

SINGULAR HERMITIAN METRICS WITH ISOLATED SINGULARITIES

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Abstract. In this paper, we study the coherence of a higher rank analogue of a multiplier ideal sheaf. Key tools of the study are Hörmander’s L^2 -estimate and a singular version of a Demailly–Skoda-type result.

§1. Introduction

Multiplier ideal sheaves for singular Hermitian metrics on line bundles are very important and have many applications in many fields. Indeed, Nadel introduced a notion of multiplier ideal sheaves and applied it to show the existence of Kähler–Einstein metrics of positive scalar curvature on certain compact complex manifolds in [19]. As another example, Siu established the invariance of plurigenera by using this notion [23]. Nadel also proved a cohomology vanishing theorem with coefficients in multiplier ideal sheaves, which is one generalization of the Kawamata–Viehweg vanishing theorem. This vanishing theorem has been applied in algebraic and complex geometry.

In his paper, Nadel showed the following celebrated result: a multiplier ideal sheaf associated with a plurisubharmonic function is coherent. An important technique of the proof is an L^2 -estimate of the $\bar{\partial}$ -equation of Hörmander type [9].

A higher rank analogue of a multiplier ideal sheaf associated with a singular Hermitian metric on a vector bundle has been also studied. This notion has been recognized as important. However, the coherence of it is only known in few cases (cf. [4, Prop. 4.1.3], [10, Th. 1.1]). If a metric has some strong positivity like Nakano positivity, it is known that the higher rank analogue of the multiplier ideal sheaf is coherent ([11, Th. 1.4], [14, Prop. 4.4], [15, Th. 1.2]). Hence, it is natural to ask whether the higher rank analogue of the multiplier ideal sheaf is coherent if the associated metric has only Griffiths positivity, which is a strictly weaker notion than Nakano positivity.

CONJECTURE 1.1. *Let (E, h) be a holomorphic vector bundle over an n -dimensional complex manifold X with a singular Hermitian metric h . If h is Griffiths semipositive, the higher rank analogue of the multiplier ideal sheaf $\mathcal{E}(h)$ is coherent.*

This conjecture seems a tough problem due to the following reasons. First, we cannot apply an L^2 -estimate of the $\bar{\partial}$ -equation directly even if h is smooth. That is because a kind of the solvability of the $\bar{\partial}$ -equation with L^2 -estimates in the optimal setting is equivalent to the Nakano positivity of the metric [8, Th. 1.1]. Second, it might be useful to get an approximating sequence $\{h_\nu\}$ of h such that the Chern curvature of h_ν is uniformly bounded below in the sense of Nakano. However, it is known that this procedure cannot be done simply by using the standard approximation defined by convolution [10, Th. 1.2].

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In order to use the technique of Hörmander's L^2 -estimate, we need to devise some settings. By imposing conditions on the singularity of h , we can give a partial answer to the conjecture.

THEOREM 1.2. *Let (E, h) be a holomorphic vector bundle over an n -dimensional complex manifold X with a Griffiths semipositive singular Hermitian metric h . If the unbounded locus $L(\det h)$ of $\det h$ is discrete, the higher rank analogue of the multiplier ideal sheaf $\mathcal{S}^m \mathcal{E}(S^m h)$ is coherent for every $m \in \mathbb{N}$. Here, $\mathcal{S}^m \mathcal{E}(S^m h)$ is the sheaf of germs of local holomorphic sections of the m th symmetric power $S^m E$ of E , which is square integrable with respect to $S^m h$ (see Definition 2.5 for the precise definition).*

The condition on h in the main theorem appears naturally when h has some kind of symmetry. Indeed, as an application of the main theorem, we get the following result.

COROLLARY 1.3. *Let E be the trivial vector bundle $E = \mathbb{B}^n \times \mathbb{C}^r$ over the unit ball $\mathbb{B}^n := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z|^2 < 1\}$, and let h be a Griffiths seminegative singular Hermitian metric on E . If $\det h$ is a radial function, that is, depending only on the radius $|z|$, $\mathcal{S}^m \mathcal{E}^*(S^m h^*)$ is coherent for every $m \in \mathbb{N}$. Especially, if h is spherically symmetric (see Definition 2.6), $\mathcal{S}^m \mathcal{E}^*(S^m h^*)$ is coherent for every $m \in \mathbb{N}$.*

§2. Preliminaries

In this section, we give properties of singular Hermitian metrics on holomorphic vector bundles. First, let us recall positivity notions for smooth Hermitian vector bundles. Let (E, h) be a smooth Hermitian vector bundle over a complex manifold X . We denote by $\Theta_{E, h}$ the Chern curvature of (E, h) and by $\tilde{\Theta}_{E, h}$ the associated Hermitian form on $T_X \otimes E$. Then (E, h) is said to be *Nakano positive* if $\tilde{\Theta}_{E, h}(\tau, \tau) > 0$ for all nonzero elements $\tau \in T_X \otimes E$. If $\tilde{\Theta}_{E, h}(v \otimes s, v \otimes s) > 0$ for all nonzero elements $v \in T_X$ and $s \in E$, (E, h) is said to be *Griffiths positive*. Corresponding negativity is defined similarly. It is clear that Nakano positivity is a stronger positivity notion than Griffiths positivity. It is also known that these notions do not coincide [6, Chap. VII, Exam. 8.4]. For Hermitian forms A and B on $T_X \otimes E$, we write $A \geq_{\text{Nak.}} B$ (resp., $A \geq_{\text{Grif.}} B$) if $A(\tau, \tau) \geq B(\tau, \tau)$ (resp., $A(v \otimes s, v \otimes s) \geq B(v \otimes s, v \otimes s)$) for any $\tau \in T_X \otimes E$ (resp., $v \in T_X$ and $s \in E$).

Next, we show positivity notions for singular Hermitian metrics. For the definition of singular Hermitian metrics, see [3, §3] or [21, Def. 1.1].

DEFINITION 2.1 ([3, Defs. 3.1 and 3.2], [21, §2]). Let (E, h) be a *singular* Hermitian bundle over a complex manifold X . Then (E, h) is said to be:

1. *Griffiths seminegative* if $\log |s|_h^2$ (or $|s|_h^2$) is plurisubharmonic for any local holomorphic section s of E .
2. *Griffiths semipositive* if the dual metric h^* on E^* is *Griffiths seminegative*.

Then we introduce the definition of unbounded loci. Let φ be a plurisubharmonic function on a complex manifold X . It is known that the *unbounded locus* of φ is defined to be the set of points $x \in X$ such that φ is unbounded on every neighborhood of x . We denote it by $L(\varphi)$. For a general singular Hermitian metric on a line bundle, the unbounded locus is defined similarly.

DEFINITION 2.2 (cf. [17, Def. 3.3]). Let (L, g) be a singular Hermitian line bundle over a complex manifold X . Suppose that g is seminegative. The *unbounded locus* $L(g)$ of g is

defined as follows: $x \in X$ is in $L(g)$ if and only if for an open coordinate U_α of x which trivializes $(L|_{U_\alpha} \cong \mathbb{C}, g|_{U_\alpha} = e^{\varphi_\alpha})$, $x \in L(\varphi_\alpha)$.

Similarly, if g is semipositive, the *unbounded locus* $L(g)$ of g is defined by $L(g) := L(g^*)$.

Note that if (L, g) has seminegative curvature, the above φ_α is a plurisubharmonic function. Thus, $L(\varphi_\alpha)$ is well defined. For another trivialization U_β , we have that $\varphi_\alpha = \log |g_{\alpha\beta}|^2 + \varphi_\beta$, where $g_{\alpha\beta}$ is a transition function of L on $U_\alpha \cap U_\beta$. Since $\log |g_{\alpha\beta}|^2$ is locally bounded, for any point $x \in U_\alpha \cap U_\beta$, $x \in L(\varphi_\alpha)$ if and only if $x \in L(\varphi_\beta)$. Hence, the unbounded locus of g can be defined globally.

We have another description of $L(g)$. We take an open trivializing covering $\{U_\alpha\}_{\alpha \in \Lambda}$ of X and set $h|_{U_\alpha} =: e^{\varphi_\alpha}$. Then $L(g) = \cup_{\alpha \in \Lambda} L(\varphi_\alpha)$. This definition is independent of the choice of open coverings for the same reason as above. Set $B(\varphi_\alpha) := U_\alpha \setminus L(\varphi_\alpha)$, where φ_α is locally bounded, and $B(g) = \cup_{\alpha \in \Lambda} B(\varphi_\alpha)$. We clearly see that $L(g) \cap B(g) = \emptyset$. Indeed, if there is an element $x \in L(g) \cap B(g)$, there exist $\alpha \in \Lambda$ and $\beta \in \Lambda$ such that $x \in L(\varphi_\alpha)$ and $x \in B(\varphi_\beta)$. Note that $x \in U_\alpha \cap U_\beta$. The above argument implies that $x \in L(\varphi_\beta)$ as well, but this contradicts the fact that $L(\varphi_\beta) \cap B(\varphi_\beta) = \emptyset$. Hence, $X = L(g) \sqcup B(g)$. It is clear that $B(g)$ is an open subset. Therefore, we see that $L(g)$ is a closed subset. If E is a vector bundle with a Griffiths semipositive singular Hermitian metric h , $(\det E, \det h)$ is semipositive as well. Thus, we can define the unbounded locus $L(\det h)$ as in Definition 2.2.

Next, we introduce an L^2 -estimate of Hörmander type. In this paper, we use the following form.

THEOREM 2.3 ([5], [6, Chap. VIII, Th. 6.1]). *Let $(X, \widehat{\omega})$ be a complete Kähler manifold, let ω be another Kähler metric which is not necessarily complete, and let $(E, h) \rightarrow X$ be a Nakano semipositive vector bundle. Then, for any $\bar{\partial}$ -closed E -valued (n, q) -form u with $q > 0$ and $\int_X \langle [\sqrt{-1}\Theta_{E,h}, \Lambda_\omega]^{-1}u, u \rangle dV_\omega < +\infty$, there exists a solution of $\bar{\partial}\alpha = u$ satisfying*

$$\int_X |\alpha|_{\omega,h}^2 dV_\omega \leq \int_X \langle [\sqrt{-1}\Theta_{E,h}, \Lambda_\omega]^{-1}u, u \rangle_{\omega,h} dV_\omega,$$

where $\langle \cdot, \cdot \rangle_{\omega,h}$ denotes the pointwise metric with respect to ω and h , $[\cdot, \cdot]$ denotes the graded Lie bracket, and $dV_\omega = \omega^n/n!$.

Then we recall the following result, which clarifies the relationship between Griffiths positivity and Nakano positivity.

THEOREM 2.4 ([7], [1], [18]). *Let (E, h) be a smooth Hermitian vector bundle. If (E, h) is Griffiths semipositive, $(S^m E \otimes \det E, S^m h \otimes \det h)$ is Nakano semipositive for every $m \in \mathbb{N}$, where $S^m E$ is the m th symmetric power of E .*

In this paper, we call this theorem a *Demailly–Skoda-type result* since this type of theorem was initially found by Demailly and Skoda [7]. This result plays a crucial role in the article. Then we introduce a notion of a higher rank analogue of a multiplier ideal sheaf.

DEFINITION 2.5 ([4, Def. 2.3.1]). Let (E, h) be a singular Hermitian vector bundle over a complex manifold X . Then we define *the higher rank analogue of the multiplier ideal sheaf* $\mathcal{E}(h)$ by

$$\mathcal{E}(h)_x = \{s \in \mathcal{O}_X(E)_x \mid |s|_h^2 \text{ is locally integrable around } x \in X\},$$

where $\mathcal{O}_X(E)$ is the locally free sheaf associated to E . Similarly, we define $\mathcal{S}^m \mathcal{E}(S^m h)$ by

$$\mathcal{S}^m \mathcal{E}(S^m h)_x = \{s \in \mathcal{O}_X(S^m E)_x \mid |s|_{S^m h}^2 \text{ is locally integrable around } x \in X\}.$$

At the last of this section, we introduce a notion of spherically symmetric singular Hermitian metrics on the trivial vector bundle over the unit ball. This is one generalization of S^1 -invariant singular Hermitian metrics over the disk, which was introduced by Berndtsson [2].

DEFINITION 2.6. Let E be the trivial vector bundle $E = \mathbb{B}^n \times \mathbb{C}^r$ over the unit ball \mathbb{B}^n , and let h be a Griffiths seminegative singular Hermitian metric. We say that h is *spherically symmetric* if for any $u \in \mathbb{C}^r$, $|u|_{h(z)} = |u|_{h(|z|)}$ for $z \in \Delta^n$. In other words, $|u|_{h(z)}$ is a radial function for any $u \in V$.

Berndtsson studied Lelong numbers and integrability indices for S^1 -invariant singular metrics and obtained many applications in [2]. In this sense, a symmetric singular Hermitian metric is an important notion. We also have another generalization such as \mathbf{T}^n -invariant or S^1 -invariant singular Hermitian metrics, where \mathbf{T}^n is the unit torus in \mathbb{C}^n . However, since they do not fit the setting of the main theorem, we do not consider them in this article.

§3. L^2 -estimates for singular Hermitian metrics

In this section, we show an L^2 -estimate for singular Hermitian metrics on holomorphic vector bundles and give a proof of the main theorem. First, we prove the following result.

THEOREM 3.1. *Let (M, ω) be a Stein manifold with a Kähler metric ω . We also let $(E = M \times \mathbb{C}^r, h)$ be the trivial holomorphic vector bundle with a Griffiths semipositive singular Hermitian metric h , and let ψ be a smooth strictly plurisubharmonic function with $\sqrt{-1} \partial \bar{\partial} \psi \geq \varepsilon \omega$ for a positive constant $\varepsilon > 0$. Then, for any $\bar{\partial}$ -closed $S^m E \otimes \det E$ -valued (n, q) -form u with finite L^2 -norm, there exists an $S^m E \otimes \det E$ -valued $(n, q-1)$ -form α such that $\bar{\partial} \alpha = u$ and*

$$\int_M |\alpha|_{\omega, S^m h \otimes \det h}^2 e^{-\psi} dV_\omega \leq \frac{1}{q\varepsilon} \int_M |u|_{\omega, S^m h \otimes \det h}^2 e^{-\psi} dV_\omega.$$

This type of result was obtained by Inayama (see [13, Th. 1.3] or [14, Th. 1.4]). Since the situation is a little bit different, we give a proof for the sake of completeness.

Proof of Theorem 3.1. Fix $m \in \mathbb{N}$. Since M is Stein, M can be embedded into \mathbb{C}^N for some $N > 0$. We may regard M as a closed submanifold in \mathbb{C}^N . Let $\iota : M \rightarrow \mathbb{C}^N$ be an inclusion map. Thanks to Siu's theorem [22], there exist an open neighborhood U of M in \mathbb{C}^N and a holomorphic retraction $p : U \rightarrow M$ such that $p \circ \iota = \text{id}_M$. Note that $(p^* E, p^* h)$ is the trivial vector bundle with a Griffiths semipositive metric $p^* h$ as well. From the results of [3, Prop. 3.1] and [21, Prop. 6.2], we obtain a sequence of smooth Hermitian metrics $\{g_\nu\}_{\nu=1}^\infty$ with Griffiths semipositive curvature increasing to $p^* h$ on any relatively compact subset in U . Take an exhaustion $\{M_j\}_{j=1}^\infty$ of M , where each M_j is a relatively compact Stein subdomain in M , $M_j \Subset M_{j+1}$, and $\cup M_j = M$. Set $\{h_\nu := \iota^* g_\nu\}_{\nu=1}^\infty$. Then $\{h_\nu\}_{\nu=1}^\infty$ is an approximate sequence with Griffiths semipositive curvature increasing to h on any relatively compact subset. Note that each h_ν is Griffiths semipositive. Due to Theorem 2.4, $S^m h_\nu \otimes \det h_\nu$ is Nakano semipositive. The Chern curvature of $S^m h_\nu \otimes \det h_\nu e^{-\psi}$ is

calculated as

$$\begin{aligned} \sqrt{-1}\Theta_{S^m h_\nu \otimes \det h_\nu} e^{-\psi} &= \sqrt{-1}\Theta_{S^m h_\nu \otimes \det h_\nu} + \sqrt{-1}\partial\bar{\partial}\psi \otimes \text{id}_{S^m E \otimes \det E} \\ &\geq_{\text{Nak.}} \varepsilon\omega \otimes \text{id}_{S^m E \otimes \det E}. \end{aligned}$$

We have that for any $S^m E \otimes \det E$ -valued (n, q) -form u ,

$$\langle [\sqrt{-1}\Theta_{S^m h_\nu \otimes \det h_\nu} e^{-\psi}, \Lambda_\omega]u, u \rangle_{\omega, S^m h_\nu \otimes \det h_\nu} e^{-\psi} \geq q\varepsilon |u|_{\omega, S^m h_\nu \otimes \det h_\nu}^2 e^{-\psi}.$$

Now, we fix M_j . By using Theorem 2.3 and [6, Chap. VIII, Rem. 4.8], we get a solution α_ν of the $\bar{\partial}$ -equation satisfying

$$\begin{aligned} \int_{M_j} |\alpha_\nu|_{\omega, S^m h_\nu \otimes \det h_\nu}^2 e^{-\psi} dV_\omega &\leq \frac{1}{q\varepsilon} \int_{M_j} |u|_{\omega, S^m h_\nu \otimes \det h_\nu}^2 e^{-\psi} dV_\omega \\ &\leq \frac{1}{q\varepsilon} \int_{M_j} |u|_{\omega, S^m h \otimes \det h}^2 e^{-\psi} dV_\omega \\ &\leq \frac{1}{q\varepsilon} \int_M |u|_{\omega, S^m h \otimes \det h}^2 e^{-\psi} dV_\omega < +\infty \end{aligned}$$

for sufficiently large ν . Here, we use the argument from [16, Lem. 2.2]. It holds that the metric on $S^m E$ induced from E is the same as the metric induced by an orthogonal projection from $E^{\otimes m}$. Hence, by the monotonicity of h_ν , it follows that $\det h_\nu \leq \det h_{\nu+1}$, $h_\nu^{\otimes m} \leq h_{\nu+1}^{\otimes m}$, and $S^m h_\nu \leq S^m h_{\nu+1}$ as a metric.

Fix sufficiently large ν_0 . We have that, for $\nu \geq \nu_0$,

$$\begin{aligned} \int_{M_j} |\alpha_\nu|_{\omega, S^m h_{\nu_0} \otimes \det h_{\nu_0}}^2 e^{-\psi} dV_\omega &\leq \int_{M_j} |\alpha_\nu|_{\omega, S^m h_\nu \otimes \det h_\nu}^2 e^{-\psi} dV_\omega < +\infty \\ &\leq \frac{1}{q\varepsilon} \int_M |u|_{\omega, S^m h \otimes \det h}^2 e^{-\psi} dV_\omega < +\infty. \end{aligned}$$

Then $\{\alpha_\nu\}_{\nu \geq \nu_0}$ forms a bounded sequence with respect to the norm $\int_{M_j} |\cdot|_{\omega, S^m h_{\nu_0} \otimes \det h_{\nu_0}}^2 e^{-\psi} dV_\omega$. We can get a weakly convergent subsequence $\{\alpha_{\nu_0, k}\}_k$. Thus, the weak limit α_j satisfies

$$\int_{M_j} |\alpha_j|_{\omega, S^m h_{\nu_0} \otimes \det h_{\nu_0}}^2 e^{-\psi} dV_\omega \leq \frac{1}{q\varepsilon} \int_M |u|_{\omega, S^m h \otimes \det h}^2 e^{-\psi} dV_\omega < +\infty.$$

Next, we fix $\nu_1 > \nu_0$. Repeating the above argument, we can choose a weakly convergent subsequence $\{\alpha_{\nu_1, k}\}_k \subset \{\alpha_{\nu_0, k}\}_k$ with respect to $\int_{M_j} |\cdot|_{\omega, S^m h_{\nu_1} \otimes \det h_{\nu_1}}^2 e^{-\psi} dV_\omega$. Then, by taking a sequence $\{\nu_n\}_n$ increasing to $+\infty$ and a diagonal sequence, we obtain a weakly convergent sequence $\{\alpha_{\nu_k, k}\}_k$ with respect to $\int_{M_j} |\cdot|_{\omega, S^m h_{\nu_\ell} \otimes \det h_{\nu_\ell}}^2 e^{-\psi} dV_\omega$ for all ℓ . Hence, α_j satisfies

$$\int_{M_j} |\alpha_j|_{\omega, S^m h \otimes \det h}^2 e^{-\psi} dV_\omega \leq \frac{1}{q\varepsilon} \int_M |u|_{\omega, S^m h \otimes \det h}^2 e^{-\psi} dV_\omega$$

thanks to the monotone convergence theorem. Since the right-hand side of the above inequality is independent of j , by using the exactly same argument, we can get an $S^m E \otimes \det E$ -valued $(n, q - 1)$ -form α satisfying $\bar{\partial}\alpha = u$ and

$$\int_M |\alpha|_{\omega, S^m h \otimes \det h}^2 e^{-\psi} dV_\omega \leq \frac{1}{q\varepsilon} \int_M |u|_{\omega, S^m h \otimes \det h}^2 e^{-\psi} dV_\omega,$$

which completes the proof. \square

We call this theorem a *singular version of a Demailly–Skoda-type result*. Indeed, $S^m h \otimes \det h$ behaves like a Nakano semipositive metric (cf. Theorem 2.4). Applying this estimate, we can prove the main theorem.

Proof of Theorem 1.2. We use the same notation as in the previous section and fix $m \in \mathbb{N}$. Since the coherence is a local property, we may assume that $X = \Delta_r^n$ is a polydisc in $\mathbb{C}^n_{\{z_1, \dots, z_n\}}$, $E = \Delta_r^n \times \mathbb{C}^r$, and $(E = \Delta_r^n \times \mathbb{C}^r, h)$ is defined over a larger polydisc $\Delta_{r'}^n$. Here, r and r' are positive constants satisfying $r < r'$, and $\Delta_r^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_i| < r \text{ for all } i \in \{1, \dots, n\}\}$. We also assume that $S^m E$ is trivial over $\Delta_{r'}^n$. We regard $\log \det h^*$ as a function on $\Delta_{r'}^n$, which is the local weight of $\det h^*$, that is, $\det h^*|_{\Delta_{r'}^n} = e^{\log \det h^*}$. We can also assume that, on this trivializing coordinate, $L(\log \det h^*) = o$ since the unbounded locus is the set of isolated points.

Let $H_{2, S^m h}^0(\Delta_r^n, S^m E)$ be the space of holomorphic sections s of $S^m E$ on Δ_r^n such that $\int_{\Delta_r^n} |s|_{S^m h}^2 d\lambda < +\infty$, where $d\lambda$ is the standard Lebesgue measure on \mathbb{C}^n . We consider the natural evaluation map $\text{ev} : H_{2, S^m h}^0(\Delta_r^n, S^m E) \otimes_{\mathbb{C}} \mathcal{O}_{\Delta_r^n} \rightarrow \mathcal{O}_{\Delta_r^n}(S^m E)$. We know that $\text{Im}(\text{ev}) =: \mathfrak{E}$ is coherent. We now prove that $\mathfrak{E}_x = \mathcal{S}^m \mathcal{E}(S^m h)_x$ for all $x \in \Delta_r^n$. Since $\mathfrak{E} \subset \mathcal{S}^m \mathcal{E}(S^m h)$, it is enough to show that $\mathcal{S}^m \mathcal{E}(S^m h)_x \subset \mathfrak{E}_x$.

Take an arbitrary element $f \in \mathcal{S}^m \mathcal{E}(S^m h)_x$. When $x \neq o$, we take a cutoff function θ around x . Here, θ is a smooth function with compact support such that $0 \leq \theta \leq 1$ and $\theta \equiv 1$ on an open neighborhood of x . We may assume that f is defined on a small open neighborhood $U \Subset \Delta_r^n \setminus \{o\}$ of x such that $\int_U |f|_{S^m h}^2 d\lambda < +\infty$ and $\text{supp}(\theta) \Subset U$. Hence, θf is defined globally. Set $u := \bar{\partial}(\theta f dz)$, where $dz = dz_1 \wedge \dots \wedge dz_n$. Note that

$$\int_{\Delta_r^n} |u|_{S^m h}^2 d\lambda = \int_U |\bar{\partial}\theta|^2 |f|_{S^m h}^2 d\lambda < +\infty.$$

Moreover, we have that

$$\int_{\Delta_r^n} |u|_{S^m h}^2 \det h e^{-(n+k) \log |z-x|^2 - |z|^2} d\lambda = \int_U |\bar{\partial}\theta|^2 |f|_{S^m h}^2 e^{-\log \det h^* - (n+k) \log |z-x|^2 - |z|^2} d\lambda < +\infty$$

for any $k \in \mathbb{N}$ since $\log \det h^*$ is bounded on U and $\bar{\partial}\theta$ is identically zero around x . Set $\eta_\delta := \log(|z-x|^2 + \delta^2)$ and $\eta = \log |z-x|^2$. We have that $\sqrt{-1} \partial \bar{\partial}((n+k)\eta_\delta + |z|^2) \geq \sqrt{-1} \sum_i dz_i \wedge d\bar{z}_i$. Applying Theorem 2.3, we get a solution α_δ of $\bar{\partial}(\alpha_\delta dz) = u$ satisfying

$$\begin{aligned} \int_{\Delta_r^n} |\alpha_\delta|_{S^m h}^2 \det h e^{-(n+k)\eta_\delta - |z|^2} d\lambda &\leq \int_{\Delta_r^n} |u|_{S^m h}^2 \det h e^{-(n+k)\eta_\delta - |z|^2} d\lambda \\ &\leq \int_{\Delta_r^n} |u|_{S^m h}^2 \det h e^{-(n+k)\eta - |z|^2} d\lambda < +\infty. \end{aligned}$$

Since the upper bound is independent of δ , thanks to the standard L^2 theory of the $\bar{\partial}$ -equation (cf. the proof of Theorem 3.1 or [11, Th. 2.3]), we get a sequence $\{\delta_j\}_{j \in \mathbb{N}}$ decreasing to 0, a sequence $\{\alpha_{\delta_j}\}_{j \in \mathbb{N}}$ converging weakly with respect to $S^m h \det h e^{-(n+k)\eta_{\delta_j} - |z|^2}$ for

all $j \in \mathbb{N}$ and the limit α satisfying $\bar{\partial}(\alpha dz) = u$ and

$$\int_{\Delta_r^n} |\alpha|_{S^m h}^2 \det h e^{-(n+k)\eta - |z|^2} d\lambda \leq \int_{\Delta_r^n} |u|_{S^m h}^2 \det h e^{-(n+k)\eta - |z|^2} d\lambda.$$

Note that $\log \det h^*$, η , and $|z|^2$ are bounded above on Δ_r^n . Thus,

$$\int_{\Delta_r^n} |\alpha|_{S^m h}^2 d\lambda < +\infty \text{ and } \int_{\Delta_r^n} \frac{|\alpha|_{S^m h}^2}{|z-x|^{2(n+k)}} d\lambda < +\infty.$$

Then $\alpha_x \in \mathcal{S}^m \mathcal{E}(S^m h)_x$. Since $S^m h$ is Griffiths semipositive, there exists a positive constant $C > 0$ such that $|\alpha|_{S^m h}^2 \geq C|\alpha|^2 = C(|\alpha_1|^2 + \dots + |\alpha_{r_m}|^2)$ on Δ_r^n , where $r_m = \text{rank}(S^m E)$ and $\alpha = {}^t(\alpha_1, \dots, \alpha_{r_m})$. Thus,

$$\int_{\Delta_r^n} \frac{|\alpha_i|^2}{|z-x|^{2(n+k)}} d\lambda < +\infty,$$

for each i . Set $F := \alpha - \theta f$. Then $F \in H_{2, S^m h}^0(\Delta_r^n, S^m E)$ and $\alpha_{i,x} \in \mathfrak{m}_x^{k+1}$, where \mathfrak{m}_x is the maximal ideal of $\mathcal{O}_{\Delta_r^n, x}$.

When $x = o$, the situation changes. Let θ be a cutoff function around the origin, which is identically 1 around o . Take $f \in \mathcal{S}^m \mathcal{E}(S^m h)_o$, U , and u in the same way. We only need to verify that the following integral is finite:

$$\int_U |\bar{\partial}\theta|^2 |f|_{S^m h}^2 e^{-\log \det h^* - (n+k) \log |z|^2 - |z|^2} d\lambda,$$

since $\log \det h^*$ is not bounded on U . Note that the support of $\bar{\partial}\theta$ is a compact subset in $U \setminus \{o\}$. We also see that $\log \det h^*$ and $\log |z|^2$ are bounded on the support of $\bar{\partial}\theta$. Hence, the above integral is finite. Then, repeating the above argument, we get a solution α of $\bar{\partial}(\alpha dz) = u$ satisfying

$$\int_{\Delta_r^n} |\alpha|_{S^m h}^2 \det h e^{-(n+k) \log |z|^2 - |z|^2} d\lambda \leq \int_{\Delta_r^n} |u|_{S^m h}^2 \det h e^{-(n+k) \log |z|^2 - |z|^2} d\lambda.$$

The rest is the same.

Eventually, in both cases, we obtain that $\theta f = \alpha - F$, that is, $f_x = \alpha_x - F_x$ for $x \in \Delta_r^n$. The above argument implies that $\alpha_x \in \mathfrak{m}_x^{k+1} \cdot \mathcal{O}_{\Delta_r^n}(S^m E)_x \cap \mathcal{S}^m \mathcal{E}(S^m h)_x$ for all $k \in \mathbb{N}$. Hence,

$$\mathcal{S}^m \mathcal{E}(S^m h)_x = \mathfrak{m}_x^{k+1} \cdot \mathcal{O}_{\Delta_r^n}(S^m E)_x \cap \mathcal{S}^m \mathcal{E}(S^m h)_x + \mathfrak{E}_x.$$

Thanks to the Artin–Rees lemma, there exists a positive integer ℓ such that, for any $k \geq \ell - 1$,

$$\mathfrak{m}_x^{k+1} \cdot \mathcal{O}_{\Delta_r^n}(S^m E)_x \cap \mathcal{S}^m \mathcal{E}(S^m h)_x = \mathfrak{m}_x^{k-\ell+1} (\mathfrak{m}_x^\ell \cdot \mathcal{O}_{\Delta_r^n}(S^m E)_x \cap \mathcal{S}^m \mathcal{E}(S^m h)_x).$$

Thus, for $k \geq \ell$,

$$\begin{aligned} \mathcal{S}^m \mathcal{E}(S^m h)_x &= \mathfrak{m}_x^{k+1} \cdot \mathcal{O}_{\Delta_r^n}(S^m E)_x \cap \mathcal{S}^m \mathcal{E}(S^m h)_x + \mathfrak{E}_x \\ &= \mathfrak{m}_x^{k-\ell+1} (\mathfrak{m}_x^\ell \cdot \mathcal{O}_{\Delta_r^n}(S^m E)_x \cap \mathcal{S}^m \mathcal{E}(S^m h)_x) + \mathfrak{E}_x \\ &\subset \mathfrak{m}_x \cdot \mathcal{S}^m \mathcal{E}(S^m h)_x + \mathfrak{E}_x \\ &\subset \mathcal{S}^m \mathcal{E}(S^m h)_x. \end{aligned}$$

Nakayama’s lemma says that $\mathcal{S}^m \mathcal{E}(S^m h)_x = \mathfrak{E}_x$, which completes the proof. □

We then give a proof of Corollary 1.3 as an application of the main theorem.

Proof of Corollary 1.3. First, we prove that $\det h$ is radial if h is spherically symmetric. Since $E = \mathbb{B}^n \times \mathbb{C}^r$ is trivial, taking a global holomorphic frame, we write $h = (h_{i\bar{j}})_{1 \leq i, j \leq r}$. Akin to [20], we compute each element $h_{i\bar{j}}$. For $v_1 = {}^t(1, 0, \dots, 0)$, $|v_1|_h^2 = h_{1\bar{1}}$ is a radial plurisubharmonic function with $h_{1\bar{1}}(z) \in [0, +\infty)$. We also have that $|v_2|_h = h_{2\bar{2}}$ is a radial plurisubharmonic function for $v_2 = {}^t(0, 1, 0, \dots, 0)$. For $v = {}^t(1, 1, 0, \dots, 0)$ and $v' = {}^t(1, \sqrt{-1}, 0, \dots, 0)$, we get $|v|_h^2 = h_{1\bar{1}} + h_{2\bar{2}} + 2\operatorname{Re}(h_{1\bar{2}})$ and $|v'|_h^2 = h_{1\bar{1}} + h_{2\bar{2}} - 2\operatorname{Im}(h_{1\bar{2}})$, respectively. We have that $|v|_h^2$ and $|v'|_h^2$ are also radial due to the assumption of h . Thus, $h_{1\bar{2}}$ is spherically symmetric. Consequently, we can say that every element $h_{i\bar{j}}$ is spherically symmetric, and so is $\log \det h$.

Therefore, it is enough to give a proof in the case that $\log \det h$ is a radial plurisubharmonic function. The rest part follows due to the standard argument below (cf. [6, Chap. I, §5]). Set $H := \{w \in \mathbb{C} \mid \operatorname{Re}(w) < 0\}$. Define the map $\exp : H \rightarrow \mathbb{B}^n \setminus \{o\}$ by $w \mapsto (e^w, 0, \dots, 0)$. Then $\log \det h \circ \exp$ is a plurisubharmonic function on H , which is independent of $\operatorname{Im}(w)$. Thus, $x \in (-\infty, 0) \mapsto \log \det h(e^x, 0, \dots, 0)$ is a convex function. We denote this map by μ . We then have that $\log \det h(z) = \mu(\log |z|)$ for $z \in \mathbb{B}^n \setminus \{o\}$. We can conclude that $L(\log \det h) \subset \{o\}$. Since the unbounded locus $L(\det h)$ of $\det h$ is isolated or empty, this corollary holds. \square

At the last of this section, we introduce an example satisfying the condition in Corollary 1.3. This type of example was introduced in [12].

EXAMPLE 3.2. Let h be a singular Hermitian metric on $E = \mathbb{B}^n \times \mathbb{C}^2$ defined by

$$h = \begin{pmatrix} |z_1|^2 + |z|^N & z_1 \\ \bar{z}_1 & 1 \end{pmatrix}$$

for sufficiently large $N > 0$. Then h is Griffiths seminegative since for any local holomorphic section $u = {}^t(u_1, u_2)$,

$$|u|_h^2 = |u_1 z_1 + u_2|^2 + |u_1|^2 |z|^N.$$

We have that $\det h = |z|^N$. Thus, $\mathcal{S}^m \mathcal{E}^*(\mathcal{S}^m h^*)$ is coherent for every $m \in \mathbb{N}$ thanks to Corollary 1.3. Note that $\mathcal{E}^*(h^*) \neq \mathcal{O}_{\mathbb{B}^2}(E^*)$ due to the assumption of N .

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Appendix. On Riemann surfaces

In this appendix, we discuss singular Hermitian metrics on a holomorphic vector bundle over a Riemann surface. If $\dim X = 1$, the situation is quite different. Actually, on Riemann surfaces, Griffiths positivity is equivalent to Nakano positivity by definition. Repeating the argument in the proof of Theorem 3.1, we have the following result.

THEOREM A.1 (cf. [14, Prop. 5.2]). *Let (M, ω) be a noncompact Riemann surface. We also let $(E = M \times \mathbb{C}^r, h)$ be the trivial vector bundle with a Griffiths semipositive singular Hermitian metric, and let ψ be a smooth strictly plurisubharmonic function with*

$\sqrt{-1}\partial\bar{\partial}\psi \geq \varepsilon\omega$ for some positive constant $\varepsilon > 0$. Then, for any E -valued $(1,1)$ -form u with finite L^2 -norm, there exists an E -valued $(1,0)$ -form α such that $\bar{\partial}\alpha = u$ and

$$\int_M |\alpha|_{\omega,h}^2 e^{-\psi} dV_\omega \leq \frac{1}{\varepsilon} \int_M |u|_{\omega,h}^2 e^{-\psi} dV_\omega.$$

This L^2 -estimate immediately implies the following theorem.

THEOREM A.2. *Let $(E, h) \rightarrow S$ be a holomorphic vector bundle with a Griffiths semipositive singular Hermitian metric h over a Riemann surface S . Then $\mathcal{E}(h)$ is coherent.*

This theorem may be already known for some experts. However, to emphasize the case that $\dim X = 1$ is special, we want to note the above result explicitly.

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