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CONFORMALLY FLAT SPACES OF CODIMENSION 2 IN A EUCLIDEAN SPACE

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1. Introduction. In a previous paper [**1**], the authors introduced and studied the notion of special conformally flat spaces and quasi-umbilical hypersurfaces. In that paper, the authors proved that every conformally flat space of codimension one in a Euclidean space is special and, conversely, every special conformally flat space can be isometrically immersed in a Euclidean space as a quasi-umbilical hypersurface.

In the present paper, the authors study the conformally flat spaces of codimension 2 in a Euclidean space. (Manifolds, mappings, functions, etc. are assumed to be sufficiently differentiable and we shall restrict ourselves only to manifolds of dimension $n > 3$.)

2. Preliminaries. We consider an n-dimensional submanifold V_n of an $(n + 2)$-dimensional Euclidean space E_{n+2} and represent it by

$$X = X(\xi^1, \xi^2, \ldots, \xi^n), \tag{2.1}$$

where X is the position vector from the origin of E_{n+2} to a point of V_n and $\{\xi^h\}$ is a local coordinate system of V_n, where here and in the sequel the indices $h, i, j, k \ldots$ run over the range $\{1, 2, \ldots, n\}$.

We put

$$X_i = \partial_i X, \quad (\partial_i = \partial/\partial\xi^i) \tag{2.2}$$

then the components of the fundamental metric tensor of V_n are given by

$$g_{ji} = X_j \cdot X_i \tag{2.3}$$

the dot denoting the inner product in E_{n+2}.

Let C and D be two mutually orthogonal unit normal vectors of V_n in E_{n+2} and let ∇_j denote the operator of covariant differentiation along V_n with respect to Levi-Civita connection. Then equations of Gauss and Weingarten are respectively written as

$$\nabla_j X_i = h_{ji}C + k_{ji}D \tag{2.4}$$

and

$$\nabla_j C = -h_j{}^i X_i + l_j D,$$
$$\nabla_j D = -k_j{}^i X_i - l_j C,$$

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where h_{ji} and k_{ji} are the second fundamental tensors with respect to C and D respectively and l_j the third fundamental tensor, $h_j{}^i$ and $k_j{}^i$ being defined by $h_j{}^i = h_{jt}g^{ti}$ and $k_j{}^i = k_{jt}g^{ti}$ respectively. The mean curvature vector is then given by

$$H = \frac{1}{n} g^{ji}\nabla_j X_i. \tag{2.6}$$

If there exists, on the submanifold V_n, two functions α, β and a unit vector field u_i such that

$$h_{ji} = \alpha g_{ji} + \beta u_j u_i, \tag{2.7}$$

then V_n is said to be *quasi-umbilical* with respect to C. In particular if $\beta = 0$ identically, then V_n is said to be umbilical with respect to C. If V_n is umbilical with respect to the mean curvature vector H, then V_n is said to be *pseudo-umbilical* in E_{n+1}.

The equations of Gauss for V_n are given by

$$R_{kji}{}^h = h_k{}^h h_{ji} - h_j{}^h h_{ki} + k_k{}^h k_{ji} - k_j{}^h k_{ki}, \tag{2.8}$$

where $R_{kji}{}^h$ is the Riemann-Christoffel curvature tensor of V_n. Denoting by $R_{ji} = R_{tji}{}^t$ and $R = g^{ji}R_{ji}$ the Ricci tensor and the scalar curvature respectively, we define a tensor field L_{ji} of type (0, 2) by

$$L_{ji} = -\frac{R_{ji}}{n-2} + \frac{Rg_{ji}}{2(n-1)(n-2)}. \tag{2.9}$$

The conformal curvature tensor $C_{kji}{}^h$ is then given by

$$C_{kji}{}^h = R_{kji}{}^h + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k{}^h g_{ji} - L_j{}^h g_{ki}, \tag{2.10}$$

where δ_h^h is the Kronecker delta and $L_k{}^h = L_{kt}g^{th}$.

A Riemannian manifold V_n is called a conformally flat space if $C_{kji}{}^h = 0$.

3. Conformally flat spaces of codimension 2. Let $\mathfrak{S}_n$ be the set of all real symmetric square matrices of order n and p be a natural number. We put

$$A^{(p)} = \text{Trace}\,(A^p), \quad A \in \mathfrak{S}_n, \tag{3.1}$$

where A^p denotes the product of A with itself p times. For any $A \in \mathfrak{S}_n$, let $x_1, x_2, \ldots, x_n$ denote eigenvalues of A. We define a real function σ on $\mathfrak{S}_n$ by

$$\sigma(A) = \sum (x_k - x_j)^2(x_i - x_h)^2, \tag{3.2}$$

where the summation is taken over all distinct k, j, i, h. We notice here that if $\sigma(A) = 0$, then at least $n - 1$ of x's are equal.

LEMMA 1. *For any $A \in \mathfrak{S}_n (n > 3)$, we have*

$$\tfrac{1}{2}\sigma(A) = -n(n-1)A^{(4)} + 4(n-1)A^{(1)}A^{(3)} + (n^2 - 3n + 3)\,(A^{(2)})^2 - 2n(A^{(1)})^2A^{(2)} + (A^{(1)})^4. \tag{3.3}$$

Proof. By a direct computation, we find that both sides of (3.3) are equal to

$$2(n-2)(n-3)\sum_{k<j} x_k^2 x_j^2 - 4(n-3)\sum_{i\neq k,j}\sum_{k<j} x_k x_j x_i^2 + 24\sum_{k<j<i<h} x_k x_j x_i x_h.$$

This proves the lemma.

THEOREM 1. *Let V_n be a conformally flat space of codimension 2 in a Euclidean $(n+2)$-space E_{n+2}. Then we have*

$$\sigma(H) = \sigma(K), \tag{3.4}$$

where $H = (h_i{}^h)$ and $K = (k_i{}^h)$.

Proof. Since the ambient space is Euclidean, the Riemann-Christoffel curvature tensor of V_n is given by (2.8); we have

$$R_{ji} = H^{(1)}h_{ji} - h_{jt}h_i{}^t + K^{(1)}k_{ji} - k_{jt}k_i{}^t, \tag{3.5}$$

$$R = (H^{(1)})^2 - H^{(2)} + (K^{(1)})^2 - K^{(2)} \tag{3.6}$$

and

$$L_{ji} = -\frac{H^{(1)}h_{ji} - h_{jt}h_i{}^t + K^{(1)}k_{ji} - k_{jt}k_i{}^t}{n-2} + \frac{Rg_{ji}}{2(n-1)(n-2)}. \tag{3.7}$$

Substituting (2.8) and (3.7) into (2.10) and transvecting the equation obtained with $h_h{}^k h^{ji}$ and $k_h{}^k k^{ji}$ respectively, we obtain

$$\begin{aligned} h_h{}^k h^{ji} C_{kji}{}^h = \frac{1}{(n-1)(n-2)} [\{&-n(n-1)H^{(4)} + 4(n-1)H^{(1)}H^{(3)} \\ &+ (n^2-3n+3)(H^{(2)})^2 - 2n(H^{(1)})^2 H^{(2)} + (H^{(1)})^4\} \\ &+ \{(n-1)(n-2)((HK)^{(1)})^2 - n(n-1)(HKHK)^{(1)} \\ &- 2(n-1)H^{(1)}K^{(1)}(HK)^{(1)} + 2(n-1)H^{(1)}(HKK)^{(1)} \\ &+ 2(n-1)K^{(1)}(HHK)^{(1)} + (H^{(1)})^2(K^{(1)})^2 - (H^{(1)})^2 K^{(2)} \\ &- H^{(2)}(K^{(1)})^2 + H^{(2)}K^{(2)}\}] \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} k_h{}^k k^{ji} C_{kji}{}^h = \frac{1}{(n-1)(n-2)} [\{&-n(n-1)K^{(4)} + 4(n-1)K^{(1)}K^{(3)} \\ &+ (n^2-3n+3)(K^{(2)})^2 - 2n(K^{(1)})^2 K^{(2)} + (K^{(1)})^4\} \\ &+ \{(n-1)(n-2)((KH)^{(1)})^2 - n(n-1)(KHKH)^{(1)} \\ &- 2(n-1)K^{(1)}H^{(1)}(KH)^{(1)} + 2(n-1)K^{(1)}(KHH)^{(1)} \\ &+ 2(n-1)H^{(1)}(KKH)^{(1)} + (K^{(1)})^2(H^{(1)})^2 - (K^{(1)})^2 H^{(2)} \\ &- K^{(2)}(H^{(1)})^2 + K^{(2)}H^{(2)}\}]. \end{aligned} \tag{3.9}$$

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Thus by Lemma 1 and the facts that

$$(HK)^{(1)} = (KH)^{(1)}, \quad (HKHK)^{(1)} = (KHKH)^{(1)},$$
$$(HKK)^{(1)} = (KKH)^{(1)}, \quad (HHK)^{(1)} = (KHH)^{(1)},$$

we obtain

$$(3.10) \qquad (h_h{}^k h^{ji} - k_h{}^k k^{ji}) C_{kji}{}^h = \frac{\sigma(H) - \sigma(K)}{2(n-1)(n-2)}.$$

This completes the proof of the theorem.

4. Applications and remarks. From Theorem 1, we have the following applications.

THEOREM 2. *Let V_n be a conformally flat space of codimension 2 in a Euclidean $(n + 2)$-space E_{n+2}. If V_n is quasi-umbilical with respect to one normal direction C, then it must be also quasi-umbilical with respect to another normal direction D.*

Proof. If V_n is quasi-umbilical with respect to C, then the second fundamental form h_{ji} with respect to the normal direction C is given in the form (2.7). Therefore the matrix $H = (h_i{}^h)$ has only two distinct eigenvalues with multiplicity $n - 1$ and 1 or n and 0. Thus we see that $\sigma(H) = 0$. Thus, from Theorem 1, we see that $\sigma(K) = 0$. This implies that the eigenvalues of $K = (k_i{}^h)$ are given in the following form

$$x, x, \ldots, x, y \quad (x\ (n-1)\text{-times}).$$

This implies that V_n is quasi-umbilical with respect to D.

From Theorem 2, we have immediately the following

COROLLARY. *Let V_n be a conformally flat space of codimension 2 in E_{n+2}. If V_n is umbilical with respect to one normal direction C, then V_n is quasi-umbilical with respect to another normal direction D.*

Remark 1. If V_n is a pseudo-umbilical submanifold of codimension 2 of E_{n+2} and *if the length of the mean curvature vector H is nowhere constant*, then V_n is automatically conformally flat (cf. [**2**]).

Remark 2. The concept of "being quasi-umbilical (or umbilical) with respect to one normal direction" is invariant under the conformal change of metric of the ambient space. But since the mean curvature vector is not invariant, the concept of "being pseudo-umbilical" is not invariant under the conformal change of metric.

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