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ON SPRINDŽUK'S CLASSIFICATION OF p-ADIC NUMBERS

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Abstract

We carry Sprindžuk's classification of the complex numbers to the field $\mathbb{Q}_p$ of p-adic numbers. We establish several estimates for the p-adic distance between p-adic roots of integer polynomials, which we apply to show that almost all p-adic numbers, with respect to the Haar measure, are p-adic $\tilde{S}$-numbers of order 1.

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1. Introduction

For an integer polynomial $P(X)$, its height, denoted by $H(P)$, is the maximum of the absolute values of its coefficients and its degree is denoted by $\deg(P)$. In 1932, Mahler [8] introduced a classification of the complex numbers based on the real numbers

$$w_n(H,\xi) = \min\{|P(\xi)| : P(x) \in \mathbb{Z}[x], \deg(P) \le n, H(P) \le H, \text{ and } P(\xi) \neq 0\}$$

for positive integers n and H. He first defined

$$w_n(\xi) = \limsup_{H\to\infty} \frac{-\log w_n(H,\xi)}{\log H}$$

and then, according to the behaviour of the sequence $(w_n(\xi))_{n\ge 1}$, he divided the set of complex numbers into four classes, called A-, S-, T-, and U-numbers. Setting $w(\xi) = \limsup_{n\to\infty}(w_n(\xi)/n)$, we say that ξ is:

- an A-number, if $w(\xi) = 0$;

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- an S-number, if $0 < w(\xi) < \infty$;
- a T-number, if $w(\xi) = \infty$ and $w_n(\xi) < \infty$ for any integer $n \geq 1$;
- a U-number, if $w(\xi) = \infty$ and $w_n(\xi) = \infty$ for some integer $n \geq 1$.

The set of A-numbers coincides with the set of complex algebraic numbers. Almost all complex numbers (in the sense of the Lebesgue measure) are S-numbers. The sets of T-numbers and of U-numbers are nonempty and of zero Hausdorff dimension. The reader is directed to [5] for further results and references on Mahler's classification.

In 1962, Sprindžuk [10] (see also [11, pages 140–142]) introduced a new classification of the complex numbers by reverting the roles played by the degree and the height in Mahler's classification. Namely, instead of fixing first a bound for the degree and letting the height tend to infinity (as we did to define the functions w_n), he fixed first a bound H for the height and let the degree tend to infinity and considered the quantities

$$\limsup_{n\to\infty} \frac{\log(-\log w_n(H,\xi))}{\log n}.$$

Sprindžuk divided the complex numbers into four disjoint classes and called the numbers in these classes $\tilde{A}$-, $\tilde{S}$-, $\tilde{T}$-, and $\tilde{U}$-numbers. He showed that the set of $\tilde{A}$-numbers is equal to the set of complex algebraic numbers and observed that, by results of Feldman, π and $\log\alpha$ are $\tilde{S}$-numbers for any algebraic number α different from 0 and 1. While $\tilde{U}$-numbers are easy to construct, the existence of $\tilde{T}$-numbers remained open for a long time, until it was confirmed in 1996 in a beautiful paper of Amou [1].

Throughout the present paper, p denotes a fixed prime number and $|\cdot|_p$ denotes the p-adic absolute value on $\mathbb{Q}$, normalized such that $|p|_p = p^{-1}$. We denote by $|\cdot|_p$ the extension of $|\cdot|_p$ to the field $\mathbb{Q}_p$ of p-adic numbers. Further, we denote by $\mathbb{C}_p$ the completion, with respect to $|\cdot|_p$, of the algebraic closure of $\mathbb{Q}_p$.

In 1935, Mahler [9] carried *mutatis mutandis* his classification of the complex numbers to the field $\mathbb{Q}_p$; see [5, Section 9.3] for references. The only difference is that the modulus $|P(\xi)|$ in the definition of $w_n(H,\xi)$ is replaced by the p-adic absolute value. As far as we are aware, Sprindžuk's classification of p-adic numbers has not been studied up to now. It is defined as follows by simply replacing the modulus by the p-adic absolute value. Given a p-adic number ξ and positive integers n and H, we define the quantities

$$w_n(H,\xi) = \min\{|P(\xi)|_p : P(x) \in \mathbb{Z}[x], \deg(P) \leq n, H(P) \leq H, \text{ and } P(\xi) \neq 0\}, \quad (1\text{-}1)$$

$$w(H,\xi) = \limsup_{n\to\infty} \frac{\log(-\log w_n(H,\xi))}{\log n}, \quad \text{and} \quad w(\xi) = \sup_{H\in\mathbb{N}} w(H,\xi).$$

Observe that $H \mapsto w(H,\xi)$ is a nondecreasing function. We call $w(\xi)$ the order of ξ. If $w(\xi)$ is finite, then we define

$$t(H,\xi) = \limsup_{n\to\infty} \frac{\log \frac{1}{w_n(H,\xi)}}{n^{w(\xi)}} \quad \text{and} \quad t(\xi) = \limsup_{H\to\infty} \frac{t(H,\xi)}{\log H}.$$

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We call $t(\xi)$ the type of ξ. If $w(\xi)$ is infinite, then we denote by $H_0(\xi)$ the smallest integer H such that $w(H,\xi)$ is infinite if such an integer exists and we set $H_0(\xi) = \infty$ otherwise. We call ξ:

- a p-adic $\tilde{A}$-number, if $0 \leq w(\xi) < 1$ or if $w(\xi) = 1$ and $t(\xi) = 0$;
- a p-adic $\tilde{S}$-number, if $1 < w(\xi) < \infty$ or if $w(\xi) = 1$ and $t(\xi) > 0$;
- a p-adic $\tilde{T}$-number, if $w(\xi) = \infty$ and $H_0(\xi) = \infty$;
- a p-adic $\tilde{U}$-number, if $w(\xi) = \infty$ and $H_0(\xi) < \infty$.

The purpose of the present paper is to establish the p-adic analogues of some of the main results on Sprindžuk's classification of complex numbers.

We start with the p-adic analogue of a result due to Sprindžuk [10] (see also [11, pages 140–142]) asserting that the set of complex $\tilde{A}$-numbers coincides with the set of complex algebraic numbers.

THEOREM 1.1. *The class of p-adic $\tilde{A}$-numbers exactly consists of the p-adic algebraic numbers. More precisely:*

(1) *the order of a p-adic algebraic number is at most equal to* 1*. If a p-adic algebraic number ξ has order* 1*, then its type is equal to* 0*;*
(2) *the order of a p-adic transcendental number is at least* 1*. If a p-adic transcendental number ξ has order* 1*, then its type is at least equal to* 1.

Sprindžuk [10] proved that almost all complex numbers, in the sense of the Lebesgue measure on $\mathbb{C}$, are $\tilde{S}$-numbers of order less than or equal to 2 and conjectured that almost all complex numbers are $\tilde{S}$-numbers of order 1. Chudnovsky [6, page 120] solved Sprindžuk's conjecture, but his proof was apparently not complete. Amou [1] supplied a complete proof of Sprindžuk's conjecture. Later, Amou [2] improved his result, which was subsequently refined by Amou and Bugeaud [3, 4].

The main purpose of the present note is to establish the p-adic analogue of Sprindžuk's conjecture, which is an immediate consequence of the following p-adic analogue of Amou and Bugeaud [3, Théorème 2]. Throughout the rest of this note, 'almost all' always refers to the Haar measure on $\mathbb{Q}_p$.

THEOREM 1.2. *Let ε be a positive real number. Then, for almost all p-adic numbers ξ, there exists a positive real constant $c(\xi,\varepsilon)$, depending only on ξ and ε, such that every integer polynomial $P(x)$ satisfies*

$$|P(\xi)|_p > \exp(-(3+\varepsilon)n\log H - (4+\varepsilon)n\log n)$$

whenever $\max\{n, H\} \geq c(\xi,\varepsilon)$*, where n and H denote the degree and the height of $P(x)$, respectively.*

We highlight the following straightforward consequence of Theorem 1.2.

COROLLARY 1.3. *Almost all p-adic numbers are p-adic $\tilde{S}$-numbers of order* 1 *and type at most* 3.

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One may expect that almost all p-adic numbers are p-adic $\tilde{S}$-numbers of order 1 and type 1, but such a statement seems to be very difficult to prove.

Besides the classification based on (1-1), there exist other ways to classify the elements of $\mathbb{Q}_p$. Namely, for a p-adic number ξ and positive integers n and H, let us consider the quantity

$$\min\{|\xi - \alpha|_p : \alpha \in \overline{\mathbb{Q}}_p, \deg(\alpha) \le n, H(\alpha) \le H, \text{ and } \xi \ne \alpha\},$$

where $\overline{\mathbb{Q}}_p$ denotes the set of elements of $\mathbb{C}_p$ which are algebraic over $\mathbb{Q}$. Here, the height $H(\alpha)$ and the degree $\deg(\alpha)$ are the height and the degree of the minimal defining polynomial of α over $\mathbb{Z}$. However, since it is more natural to approximate p-adic numbers by p-adic numbers (and not by numbers in a larger field), we should rather consider the quantity

$$w_n^*(H, \xi) = \min\{|\xi - \alpha|_p : \alpha \in \overline{\mathbb{Q}}_p \cap \mathbb{Q}_p, \deg(\alpha) \le n, H(\alpha) \le H, \text{ and } \xi \ne \alpha\}.$$

Then a difficulty occurs, since the fact that an integer polynomial $P(x)$ is p-adically small at a p-adic number ξ does not straightforwardly imply that $P(x)$ has a root in $\mathbb{Q}_p$ which is p-adically close to ξ; see, for example, [5, Section 9.3] for a discussion. By using the quantities $w_n^*(H, \xi)$ in place of $w_n(H, \xi)$, we can divide the set of p-adic numbers into four classes, called $\tilde{A}^*$-, $\tilde{S}^*$-, $\tilde{T}^*$-, and $\tilde{U}^*$-numbers. It then follows from Theorem 1.2 that almost all p-adic numbers are p-adic $\tilde{S}^*$-numbers.

It is claimed in [5, page 167] that any two algebraically dependent complex numbers belong to the same class in Sprindžuk's classification. A (tentative) proof is left as part of Exercise 8.1. However, the hint given is not sufficient since the constants c_1 and c_3 occurring in (3.3) of [5, Ch. 3] heavily depend on the degree n. Consequently, it may be the case that, unlike Mahler's classification, Sprindžuk's classification does not enjoy a strong invariance property. We leave this as an open problem.

Problem 1.4. Do there exist two algebraically dependent (transcendental) p-adic (respectively, complex) numbers which do not belong to the same class in Sprindžuk's classification?

The proof of Theorem 1.2 rests on several estimates for the p-adic distance between p-adic roots of integer polynomials, stated and proved in Section 2. In Section 3, we apply these results to establish Theorems 1.1 and 1.2.

2. Lower bounds for the p-adic distance between roots of integer polynomials

Let $P(x) = a_n(x - \alpha_1) \cdots (x - \alpha_n)$ be a nonzero integer polynomial. The Mahler measure of $P(x)$, denoted by $M(P)$, is the quantity

$$M(P) = |a_n| \prod_{i=1}^{n} \max\{1, |\alpha_i|\}.$$

The Mahler measure is a multiplicative function, which satisfies (see [5, pages 219–220])

$$M(P) \le \sqrt{n+1}\, H(P). \tag{2-1}$$

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Let $P(x) = a_n x^n + \cdots + a_0$ be an integer polynomial of degree $n \geq 1$ and coprime coefficients. Denote the roots of $P(x)$ in $\mathbb{C}_p$ by $\alpha_1, \ldots, \alpha_n$. Then (see [12, Lemma 3.1])

$$|a_n|_p \prod_{i=1}^{n} \max\{1, |\alpha_i|_p\} = 1. \tag{2-2}$$

Let E_n $(n = 0, 1, 2, \ldots)$ be subsets of $\mathbb{Q}_p$ and assume that $\sum_{n=0}^{\infty} \lambda_p(E_n)$ converges, where λ_p denotes the Haar measure on $\mathbb{Q}_p$. By a classical covering argument (see, for example, [5, Lemma 1.3]),

$$\lambda_p\Big(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n\Big) = 0. \tag{2-3}$$

Our first result in this section is a p-adic analogue of [3, Théorème 1]. We also refer the reader to [5, Appendix A] and [3] for further references. It is crucial for our application that, in Theorems 2.1–2.3, the dependence on n occurs through the quantity n^n and not through the quantity 2^{n^2}.

Theorem 2.1. *Let $P(x)$ be an integer polynomial of degree $n \geq 2$. Let α be a root of $P(x)$ in $\mathbb{C}_p$ of degree n_1 and of multiplicity s_1. If β is a root of $P(x)$ in $\mathbb{C}_p$, distinct from α and of multiplicity s_2, then*

$$|\alpha - \beta|_p \geq \left(\binom{n+1}{s_1+1} H(P)\right)^{-n_1/s_2} M(P)^{1/s_2 - n/(s_1 s_2)} \max\{1, |\alpha|_p\} \max\{1, |\beta|_p\},$$

if $s_2 \geq s_1$, while

$$|\alpha - \beta|_p \geq \left(\binom{n+1}{s+1} H(P)\right)^{-n/(2s^2)} M(P)^{1/(2s) - n/(2s^2)} \max\{1, |\alpha|_p\}^{3/2} \max\{1, |\beta|_p\}^{3/2},$$

if $s_1 = s_2 = s$. In particular,

$$|\alpha - \beta|_p \geq 2^{-3n/(2s_1 s_2)} n^{-n(2s_1+3)/(2s_1 s_2)} H(P)^{-2n/(s_1 s_2)}, \tag{2-4}$$

if $s_2 \geq s_1$, while

$$|\alpha - \beta|_p \geq 2^{-n/s} n^{-n(2s+3)/(4s^2)} H(P)^{1/(2s) - n/s^2} \max\{1, |\alpha|_p\}^{3/2} \max\{1, |\beta|_p\}^{3/2},$$

if $s_1 = s_2 = s$.

We prove Theorem 2.1 by adapting the method of the proof of [3, Théorème 1] to the p-adic case.

Proof of Theorem 2.1. We adapt the proof of [3, Théorème 1] to the p-adic case. Let $Q(x) = a_1(x - \alpha_1) \cdots (x - \alpha_{n_1})$ be a separable integer polynomial with $\alpha = \alpha_1$. Note that $\alpha_1, \ldots, \alpha_{n_1}$ are in $\mathbb{C}_p$. Then $Q(x)$ divides $P(x)$ in $\mathbb{Z}[x]$ and the integer polynomials $Q(x)$

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and $P^{(s_1)}(x)/s_1!$ are relatively prime. Since their resultant, denoted by $\mathrm{Res}(Q, P^{(s_1)}/s_1!)$, is a nonzero integer,

$$\Big|\mathrm{Res}\Big(Q, \frac{P^{(s_1)}}{s_1!}\Big)\Big|_p^{-1} \le \Big|\mathrm{Res}\Big(Q, \frac{P^{(s_1)}}{s_1!}\Big)\Big| \le |a_1|^{n-s_1} \prod_{i=1}^{n_1} \frac{|P^{(s_1)}(\alpha'_i)|}{s_1!}$$
$$\le \binom{n+1}{s_1+1}^{n_1} H(P)^{n_1} M(Q)^{n-s_1},$$

where $\alpha'_1, \ldots, \alpha'_{n_1}$ are the complex roots of $Q(x)$. Hence,

$$1 \le \binom{n+1}{s_1+1}^{n_1} H(P)^{n_1} M(Q)^{n-s_1} |a_1|_p^{n-s_1} \prod_{i=1}^{n_1} \Big|\frac{P^{(s_1)}(\alpha_i)}{s_1!}\Big|_p. \tag{2-5}$$

We have

$$\Big|\frac{P^{(s_1)}(\alpha_i)}{s_1!}\Big|_p \le \max\{1, |\alpha_i|_p\}^{n-s_1} \quad (i = 2, \ldots, n_1). \tag{2-6}$$

Denoting the leading coefficient of $P(x)$ by a,

$$\Big|\frac{P^{(s_1)}(\alpha)}{s_1!}\Big|_p = |a|_p \prod_{\substack{\gamma\ne\alpha \\ P(\gamma)=0}} |\alpha-\gamma|_p \le |a|_p |\alpha-\beta|_p^{s_2} \max\{1, |\alpha|_p\}^{n-s_1-s_2} \prod_{\substack{\gamma\ne\alpha,\gamma\ne\beta \\ P(\gamma)=0}} \max\{1, |\gamma|_p\}. \tag{2-7}$$

Here and below, the roots γ of $P(x)$ are counted with their multiplicities. Combining (2-5)–(2-7) and using (2-2) and the inequality $M(Q) \le M(P)^{1/s_1}$,

$$|\alpha-\beta|_p \ge \Big(\binom{n+1}{s_1+1} H(P)\Big)^{-n_1/s_2} M(P)^{1/s_2 - n/(s_1 s_2)} \max\{1, |\alpha|_p\} \max\{1, |\beta|_p\} \quad (s_2 \ge s_1).$$

Thus, using (2-1) and the inequalities $n_1 \le n/s_1$ and $\binom{n+1}{s_1+1} \le (n+1)(n/2)^{s_1}$,

$$|\alpha-\beta|_p \ge 2^{-3n/(2s_1s_2)} n^{-n(2s_1+3)/(2s_1s_2)} H(P)^{-2n/(s_1s_2)} \quad (s_2 \ge s_1).$$

If $s_1 = s_2 = s$, we can choose $Q(x)$ such that $\alpha_2 = \beta$. We then have the analogue of (2-7) for $|P^{(s)}(\beta)/s!|_p$, namely the upper bound

$$\Big|\frac{P^{(s)}(\beta)}{s!}\Big|_p \le |a|_p |\alpha-\beta|_p^{s} \max\{1, |\beta|_p\}^{n-2s} \prod_{\substack{\gamma\ne\alpha,\gamma\ne\beta \\ P(\gamma)=0}} \max\{1, |\gamma|_p\}. \tag{2-8}$$

Combining (2-5), (2-7), (2-8), and (2-6) for $i = 3, \ldots, n_1$ and using (2-2) and the inequalities $n_1 \le n/s$ and $M(Q) \le M(P)^{1/s}$,

$$|\alpha-\beta|_p \ge \Big(\binom{n+1}{s+1} H(P)\Big)^{-n/(2s^2)} M(P)^{1/(2s) - n/(2s^2)} \max\{1, |\alpha|_p\}^{3/2} \max\{1, |\beta|_p\}^{3/2}.$$

Hence, using (2-1) and the inequality $\binom{n+1}{s+1} \le (n+1)(n/2)^s$,

$$|\alpha-\beta|_p \ge 2^{-n/s} n^{-n(2s+3)/(4s^2)} H(P)^{1/(2s) - n/s^2} \max\{1, |\alpha|_p\}^{3/2} \max\{1, |\beta|_p\}^{3/2}.$$

This completes the proof of Theorem 2.1.

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Our second result in this section is a p-adic analogue of Diaz and Mignotte [7, Lemme].

THEOREM 2.2. *Let $P(x)$ be an integer polynomial of degree $n \geq 1$ and ξ be a number in $\mathbb{C}_p$. Let α be a root of $P(x)$ in $\mathbb{C}_p$ such that $|\xi - \alpha|_p \leq |\xi - t|_p$ for any root t of $P(x)$ in $\mathbb{C}_p$. Let s and d denote the multiplicity of α as a root of $P(x)$ and the degree of α, respectively. Then*

$$|\xi - \alpha|_p^s \leq \left(\binom{n+1}{s+1} H(P)\right)^d M(P)^{n/s-1} |P(\xi)|_p.$$

In particular,

$$|\xi - \alpha|_p^s \leq 2^n n^{n+3n/(2s)} H(P)^{2n/s-1} |P(\xi)|_p. \tag{2-9}$$

Proof of Theorem 2.2. We adapt the proof of Diaz and Mignotte [7, Lemme] to the p-adic case. Let $Q(x) = a(x - \alpha_1) \cdots (x - \alpha_d)$ be the minimal polynomial of $\alpha = \alpha_1$ over $\mathbb{Z}$. (Note that $\alpha_1, \ldots, \alpha_d$ are in $\mathbb{C}_p$.) Then $Q^s(x)$ divides $P(x)$ in $\mathbb{Z}[x]$ and the integer polynomials $Q(x)$ and $P^{(s)}(x)/s!$ are relatively prime. Hence, their resultant is a nonzero integer. Thus, as in the proof of Theorem 2.1,

$$\left|\mathrm{Res}\left(Q, \frac{P^{(s)}}{s!}\right)\right|_p^{-1} \leq \left|\mathrm{Res}\left(Q, \frac{P^{(s)}}{s!}\right)\right| \leq \left(\binom{n+1}{s+1} H(P)\right)^d M(Q)^{n-s}.$$

So,

$$1 \leq \left(\binom{n+1}{s+1} H(P)\right)^d M(Q)^{n-s} |a|_p^{n-s} \prod_{i=1}^{d} \left|\frac{P^{(s)}(\alpha_i)}{s!}\right|_p. \tag{2-10}$$

By the hypothesis of the theorem, $|\xi - \alpha|_p \leq |\xi - \gamma|_p$ for any root γ of $P(x)$ in $\mathbb{C}_p$. This implies that

$$|\gamma - \alpha|_p \leq \max\{|\gamma - \xi|_p, |\xi - \alpha|_p\} = |\xi - \gamma|_p$$

for any root γ of $P(x)$ in $\mathbb{C}_p$. Denoting the leading coefficient of $P(x)$ by b,

$$\left|\frac{P^{(s)}(\alpha)}{s!}\right|_p = |b|_p \prod_{\substack{t \neq \alpha \\ P(t)=0}} |\alpha - t|_p \leq |P(\xi)|_p |\xi - \alpha|_p^{-s}. \tag{2-11}$$

We have

$$\left|\frac{P^{(s)}(\alpha_i)}{s!}\right|_p \leq \max\{1, |\alpha_i|_p\}^{n-s} \quad (i = 2, \ldots, d). \tag{2-12}$$

Combining (2-10)–(2-12) and using (2-2) and the inequality $M(Q) \leq M(P)^{1/s}$,

$$|\xi - \alpha|_p^s \leq \left(\binom{n+1}{s+1} H(P)\right)^d M(P)^{n/s-1} |P(\xi)|_p.$$

Hence, using (2-1) and the inequalities $d \leq n/s$ and $\binom{n+1}{s+1} \leq (n+1)(n/2)^s$,

$$|\xi - \alpha|_p^s \leq 2^n n^{n+3n/(2s)} H(P)^{2n/s-1} |P(\xi)|_p.$$

This completes the proof of Theorem 2.2.

Our last result in this section is a p-adic analogue of [5, Theorem A.1].

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THEOREM 2.3. *Let $P(x)$ and $Q(x)$ be integer polynomials of degrees $n \geq 1$ and $m \geq 1$, respectively. Let α be a root of $P(x)$ in $\mathbb{C}_p$ and β be a root of $Q(x)$ in $\mathbb{C}_p$. Denote the multiplicity of α as a root of $P(x)$ and that of β as a root of $Q(x)$ by s and t, respectively. Assume that $P(\beta) \neq 0$. Then*

$$|P(\beta)|_p \geq (n+1)^{-m/t}(m+1)^{-n/(2t)} H(P)^{-m/t} H(Q)^{-n/t} \max\{1, |\beta|_p\}^n$$

and

$$|\alpha - \beta|_p \geq (n+1)^{-m/(st)}(m+1)^{-n/(2st)} H(P)^{-m/(st)} H(Q)^{-n/(st)} \max\{1, |\alpha|_p\} \max\{1, |\beta|_p\}. \tag{2-13}$$

Proof of Theorem 2.3. Let $Q_1(x) = b(x-\beta_1)\cdots(x-\beta_{q_1})$ be the minimal polynomial of $\beta = \beta_1$ over $\mathbb{Z}$. (Note that $\beta_1, \ldots, \beta_{q_1}$ are in $\mathbb{C}_p$.) Then $Q_1^t(x)$ divides $Q(x)$ in $\mathbb{Z}[x]$ and the polynomials $Q_1(x)$ and $P(x)$ are relatively prime. Hence, the resultant of the polynomials $Q_1(x)$ and $P(x)$ is a nonzero integer. Thus,

$$|\mathrm{Res}(Q_1, P)|_p^{-1} \leq |\mathrm{Res}(Q_1, P)| \leq (n+1)^{q_1} H(P)^{q_1} M(Q_1)^n.$$

So,

$$1 \leq (n+1)^{q_1} H(P)^{q_1} M(Q_1)^n |b|_p^n \prod_{i=1}^{q_1} |P(\beta_i)|_p. \tag{2-14}$$

We have

$$|P(\beta_i)|_p \leq \max\{1, |\beta_i|_p\}^n \quad (i = 2, \ldots, q_1). \tag{2-15}$$

Combining (2-14) and (2-15) and using (2-2),

$$1 \leq |P(\beta)|_p (n+1)^{q_1} H(P)^{q_1} \Big(\frac{M(Q_1)}{\max\{1, |\beta|_p\}}\Big)^n.$$

Hence, using the inequalities (2-1), $q_1 \leq m/t$, and $M(Q_1) \leq M(Q)^{1/t}$,

$$|P(\beta)|_p \geq (n+1)^{-m/t}(m+1)^{-n/(2t)} H(P)^{-m/t} H(Q)^{-n/t} \max\{1, |\beta|_p\}^n. \tag{2-16}$$

Furthermore, denoting the leading coefficient of $P(x)$ by a_n,

$$|P(\beta)|_p = |a_n|_p |\beta - \alpha|_p^s \prod_{\substack{\gamma \neq \alpha \\ P(\gamma)=0}} |\beta - \gamma|_p \leq |a_n|_p |\beta - \alpha|_p^s \max\{1, |\beta|_p\}^{n-s} \prod_{\substack{\gamma \neq \alpha \\ P(\gamma)=0}} \max\{1, |\gamma|_p\}.$$

Thus, using (2-2),

$$|P(\beta)|_p \leq |\beta - \alpha|_p^s \max\{1, |\beta|_p\}^{n-s} \max\{1, |\alpha|_p\}^{-s}. \tag{2-17}$$

We infer from (2-16) and (2-17) that

$$|\alpha - \beta|_p \geq (n+1)^{-m/(st)}(m+1)^{-n/(2st)} H(P)^{-m/(st)} H(Q)^{-n/(st)} \max\{1, |\alpha|_p\} \max\{1, |\beta|_p\}.$$

This completes the proof of Theorem 2.3.

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3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let ξ be a p-adic algebraic number of degree m and let $Q(x)$ be the minimal polynomial of ξ over $\mathbb{Z}$. Let n and H be positive integers and let $P(x)$ be an integer polynomial with $\deg(P) \leq n$, $H(P) \leq H$, and $P(\xi) \neq 0$. Then, by Theorem 2.3,

$$|P(\xi)|_p \geq (n+1)^{-m}(m+1)^{-n}H^{-m}H(Q)^{-n}.$$

With the notation of Section 1, we get $w(H, \xi) \leq 1$ and thus $w(\xi) \leq 1$. Furthermore, if $w(\xi) = 1$, then we get $t(\xi) = 0$. This proves assertion (1) of Theorem 1.1.

Let ξ be a p-adic transcendental number with $|\xi|_p = p^{-h}$ and let n and H be positive integers. As proved by Mahler [9], there exists an integer polynomial $P(x)$ with $\deg(P) \leq n$ and $H(P) \leq H$ such that

$$0 \neq |P(\xi)|_p \leq p^{-nt+1}(H+1)^{-n-1},$$

where $t = \min\{0, h\}$. We get $w(H, \xi) \geq 1$ and thus $w(\xi) \geq 1$. Furthermore, if $w(\xi) = 1$, then we get $t(\xi) \geq 1$. This proves assertion (2) of Theorem 1.1.

Proof of Theorem 1.2. Let ε be a positive real number. Let B be a ball of radius 1 in $\mathbb{Q}_p$ and let $n_0 \geq 4$ be an integer which we will determine later. By Sprindžuk's theorem [11, pages 89 and 112], for almost all p-adic numbers ξ in B, in the sense of the Haar measure on $\mathbb{Q}_p$, the inequality

$$|P(\xi)|_p \leq \exp(-(2+\varepsilon)n \log H)$$

is satisfied by only a finite number of integer polynomials $P(x)$ of degree $n < n_0$ and of height H. Denote by E the set of p-adic numbers ξ in B such that the inequality

$$|P(\xi)|_p \leq \exp(-(3+\varepsilon)n \log H - (4+\varepsilon)n \log n)$$

is satisfied by infinitely many integer polynomials $P(x)$ of degree $n \geq n_0$ and of height H. Hence, in order to prove the theorem, it is sufficient to show that the Haar measure of E is equal to zero. For any positive integers n, s, and H with $1 \leq s \leq n$ and $n \geq n_0$, we denote by $A(n, H, s)$ the set of numbers α in $\mathbb{C}_p$ which are the roots, with multiplicity s, of some integer polynomials $P(x)$ of degree n and of height H. For any positive integers n, s, and H with $1 \leq s \leq n$ and $n \geq n_0$, let $E(n, H, s)$ denote the set of p-adic numbers ξ for which there exists an algebraic number α in $A(n, H, s)$ such that

$$|\xi - \alpha|_p \leq \exp\bigg(\bigg(\frac{2}{s^2} - \frac{3+\varepsilon}{s}\bigg)n \log H + \bigg(\frac{3}{2s^2} - \frac{3+\varepsilon}{s}\bigg)n \log n + \frac{n}{s}\log 2\bigg). \tag{3-1}$$

Setting $E(n, H) := E(n, H, 1) \cup \cdots \cup E(n, H, n)$, we observe that (2-9) in Theorem 2.2 ensures that every p-adic number ξ in E belongs to infinitely many sets $E(n, H)$. Thus,

$$E \subseteq \bigcap_{N=n_0}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{H=1}^{\infty} E(n, H) \cup \bigcap_{H_0=1}^{\infty} \bigcup_{n=n_0}^{\infty} \bigcup_{H=H_0}^{\infty} E(n, H). \tag{3-2}$$

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By [3, Lemma 1], the number of elements of $A(n, H, s)$ is bounded from above by

$$\mathrm{Card} A(n, H, s) \le \min\{2^{3n} H^n, 2^{7n^2/s^2} H^{2n/s^2}\}. \tag{3-3}$$

Let us denote the number in the right-hand side of (3-1) by ρ and let $D(n, H, s)$ denote the set of numbers in $\mathbb{C}_p$ whose distance to the ball B is less than ρ. For any positive integers n and s with $n \ge 4$ and $3 \le s \le n$, it follows from (2-4) and (2-13) that

$$|\alpha - \beta|_p \ge 2^{-n/s} n^{-5n/(2s)} H^{-2n/s} \tag{3-4}$$

holds for any α, β in $A(n, H, s)$ with $\alpha \neq \beta$. Let us denote the number in the right-hand side of (3-4) by δ. We observe that $\delta \ge 4\rho$ holds for sufficiently large n. Hence, the set $D(n, H, s)$ can be covered by at most $[2\delta^{-1}] + 2$ open balls of radius δ. For $3 \le s \le n$, this yields for sufficiently large n the bound

$$\mathrm{Card}(A(n, H, s) \cap D(n, H, s)) \le 2^{3n/s} n^{5n/(2s)} H^{2n/s}. \tag{3-5}$$

In order to bound the Haar measure of $E(n, H, s)$, we separate three cases with respect to the value of s:

$$s = 1, 2; \quad 3 \le s < n\sqrt{7/\log n}; \quad n\sqrt{7/\log n} \le s \le n.$$

For $s = 1, 2$, the first upper bound of (3-3) and (3-1) yield

$$\lambda_p(E(n, H, s)) \le \exp\Big(-\frac{n}{s}\Big(\varepsilon \log H + \Big(\frac{3}{2} + \varepsilon\Big)\log n - 4\log 2\Big)\Big).$$

For $3 \le s < n\sqrt{7/\log n}$, (3-5) and (3-1) imply that for sufficiently large n,

$$\lambda_p(E(n, H, s)) \le \exp\Big(-\frac{n}{s}\Big(\Big(\frac{1}{3} + \varepsilon\Big)\log H + \varepsilon \log n - 4\log 2\Big)\Big).$$

For $n\sqrt{7/\log n} \le s \le n$, the second upper bound of (3-3) and (3-1) yield for sufficiently large n that

$$\lambda_p(E(n, H, s)) \le \exp\Big(-\frac{n}{s}\Big(\Big(3 + \varepsilon - \frac{4}{s}\Big)\log H + \Big(3 + \varepsilon - \frac{3}{2s} - \frac{s}{n}\log 2\Big)\log n - \log 2\Big)\Big).$$

In all these three cases, choosing n_0 sufficiently large, we have for $n \ge n_0$ the inequality

$$\lambda_p(E(n, H, s)) \le (nH)^{-(2+\eta)},$$

where η is an appropriate positive real number depending on ε. Hence,

$$\lambda_p(E(n, H)) \le (nH)^{-(1+\eta)}$$

and the series

$$\sum_{n=n_0}^{\infty} \sum_{H=1}^{\infty} \lambda_p(E(n, H))$$

converges. Then (2-3) and (3-2) imply that the Haar measure of E is zero. This completes the proof of Theorem 1.2. □

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