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The effects of diffusion on the principal eigenvalue for age-structured models with random diffusion

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In this paper, we study the principal spectral theory of age-structured models with random diffusion. First, we provide an equivalent characteristic for the principal eigenvalue, the strong maximum principle and a positive strict super-solution. Then, we use the result to investigate the effects of diffusion rate on the principal eigenvalue. Finally, we study how the principal eigenvalue affects the global dynamics of the KPP model and verify that the principal eigenvalue being zero is a critical value.

Keywords: Age structure; global dynamics; principal eigenvalue; random diffusion; strong maximum principle

2010 *Mathematics Subject Classification:* 35K57; 92D25; 47A10

1. Introduction

In modelling the population dynamics of biological species and the transmission dynamics of infectious diseases, it is very important to consider the age structure as well as the spatial movement of the populations. In the last few decades, various age-structured models with random diffusion have been constructed and studied in the literature, see [**1**, **2**, **7**, **8**, **15**–**18**, **24**, **25**, **39**–**49**]. Among some of them, for example [**1**], [**15**] and [**7**, **8**], the principal eigenvalue serves as a crucial tool in investigating such equations. In this paper, we develop the principal spectral theory for age-structured models with random diffusion. More precisely, we are interested in the effects of diffusion on the principal eigenvalue for the following age-structured models with random diffusion

$$\begin{cases} \dfrac{\partial u(a,x)}{\partial a} = \mathrm{d}\Delta u(a,x) - \mu(a,x)u(a,x) - \lambda u(a,x), & (a,x) \in (0,a^+) \times \Omega, \\ u = 0, & (a,x) \in (0,a^+) \times \partial\Omega, \\ u(0,x) = \displaystyle\int_0^{a^+} \beta(a,x)u(a,x)\,\mathrm{d}a, & x \in \Omega, \end{cases} \tag{1.1}$$

where $u(a,x)$ denotes the density of the population with age $a \in [0,a^+]$ at location $x \in \overline{\Omega}$, in which $a^+ < \infty$ represents the maximum age and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. Here, $d > 0$ represents the diffusion rate. We assume that the birth rate $\beta(a, x)$ is positive everywhere and the death rate $\mu(a, x)$ is nonnegative, and both belong to $C^{1,2}([0, a^+] \times \overline{\Omega})$. Define

$$\underline{\mu}(a) := \inf_{x\in\overline{\Omega}} \mu(a, x), \quad \overline{\mu}(a) := \sup_{x\in\overline{\Omega}} \mu(a, x),$$
$$\underline{\beta}(a) := \inf_{x\in\overline{\Omega}} \beta(a, x), \quad \overline{\beta}(a) := \sup_{x\in\overline{\Omega}} \beta(a, x).$$

Actually, the effects of diffusion on the principal eigenvalue for both elliptic and parabolic types of operators in the time periodic case have been studied, see [**3**, **23**, **27**, **31**] and references cited therein. However, we cannot directly apply these results to our case, since our eigenvalue problem has an initial-boundary condition. The reason is that such a function space satisfying this integral condition is unknown and heavily depends on the birth rate β. We outcome this difficulty by exploiting the solvability of the age-structured model without random diffusion which allows us to construct appropriate sub/super-solutions for studying the effects of diffusion.

The paper is organized as follows. In § 2, we recall some existing theory on the principal eigenvalue for age-structured models with random diffusion and give an equivalent characterization of the principal eigenvalue, the strong maximum principle and a positive strict super-solution. In § 3, we first analyse the spectral bound $s(\mathcal{B})$ of $\mathcal{B}$ defined in (3.1) which corresponds to age-structured models without random diffusion defined in (3.12), and then use global implicit function theorem to investigate the solvability of (3.12) which is a key ingredient in constructing the sub/super-solutions of $\mathcal{A}$ defined in (2.1) later. In § 4, we study the effects of the diffusion rate on the principal eigenvalue of $\mathcal{A}$. In § 5, we discuss the existence, uniqueness and stability of the age-structured KPP type models with random diffusion via the sign of the principal eigenvalue of the linearized operator. Finally, we provide a discussion in § 6.

2. Preliminaries

We recall the theory of resolvent positive operators in a Banach space Q, the readers can refer to [**37**, **38**]. A linear operator $C : D \to Q$, defined on a linear subspace D of Q, is said to be *positive* if $Cx \in Q_+$ for all $x \in D \cap Q_+$ and C is not the 0 operator, where Q_+ is closed convex cone that is normal and generating.

Definition 2.1. A closed operator A in Q is said to be *resolvent positive* if the resolvent set of A, $\rho(A)$, contains a ray (ω, ∞) and $(\lambda I - A)^{-1}$ is a positive operator (i.e. maps Q_+ into Q_+) for all $\lambda > \omega$.

Definition 2.2. We define the *spectral bound* of a closed operator A as

$$s(A) = \sup\{\operatorname{Re}\lambda \in \mathbb{C}; \lambda \in \sigma(A)\},$$

the *real spectral bound* of A as

$$s_{\mathbb{R}}(A) = \sup\{\lambda \in \mathbb{R}; \lambda \in \sigma(A)\},$$

and the *spectral radius* of A as

$$r(A) = \sup\{|\lambda|; \lambda \in \sigma(A)\},$$

where $\sigma(A) = \mathbb{C} \setminus \rho(A)$ represents the spectrum of A.

In order to deal with the problem, we introduce the state space $E = L^p((0,a^+) \times \Omega)$ and $X = L^p(\Omega)$, where $p > N$ and define an operator $\mathcal{A} : E \to E$ by

$$\mathcal{A}\phi(a,x) = \mathrm{d}\Delta\phi(a,x) - \frac{\partial \phi(a,x)}{\partial a} - \mu(a,x)\phi(a,x), \quad \forall \phi \in D(\mathcal{A}),$$

$$D(\mathcal{A}) = \left\{ \phi(a,x) \middle| \phi, \mathcal{A}\phi \in E, \phi|_{\partial\Omega} = 0, \phi(0,x) = \int_0^{a^+} \beta(a,x)\phi(a,x)\,\mathrm{d}a \right\}. \quad (2.1)$$

The eigenvalue problem of $\mathcal{A}$ is explicitly written as

$$\begin{cases} \dfrac{\partial u(a,x)}{\partial a} = \mathrm{d}\Delta u(a,x) - \mu(a,x)u(a,x) - \lambda u(a,x), & (a,x) \in (0,a^+) \times \Omega, \\ u = 0, & (a,x) \in (0,a^+) \times \partial\Omega, \\ u(0,x) = \displaystyle\int_0^{a^+} \beta(a,x)u(a,x)\,\mathrm{d}a \in X, & x \in \Omega. \end{cases} \quad (2.2)$$

Define an evolution family $\{\mathcal{U}(\tau,a)\}_{\{0 \leqslant \tau \leqslant a \leqslant a^+\}}$ on X for the following equations

$$\begin{cases} \dfrac{\partial u(a,x)}{\partial a} = \mathrm{d}\Delta u(a,x) - \mu(a,x)u(a,x), & (a,x) \in (0,a^+) \times \Omega, \\ u = 0, & (a,x) \in (0,a^+) \times \partial\Omega, \\ u(\tau,x) = \phi(x) \in X, & x \in \Omega. \end{cases} \quad (2.3)$$

Note that the existence of the evolution family $\mathcal{U}$ is from the theory of classical parabolic equations, see [**15**, lemma 1]. Next, define an operator $\mathcal{M}_\lambda : X \to X$ by

$$\mathcal{M}_\lambda \phi = \int_0^{a^+} \beta(a,x)\,\mathrm{e}^{-\lambda a}\mathcal{U}(0,a)\phi\,\mathrm{d}a, \quad \phi \in X. \quad (2.4)$$

In fact such $\mathcal{M}_\lambda$ is obtained by plugging the solution of the characteristic equation of $\mathcal{A}$ into the boundary condition, the readers can refer to [**15**, **47**] for the derivation. Moreover, $\mathcal{M}_\lambda$ is a compact and nonsupporting operator, see [**15**], where nonsupporting is a generalization of strong positivity when working on a Banach space Q with a positive cone Q_+ which has empty interior, for example the L^p space (see [**35**] or [**28**] for a complete definition). In the next theorem, a function is *strictly positive* if it is a quasi-interior point of Q_+ which is defined in the following (2.5). Now we recall some existing results on the principal eigenvalue of $\mathcal{A}$.

THEOREM 2.3 [**15**, theorem 3].

(*i*) *The spectrum $\sigma(\mathcal{A})$ consists of all eigenvalues of operator $\mathcal{A}$, and the intersection of any finite strip paralleling the y axis with $\sigma(\mathcal{A})$ contains at most a finite number of eigenvalues of $\mathcal{A}$;*

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(*ii*) *$\mathcal{A}$ has only one real eigenvalue λ_0 which is algebraically simple, and it is greater than the real part of any other eigenvalue of $\mathcal{A}$;*

(*iii*) *The eigenfunction ϕ corresponding to λ_0 is given by $\phi(a,x) = e^{-\lambda_0 a}\mathcal{U}(0,a)\phi_0(x)$, where ϕ_0 is nontrivial fixed point of $\mathcal{M}_{\lambda_0}$. Moreover, ϕ_0 is a quasi-interior point of X; i.e.*

$$\langle f, \phi_0 \rangle > 0, \quad \forall f \in X^*,\ f \geqslant 0,\ f \neq 0, \tag{2.5}$$

where X^* represents the space of linear functionals on X and $\langle \cdot, \cdot \rangle$ denotes the usual dual product.

(*iv*) *λ_0 satisfies the following equation*

$$r(\mathcal{M}_{\lambda_0}) = r\left(\int_0^{a^+} \beta(a,x)\, e^{-\lambda_0 a}\mathcal{U}(0,a)\, da\right) = 1; \tag{2.6}$$

(*v*) *$s(\mathcal{A}) = \omega_0(\mathcal{A}) = \lambda_0$, where $\omega_0(\mathcal{A})$ denotes the growth bound of $\mathcal{A}$.*

Lemma 2.4 [**47**, lemma 2.6]. *The mapping $\lambda \to r(\mathcal{M}_\lambda) : \mathbb{R} \to (0,\infty)$ is continuous and strictly decreasing and* $\lim_{\lambda\to\infty} r(\mathcal{M}_\lambda) = 0$, $\lim_{\lambda\to-\infty} r(\mathcal{M}_\lambda) = \infty$.

We would like to mention that in a series of papers, Walker [**39**–**47**] investigated various issues related to the existence of positive nontrivial steady states of age-structured models with nonlinear diffusion and (non)linear birth and death rates, where he employed bifurcations methods to treat these problems by assuming the maximum regularity of the diffusion operator [**5**, **32**]. In most of his papers, similar results on the spectrum of $\mathcal{A}$ were also obtained.

To obtain the strong maximum principle and sub/super-solutions, we introduce the following Sobolev space:

$$Z = W_p^{1,2}((0,a^+) \times \Omega) \quad \text{with} \quad p > N.$$

Thus, the standard Sobolev embedding theorem guarantees that $\phi \in C^{\theta/2,1+\theta}([0,a^+] \times \overline{\Omega})$ for some $\theta \in (0,1)$ if $\phi \in Z$. Hence in the following context, $\phi \in Z$ and thus $\phi \in C^{\theta/2,1+\theta}([0,a^+] \times \overline{\Omega})$ for some $\theta \in (0,1)$ when we mention that $\phi(a,x) = \mathrm{e}^{-\lambda_0 a}\mathcal{U}(0,a)\phi_0(x)$, where $\phi_0 \in L^p(\Omega)$.

Definition 2.5. A function $\overline{w} \in Z$ is called a *super-solution* of $\mathcal{A}$ if $\overline{w}$ satisfies

$$\begin{cases} \dfrac{\partial \overline{w}}{\partial a} \geqslant \mathrm{d}\Delta \overline{w} - \mu \overline{w}, & (a,x) \in (0,a^+] \times \Omega, \\ \overline{w} \geqslant 0, & (a,x) \in (0,a^+] \times \partial\Omega, \\ \overline{w}(0,x) \geqslant \int_0^{a^+} \beta(a,x)\overline{w}(a,x)\,\mathrm{d}a, & x \in \Omega. \end{cases} \tag{2.7}$$

The function $\overline{w}$ is called a *strict super-solution* if it is a super-solution but not a solution. A *sub-solution* $\underline{w}$ is defined by reversing the inequality signs in (2.7).

Definition 2.6. We say that $\mathcal{A}$ admits the *strong maximum principle* if $w \in Z$ satisfying (2.7) implies that $w > 0$ in $[0,a^+] \times \Omega$ unless $w \equiv 0$.

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With the above preparations, we are ready to give an equivalent characteristic for the principal eigenvalue, the strong maximum principle and a positive strict super-solution. We should mention that such an equivalent characteristic was partially proved by Delgado *et al.* [7, 8].

PROPOSITION 2.7. *The following statements are equivalent:*

(*i*) $\mathcal{A}$ *admits the strong maximum principle property;*

(*ii*) $\lambda_0 < 0$*;*

(*iii*) $\mathcal{A}$ *has a strict super-solution which is positive in* $[0, a^+] \times \Omega$.

Proof. The idea of the proof below traced back to [8]. For completeness and for the reader's convenience, we include necessary modifications and provide a detailed proof. (i) $\Rightarrow$ (ii). Suppose that $\lambda_0 \geqslant 0$. Then for the corresponding principal eigenfunction $\phi > 0$ in $[0, a^+] \times \Omega$, it is obvious that $-\phi < 0$ and solves the following characteristic equation for $\lambda = \lambda_0$:

$$\begin{cases} \dfrac{\partial(-\phi)}{\partial a} - \mathrm{d}\Delta(-\phi) + \mu(-\phi) = -\lambda_0(-\phi) \geqslant 0, & (a,x) \in (0, a^+] \times \Omega, \\ -\phi = 0, & (a,x) \in (0, a^+] \times \partial\Omega, \\ -\phi(0,x) = \int_0^{a^+} \beta(a,x)(-\phi)(a,x)\,\mathrm{d}a, & x \in \Omega. \end{cases} \tag{2.8}$$

Thus, applying the strong maximum principle in definition 2.6 to $-\phi$, we find $-\phi > 0$ in $[0, a^+] \times \Omega$, a contradiction with the positivity of ϕ. (ii) $\Rightarrow$ (iii). It is obvious that the corresponding principal eigenfunction $\phi > 0$ is a strict super-solution of $\mathcal{A}$. (iii) $\Rightarrow$ (i). Let $\psi \in Z$ satisfy (2.7) with $\psi \not\equiv 0$ and ϕ be a strict super-solution of $\mathcal{A}$ which is positive. Assume by contradiction that there exists $(a_0, x_0) \in [0, a^+] \times \overline{\Omega}$ such that $\psi(a_0, x_0) = \min_{[0,a^+]\times\overline{\Omega}} \psi \leqslant 0$. Then consider the set

$$\Gamma := \{\epsilon \in \mathbb{R} : \psi + \epsilon\phi \geqslant 0 \,\mathrm{in}\, [0, a^+] \times \overline{\Omega}\}.$$

Denote by $\epsilon_0 = \min \Gamma$ and $u_0 = \psi + \epsilon_0\phi$. It is clear that $\epsilon_0 \geqslant 0$ by the assumption of $\psi(a_0, x_0) \leqslant 0$ and that $u_0 \geqslant 0$. Then by the definition of a strict super-solution, we have

$$\begin{cases} \dfrac{\partial u_0}{\partial a} - \mathrm{d}\Delta u_0 + \mu u_0 \geqslant 0, & (a,x) \in (0, a^+] \times \Omega, \\ u_0 \geqslant 0, & (a,x) \in (0, a^+] \times \partial\Omega, \\ u_0(0,x) \geqslant \int_0^{a^+} \beta(a,x) u_0(a,x)\,\mathrm{d}a, & x \in \Omega. \end{cases} \tag{2.9}$$

It follows that u_0 is a nonnegative strict super-solution. Observe that by [6, proposition 13.1] we have that $u_0(a,x) > 0$ in $(0, a^+) \times \Omega$. Then $u_0(0,x) > 0$ and thus by [6, theorem 13.5], u_0 is strictly positive, in the sense that it is positive and its normal derivative at $\partial\Omega$ is negative. This contradicts the fact that ϵ_0 is the infimum of Γ. Hence $\psi > 0$ in $[0, a^+] \times \Omega$ which concludes (i). □

3. The spectrum without diffusion

Define an operator $\mathcal{B} : E \to E$ by

$$\mathcal{B}\phi(a,x) = -\frac{\partial \phi(a,x)}{\partial a} - \mu(a,x)\phi(a,x), \quad \forall \phi \in D(\mathcal{B}),$$

$$D(\mathcal{B}) = \left\{ \phi(a,x) \middle| \phi, \mathcal{B}\phi \in E, \phi|_{\partial\Omega} = 0, \phi(0,x) = \int_0^{a^+} \beta(a,x)\phi(a,x)\, \mathrm{d}a \right\}. \quad (3.1)$$

Proposition 3.1. $(\alpha I - \mathcal{B})^{-1}$ *exists when* $Re\,\alpha > \alpha_0$, *where* $\alpha_0 \in \mathbb{R}$ *satisfies*

$$r(\mathcal{G}_{\alpha_0}) = r\left(\int_0^{a^+} \beta(a,x)\, e^{-\alpha_0 a}\Pi(0,a,x)\, da \right) = 1, \quad (3.2)$$

in which $\Pi(\gamma,a,x) := e^{-\int_\gamma^a \mu(s,x)\, ds}$ *and* $\mathcal{G}_\alpha : X \to X$ *is a linear bounded operator defined in (3.7). Moreover,* $s(\mathcal{B}) = \alpha_0$ *and* α_0 *also satisfies the following equation,*

$$\max_{x\in\overline{\Omega}} \int_0^{a^+} \beta(a,x)\, e^{-\alpha_0 a}\Pi(0,a,x)\, da = 1. \quad (3.3)$$

Proof. Writing the equation $(\alpha I - \mathcal{B})\phi = \psi \in E$ explicitly, we obtain

$$\begin{cases} \dfrac{\partial \phi(a,x)}{\partial a} = -(\alpha + \mu(a,x))\phi(a,x) + \psi(a,x), & (a,x) \in (0,a^+) \times \Omega, \\ \phi(a,x) = 0, & (a,x) \in (0,a^+) \times \partial\Omega, \\ \phi(0,x) = \int_0^{a^+} \beta(a,x)\phi(a,x)\, \mathrm{d}a, & x \in \Omega. \end{cases} \quad (3.4)$$

Solving the equation, one has

$$\phi(a,x) = \mathrm{e}^{-\alpha a}\Pi(0,a,x)\phi(0,x) + \int_0^a \mathrm{e}^{-\alpha(a-\gamma)}\Pi(\gamma,a,x)\psi(\gamma,x)\, \mathrm{d}\gamma, \quad (3.5)$$

and accordingly

$$\begin{aligned} &\phi(0,x) - \int_0^{a^+} \beta(a,x)\, \mathrm{e}^{-\alpha a}\Pi(0,a,x)\phi(0,x)\, \mathrm{d}a \\ &\quad = \int_0^{a^+} \beta(a,x) \int_0^a \mathrm{e}^{-\alpha(a-\gamma)}\Pi(\gamma,a,x)\psi(\gamma,x)\, \mathrm{d}\gamma\, \mathrm{d}a, \end{aligned}$$

which is equivalent to

$$(I - \mathcal{G}_\alpha)\phi(0,x) = \int_0^{a^+} \beta(a,x) \int_0^a \mathrm{e}^{-\alpha(a-\gamma)}\Pi(\gamma,a,x)\psi(\gamma,x)\, \mathrm{d}\gamma\, \mathrm{d}a. \quad (3.6)$$

where

$$[\mathcal{G}_\alpha g](x) = \int_0^{a^+} \beta(a,x)\, \mathrm{e}^{-\alpha a}\Pi(0,a,x)g(x)\, \mathrm{d}a, \quad g \in X. \quad (3.7)$$

Thus if $1 \in \rho(\mathcal{G}_\alpha)$, then

$$\phi(0,x) = (I-\mathcal{G}_\alpha)^{-1}\left[\int_0^{a^+}\beta(a,x)\int_0^a \mathrm{e}^{-\alpha(a-\gamma)}\Pi(\gamma,a,x)\psi(\gamma,x)\,\mathrm{d}\gamma\,\mathrm{d}a\right], \qquad (3.8)$$

which implies that

$$\begin{aligned}\phi(a,x) = {} & \mathrm{e}^{-\alpha a}\Pi(0,a,x)(I-\mathcal{G}_\alpha)^{-1} \\ & \times\left[\int_0^{a^+}\beta(a,x)\int_0^a \mathrm{e}^{-\alpha(a-\gamma)}\Pi(\gamma,a,x)\psi(\gamma,x)\,\mathrm{d}\gamma\,\mathrm{d}a\right] \\ & + \int_0^a \mathrm{e}^{-\alpha(a-\gamma)}\Pi(\gamma,a,x)\psi(\gamma,x)\,\mathrm{d}\gamma. \end{aligned} \qquad (3.9)$$

It follows that $\alpha \in \rho(\mathcal{B})$ and thus $(\alpha I-\mathcal{B})^{-1}$ exists. Now the problem is inverted into finding such α satisfying $1 \in \rho(\mathcal{G}_\alpha)$. By assumptions on β and μ, we have

$$\mathcal{G}_\alpha g \geqslant \int_0^{a^+}\underline{\beta}(a)\,\mathrm{e}^{-\alpha a}\tilde{\Pi}(0,a)\,\mathrm{d}a g, \quad g \in X, \qquad (3.10)$$

where $\tilde{\Pi}(\gamma,a) := \mathrm{e}^{-\int_\gamma^a \overline{\mu}(s)\,\mathrm{d}s}$. Now we define

$$\mathcal{H}_\alpha := \int_0^{a^+}\underline{\beta}(a)\,\mathrm{e}^{-\alpha a}\tilde{\Pi}(0,a)\,\mathrm{d}a,$$

then one has from (3.10) that $\mathcal{G}_\alpha \geqslant \mathcal{H}_\alpha$ in the sense of positive operators (actually $\mathcal{H}_\alpha$ is a function of α) and that $r(\mathcal{G}_\alpha)$ is a strictly decreasing continuous function with respect to α, see [**21**, lemmas 3.3 and 3.4]. Recall from the theory of classical age-structured models that there is a unique $\alpha_1 \in \mathbb{R}$ such that

$$\int_0^{a^+}\underline{\beta}(a)\,\mathrm{e}^{-\alpha_1 a}\tilde{\Pi}(0,a)\,\mathrm{d}a = 1,$$

i.e. $\mathcal{H}_{\alpha_1} = 1$. Now by the positive operators theory, we have immediately $r(\mathcal{G}_{\alpha_1}) \geqslant r(\mathcal{H}_{\alpha_1}) = \mathcal{H}_{\alpha_1} = 1$, there exists a unique $\alpha_0 \in \mathbb{R}$ satisfying $r(\mathcal{G}_{\alpha_0}) = 1$. Note for any $\alpha \in \mathbb{C}$, when $\operatorname{Re}\alpha > \alpha_0$ we have $r(\mathcal{G}_{\operatorname{Re}\alpha}) < r(\mathcal{G}_{\alpha_0}) = 1$, $(I-\mathcal{G}_{\operatorname{Re}\alpha})^{-1}$ exists. It follows that $\alpha \in \rho(\mathcal{B})$ when $\operatorname{Re}\alpha > \alpha_0$, which implies that $\rho(\mathcal{B})$ contains a ray (α_0,∞) and $(\alpha I-\mathcal{B})^{-1}$ is obviously a positive operator by (3.9) for all $\alpha > \alpha_0$. Thus $\mathcal{B}$ is a resolvent positive operator. Furthermore, α_0 is larger than all other real spectral values in $\sigma(\mathcal{B})$. It follows that $\alpha_0 = s_{\mathbb{R}}(\mathcal{B})$. But since E is a Banach space with a normal and generating cone $E_+ := L^p_+((0,a^+)\times\Omega)$ and $s(\mathcal{B}) \geqslant \alpha_0 > -\infty$ due to $\alpha_0 \in \sigma(\mathcal{B})$, we can conclude from [**37**, theorem 3.5] that $s(\mathcal{B}) = s_{\mathbb{R}}(\mathcal{B}) = \alpha_0$. Note that $\mathcal{G}_\alpha$ is actually a positive multiplication operator in X. We can obtain the spectral radius $r(\mathcal{G}_\alpha)$ of $\mathcal{G}_\alpha$ easily via

$$r(\mathcal{G}_\alpha) = \max_{x\in\overline{\Omega}} \int_0^{a^+} \beta(a,x)\,\mathrm{e}^{-\alpha a}\Pi(0,a,x)\,\mathrm{d}a.$$

Thus α_0 satisfies (3.3). Denote

$$\alpha_{\min} := \min_{x\in\overline{\Omega}} \int_0^{a^+} \beta(a,x)\,\mathrm{e}^{-\alpha a}\Pi(0,a,x)\,\mathrm{d}a.$$

It is easy to see from [**26**, proposition 2.7] that $\sigma_e(\mathcal{G}_\alpha) = \sigma(\mathcal{G}_\alpha) = \cup_{x\in\overline{\Omega}}\mathcal{G}_\alpha(x) = [\alpha_{\min}, r(\mathcal{G}_\alpha)]$, where

$$\mathcal{G}_\alpha(x) = \int_0^{a^+} \beta(a,x)\,\mathrm{e}^{-\alpha a}\Pi(0,a,x)\,\mathrm{d}a, \tag{3.11}$$

and $\sigma_e(A)$ represents the essential spectrum of A. □

Next we give a key proposition on the solvability of the following equation, which is important in studying the effects of diffusion rate on the principal eigenvalue:

$$\begin{cases} \dfrac{\partial u(a,x)}{\partial a} = -\alpha u(a,x) - \mu(a,x)u(a,x), & (a,x)\in(0,a^+)\times\Omega,\\ u(a,x) = 0, & (a,x)\in(0,a^+)\times\partial\Omega,\\ u(0,x) = \int_0^{a^+}\beta(a,x)u(a,x)\,\mathrm{d}a, & x\in\Omega. \end{cases} \tag{3.12}$$

PROPOSITION 3.2. *There exists a C^2 continuously differentiable function $x \to \alpha(x) : \overline{\Omega} \to \mathbb{R}$ such that equation (3.12) has positive solutions and*

$$\int_0^{a^+} \beta(a,x)\, e^{-\alpha(x)a}\Pi(0,a,x)\, da = 1, \quad \forall x\in\overline{\Omega}.$$

Moreover, $\alpha(x) \leqslant \alpha_0$ for all $x\in\overline{\Omega}$.

Proof. Solving (3.12) explicitly, we obtain a formal positive solution, $u(a,x) = \mathrm{e}^{-\alpha a}\Pi(0,a,x)\phi(x)$ provided with $\phi > 0$ and $\phi \equiv 0$ at $\partial\Omega$. Then plugging it into the integral initial condition we get

$$\int_0^{a^+} \beta(a,x)\,\mathrm{e}^{-\alpha a}\Pi(0,a,x)\,\mathrm{d}a = 1.$$

Now define

$$\mathcal{G}(\alpha,x) := \mathcal{G}_\alpha(x) = \int_0^{a^+} \beta(a,x)\,\mathrm{e}^{-\alpha a}\Pi(0,a,x)\,\mathrm{d}a.$$

It is easy to check that $\mathcal{G} : \mathbb{R}\times\overline{\Omega} \to (0,\infty)$ is a $C^{1,2}$ continuously differentiable function with respect to α and x due to $C^{1,2}$ continuous differentiability of β and

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μ. Moreover,

$$\frac{\partial \mathcal{G}}{\partial \alpha} = -\int_0^{a^+} \beta(a,x) a \mathrm{e}^{-\alpha a} \Pi(0,a,x)\,\mathrm{d}a < 0, \quad \forall x \in \overline{\Omega}, \tag{3.13}$$

$$a\frac{\partial \mathcal{G}}{\partial x_i} = \int_0^{a^+} \frac{\partial \beta(a,x)}{\partial x_i} \mathrm{e}^{-\alpha a} \Pi(0,a,x)\,\mathrm{d}a - \int_0^{a^+}\int_0^a \beta(a,x)\,\mathrm{e}^{-\alpha a} \frac{\partial \mu(s,x)}{\partial x_i} \Pi(0,a,x)\,\mathrm{d}s\,\mathrm{d}a, \quad i = 1, \ldots, N. \tag{3.14}$$

It follows by implicit function theorem that (i) $\mathcal{G}$ is locally solvable for $x \in \overline{\Omega}$ due to (3.13), i.e. for each $(\alpha_0, x_0) \in O_{\mathcal{G}} := \{(\alpha, x) \in \mathbb{R} \times \overline{\Omega} : \mathcal{G}(\alpha, x) = 1\}$, there are open neighbourhoods N_{α_0} and N_{x_0} of α_0 and x_0 respectively, and a continuous map α of N_{x_0} into N_{α_0} such that for $x \in N_{x_0}, \alpha = \alpha(x)$ is the unique solution in N_{α_0} of $\mathcal{G}(\alpha, x) = 1$. Next, let A be any family of compact subset of $\overline{\Omega}$ such that for each compact subset C of $\overline{\Omega}$, there is an $S \in A$ such that $C \subseteq S$ and similarly, let B denote any collection of compact subsets of $\mathbb{R}$ with the property that for any compact set D in $\mathbb{R}$, there is $T \in B$ such that $D \subseteq T$. Note that due to $\mu, \beta \in C^{1,2}([0, a^+] \times \overline{\Omega})$, we have from (3.13) and (3.14) that

$$\left|\frac{\partial \alpha}{\partial x}\right| = \left|\frac{\partial \mathcal{G}}{\partial x}\right| \Big/ \left|\frac{\partial \mathcal{G}}{\partial \alpha}\right| \leqslant C,$$

where $|\partial\alpha/\partial x|$ ($|\partial\mathcal{G}/\partial x|$ respectively) denotes the length of vector $\partial\alpha/\partial x = (\partial\alpha/\partial x_1, \ldots, \partial\alpha/\partial x_N)$ ($\partial\mathcal{G}/\partial x = (\partial\mathcal{G}/\partial x_1, \ldots, \partial\mathcal{G}/\partial x_N)$ respectively) in the usual sense. It follows by the mean value theorem that we can extend α up to the boundary of N_{x_0}. In fact if any sequence $\{x_k\} \in N_{x_0}$ converges to $b \in \partial N_{x_0}$, then

$$|\alpha(x_k) - \alpha(x_l)| \leqslant \left|\frac{\partial \alpha(\xi)}{\partial x}\right| |x_k - x_l|,$$

for some $\xi \in (x_k, x_l)$, which implies that $\{\alpha(x_k)\}$ is a Cauchy sequence. Thus $\{\alpha(x_k)\}$ converges to $\alpha(b)$ by the continuity of α. Hence by the above argument we have (ii) for each $S \in A$, there is a $T \in B$ such that $x \in S, \alpha \in \mathbb{R}$ and $\mathcal{G}(\alpha, x) = 1$ imply that $\alpha \in T$. Finally, since $\partial\mathcal{G}/\partial a < 0$ for all $x \in \overline{\Omega}$, then (iii) for some $x_0 \in \overline{\Omega}$, there is exactly one α_0 such that $\mathcal{G}(\alpha_0, x_0) = 1$. In actually, (iii) implies that the extension of α is unique. Now we have verified the three hypotheses (i), (ii) and (iii) satisfying the global implicit function theorem, see [**34**, theorem 1] or [**33**]. It follows that we have a unique $\alpha : \overline{\Omega} \to \mathbb{R}$ such that $\mathcal{G}(\alpha(x), x) = 1$ for all $x \in \overline{\Omega}$ and α is C^2 continuously differentiable since β and μ are C^2 continuously differentiable by assumptions. Moreover, it is obvious $\alpha(x) \leqslant \alpha_0$ by (3.3). In fact, $\alpha_0 = \max_{x\in\overline{\Omega}} \alpha(x)$. Thus the proposition is desired. □

Proposition 3.3. $\lambda_0 \geqslant \alpha_0$.

Proof. Note that λ_0 and α_0 satisfy (2.6) and (3.2), respectively. It is easy to see that we obtain the classical heat equation in Ω with Dirichlet boundary condition

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after we subtract (2.3) by the following equation

$$\begin{cases} \dfrac{\partial u(a,x)}{\partial a} = -\mu(a,x)u(a,x), & (a,x) \in (0,a^+) \times \Omega, \\ u(a,x) = 0, & (a,x) \in (0,a^+) \times \partial\Omega, \\ u(0,x) = \phi \in X, & x \in \Omega. \end{cases} \tag{3.15}$$

Since the semigroup generated by a Laplace operator in X is strongly positive, we have $\mathcal{U}(0,a) \gg \Pi(0,a,x)$, i.e. $0 < \phi \in X \Rightarrow \mathcal{U}(0,a)\phi(x) > \Pi(0,a,x)\phi(x)$. It follows that $\mathcal{M}_\lambda \gg \mathcal{G}_\lambda$, then by the theory of positive operators, we have $r(\mathcal{M}_\lambda) \geqslant r(\mathcal{G}_\lambda)$. Hence the result is desired. □

4. Effects of diffusion rate

In this section, we study the effects of diffusion rate d on the principal eigenvalue λ_0 of $\mathcal{A}$. We would like to emphasize that unlike elliptic-type operators studied in the literature, the principal eigenvalue for our parabolic-type operator $\mathcal{A}$ does not admit the usual L^2 variational formula due to the presence of the age derivative in the operator. Thus proposition 2.7 remedies the situation and plays crucial roles in the following results. First we give a property on the continuous dependence λ_0 on the birth rate β and death rate μ. We write $\lambda_0(\beta,\mu)$ for λ_0 to highlight the dependence on β and μ.

Proposition 4.1. *$\lambda_0(\beta,\mu)$ is strictly increasing and decreasing with respect to β and μ, respectively.*

Proof. We have known that $\mathcal{M}_\lambda$ is compact and nonsupporting, then by Krein–Rutman theorem, see [**35**], $r(\mathcal{M}_\lambda)$ is the principal eigenvalue of $\mathcal{M}_\lambda$. Now, if $\beta_1 > \beta_2$, it follows that $\mathcal{M}_\lambda(\beta_1) > \mathcal{M}_\lambda(\beta_2)$ in the operator sense which implies that $r(\mathcal{M}_\lambda(\beta_1)) > r(\mathcal{M}_\lambda(\beta_2))$ by [**28**, theorem 4.3]. Thus by lemma 2.4, we have $\lambda_0(\beta_1,\mu) > \lambda_0(\beta_2,\mu)$. Similarly, when $\mu_1 > \mu_2$, since $\mathcal{U}(0,a)$ is strongly positive in X, we have $\mathcal{U}_{\mu_1}(0,a) < \mathcal{U}_{\mu_2}(0,a)$ in the operator sense, which implies $\mathcal{M}_\lambda(\mu_1) < \mathcal{M}_\lambda(\mu_2)$. Then it follows that $r(\mathcal{M}_\lambda(\mu_1)) < r(\mathcal{M}_\lambda(\mu_2))$, hence $\lambda_0(\beta,\mu_1) < \lambda_0(\beta,\mu_2)$ by lemma 2.4. In summary, the result is desired. □

In the next main theorem, we write λ_0^d for λ_0 to highlight the dependence on d.

Theorem 4.2. *The function $d \to \lambda_0^d$ is continuous on $(0,\infty)$ and satisfies*

$$\lambda_0^d \to \begin{cases} \alpha_0 & as \quad d \to 0^+, \\ -\infty & as \quad d \to \infty. \end{cases} \tag{4.1}$$

Proof. Since λ_0^d is an isolated eigenvalue, the continuity of $d \to \lambda_0^d$ follows from the classical perturbation theory, see [**22**]. For the limits, we first claim that for every

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$\epsilon > 0$, there exists $d_\epsilon > 0$ such that

$$\lambda_0^d \leqslant \alpha_0 + \epsilon, \quad \forall d \in (0, d_\epsilon). \tag{4.2}$$

Consider the following equation,

$$\begin{cases} \dfrac{\partial \psi(a,x)}{\partial a} = -(\alpha(x) + \mu(a,x))\psi(a,x), & (a,x) \in (0,a^+) \times \Omega, \\ \psi(a,x) = 0, & (a,x) \in (0,a^+) \times \partial\Omega, \\ \psi(0,x) = \int_0^{a^+} \beta(a,x)\psi(a,x)\,\mathrm{d}a, & x \in \Omega. \end{cases} \tag{4.3}$$

By proposition 3.2, we know that equation (4.3) has a positive continuously differentiable solution $\psi(a,x) = \mathrm{e}^{-\alpha(x)a}\Pi(0,a,x)\psi_0(x)$ provided $\psi_0 > 0$ with C^2 regularity in $x \in \Omega$. Moreover, by proposition 3.2 and some computations, we have

$$\nabla\psi = -\nabla\alpha\psi - \int_0^a \nabla\mu(s,\cdot)\,\mathrm{d}s\psi + \mathrm{e}^{-\alpha(x)a}\Pi(0,a,x)\nabla\psi_0,$$

which implies

$$\begin{aligned} \Delta\psi = -\Delta\alpha\psi - \nabla\alpha\cdot\nabla\psi - \int_0^a \Delta\mu(s,\cdot)\,\mathrm{d}s\psi - \int_0^a \nabla\mu(s,\cdot)\,\mathrm{d}s\cdot\nabla\psi \\ - \left(\nabla\alpha\cdot\nabla\psi_0 + \int_0^a \nabla\mu(s,\cdot)\,\mathrm{d}s\cdot\nabla\psi_0 - \Delta\psi_0\right)\mathrm{e}^{-\alpha(x)a}\Pi(0,a,x). \end{aligned} \tag{4.4}$$

It follows that

$$\max_{(a,x)\in[0,a^+]\times\overline{\Omega}} \Delta\psi \leqslant C(\|\alpha\|_{C^2(\overline{\Omega})}, \|\mu\|_{C^{1,2}([0,a^+]\times\overline{\Omega})}, \|\psi_0\|_{C^2(\overline{\Omega})}, a^+), \tag{4.5}$$

where C denotes a constant only depending on α, μ, ψ_0 and a^+. It follows that $\Delta\psi$ is bounded above in $[0,a^+]\times\overline{\Omega}$. Now since ψ is also bounded in $[0,a^+]\times\overline{\Omega}$, we can normalize ψ such that $\min_{(a,x)\in[0,a^+]\times\overline{\Omega}}\psi = 1$ by choosing a specific ψ_0. Thus for each $\epsilon > 0$, there exists $d_\epsilon > 0$ such that for each $d \in (0, d_\epsilon)$, there holds

$$\mathrm{d}\Delta\psi - \epsilon\psi \leqslant 0, \not\equiv 0.$$

Now consider the following auxiliary eigenvalue problem

$$\begin{cases} \dfrac{\partial \phi}{\partial a} - \mathrm{d}\Delta\phi + \mu\phi + (\alpha_0 + \epsilon)\phi = -\lambda\phi, & (a,x) \in (0,a^+) \times \Omega, \\ \phi(a,x) = 0, & (a,x) \in (0,a^+) \times \partial\Omega, \\ \phi(0,x) = \int_0^{a^+} \beta(a,x)\phi(a,x)\,\mathrm{d}a, & x \in \Omega. \end{cases} \tag{4.6}$$

Denote by $\underline{\lambda}_0$ the principal eigenvalue of (4.6). Then it is easy to check that ψ is a strict super-solution of (4.6). Indeed,

$$\frac{\partial\psi}{\partial a} - \mathrm{d}\Delta\psi + \mu\psi + (\alpha_0 + \epsilon)\psi = (\alpha_0 - \alpha(x))\psi + \epsilon\psi - \mathrm{d}\Delta\psi \geqslant 0, \not\equiv 0,$$

when $d \in (0, d_\epsilon)$ and we used $\alpha_0 \geqslant \alpha(x)$ by proposition 3.2. Now it follows that $\underline{\lambda}_0 < 0$ by proposition 2.7 which implies that $\lambda_0^d < \alpha_0 + \epsilon$. Next from proposition

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3.3, we have $\alpha_0 \leqslant \lambda_0^d$. Setting $d \to 0^+$, we find

$$\alpha_0 \leqslant \liminf_{d\to 0^+} \lambda_0^d \leqslant \limsup_{d\to 0^+} \lambda_0^d \leqslant \alpha_0 + \epsilon, \quad \forall \epsilon > 0,$$

which leads to $\lambda_0^d \to \alpha_0$ as $d \to 0^+$. Finally, to show $\lambda_0^d \to -\infty$ as $d \to \infty$, let us consider the eigenvalue problem of classical Laplace equation with Dirichlet boundary condition in Ω:

$$\begin{cases} -\Delta\phi = \lambda\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{4.7}$$

It is well known from the classical Laplace equation that the principal eigenvalue of $-\Delta$ with Dirichlet boundary condition exists and is positive. Let $\lambda^0 > 0$ be the principal eigenvalue of $-\Delta$ and $\Psi^0 \in X_+$ be an associated eigenfunction. Next let $\Psi^1 \in L^p(0, a^+)$ be the solution of the following equation (note that the existence is guaranteed by the theory of classical age-structured models, see [**50**]):

$$\begin{cases} \dfrac{\partial \Psi^1(a)}{\partial a} = -(\lambda^1 + \underline{\mu}(a))\Psi^1(a), \\ \Psi^1(0) = \displaystyle\int_0^{a^+} \overline{\beta}(a)\Psi^1(a)\,\mathrm{d}a, \end{cases} \tag{4.8}$$

where λ^1 satisfies

$$\int_0^{a^+} \overline{\beta}(a)\mathrm{e}^{-\lambda^1 a}\mathrm{e}^{-\int_0^a \underline{\mu}(s)\,\mathrm{d}s}\,\mathrm{d}a = 1.$$

Now let $\lambda_d = -d\lambda^0 + \lambda^1$ and $\Psi(a,x) = \Psi^0(x)\Psi^1(a)$. Consider the following auxiliary eigenvalue problem:

$$\begin{cases} \dfrac{\partial \phi}{\partial a} - \mathrm{d}\Delta\phi + \mu\phi + \lambda_d\phi = -\lambda\phi, & (a,x) \in (0,a^+) \times \Omega, \\ \phi(a,x) = 0, & (a,x) \in (0,a^+) \times \partial\Omega, \\ \phi(0,x) = \displaystyle\int_0^{a^+} \beta(a,x)\phi(a,x)\,\mathrm{d}a, & x \in \Omega. \end{cases} \tag{4.9}$$

Denote by $\overline{\lambda}_0$ the principal eigenvalue of (4.9). It is easy to see that

$$\begin{cases} \dfrac{\partial \Psi}{\partial a} - \mathrm{d}\Delta\Psi + \mu\Psi + \lambda_d\Psi \geqslant \Psi^0\left[\dfrac{\partial \Psi^1}{\partial a} + \underline{\mu}\Psi^1 + \lambda^1\Psi^1\right] & \\ \quad +d\lambda^0\Psi^0\Psi^1 - d\lambda^0\Psi^0\Psi^1 = 0, & (a,x) \in (0,a^+) \times \Omega, \\ \Psi(a,x) = \Psi^1(a)\Psi^0(x) = 0, & (a,x) \in (0,a^+) \times \partial\Omega, \\ \Psi(0,x) = \Psi^0(x)\displaystyle\int_0^{a^+} \overline{\beta}(a)\Psi^1(a)\,\mathrm{d}a & \\ \quad \geqslant \displaystyle\int_0^{a^+} \beta(a,x)\Psi(a,x)\,\mathrm{d}a, & x \in \Omega. \end{cases}$$

Thus Ψ is a strict super-solution of (4.9). It follows by proposition 2.7 that $\overline{\lambda}_0 < 0$, which implies that $\lambda_0^d < \lambda_d = -d\lambda^0 + \lambda^1$. Setting $d \to \infty$, we reach at $\lambda_0^d \to -\infty$ as $d \to \infty$. Hence the result is desired. □

Next we give the strict monotonicity of λ_0^d with respect to d under some specific conditions in the following.

THEOREM 4.3. *If* $\mu(a,x)=\mu_1(a)+\mu_2(x)$ *and* $\beta(a,x)\equiv\beta(a)$, *where* $\mu_1\in L^2(0,a^+), \mu_2\in L^2(\Omega)$ *and* $\beta\in L^2(0,a^+)$, *then* $d\to\lambda_0^d$ *is strictly decreasing.*

Proof. We write $\mathcal{A}=\mathcal{T}+L$, where

$$[Lv](x)=\mathrm{d}\Delta v(x)-\mu_2(x)v(x),\quad v\in W_2^2(\Omega),$$
$$[\mathcal{T}v](a)=-\frac{\partial v(a)}{\partial a}-\mu_1(a)v(a),\quad v\in W_2^1(0,a^+)$$

with

$$D(\mathcal{T}):=\left\{\phi(a)\big|\phi,\phi'\in L^2(0,a^+),\phi(0)=\int_0^{a^+}\beta(a)\phi(a)\,\mathrm{d}a\right\}.$$

Let $(\lambda_1^d(L),v_2)$ to be the principal eigenpair of $-L$. Then by the classical theory for second order elliptic PDEs or the usual L^2 variational structure, we know that $d\to\lambda_1^d(L)$ is strictly increasing. Now define $v_1(a)$ be the solution of characteristic equation $\mathcal{T}v_1=\lambda_1v_1$ (note that the existence of (λ_1,v_1) is guaranteed by the theory of age-structured models, see [**50**]). It follows that $\lambda_0^d=-\lambda_1^d(L)+\lambda_1$ is the principal eigenvalue of $\mathcal{A}$ with the principal eigenfunction $v_2(x)v_1(a)$. As $d\to\lambda_1^d(L)$ is strictly increasing, so $d\to\lambda_0^d$ is strictly decreasing. □

REMARK 4.4. Observe that in theorem 4.3 we define directly the principal eigenfunction v_2v_1 in a larger space $W_2^{1,2}((0,a^+)\times\Omega)$ which contains Z, since we do not need proposition 2.7 to prove the conclusion any more.

5. Global dynamics

In this section we are interested in the global dynamics of the following equation:

$$\begin{cases}\dfrac{\partial u(t,a,x)}{\partial t}+\dfrac{\partial u(t,a,x)}{\partial a}=\mathrm{d}\Delta u(t,a,x) & \\ \quad-\mu(a,x)u(t,a,x)+f(a,x,u(t,a,x)), & (t,a,x)\in(0,\infty)\times(0,a^+)\times\Omega,\\ u(t,0,x)=\displaystyle\int_0^{a^+}\beta(a,x)u(t,a,x)\,\mathrm{d}a, & (t,x)\in(0,\infty)\times\Omega,\\ u(t,a,x)=0, & (t,a,x)\in(0,\infty)\times(0,a^+)\times\partial\Omega,\\ u(0,a,x)=u_0(a,x), & (a,x)\in(0,a^+)\times\Omega,\end{cases}\tag{5.1}$$

where f is a KPP type of nonlinearity. Such type of equations naturally appears in some ecological problems when in addition to the dispersion of the individuals in the environment, the birth and death of individuals are also modelled, see [**14**, **19**, **29**, **30**].

On the nonlinear function f we assume that

(i) $f\in C([0,a^+]\times\overline{\Omega}\times[0,\infty))$ and is differentiable with respect to u;

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(ii) $f_u(\cdot, \cdot, 0)$ is Lipschitz;

(iii) $f(\cdot, \cdot, 0) \equiv 0$ and $f(a, x, u)/u$ is decreasing with respect to u;

(iv) There exists $K > 0$ such that $f(a, x, u) \leqslant 0$ for all $u \geqslant K$ and all a, x.

A typical example of such a nonlinearity is given as $f(a, x, s) = s(k(a, x) - s)$. In the following we will only consider this case for the convenience; namely,

$$\begin{cases} \dfrac{\partial u(a,x)}{\partial a} = \mathrm{d}\Delta u(a,x) - \mu(a,x)u(a,x) & \\ \quad +u(a,x)(k(a,x) - u(a,x)), & (a,x) \in (0, a^+) \times \Omega, \\ u(a,x) = 0, & (a,x) \in (0, a^+) \times \partial\Omega, \\ u(0,x) = \displaystyle\int_0^{a^+} \beta(a,x)u(a,x)\,\mathrm{d}a, & x \in \Omega, \end{cases} \tag{5.2}$$

where $k(a, x) \leqslant K$ for any $(a, x) \in [0, a^+] \times \Omega$.

Definition 5.1. We call u is a *super-solution (resp. sub-solution)* of (5.2) if $=$ is replaced by $\geqslant$ (resp. $\leqslant$) in (5.2).

Now let us prove the comparison principle for (5.2).

Lemma 5.2. *Let u be a strictly positive sub-solution of (5.2) and v be a strictly positive super-solution of (5.2). Then $u \leqslant v$ in $[0, a^+] \times \overline{\Omega}$.*

Proof. Let $\alpha_* := \sup\{\alpha > 0 : \alpha u \leqslant v \,\mathrm{in}\, [0, a^+] \times \overline{\Omega}\}$. By assumptions on u and v, the number α_* is well defined and positive. If $\alpha_* \geqslant 1$, then we are done. So we assume that $\alpha_* < 1$. Set $w := v - \alpha_* u$. Then $w \geqslant 0$ and there exists $(a_0, x_0) \in [0, a^+] \times \overline{\Omega}$ such that $w(a_0, x_0) = 0$. Obviously, w satisfies

$$\begin{aligned} \frac{\partial w(a,x)}{\partial a} &\geqslant \mathrm{d}\Delta w(a,x) - \mu(a,x)w(a,x) + v(k - v) - \alpha_* u(k - u) \\ &> \mathrm{d}\Delta w(a,x) - \mu(a,x)w(a,x) + v(k - v) - \alpha_* u(k - \alpha_* u) \\ &\quad \times (a,x) \in (0, a^+) \times \Omega, \end{aligned} \tag{5.3}$$

where we used $\alpha_* < 1$ in the second inequality. Considering the above inequality at (a_0, x_0) with $a_0 > 0$, we immediately deduce a contradiction by $\partial w(a_0, x_0)/\partial a \leqslant 0$ and $\Delta w(a_0, x_0) \geqslant 0$. If $a_0 = 0$, from the integral boundary condition, we have

$$\int_0^{a^+} \beta(a, x_0) w(a, x_0)\,\mathrm{d}a = 0,$$

which by the positivity of β and $w \geqslant 0$ implies $w(a, x_0) = 0$ for all $a \in [0, a^+]$. Then integrating (5.3) from 0 to a^+ at $x = x_0$, we still have the contradiction as above. Thus $\alpha_* \geqslant 1$ and the proof is completed. □

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We denote by λ_0^k the eigenvalue of the following linearized operator $\mathcal{A}_k$ which is obtained by linearizing (5.2) at $u = 0$:

$$\begin{cases} \dfrac{\partial u(a,x)}{\partial a} = \mathrm{d}\Delta u(a,x) - \mu(a,x)u(a,x) + k(a,x)u(a,x), & (a,x) \in (0,a^+) \times \Omega, \\ u(a,x) = 0, & (a,x) \in (0,a^+) \times \partial\Omega, \\ u(0,x) = \displaystyle\int_0^{a^+} \beta(a,x)u(a,x)\,\mathrm{d}a, & x \in \Omega. \end{cases} \tag{5.4}$$

Proposition 5.3. *There exists at a unique positive nontrivial solution $u^*(a,x)$ of (5.2) when $\lambda_0^k > 0$.*

Proof. First we construct a pair of sub-solution and super-solution for equation (5.2) to establish the existence. Taking $\underline{u}(a,x) := \epsilon\phi(a,x)$, where ϕ is the positive eigenfunction associated with λ_0^k. Let us check that it is a sub-solution of (5.2):

$$\frac{\partial \underline{u}(a,x)}{\partial a} - \mathrm{d}\Delta \underline{u}(a,x) + \mu(a,x)\underline{u}(a,x) - \underline{u}(a,x)k(a,x) \leqslant -\underline{u}^2(a,x),$$

provided $\epsilon\phi(a,x) \leqslant \lambda_0^k$ which is true by taking $\epsilon > 0$ sufficiently small if $\lambda_0^k > 0$, since $\phi(a,x)$ is bounded. Furthermore,

$$\underline{u}(0,x) = \epsilon\phi(0,x) = \epsilon\int_0^{a^+} \beta(a,x)\phi(a,x)\,\mathrm{d}a = \int_0^{a^+} \beta(a,x)\underline{u}(a,x)\,\mathrm{d}a$$

holds for any $x \in \Omega$ and $\underline{u}(a,x) = \epsilon\phi(a,x) = 0$ for $x \in \partial\Omega$. Next, we construct a super-solution of (5.2), motivated by Delgado *et al.* [**7**, theorem 14]. Define

$$F_q(a) := qa - \int_0^a \underline{\mu}(s)\,\mathrm{d}s, \quad q \in \mathbb{R},$$

and take $q \in \mathbb{R}$ sufficiently large so that

$$\int_0^{a^+} \mathrm{e}^{F_q(a)}\,\mathrm{d}a \geqslant \frac{1}{\hat{\beta}}, \quad q > K, \tag{5.5}$$

where $\hat{\beta} := \sup_{(a,x)\in[0,a^+]\times\overline{\Omega}} \beta(a,x)$. Consider the function

$$G(x) := \int_0^{a^+} \frac{\mathrm{e}^{F_q(a)}}{1 + x\int_0^a \mathrm{e}^{F_q(s)}\,\mathrm{d}s}\,\mathrm{d}a.$$

Observe that G is a continuous function and by (5.5) we have that

$$\lim_{x\to 0} G(x) \geqslant \frac{1}{\hat{\beta}}, \quad \lim_{x\to\infty} G(x) = 0.$$

So there exists $y_0 > 0$ such that $G(y_0) = 1/\hat{\beta}$, i.e.

$$\int_0^{a^+} \frac{\mathrm{e}^{F_q(a)}}{1 + y_0\int_0^a \mathrm{e}^{F_q(s)}\,\mathrm{d}s}\,\mathrm{d}a = \frac{1}{\hat{\beta}}. \tag{5.6}$$

Define $Y(a)$ the unique solution of the differential equation

$$\frac{\partial y(a)}{\partial a} + \underline{\mu}(a)y = qy - y^2, \quad y(0) = y_0,$$

where y_0 is defined by (5.6). Solving the above equation, we obtain that

$$Y(a) = \frac{\mathrm{e}^{F_q(a)}}{1/y_0 + \int_0^a \mathrm{e}^{F_q(s)}\,\mathrm{d}s}. \tag{5.7}$$

Take $\overline{u}(a,x) := MY(a)$, with M a positive large constant. It can be proven that $\overline{u}$ is a super-solution of (5.2) for q large. Indeed, $\overline{u} > 0$ on $\partial\Omega$ and

$$\frac{\partial \overline{u}}{\partial a} - \mathrm{d}\Delta\overline{u} + \mu\overline{u} - k\overline{u} \geqslant -\overline{u}^2,$$

provided that $M \geqslant 1$ and

$$\int_0^{a^+} \beta(a,x)\overline{u}(a,x)\,\mathrm{d}a \leqslant My_0\hat{\beta}\int_0^{a^+} \frac{\mathrm{e}^{F_q(a)}}{1 + y_0\int_0^a \mathrm{e}^{F_q(s)}\,\mathrm{d}s}\,\mathrm{d}a = My_0 = \overline{u}(0,x).$$

Now it is clear that we can choose $\epsilon > 0$ and $M > 0$ such that $\underline{u} \leqslant \overline{u}$. Then by a basic iterative scheme we obtain the existence of a positive nontrivial solution u of (5.2), see [**4**, theorem A.1] or [**7**, theorem 14]. For the uniqueness, let u and v be two nonnegative bounded solutions of (5.2). Since they are bounded and strictly positive, the following quantity is well defined:

$$\gamma^* := \inf\{\gamma > 0 | \gamma u \geqslant v\}.$$

We claim that $\gamma^* \leqslant 1$. Indeed, assume by contradiction that $\gamma^* > 1$. Based on $\gamma^* u \geqslant v$, we consider two cases: (i) $\gamma^* u \geqslant v, \gamma^* u \not\equiv v$. Then from the initial integral boundary condition with positivity of β everywhere, we have $\gamma^* u_0 := \gamma^* u(0,x;u_0) \geqslant v(0,x;v_0) =: v_0$ and $\gamma^* u_0 \not\equiv v_0$. Now consider the following reaction-diffusion problem

$$\begin{aligned}\frac{\partial u}{\partial a} &= \mathrm{d}\Delta u(a,x) - \mu(a,x)u(a,x) + u(a,x)(k(a,x) - u(a,x)),\\ &\quad (a,x) \in (0,a^+) \times \Omega.\end{aligned} \tag{5.8}$$

By [**13**, theorem 2.2], solutions of equation (5.8) have strong monotone property, i.e. for $\phi, \psi \in X_+$ with $\phi \geqslant \psi, \phi \not\equiv \psi, u(a,x;\phi) \gg u(a,x;\psi), a > 0$ at which both $u(a,x;\phi)$ and $u(a,x;\psi)$ exist, where u is the solution of (5.8). Then we have

$$u(a,x;\gamma^* u_0) \gg u(a,x;v_0). \tag{5.9}$$

On the other hand, let $w(a,x) = \gamma^* u(a,x;u_0)$. Then $w(0,x) = \gamma^* u_0$ and

$$\begin{aligned}\frac{\partial w}{\partial a} &= \mathrm{d}\Delta w(a,x) - \mu(a,x)w(a,x) + w(a,x)[k(a,x) - u(a,x;u_0)]\\ &= \mathrm{d}\Delta w(a,x) - \mu(a,x)w(a,x) + w(a,x)[k(a,x) - w(a,x)]\\ &\quad + w(a,x)[w(a,x) - u(a,x;u_0)]\\ &\geqslant \mathrm{d}\Delta w(a,x) - \mu(a,x)w(a,x) + w(a,x)[k(a,x) - w(a,x)] + \delta_0\end{aligned}$$

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for some $\delta_0 > 0$ and $0 \leqslant a < a^+$ since $\gamma^* > 1$. Now by the comparison principle, we have

$$\gamma^* u(a, x; u_0) \geqslant u(a, x; \gamma^* u_0) \tag{5.10}$$

for some $0 < a < a^+$. Now combing (5.9) and (5.10), we have

$$\gamma^* u(a, x; u_0) \gg u(a, x; v_0),$$

which is a contradiction with the definition of γ^*. Hence it follows that $\gamma^* \leqslant 1$. Next let us consider case (ii) $\gamma^* u \equiv v$. But

$$\begin{aligned} 0 &= -\frac{\partial v}{\partial a} + \mathrm{d}\Delta v(a, x) - \mu v + v(k - v) \\ &= -\frac{\partial(\gamma^* u)}{\partial a} + \mathrm{d}\Delta \gamma^* u(a, x) - \gamma^* \mu u + \gamma^* u(k - \gamma^* u) \\ &= -\gamma^* u(k - u) + \gamma^* u(k - \gamma^* u) < 0, \end{aligned} \tag{5.11}$$

which is a contradiction due to $\gamma^* > 1$. Thus we still have $\gamma^* \leqslant 1$. In summary we conclude as a consequence $u \geqslant v$. Now switch the role u and v in the above argument, we also have $v \geqslant u$, which shows the uniqueness of the solution. □

THEOREM 5.4. *The nontrivial equilibrium u^* is stable in the sense of $u(t, a, x) \to u^*(a, x)$ pointwise as $t \to \infty$ if $\lambda_0^k > 0$, where $u(t, a, x)$ is a solution of (5.1) with initial data $u_0 \geqslant 0$ and $u_0 \neq 0$.*

Proof. The existence of a solution $u(t, a, x)$ for (5.1) defined for all time t follows from a standard semigroup method by writing equation (5.1) as an abstract Cauchy problem and based on the Lipschitz assumption on f, see [**1**], thus we omit it. Next, the proof of stability is motivated by [**4**, theorem 1.7]. Note $u_0 \geqslant 0$ and $u_0 \neq 0$, using the comparison principle, there exists a positive constant δ such that $u(1, a, x) > \delta$ in $[0, a^+] \times \Omega$. Since $\lambda_0^k > 0$, we can build a bounded continuous function $\underline{u}$ so that $\epsilon \underline{u}$ is a sub-solution of (5.1) for ϵ small enough. Since $u(1, a, x) \geqslant \delta$ and $\underline{u}$ is bounded, by choosing ϵ smaller if necessary we achieve also that $\epsilon \underline{u} \leqslant u(1, a, x)$. Now let us denote $\underline{U}(t, a, x)$ the solution of (5.1) with initial data $\epsilon \underline{u}$. By construction, using a standard argument, $\underline{U}(t, a, x)$ is a non-decreasing function of the time and $\underline{U}(t, a, x) \leqslant u(t + 1, a, x)$. On the other hand, $MY(a)$ which is defined in the proof in proposition 5.3 is a super-solution of (5.1) and u_0 is bounded, we have also $u(t, a, x) \leqslant \overline{U}(t, a, x)$ if necessary choosing M large enough, where $\overline{U}(t, a, x)$ denotes the solution of (5.1) with initial data $\overline{U}(0, a, x) = MY(a) \geqslant u_0$. A standard argument using the comparison principle shows that $\overline{U}$ is a non-increasing function of t. Thus we have for all time t,

$$\epsilon \underline{u} \leqslant \underline{U}(t, a, x) \leqslant u(t + 1, a, x) \leqslant \overline{U}(t + 1, a, x).$$

Since $\underline{U}(t, a, x)$ (respectively $\overline{U}(t, a, x)$) is an uniformly bounded monotonic function of t, $\underline{U}$ (resp. $\overline{U}$) converges pointwise to $\underline{p}$ (resp. $\overline{p}$) which is a solution of (5.2). From $\underline{U} \neq 0$, using the uniqueness of a non-trivial solution of (5.2), we deduce that $\underline{p} \equiv \overline{p} = u^* \neq 0$ and therefore, $u(t, a, x) \to u^*$ pointwise as $t \to \infty$. □

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Finally, let us give a similar result on the long-time dynamics of (5.1) in terms of diffusion rate d.

Theorem 5.5. *Equation* (5.1) *admits a unique equilibrium* u^* *that is stable for each* $0 < d \ll 1$ *if* $\alpha_2 > 0$, *where* α_2 *satisfies*

$$\max_{x\in\overline{\Omega}} \int_0^{a^+} \beta(a,x) e^{-\alpha_2 a} e^{\int_0^a (k(s,x)-\mu(s,x))\, ds}\, da = 1. \tag{5.12}$$

Proof. Note that the linearized operator $\mathcal{A}_k$ defined in (5.4) also satisfies all the properties of $\mathcal{A}$ discussed in § 4. Then by theorem 4.2, $\lambda_0^d(\mathcal{A}_k) > 0$ for all $0 < d \ll 1$ if $\alpha_2 > 0$, then the result follows from proposition 5.3 and theorem 5.4. □

At the end of this section, we investigate the asymptotic behaviour of the equilibrium u^* with respect to diffusion rate d. In order to highlight the dependence of u^* on d we denote u^* by u_d^*. Before proceeding, we first give a lemma on the solution of (5.2) without random diffusion, that is

$$\begin{cases} \dfrac{\partial v(a,x)}{\partial a} = -\mu(a,x)v(a,x) + v(a,x)(k(a,x) - v(a,x)), & (a,x) \in (0,a^+) \times \Omega, \\ v(a,x) = 0, & (a,x) \in (0,a^+) \times \partial\Omega, \\ v(0,x) = \displaystyle\int_0^{a^+} \beta(a,x)v(a,x)\,\mathrm{d}a, & x \in \Omega. \end{cases} \tag{5.13}$$

Lemma 5.6. *Suppose* $\alpha_2 > 0$, *then for each* $x \in \Omega$, *equation (5.13) has a unique positive solution, denoted by* $v^*(a,x)$, *which is continuous in* x, *where* α_2 *satisfies (5.12).*

Proof. First imitating the proofs in propositions 3.1 and 3.2, we can show that for such α_2 satisfying (5.12), there exists a positive solution v to the following equation

$$\begin{cases} \dfrac{\partial v(a,x)}{\partial a} = -\alpha_2 v(a,x) - (\mu(a,x) - k(a,x))v(a,x), & (a,x) \in (0,a^+) \times \Omega, \\ v(a,x) = 0, & (a,x) \in (0,a^+) \times \partial\Omega, \\ v(0,x) = \displaystyle\int_0^{a^+} \beta(a,x)v(a,x)\,\mathrm{d}a, & x \in \Omega. \end{cases} \tag{5.14}$$

Then it is easy to see by the proof of proposition 5.3 that $\underline{v} := \epsilon v$ is a sub-solution of (5.13) when $\alpha_2 > 0$ by taking $\epsilon > 0$ sufficiently small. Meanwhile, $\overline{v} := MY(a)$ defined in (5.7) for M sufficiently large is also a super-solution of (5.13). Now it is clear that we can choose $\epsilon > 0$ and $M > 0$ such that $\underline{v} \leqslant \overline{v}$. Then by a basic iterative scheme as in proposition 5.3 we obtain the existence of a positive nontrivial solution v^* of (5.13). □

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Theorem 5.7. *If $\alpha_2 > 0$ and v^* is from lemma 5.6, we have the following asymptotic result:*

$$\lim_{d\to 0^+} u_d^*(a,x) = v^*(a,x) \quad \textit{uniformly in} \quad (a,x)\in[0,a^+]\times\overline{\Omega}, \tag{5.15}$$

where u_d^* is given in theorem 5.5.

Proof. The proof is motivated by [**36**, theorem 6.3]. We claim that for each $0 < \epsilon \ll 1$, there exists a d_ϵ such that for each $d\in(0,d_\epsilon)$ there holds

$$v^*(a,x) - \epsilon \leqslant u_d^*(a,x) \leqslant v^*(a,x)+\epsilon, \quad (a,x)\in[0,a^+]\times\overline{\Omega}.$$

Let us prove the lower bound, the upper bound follows from similar arguments. Let $0<\epsilon\ll 1$. Since $\min_{[0,a^+]\times\overline{\Omega}} v^* > 0$, there exists $\delta=\delta(\epsilon)>0$ such that

$$v(a,x) := (1-\delta)v^*(a,x) \geqslant v^*(a,x)-\epsilon > 0, \quad (a,x)\in[0,a^+]\times\overline{\Omega}.$$

Note that for each $(a,x)\in(0,a^+)\times\Omega$,

$$\begin{aligned}
&-\frac{\partial v(a,x)}{\partial a} + \mathrm{d}\Delta v(a,x) - \mu(a,x)v(a,x) + v(a,x)(k(a,x)-v(a,x))\\
&= -(1-\delta)\frac{\partial v^*(a,x)}{\partial a} + d(1-\delta)\Delta v^*(a,x) - (1-\delta)\mu(a,x)v^*(a,x)\\
&\quad + (1-\delta)v^*(a,x)(k(a,x)-v^*(a,x)) + v(t,x)(k(a,x)-v(a,x))\\
&\quad - (1-\delta)v^*(a,x)(k(a,x)-v^*(a,x))\\
&= d(1-\delta)\Delta v^*(a,x) + v(t,x)(k(a,x)-v(a,x))\\
&- (1-\delta)v^*(a,x)(k(a,x)-v^*(a,x)),
\end{aligned}$$

and we have seen in (4.5) that $\mathrm{d}\Delta v^*\to 0$ as $d\to 0^+$ uniformly in $(a,x)\in[0,a^+]\times\overline{\Omega}$. Since for each $(a,x)\in[0,a^+]\times\overline{\Omega}$,

$$v(t,x)(k(a,x)-v(a,x)) - (1-\delta)v^*(a,x)(k(a,x)-v^*(a,x)) > 0,$$

there holds

$$\inf_{(a,x)\in[0,a^+]\times\overline{\Omega}} v(t,x)(k(a,x)-v(a,x)) - (1-\delta)v^*(a,x)(k(a,x)-v^*(a,x)) > 0.$$

Hence, there exists $d_\epsilon>0$ such that for each $d\in(0,d_\epsilon)$, there holds

$$\begin{aligned}
\frac{\partial v(a,x)}{\partial a} &< \mathrm{d}\Delta v(a,x) - \mu(a,x)v(a,x) + v(a,x)(k(a,x)\\
&\quad - v(a,x)), \quad (a,x)\in(0,a^+)\times\Omega.
\end{aligned} \tag{5.16}$$

It remains to show that for each $d\in(0,d_\epsilon)$, there holds $v(a,x)\leqslant u_d^*(a,x)$ for all $(a,x)\in[0,a^+]\times\overline{\Omega}$. To do this, let us fix any $d\in(0,d_\epsilon)$ and define

$$\alpha_* = \inf\{\alpha>0 : v(a,x)\leqslant \alpha u_d^*(a,x)\,\text{for all}\,(a,x)\in[0,a^+]\times\overline{\Omega}\}.$$

Since $\min_{[0,a^+]\times\overline{\Omega}} u_d^* > 0$ and $v(a,x)$ is bounded, α_* is well-defined and positive. Due to the continuity of $v(a,x)$ and $u_d^*(a,x)$, there holds $v(a,x)\leqslant \alpha_* u_d^*(a,x)$

for all $(a, x) \in [0, a^+] \times \overline{\Omega}$. Moreover, there exists $(a_0, x_0) \in [0, a^+] \times \overline{\Omega}$ such that $v(a_0, x_0) = \alpha_* u_d^*(a_0, x_0)$. Clearly, if $\alpha_* \leqslant 1$, we are done. Therefore, let us assume $\alpha_* > 1$. By (5.16) and the equation satisfied by $u_d^*(a, x)$, we see that $w(a, x) := v(a, x) - \alpha_* u_d^*(a, x)$ satisfies

$$\begin{aligned} \frac{\partial w(a,x)}{\partial a} &< \mathrm{d}\Delta w(a,x) - \mu(a,x)w(a,x) + v(a,x)(k(a,x) - v(a,x)) \\ &\quad - \alpha_* u_d^*(a,x)(k(a,x) - u_d^*(a,x)) \end{aligned} \tag{5.17}$$

for all $(a, x) \in (0, a^+) \times \Omega$. However, if $a_0 \in (0, a^+)$, we have $\partial w(a_0, x_0)/\partial a \geqslant 0, \mathrm{d}\Delta w(a_0, x_0) \leqslant 0$ and

$$v(a_0, x_0)(k(a_0, x_0) - v(a_0, x_0)) - \alpha_* u_d^*(a_0, x_0)(k(a_0, x_0) - u_d^*(a_0, x_0)) < 0,$$

where we used $\alpha_* > 1$ so that $\alpha_* u_d^*(a_0, x_0) > u_d^*(a_0, x_0)$ and hence we arrive at

$$\begin{aligned} \frac{\partial w(a_0,x_0)}{\partial a} &< \mathrm{d}\Delta w(a_0,x_0) - \mu(a_0,x_0)w(a_0,x_0) \\ &\quad + v(a_0,x_0)(k(a_0,x_0) - v(a_0,x_0)) - \alpha_* u_d^*(a_0,x_0)(k(a_0,x_0) - u_d^*(a_0,x_0)), \end{aligned}$$

which leads to a contradiction. Hence $\alpha_* \leqslant 1$ which is desired. Now if $a_0 = 0$, it follows from the integral boundary condition that

$$\int_0^{a^+} \beta(a, x_0) w(a, x_0)\, \mathrm{d}a = 0.$$

The positivity of β implies $w(a, x_0) \equiv 0$ for all $a \in [0, a^+]$. Then integrating (5.17) from 0 to a^+ at $x = x_0$, we still have the contradiction as above. Thus $\alpha_* \leqslant 1$ and the proof is completed. □

6. Discussion

In this paper, we studied the spectrum theory for age-structured models with random diffusion. We provided an equivalent characteristic for the principal eigenvalue, strong maximum principle and a positive strict super-solution. Then we used it to examine the effects of diffusion rate on the principal eigenvalue. Finally, we investigated the existence, uniqueness and stability of a diffusive age-structured model with KPP type of nonlinearity and verified that the principal eigenvalue being zero is a threshold. It is also interesting to study the effects of advection term on the principal eigenvalue if the equation contains an advection term. In addition, we also studied the principal spectral theory, limiting properties and global dynamics for age-structured models with nonlocal diffusion, see [**20**]. We expect that analysis on the principal eigenvalue and constructions of sub/super-solutions can be applied to study the existence of travelling wave solutions and spreading speeds of age-structured models with random diffusion, see [**9**–**12**]. We leave these for future consideration.

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