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THE EFFECT OF AN ENCLOSED AIR CAVITY ON A RECTANGULAR DRUM

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Abstract

The effect of an enclosed air cavity on the natural vibration frequencies of a rectangular membrane is investigated. The modes specified by an even integer are not affected. For the odd-odd modes, the frequency equation is found *via* a Green's function formulation and is solved to first order in a parameter representing the effect of the cavity of the rectangular drum. The frequencies are raised, with the fundamental being most affected. In the case of degeneracies, each degenerate mode contributes to the frequency shift, but the degeneracy itself is not broken to first order.

1. Introduction

The eigenfrequencies of a freely vibrating rectangular membrane with fixed edges were found as long ago as 1829 by Poisson ([9], section 5). Unlike the vibrating string, the overtones do not form a harmonic series, because the square root of a sum of squares of integers is not in general an integer. In this paper we calculate the effect on the eigenfrequencies of the compressibility of air entrapped in an enclosure to which the membrane is affixed along its rim, thus forming a rectangular drum.

The problem of the effect of the air enclosed within the shell of a circular kettle-drum was investigated by Morse ([6], page 157). Only the circularly-symmetric modes are altered; thus that formulation is essentially one-dimensional and leads to a fairly simple frequency equation in terms of ordinary Bessel functions ([5], page 93). More recently, the author [2] has analysed the effect of an air cavity on an annular drum: Bessel functions of the second kind are then also involved.

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The case of the rectangular membrane with air cavity is inherently more complicated because it is genuinely a two-dimensional problem. This is reflected in the frequency equation (11), derived in Section 3 below, which involves an infinite summation rather than a single functional equation. Nevertheless, in Section 4 we calculate an analytic expression for the first-order correction to the frequencies. The extra contributions in the case of degenerate modes are obtained in Section 5. In particular, it is found that the degeneracies are not broken to first order.

Considerations such as these may lead to more accurate criteria for the design of membranous devices with entrapped gases such as transducers, pressure instruments, and aerodynamic bulkheads: to avoid resonances, the frequencies of applied vibrations should not coincide with natural vibration frequencies.

2. Basic equations

For adiabatic alternations of pressure, the space-dependent part of the amplitude u of a vibrating membrane stretched with tension T may be shown, following the method of Kinsler and Frey ([5], page 91), to satisfy the equation

$$\nabla^2 u + k^2 u = \frac{\gamma P_0}{T V_0} \int u \, dA \tag{1}$$

with $k = \omega/c$, where ω is the angular frequency of vibration and c is the constant speed of free waves in the membrane. Further, γ is the ratio of specific heat at constant pressure to specific heat at constant volume of the entrapped air which has equilibrium pressure P_0 and equilibrium volume V_0. The integration on the right-hand side of (1) extends over the area of the membrane which has fixed edges attached to the cavity walls, and represents the change in volume of the drum cavity due to the membrane displacement [6].

In the absence of the cavity, the freely vibrating rectangular membrance $(0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b)$ has the well-known solutions ([5], page 84)

$$u_0 = \sin k_1 x \sin k_2 y, \tag{2a}$$

$$k^2 = k_1^2 + k_2^2, \tag{2b}$$

$$k_1 = m\pi/a, \quad k_2 = n\pi/b; \quad m, n = 1, 2, 3, \ldots, \tag{2c}$$

so

$$\omega = \pi c \left(m^2/a^2 + n^2/b^2 \right)^{1/2}. \tag{3}$$

With cavity present, $\gamma \neq 0$, but for the cases of even integer values of m and/or n in (2c) the solution u_0 (equation (2)) for u still holds since the right-hand side of

(1) still vanishes because the integral itself is zero (*cf.* Rayleigh [10], page 310). It remains therefore to consider solutions to the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = \frac{\gamma P_0}{T V_0} \int_0^a dx \int_0^b dy\, u, \tag{4a}$$

with the boundary condition of fixed edges of the drum:

$$u = 0 \quad \text{on } x = 0, a; \quad y = 0, b, \tag{4b}$$

for the case for which, when γ is small, (ak_1/π) and (bk_2/π) in (2b) are both near odd integer values.

3. Green's function formulation and solution

The Green's function for the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2 + k^2$ on the left-hand side of equation (4a) has the usual double series expansion involving eigenfunctions of the operator satisfying the boundary conditions (4b). (See Morse and Feshbach [7], page 1365.) One of these series may be summed explicitly (*cf.* Jackson [4], page 89) to yield a more manageable single-sum Green's function, normalized so that

$$(\nabla^2 + k^2)G = \delta(x - x')\delta(y - y'), \tag{5}$$

$$\begin{aligned}(a/2)G(x, x'; y, y') &= \sum_{l=1}^{L} \sin\frac{l\pi x}{a} \sin\frac{l\pi x'}{a} \frac{\sin K_l y_< \sin K_l(y_> - b)}{K_l \sin K_l b} \\ &+ \sum_{l=L+1}^{\infty} \sin\frac{l\pi x}{a} \sin\frac{l\pi x'}{a} \frac{\sinh|K_l| y_< \sinh|K_l|(y_> - b)}{|K_l| \sinh|K_l| b},\end{aligned} \tag{6}$$

where $y_<$ and $y_>$ are respectively the smaller and larger of y and y',

$$K_l = \left[k^2 - l^2\pi^2/a^2\right]^{1/2}, \tag{7}$$

and L is the greatest integer less than ka/π (*cf.* Morse and Ingard [8], page 500).

Insofar as the right-hand side of equation (4a) is a constant, we use the Green's function (6) to solve the equation

$$(\nabla^2 + k^2)u = 1. \tag{8}$$

Thus

$$u(x, y) = \int_0^a dx' \int_0^b dy'\, G(x, x'; y, y'). \tag{9}$$

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We find that only odd-integer values in the sum (6) contribute, consistent with the comment after equations (4), and that the amplitude u is given by

$$(\pi/4)u = \sum_{m=0}^{M} \frac{\sin(2m+1)\pi x/a}{(2m+1)K_{2m+1}^2 \sin(K_{2m+1}b)} \times\left[\sin K_{2m+1}(y-b) - \sin K_{2m+1}y + \sin K_{2m+1}b\right] - \sum_{m=M+1}^{\infty} \frac{\sin(2m+1)\pi x/a}{(2m+1)|K_{2m+1}|^2 \sinh |K_{2m+1}|b} \times\left[\sinh|K_{2m+1}|(y-b) - \sinh|K_{2m+1}|y + \sinh|K_{2m+1}|b\right] \quad (10)$$

where $m = 0, 1, 2, \ldots$, and $2M + 1$ is the greatest odd integer less than ka/π.

The solution u given by (10) is substituted into equation (4a), where the left-hand side has the value 1, by (8). When the integration on the right-hand side is performed, an equation for k is obtained. After some manipulation, this frequency equation may be written in the form

$$\sum_{m=0}^{M} \frac{1 - \tan\theta_{2m+1}/\theta_{2m+1}}{(2m+1)^2\theta_{2m+1}^2} - \sum_{m=M+1}^{\infty} \frac{1 - \tanh\chi_{2m+1}/\chi_{2m+1}}{(2m+1)^2\chi_{2m+1}^2} = \frac{1}{\alpha}, \quad (11)$$

where

$$\theta_{2m+1} = K_{2m+1}b/2, \qquad \chi_{2m+1} = |K_{2m+1}|b/2 \quad (12)$$

and

$$\alpha = \frac{2\gamma P_0}{\pi^2 T V_0} ab^3 \quad (13)$$

are all dimensionless quantities. The quantity α is the air cavity parameter which is a measure of the effect of the entrapped air in the drum.

4. First-order effect

Equation (11) is rather difficult to grasp and analyze in general, but for small α the first-order corrections to eigenfrequencies may be calculated explicitly.

If α is very small, evidently solutions of (11) exist for which $\cos\theta_{2m+1}$ is very small. Thus we look for solutions to

$$\frac{\alpha\tan\theta_{2m+1}}{(2m+1)^2\theta_{2m+1}^3} = -1, \quad (14)$$

where

$$\theta_{2m+1} = (2n+1)\pi/2 + \varepsilon, \quad n = 0, 1, 2, \ldots, \quad (15)$$

where ε is a small correction of first order in α, and higher-order corrections to (14) are neglected. Substitution of (15) into (14) yields, to first order,

$$\varepsilon = \frac{8\alpha}{\pi^3(2m+1)^2(2n+1)^3}. \tag{16}$$

By (7) and (12), then

$$k^2 = \pi^2\left[\frac{(2m+1)^2}{a^2} + \frac{(2n+1)^2}{b^2}\right] + \frac{32\alpha}{\pi^2 b^2}\frac{1}{(2m+1)^2(2n+1)^2} \tag{17}$$

where

$$\frac{\alpha}{b^2} \equiv \beta = \frac{2\gamma P_0}{\pi^2 T V_0}(ab). \tag{18}$$

As $\alpha \to 0$, k^2 in (17) tends, as desired, to the odd-odd mode value for the free membrane without cavity, equation (2).

The ratio of cavity-loaded frequency to free frequency for any particular mode is then given to first-order by

$$\frac{f^{(\alpha)}_{(2m+1,2n+1)}}{f^{(\alpha=0)}_{(2m+1,2n+1)}} = 1 + \frac{16}{\pi^4}\frac{\beta}{(2m+1)^2(2n+1)^2\left[(2m+1)^2/a^2 + (2n+1)^2/b^2\right]}. \tag{19}$$

This increases with increasing β, as expected, and the higher-order drum modes are evidently less affected. The fundamental mode ratio is the most affected, just as for the circular case [5].

We note that equation (19) is unchanged under simultaneous interchange of m, a with n, b values. This symmetry must be explicit in any physical consequence such as (19), even though for convenience of analysis the coordinates x and y were treated on a different footing in the "once-summed" Green's function (6).

It is always of interest in vibrating systems to investigate the ratios of higher mode frequencies to fundamental mode frequency. In particular, we evaluate the ratio of the odd-odd $(2m+1, 2n+1)$ mode frequency to lowest $(1,1)$ mode frequency in the presence of the drum air cavity, compared with this ratio in the free case (given by (2), (3)). Thus

$$\begin{aligned} R &\equiv \frac{f^{(\alpha)}_{(2m+1,2n+1)}}{f^{(\alpha)}_{(1,1)}} \Big/ \frac{f^{(\alpha=0)}_{(2m+1,2n+1)}}{f^{(\alpha=0)}_{(1,1)}} \\ &= 1 - \frac{16}{\pi^4}\beta\frac{1}{(1/a^2 + 1/b^2)} \\ &\quad \times \left\{1 - \frac{1/a^2 + 1/b^2}{(2m+1)^2(2n+1)^2\left[(2m+1)^2/a^2 + (2n+1)^2/b^2\right]}\right\}, \end{aligned} \tag{20}$$

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to first order in β. This ratio R is a little less than 1, decreasing with increasing (but still small) β (equation (18)).

It only remains to check that the condition is satisfied that m must be less than or equal to M where $2M + 1$ is the greatest integer less than ka/π (see equation (11)). By (17), in which the correction to the solution k^2 is positive, $ka/\pi > (2m + 1)$. Thus $m \leqslant M$, so equation (14) does indeed always correspond to a term in the first sum of equation (11).

5. Degeneracies

Equation (17) holds if there are no degeneracies, *i.e.* different modes with the same frequency. This will be so if b^2 and a^2 are incomensurable [11], *i.e.* b^2/a^2 is irrational. In any case, the fundamental mode $m = 0$, $n = 0$ is always a non-degenerate (singlet) mode.

In the case of degeneracies, the preceding analysis of equation (11) follows through to yield the equation (17) where the correction term is replaced by

$$\delta = \frac{32}{\pi^2}\beta\sum{}^{1} \frac{1}{(2m+1)^2(2n+1)^2}, \tag{21}$$

where the sum Σ^1 is over those distinct pairs (m, n) for which the sum of sides-weighted odd squares

$$(2m+1)^2/a^2 + (2n+1)^2/b^2 \tag{22}$$

has the same value. Thus each mode in a degenerate set contributes its increase to the total frequency shift due to the presence of the cavity, but the degeneracy is not broken.

To be specific, we consider the case of a square, $b = a$. Then if $m \neq n$, the corresponding frequency has (at least) a doublet degeneracy, corresponding to interchange of m and n, and leading, *via* (21), to an extra factor of 2 in the second term of equation (17). Higher order degeneracies exist. A well-known example is the triplet case

$$1^2 + 7^2 = 7^2 + 1^2 = 5^2 + 5^2. \tag{23}$$

It is interesting to note that the 3rd mode ($m = n = 2$) of this set gives a different contribution from the first two modes ($m, n = 0, 3$) in the sum (21). In general, all the possible degeneracies and their multiplicities for the square in the free case are known (see [3], sections 16.9–16.10). For the square membrane with cavity, the doublet degeneracy corresponding to $m \neq n$ mentioned above will not be split even for larger parameter α, by symmetry: it merely corresponds to interchange of axes.

The rectangular case is more complicated, and only partial results for degeneracies in the free case are known (see *e.g.* [1], sections 37–42). It is important to note that, despite the frequency shifts, the degeneracy is not broken to first order. It is likely that there would be splitting at higher orders of perturbation.

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