nabla K(a)|)\alpha \operatorname{sign}(\chi_a)\lambda \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial \lambda} + (1 - \varphi(\lambda^\gamma |\nabla K(a)|))\alpha \frac{1}{\lambda} \frac{\partial P\widetilde{\delta}_{(a,\lambda)}}{\partial a} \frac{\nabla K(a)}{|\nabla K(a)|}.$$

We claim that $W_2$ satisfies (i) of Theorem 3.1. Indeed, using the expansion of Propositions 2.5 and 2.8, we have

https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

$$\begin{aligned}
&\langle \partial J(u), W_2(u)\rangle \\
&\quad = 2\alpha^2 J(u)\varphi(\lambda^\gamma|\nabla K(a)|)\,\mathrm{sign}(\chi_a) \\
&\qquad \times \begin{cases} -\dfrac{n-2s}{2}c_1\dfrac{H(a,a)}{\lambda^{n-2s}} & \text{if } n<2+2s \\ \dfrac{n-2s}{n}\dfrac{c_2}{K(a)}\dfrac{\Delta K(a)}{\lambda^2} & \text{if } n>2+2s \\ \dfrac{n-2s}{n}\dfrac{c_2}{K(a)}\dfrac{\Delta K(a)}{\lambda^2} - \dfrac{n-2s}{2}c_1\dfrac{H(a,a)}{\lambda^{n-2s}} & \text{if } n=2+2s \end{cases} \\
&\qquad + \varphi(\lambda^\gamma|\nabla K(a)|)\Big(o\Big(\frac{1}{\lambda^2} + \frac{1}{(\lambda d(a,\partial\Omega))^{n-2s}}\Big)\Big) \\
&\qquad - 2\alpha^2 J(u)(1-\varphi(\lambda^\gamma|\nabla K(a)|))\Big(c_3\frac{|\nabla K(a)|}{K(a)\lambda} + O\Big(\frac{1}{\lambda^2}\Big) + O\Big(\frac{1}{\lambda^{n+1-2s}}\Big)\Big).
\end{aligned}$$

Observe that

$$\lambda^\gamma|\nabla K(a)| \le \frac{2\eta_0}{\varepsilon^\gamma} \mapsto |\nabla K(a)| \le 2\eta_0 \quad \text{and} \quad \frac{|\nabla K(a)|}{\lambda} = O\Big(\frac{1}{\lambda^{\gamma+1}}\Big).$$

Therefore,

$$\begin{aligned}
&\langle \partial J(u), W_2(u)\rangle \\
&\quad \le -c\Big[\varphi(\lambda^\gamma|\nabla K(a)|)\Big(\frac{1}{\lambda^2} + \frac{1}{\lambda^{n-2s}} + \frac{|\nabla K(a)|}{\lambda}\Big) \\
&\qquad + (1-\varphi(\lambda^\gamma|\nabla K(a)|))\Big(\frac{|\nabla K(a)|}{\lambda} + O\Big(\frac{1}{\lambda^2}\Big) + O\Big(\frac{1}{\lambda^{n+1-2s}}\Big)\Big)\Big].
\end{aligned}$$

Observe now that if $\lambda^\gamma|\nabla K(a)| \ge \eta_0/\varepsilon^\gamma$, we have

$$\frac{1}{\lambda^2} = o\Big(\frac{|\nabla K(a)|}{\lambda}\Big) \quad \text{and} \quad \frac{1}{\lambda^{n+1-2s}} = o\Big(\frac{|\nabla K(a)|}{\lambda}\Big).$$

Indeed, if $\gamma = 1$, then $1/\lambda^{n+1-2s} = O(1/\lambda^2) = o(|\nabla K(a)|/\lambda)$ as $\varepsilon$ small. If $\gamma = n-2s$, then $1/\lambda^2 = O(1/\lambda^{n+1-2s}) = o(|\nabla K(a)|/\lambda)$ as $\varepsilon$ small. Therefore,

$$\langle \partial J(u), W_2(u)\rangle \le -c\Big(\frac{|\nabla K(a)|}{\lambda} + \frac{1}{\lambda^2} + \frac{1}{\lambda^{n+1-2s}}\Big).$$

The pseudo-gradient $W_2$ satisfies (i) of Theorem 3.1. By construction $\lambda(t)$ increases if and only if the concentration point $a(t)$ is near a critical point $y$ in $\mathcal{K}^+$.

The required vector field $W$ of Theorem 3.1 is defined by a convex combination of $W_1$ and $W_2$. It satisfies (i), (iii) and (iv) of Theorem 3.1. Inequality (ii) follows from (i) and the estimate of $|\bar{v}|$ given in Proposition 2.11. This completes the proof of Theorem 3.1. □

Corollary 3.2. *Under conditions (A) and (nd), the critical points at infinity of $J$ in $V(1,\varepsilon)$ are*

$$(y)_\infty := \frac{1}{K(y)^{(n-2s)/n}} P\widetilde{\delta_{(y,\infty)}},\ y \in \mathcal{K}^+.$$

*The Morse index of $(y)_\infty$ equals $n - \mathrm{ind}(K,y)$.*

https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

PROOF. The characterization of the critical points at infinity of $J$ in $V(1,\varepsilon)$ follows from Theorem 3.1. Concerning the Morse index of a critical point at infinity, the claim follows from the expansion of $J(\alpha P\widetilde{\delta}_{(a,\lambda)} + \bar{v})$ when $a$ approaches $y$ with $y \in \mathcal{K}^+$. Arguing as [4, Lemma 4.2], we can find a change of variables

$$(a,\lambda) \longmapsto (a',\lambda')$$

such that

$$J(\alpha P\widetilde{\delta}_{(a,\lambda)} + \bar{v}) = \frac{S}{K(a)^{(n-2s)/n}}\Big(1 + \frac{1}{\lambda^{\beta}}\Big),$$

where $\beta = 2$ if $n \geq 2 + 2s$ and $\beta = n - 2s$ if $n < 2 + 2s$. Therefore, the Morse index of $J$ at $(y)_\infty$ corresponds to the Morse index of $1/K(a)$ at $y$. This finishes the proof of Corollary 3.2. □

## 4. Proof of the existence theorems

**4.1. Proof of Theorems 1.1 and 1.2.** Let $\alpha_0 > 0$ such that $S^{2s/n} + \alpha_0 < (2S)^{2s/n}$. Using Corollary 2.4, there exists $\delta_0 > 0$ such that if $\|K - 1\|_{L^\infty(\bar{\Omega})} \leq \delta_0$, then the values of $J$ at all critical points at infinity in $V(1,\varepsilon)$ are below $S^{2s/n} + \alpha_0/4$ and the values of $J$ at the remaining critical points at infinity are above $S^{2s/n} + \alpha_0$.

For any critical point at infinity $(y)_\infty, y \in \mathcal{K}^+$, we denote by $W_u^\infty(y)_\infty$ the unstable manifold at infinity of $(y)_\infty$ with respect to the gradient vector field $(-\partial J)$. According to [2, pages 356–357] and [10, Lemma 10], $W_u^\infty(y)_\infty$ is identified by $W_s(y)$, where $W_s(y)$ is the classical stable manifold of the (true) critical point $y$ with respect to $(-\partial K)$. Therefore, $\dim W_u^\infty(y)_\infty = n - \operatorname{ind}(K,y)$. Let us set

$$M_{k_0}^\infty = \bigcup_{y \in \mathcal{K}^+, n - \operatorname{ind}(K,y) \leq k_0 - 1} W_u^\infty(y)_\infty.$$

$M_{k_0}^\infty$ defines a stratified set of top dimension less than $k_0 - 1$. Moreover,

$$M_{k_0}^\infty \subset J_{S^{2s/n} + \alpha_0/4} = \Big\{u \in \Sigma^+, J(u) \leq S^{2s/n} + \frac{\alpha_0}{4}\Big\}.$$

We introduce the following proposition.

PROPOSITION 4.1. *If $J$ has no critical point in $\Sigma^+$, then the set $M_{k_0}^\infty$ is contractible in $J_{S^{2s/n} + \alpha_0/4}$.*

PROOF OF PROPOSITION 4.1. Let

$$J_1(u) = \frac{|u|^2}{\big(\int_\Omega u^{2n/(n-2s)}\big)^{(n-2s)/n}}, \quad u \in \Sigma^+.$$

By [21] we know that

$$S^{2s/n} = \inf_{u \in^+} J_1(u).$$

Using the fact that

$$J(u) = J_1(u)(1 + O(\|K - 1\|_{L^\infty(\bar{\Omega})})),$$

https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

we derive, for $\|K-1\|_{L^\infty(\bar{\Omega})}$ small enough,

$$J_{S^{2s/n}+\alpha_0/4} \subset J_{1\,S^{2s/n}+\alpha_0/2} \subset J_{S^{2s/n}+3\alpha_0/4}.$$

Using the fact that $J$ has no critical point in $\Sigma^+$ and no critical value at infinity between $S^{2s/n}+\alpha_0/4$ and $S^{2s/n}+3\alpha_0/4$, we get

$$J_{S^{2s/n}+3\alpha_0/4} \simeq J_{S^{2s/n}+\alpha_0/4} \quad \text{and therefore } J_{1\,S^{2s/n}+\alpha_0/2} \simeq J_{S^{2s/n}+\alpha_0/4}. \tag{4-1}$$

Here $\simeq$ denotes retraction by deformation. The following lemma shows that $J_{1\,S^{2s/n}+\alpha_0/2}$ and $\Omega$ are topologically the same.

Lemma 4.2. *For $\alpha_0>0$ small enough, $J_{1\,S^{2s/n}+\alpha_0/2}$ is homotopy equivalent to $\Omega$.*

Proof. For any $\lambda>0$ we define

$$\begin{aligned} f_\lambda : \Omega &\longrightarrow \Sigma^+ \\ a &\longmapsto \frac{P\widetilde{\delta}_{(a,\lambda)}}{|P\widetilde{\delta}_{(a,\lambda)}|}. \end{aligned}$$

Since $\lim_{\lambda\to+\infty} J_1(P\widetilde{\delta}_{(a,\lambda)}/|P\widetilde{\delta}_{(a,\lambda)}|) = S^{2s/n}$, uniformly with respect to $a\in\Omega$, then for $\lambda_0$ large enough, $f_{\lambda_0}$ maps $\Omega$ into $J_{1\,S^{2s/n}+\alpha_0/2}$.

In order to construct a continuous map $r : J_{1\,S^{2s/n}+\alpha_0/2} \longrightarrow \Omega$ satisfying $r\circ f_{\lambda_0}\sim \mathrm{id}_\Omega$, we first state the following claim:

$$J_{1\,S^{2s/n}+\alpha_0/2} \subset V(1,\varepsilon) \quad \text{for } \alpha_0 \text{ small enough.} \tag{4-2}$$

Indeed, if not, there exists a sequence $(w_k)_k$ in $\Sigma^+$ such that $\lim_{k\to+\infty} J_1(w_k) = S^{2s/n}$ and $w_k\notin V(1,\varepsilon)$, for all $k$. Since $(w_k)$ is a minimizing sequence, it satisfies $\lim_{k\to+\infty}\partial J_1(w_k)=0$. Using these facts and the result of Proposition 2.1, we derive that $(w_k)_k$ converges in $\Sigma^+$ and therefore the infimum of $J_1$ on $\Sigma^+$ is attained. This is a contradiction with the result of [21], where it is proved that the infimum of $J_1$ is never achieved in the bounded domain cases. Hence (4-2) is valid.

Now, from the result of Proposition 2.2, we know that for any $u\in V(1,\varepsilon)$, the minimization problem

$$\min\{|u-\alpha P\widetilde{\delta}_{(a,\lambda)}|, \alpha>0, a\in\Omega, \lambda>0\}$$

has a unique solution $(\bar{\alpha}(u),\bar{a}(u),\bar{\lambda}(u))$ up to permutation. Let

$$\begin{aligned} \bar{a} : V(1,\varepsilon) &\longrightarrow \Omega \\ u &\longmapsto \bar{a}(u) \end{aligned}$$

and let $r=\bar{a}\circ\imath$ where $\imath : J_{1\,S^{2s/n}+\alpha_0/2} \hookrightarrow V(1,\varepsilon)$ is the natural injection. Clearly we have $r\circ f_{\lambda_0}=\mathrm{id}_\Omega$. The result of Lemma 4.2 follows. □

Using Lemma 4.2 and (4-1), the result of Proposition 4.1 follows, since by assumption of Theorems 1.1 and 1.2, $\Omega$ is a contractible domain. □

https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

Without loss of generality, we may assume that $k_0 - 1 = \dim M_{k_0}^\infty$. Let $C(M_{k_0}^\infty)$ be a contraction of dimension $k_0$ of $M_{k_0}^\infty$ in $J_{S^{2s/n}+\alpha_0/4}$. We are arguing by contradiction and supposition that $J$ has no critical points in $\Sigma^+$. Using the deformation lemma of [3] and a dimension argument, we have

$$C(M_{k_0}^\infty) \simeq \bigcup_{y\in\mathcal{K}^+, n-\mathrm{ind}(K,y)\le k_0} W_u^\infty(y)_\infty.$$

Therefore by assumption ($a$) of Theorem 1.1, we obtain

$$C(M_{k_0}^\infty) \simeq M_{k_0}^\infty.$$

Thus, by applying an Euler–Poincaré characteristic argument, we get

$$1 = \sum_{y\in\mathcal{K}^+, n-\mathrm{ind}(K,y)\le k_0-1} (-1)^{n-\mathrm{ind}(K,y)}.$$

This contradicts assumption ($b$) of Theorem 1.1. The proof of Theorems 1.1 and 1.2 follows.

**4.2. Proof of Theorem 1.3.** Assume that $J$ has no critical points in $\Sigma^+$. Using (4-1), we have

$$\chi(J_{S^{2s/n}+\alpha_0/4}) = \chi(J_{1S^{2s/n}+\alpha_0/2}).$$

Since

$$J_{S^{2s/n}+\alpha_0/4} \simeq \bigcup_{y\in\mathcal{K}^+} W_u^\infty(y)_\infty,$$

we get

$$\chi(J_{1S^{2s/n}+\alpha_0/2}) = \sum_{y\in\mathcal{K}^+} (-1)^{n-\mathrm{ind}(K,y)}.$$

The result of Lemma 4.2 completes the proof of Theorem 1.3.

## References


[1] A. Bahri, *Critical Points at Infinity in Some Variational Problems*, Pitman Research Notes in Mathematics Series, 182 (Longman Scientific & Technical, Harlow, 1989).

[2] A. Bahri, ‘An invariant for Yamabe-type flows with applications to scalar curvature problems in high dimensions’, *Duke Math. J.* **81** (1996), 323–466.

[3] A. Bahri and P. Rabinowitz, ‘Periodic orbits of hamiltonian systems of three body type’, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **8** (1991), 561–649.

[4] R. Ben Mahmoud and H. Chtioui, ‘Prescribing the scalar curvature problem on higher-dimensional manifolds’, *Discrete Contin. Dyn. Syst.* **32**(5) (2012), 1857–1879.

[5] C. Brändle, E. Colorado, A. de Pablo and U. Sánchez, ‘A concave-convex elliptic problem involving the fractional Laplacian’, *Proc. R. Soc. Edinburgh Sect. A* **143** (2013), 39–71.

[6] A. Chang and M. Gonzalez, ‘Fractional Laplacian in conformal geometry’, *Adv. Math.* **226**(2) (2011), 1410–1432.

[7] H. Chtioui and W. Abdelhedi, ‘On a fractional Nirenberg problem on $n$-dimensional spheres: existence and multiplicity results’, *Bull. Sci. Math.* **140**(6) (2016), 617–628.


https://doi.org/10.1017/S144678871900048X Published online by Cambridge University Press

[8] H. Chtioui, W. Abdelhedi and H. Hajaiej, ‘A complete study of the lack of compactness and existence results of a fractional Nirenberg equation via a flatness hypothesis: Part I’, *Anal. PDE* **9**(6) (2016), 1285–1315.
[9] H. Chtioui, W. Abdelhedi and H. Hajaiej, ‘The Bahri–Coron theorem for fractional Yamabe-type problems’, *Adv. Nonlinear Stud.* **18**(2) (2018), 393–407.
[10] J. M. Coron and A. Bahri, ‘On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of topology of the domain’, *Comm. Pure Appl. Math.* **41** (1988), 255–294.
[11] Y. Li, T. Jin and J. Xiong, ‘On a fractional Nirenberg problem, part I: Blow up analysis and compactness of solutions’, *J. Eur. Math. Soc. (JEMS)* **16** (2014), 1111–1171.
[12] Y. Li, T. Jin and J. Xiong, ‘On a fractional Nirenberg problem, part II: Existence of solutions’, *Int. Math. Res. Not. IMRN (6)* **2015** (2015), 1555–1589.
[13] Y. Li, T. Jin and J. Xiong, ‘The Nirenberg problem and its generalizations: a unified approach’, *Math. Ann.* **369**(1–2) (2017), 109–151.
[14] J. Palatucci, E. Di Nezza and E. Valdinoci, ‘Hitchhiker’s guide to the fractional Sobolev spaces’, *Bull. Sci. Math.* **136** (2012), 521–573.
[15] K. Sharaf, ‘An infinite number of solutions for an elliptic problem with power nonlinearity’, *Differential Integral Equations* **30** (2017), 133–144.
[16] L. Silvestre and L. Caffarelli, ‘An extension problem related to the fractional Laplacian’, *Comm. Partial Differential Equations* **32** (2007), 1245–1260.
[17] M. Struwe, ‘A global compactness result for elliptic boundary value problem involving limiting nonlinearities’, *Math. Z.* **187** (1984), 511–517.
[18] J. Tan, ‘Positive solutions of nonlinear problems involving the square root of the Laplacian’, *Adv. Math.* **224** (2010), 2052–2093.
[19] J. Tan, ‘The Brezis–Nirenberg type problem involving the square root of the Laplacian’, *Calc. Var. Partial Differential Equations* **42** (2011), 21–41.
[20] J. Tan, ‘Positive solutions for non local elliptic problems’, *Discrete Contin. Dyn. Syst.* **33** (2013), 837–859.
[21] N. Tavoularis and A. Cotsiolis, ‘Best constants for Sobolev inequalities for higher order fractional derivatives’, *J. Math. Anal. Appl.* **295** (2004), 225–236.
[22] J. Torrea and P. Stinga, ‘Extension problem and Harnack’s inequality for some fractional operators’, *Comm. Partial Differential Equations* **35** (2010), 2092–2122.
[23] Y. Zheng, Y. Chen and C. Liu, ‘Existence results for the fractional Nirenberg problem’, *J. Funct. Anal.* **270**(11) (2016), 4043–4086.

AZEB ALGHANEMI, Department of Mathematics,
King Abdulaziz University, P.O. Box 80230, Jeddah, Kingdom of Saudi Arabia
e-mail: aalghanemi@kau.edu.sa

HICHEM CHTIOUI, Department of Mathematics,
Sfax University, Faculty of Sciences of Sfax, 3018 Sfax, Tunisia
e-mail: Hichem.Chtioui@fss.rnu.tn